School of Mechanical Engineering

Course Code : BTME3061

Course Name: FINITE ELEMENT ANALYSIS

FINITE ELEMENT ANALYSIS- CO-ORDINATE TRANSFORMATIONS

Name of the Faculty: Mr. MANOJ KUMAR SHUKLA

Program Name: B.Tech (ME)

Lecture Objective-

1-D elements

- coordinate transformation
- 1-D elements
  - linear basis functions
  - quadratic basis functions
  - cubic basis functions

2-D elements

- coordinate transformation
- triangular elements
  - linear basis functions
  - quadratic basis functions
- rectangular elements
  - linear basis functions
  - quadratic basis functions

We wish to approximate a function u(x) defined in an interval [a,b] by some set of basis functions

$