

Lecture Notes

on

Binomial Heap



July 2020
(Be safe and stay at home)

Priority queue data type

A min-oriented priority queue supports the following core operations:

- MAKE-HEAP(): create an empty heap.
- INSERT(H, x): insert an element x into the heap.
- EXTRACT-MIN(H): remove and return an element with the smallest key.
- DECREASE-KEY(H, x, k): decrease the key of element x to k .

The following operations are also useful:

- IS-EMPTY(H): is the heap empty?
- FIND-MIN(H): return an element with smallest key.
- DELETE(H, x): delete element x from the heap.
- MELD(H_1, H_2): replace heaps H_1 and H_2 with their union.

Note. Each element contains a key (duplicate keys are permitted) from a totally-ordered universe.

Priority queue applications

Applications.

- A* search.
- Heapsort.
- Online median.
- Huffman encoding.
- Prim's MST algorithm.
- Discrete event-driven simulation.
- Network bandwidth management.
- Dijkstra's shortest-paths algorithm.
- ...



<http://younginc.site11.com/source/5895/fos0092.html>



SECTION 2.4

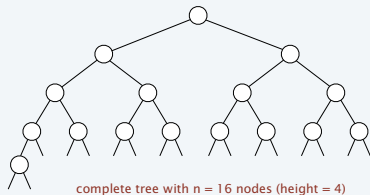
PRIORITY QUEUES

- ▶ *binary heaps*
- ▶ *d-ary heaps*
- ▶ *binomial heaps*
- ▶ *Fibonacci heaps*

Complete binary tree

Binary tree. Empty or node with links to two disjoint binary trees.

Complete tree. Perfectly balanced, except for bottom level.



Property. Height of complete binary tree with n nodes is $\lfloor \log_2 n \rfloor$.

Pf. Height increases (by 1) only when n is a power of 2. ■

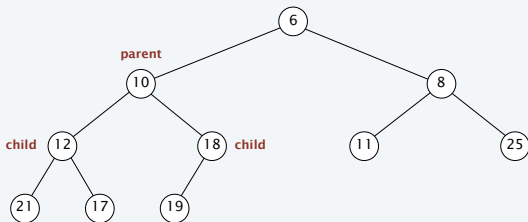
A complete binary tree in nature



Binary heap

Binary heap. Heap-ordered complete binary tree.

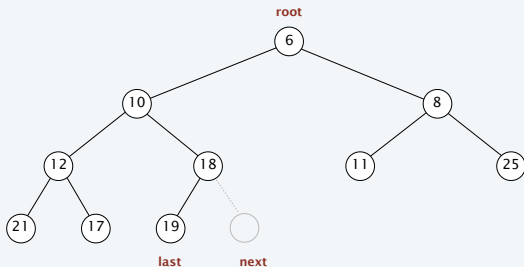
Heap-ordered tree. For each child, the key in child \geq key in parent.



Explicit binary heap

Pointer representation. Each node has a pointer to parent and two children.

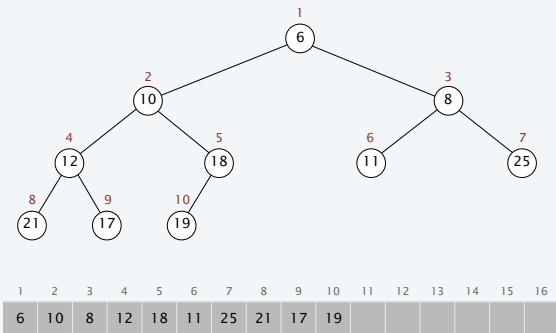
- Maintain number of elements n .
- Maintain pointer to root node.
- Can find pointer to last node or next node in $O(\log n)$ time.



Implicit binary heap

Array representation. Indices start at 1.

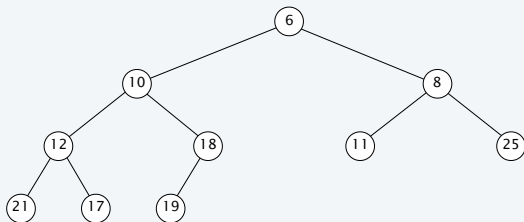
- Take nodes in **level** order.
- Parent of node at k is at $\lfloor k / 2 \rfloor$.
- Children of node at k are at $2k$ and $2k + 1$.



Binary heap demo

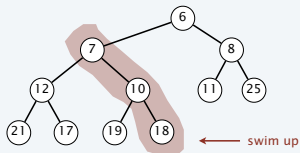
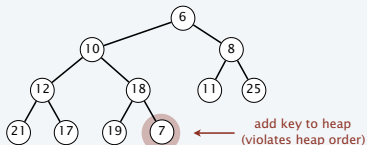


heap ordered



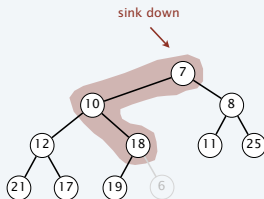
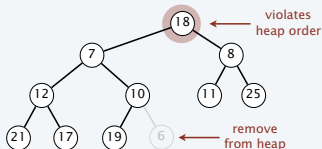
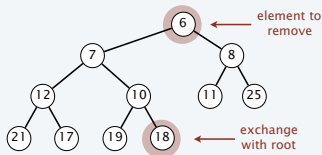
Binary heap: insert

Insert. Add element in new node at end; repeatedly exchange new element with element in its parent until heap order is restored.



Binary heap: extract the minimum

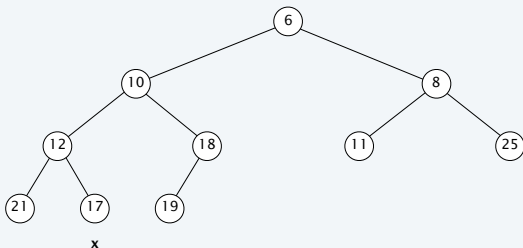
Extract min. Exchange element in root node with last node; repeatedly exchange element in root with its smaller child until heap order is restored.



Binary heap: decrease key

Decrease key. Given a **handle** to node, repeatedly exchange element with its parent until heap order is restored.

decrease key of node **x** to 11



Binary heap: analysis

Theorem. In an **implicit** binary heap, any sequence of m INSERT, EXTRACT-MIN, and DECREASE-KEY operations with n INSERT operations takes $O(m \log n)$ time.

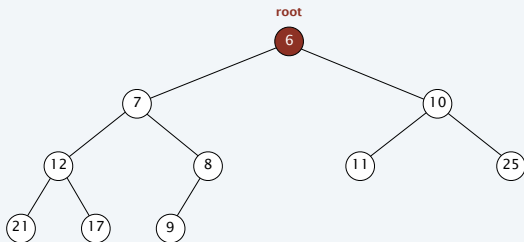
Pf.

- Each heap op touches nodes only on a path from the root to a leaf; the height of the tree is at most $\log_2 n$.
- The total cost of expanding and contracting the arrays is $O(n)$. ■

Theorem. In an **explicit** binary heap with n nodes, the operations INSERT, DECREASE-KEY, and EXTRACT-MIN take $O(\log n)$ time in the worst case.

Binary heap: find-min

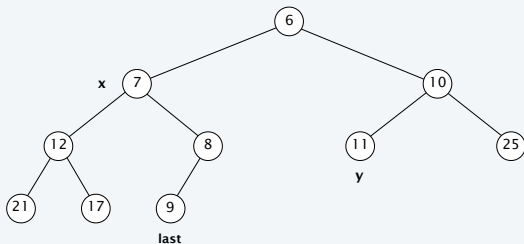
Find the minimum. Return element in the root node.



Binary heap: delete

Delete. Given a **handle** to a node, exchange element in node with last node; either swim down or sink up the node until heap order is restored.

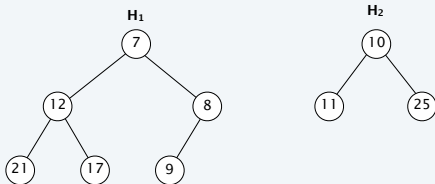
delete node **x** or **y**



Binary heap: meld

Meld. Given two binary heaps H_1 and H_2 , merge into a single binary heap.

Observation. No easy solution: $\Omega(n)$ time apparently required.

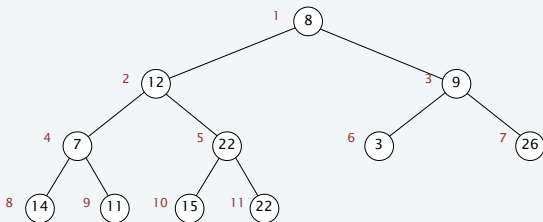


Binary heap: heapify

Heapify. Given n elements, construct a binary heap containing them.

Observation. Can do in $O(n \log n)$ time by inserting each element.

Bottom-up method. For $i = n$ to 1, repeatedly exchange the element in node i with its smaller child until subtree rooted at i is heap-ordered.



8	12	9	7	22	3	26	14	11	15	22
1	2	3	4	5	6	7	8	9	10	11


Binary heap: heapify

Theorem. Given n elements, can construct a binary heap containing those n elements in $O(n)$ time.

Pf.

- There are at most $\lceil n / 2^{h+1} \rceil$ nodes of height h .
- The amount of work to sink a node is proportional to its height h .
- Thus, the total work is bounded by:

$$\sum_{h=0}^{\lfloor \log_2 n \rfloor} \lceil n / 2^{h+1} \rceil h \leq \sum_{h=0}^{\lfloor \log_2 n \rfloor} n h / 2^h \leq 2n \blacksquare$$

$$\sum_{i=1}^k \frac{i}{2^i} = 2 - \frac{k}{2^k} - \frac{1}{2^{k-1}} \leq 2$$


Corollary. Given two binary heaps H_1 and H_2 containing n elements in total, can implement MELD in $O(n)$ time.

Priority queues performance cost summary

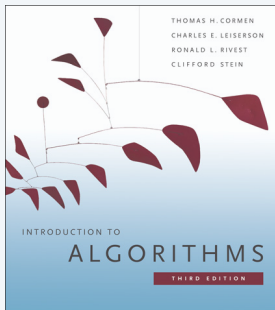
operation	linked list	binary heap
MAKE-HEAP	$O(1)$	$O(1)$
ISEMPTY	$O(1)$	$O(1)$
INSERT	$O(1)$	$O(\log n)$
EXTRACT-MIN	$O(n)$	$O(\log n)$
DECREASE-KEY	$O(1)$	$O(\log n)$
DELETE	$O(1)$	$O(\log n)$
MELD	$O(1)$	$O(n)$
FIND-MIN	$O(n)$	$O(1)$

Priority queues performance cost summary

Q. Reanalyze so that EXTRACT-MIN and DELETE take $O(1)$ amortized time?

operation	linked list	binary heap	binary heap †
MAKE-HEAP	$O(1)$	$O(1)$	$O(1)$
ISEMPTY	$O(1)$	$O(1)$	$O(1)$
INSERT	$O(1)$	$O(\log n)$	$O(\log n)$
EXTRACT-MIN	$O(n)$	$O(\log n)$	$O(1)^\dagger$
DECREASE-KEY	$O(1)$	$O(\log n)$	$O(\log n)$
DELETE	$O(1)$	$O(\log n)$	$O(1)^\dagger$
MELD	$O(1)$	$O(n)$	$O(n)$
FIND-MIN	$O(n)$	$O(1)$	$O(1)$

† amortized



CHAPTER 6 (2ND EDITION)

PRIORITY QUEUES

- ▶ *binary heaps*
- ▶ *d-ary heaps*
- ▶ *binomial heaps*
- ▶ *Fibonacci heaps*

Priority queues performance cost summary

operation	linked list	binary heap	d-ary heap
MAKE-HEAP	$O(1)$	$O(1)$	$O(1)$
ISEMPTY	$O(1)$	$O(1)$	$O(1)$
INSERT	$O(1)$	$O(\log n)$	$O(\log_d n)$
EXTRACT-MIN	$O(n)$	$O(\log n)$	$O(d \log_d n)$
DECREASE-KEY	$O(1)$	$O(\log n)$	$O(\log_d n)$
DELETE	$O(1)$	$O(\log n)$	$O(d \log_d n)$
MELD	$O(1)$	$O(n)$	$O(n)$
FIND-MIN	$O(n)$	$O(1)$	$O(1)$

Goal. $O(\log n)$ INSERT, DECREASE-KEY, EXTRACT-MIN, and **MELD**.

mergeable heap

Binomial heaps

Programming
Techniques

S.L. Graham, R.L. Rivest
Editors

A Data Structure for Manipulating Priority Queues

Jean Vuillemin
Université de Paris-Sud

A data structure is described which can be used for representing a collection of priority queues. The primitive operations are insertion, deletion, union, update, and search for an item of earliest priority.

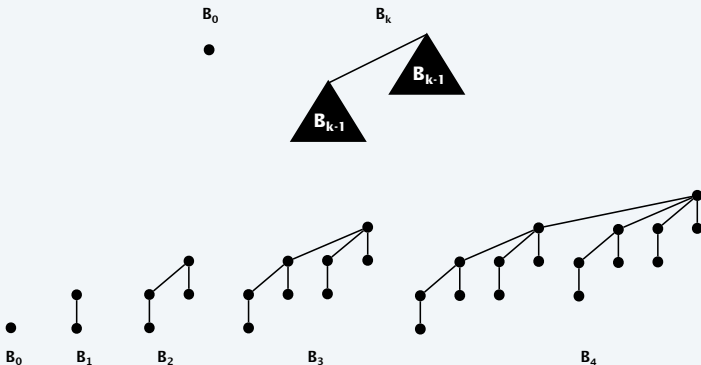
Key Words and Phrases: data structures, implementation of set operations, priority queues, mergeable heaps, binary trees

CR Categories: 4.34, 5.24, 5.25, 5.32, 8.1

Binomial tree

Def. A binomial tree of order k is defined recursively:

- Order 0: single node.
- Order k : one binomial tree of order $k - 1$ linked to another of order $k - 1$.

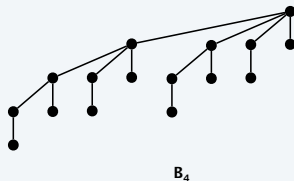
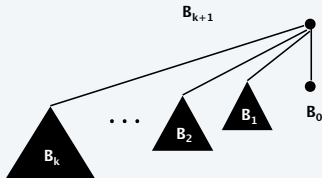


Binomial tree properties

Properties. Given an order k binomial tree B_k ,

- Its height is k .
- It has 2^k nodes.
- It has $\binom{k}{i}$ nodes at depth i .
- The degree of its root is k .
- Deleting its root yields k binomial trees B_{k-1}, \dots, B_0 .

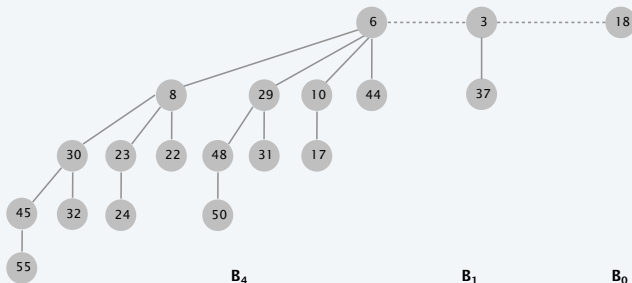
Pf. [by induction on k]



Binomial heap

Def. A **binomial heap** is a sequence of binomial trees such that:

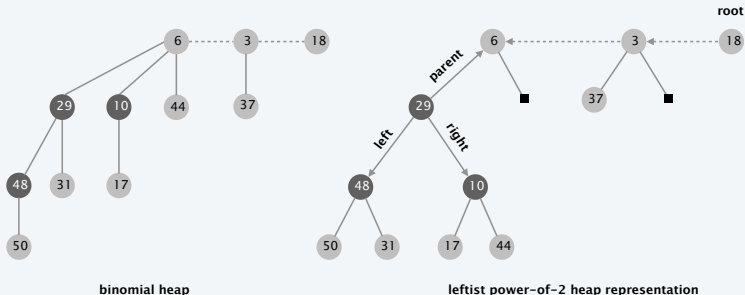
- Each tree is heap-ordered.
- There is either 0 or 1 binomial tree of order k .



Binomial heap representation

Binomial trees. Represent trees using left-child, right-sibling pointers.

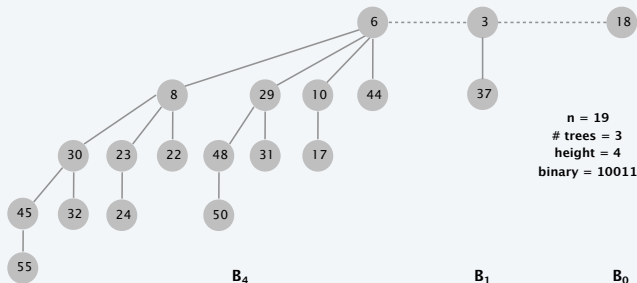
Roots of trees. Connect with singly-linked list, with degrees decreasing from left to right.



Binomial heap properties

Properties. Given a binomial heap with n nodes:

- The node containing the min element is a root of B_0, B_1, \dots , or B_k .
- It contains the binomial tree B_i iff $b_i = 1$, where $b_k \cdot b_2 b_1 b_0$ is binary representation of n .
- It has $\leq \lfloor \log_2 n \rfloor + 1$ binomial trees.
- Its height $\leq \lfloor \log_2 n \rfloor$.

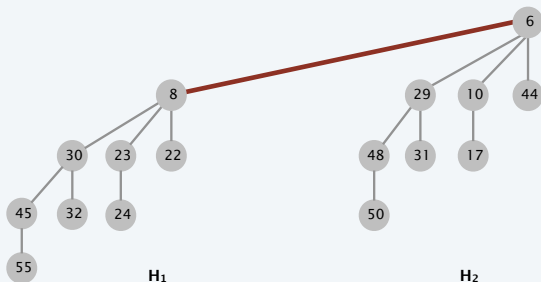


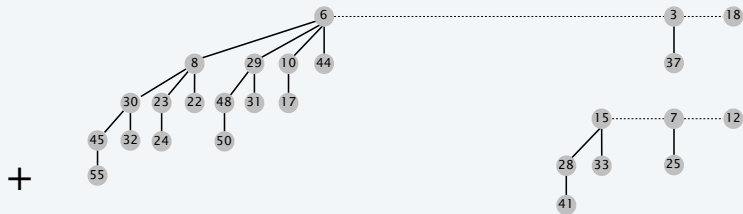
Binomial heap: meld

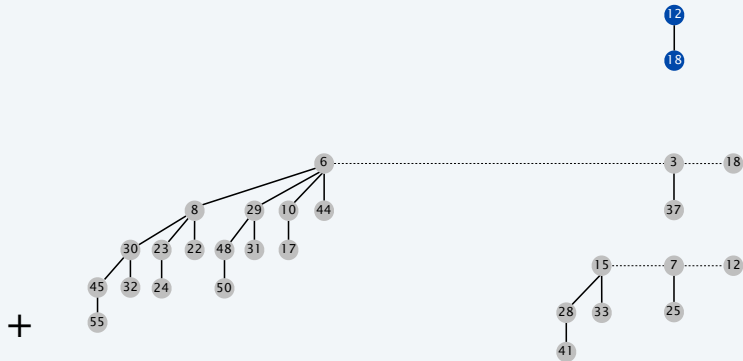
Meld operation. Given two binomial heaps H_1 and H_2 , (destructively) replace with a binomial heap H that is the union of the two.

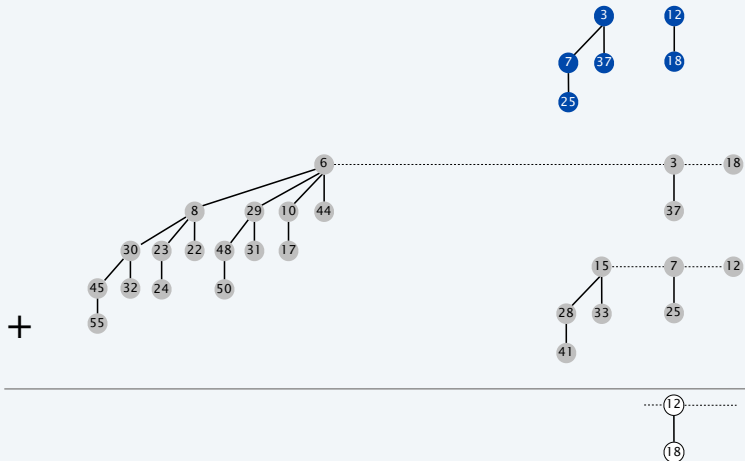
Warmup. Easy if H_1 and H_2 are both binomial trees of order k .

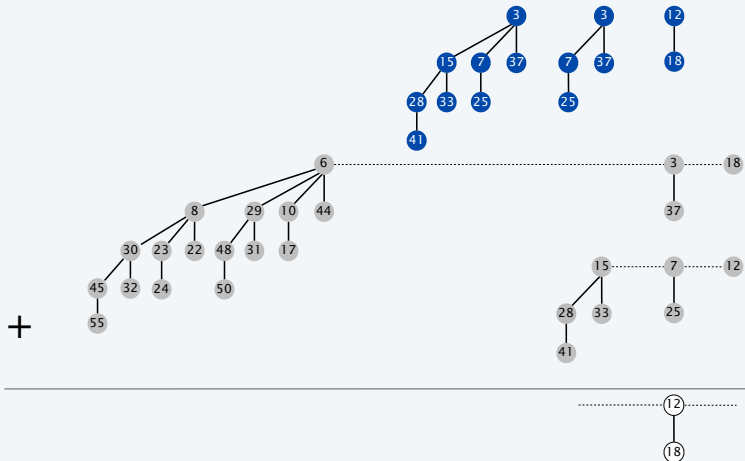
- Connect roots of H_1 and H_2 .
- Choose node with smaller key to be root of H .

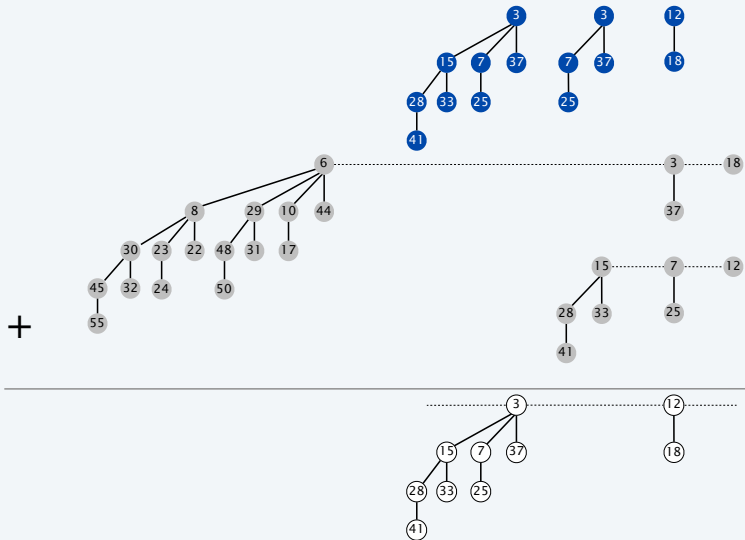


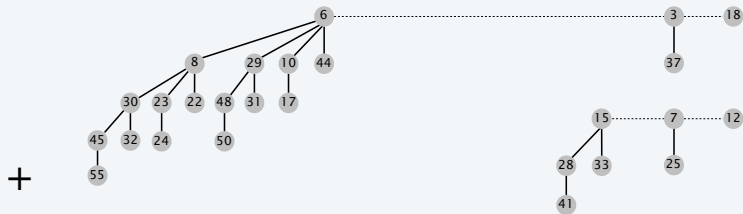












$19 + 7 = 26$

		1	1	1	
	1	0	0	1	1
+	0	0	1	1	1
	1	1	0	1	0

Binomial heap: meld

Meld operation. Given two binomial heaps H_1 and H_2 , (destructively) replace with a binomial heap H that is the union of the two.

Solution. Analogous to binary addition.

Running time. $O(\log n)$.

Pf. Proportional to number of trees in root lists $\leq 2 (\lceil \log_2 n \rceil + 1)$. ■

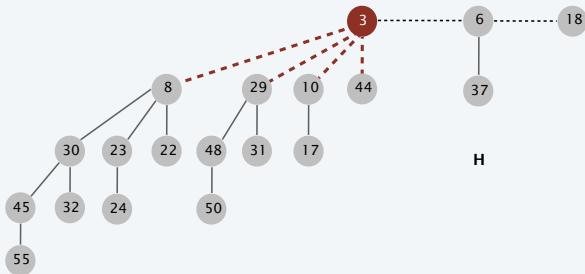
$$19 + 7 = 26$$

			1	1	1	
	1	0	0	1	1	
+	0	0	1	1	1	
	1	1	0	1	0	

Binomial heap: extract the minimum

Extract-min. Delete the node with minimum key in binomial heap H .

- Find root x with min key in root list of H , and delete.

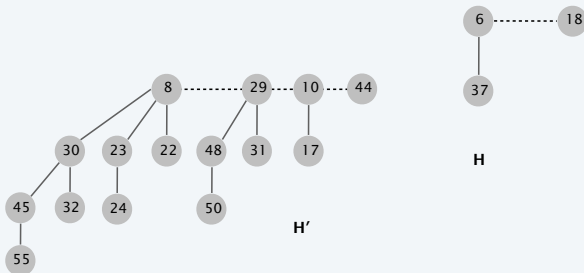


Binomial heap: extract the minimum

Extract-min. Delete the node with minimum key in binomial heap H .

- Find root x with min key in root list of H , and delete.
- $H' \leftarrow$ broken binomial trees.
- $H \leftarrow \text{MELD}(H', H)$.

Running time. $O(\log n)$.

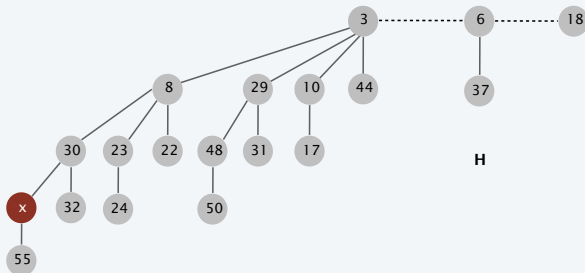


Binomial heap: decrease key

Decrease key. Given a handle to an element x in H , decrease its key to k .

- Suppose x is in binomial tree B_k .
- Repeatedly exchange x with its parent until heap order is restored.

Running time. $O(\log n)$.

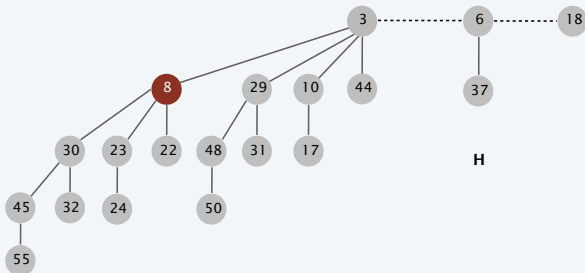


Binomial heap: delete

Delete. Given a handle to an element x in a binomial heap, delete it.

- $\text{DECREASE-KEY}(H, x, -\infty)$.
- $\text{DELETE-MIN}(H)$.

Running time. $O(\log n)$.

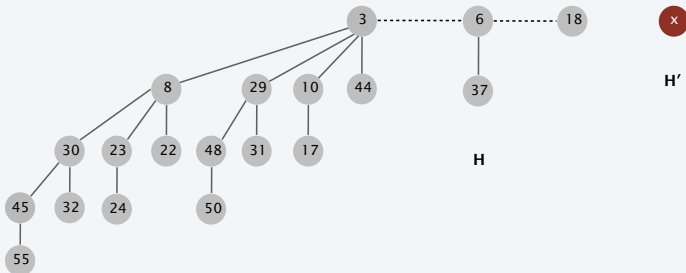


Binomial heap: insert

Insert. Given a binomial heap H , insert an element x .

- $H' \leftarrow \text{MAKE-HEAP}()$.
- $H' \leftarrow \text{INSERT}(H', x)$.
- $H \leftarrow \text{MELD}(H', H)$.

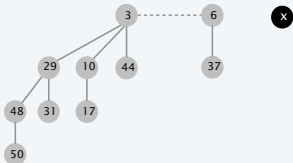
Running time. $O(\log n)$.



Binomial heap: sequence of insertions

Insert. How much work to insert a new node x ?

- If $n = \dots\dots 0$, then only 1 credit.
- If $n = \dots\dots 01$, then only 2 credits.
- If $n = \dots\dots 011$, then only 3 credits.
- If $n = \dots\dots 0111$, then only 4 credits.



Observation. Inserting one element can take $\Omega(\log n)$ time.

if $n = 11\dots 111$

Theorem. Starting from an empty binomial heap, a sequence of n consecutive INSERT operations takes $O(n)$ time.

Pf. $(n/2)(1) + (n/4)(2) + (n/8)(3) + \dots \leq 2n$. ■

$$\sum_{i=1}^k \frac{i}{2^i} = 2 - \frac{k}{2^k} - \frac{1}{2^{k-1}} \leq 2$$

Binomial heap: amortized analysis

Theorem. In a binomial heap, the amortized cost of INSERT is $O(1)$ and the worst-case cost of EXTRACT-MIN and DECREASE-KEY is $O(\log n)$.

Pf. Define potential function $\Phi(H_i) = \text{trees}(H_i) = \#$ trees in binomial heap H_i .

- $\Phi(H_0) = 0$.
- $\Phi(H_i) \geq 0$ for each binomial heap H_i .

Case 1. [INSERT]

- Actual cost $c_i = \text{number of trees merged} + 1$.
- $\Delta\Phi = \Phi(H_i) - \Phi(H_{i-1}) = 1 - \text{number of trees merged}$.
- Amortized cost $= \hat{c}_i = c_i + \Phi(H_i) - \Phi(H_{i-1}) = 2$.

Binomial heap: amortized analysis

Theorem. In a binomial heap, the amortized cost of INSERT is $O(1)$ and the worst-case cost of EXTRACT-MIN and DECREASE-KEY is $O(\log n)$.

Pf. Define potential function $\Phi(H_i) = \text{trees}(H_i) = \#$ trees in binomial heap H_i .

- $\Phi(H_0) = 0$.
- $\Phi(H_i) \geq 0$ for each binomial heap H_i .

Case 2. [DECREASE-KEY]

- Actual cost $c_i = O(\log n)$.
- $\Delta\Phi = \Phi(H_i) - \Phi(H_{i-1}) = 0$.
- Amortized cost $= \hat{c}_i = c_i = O(\log n)$.

Binomial heap: amortized analysis

Theorem. In a binomial heap, the amortized cost of INSERT is $O(1)$ and the worst-case cost of EXTRACT-MIN and DECREASE-KEY is $O(\log n)$.

Pf. Define potential function $\Phi(H_i) = \text{trees}(H_i) = \#$ trees in binomial heap H_i .

- $\Phi(H_0) = 0$.
- $\Phi(H_i) \geq 0$ for each binomial heap H_i .

Case 3. [EXTRACT-MIN or DELETE]

- Actual cost $c_i = O(\log n)$.
- $\Delta\Phi = \Phi(H_i) - \Phi(H_{i-1}) \leq \Phi(H_i) \leq \lfloor \log_2 n \rfloor$.
- Amortized cost = $\hat{c}_i = c_i + \Phi(H_i) - \Phi(H_{i-1}) = O(\log n)$. ■

Priority queues performance cost summary

operation	linked list	binary heap	binomial heap	binomial heap
MAKE-HEAP	$O(1)$	$O(1)$	$O(1)$	$O(1)$
ISEMPTY	$O(1)$	$O(1)$	$O(1)$	$O(1)$
INSERT	$O(1)$	$O(\log n)$	$O(\log n)$	$O(1)^\dagger$
EXTRACT-MIN	$O(n)$	$O(\log n)$	$O(\log n)$	$O(\log n)$
DECREASE-KEY	$O(1)$	$O(\log n)$	$O(\log n)$	$O(\log n)$
DELETE	$O(1)$	$O(\log n)$	$O(\log n)$	$O(\log n)$
MELD	$O(1)$	$O(n)$	$O(\log n)$	$O(1)^\dagger$
FIND-MIN	$O(n)$	$O(1)$	$O(\log n)$	$O(1)$

homework

† amortized

Hopeless challenge. $O(1)$ INSERT, DECREASE-KEY and EXTRACT-MIN. Why?

Challenge. $O(1)$ INSERT and DECREASE-KEY, $O(\log n)$ EXTRACT-MIN.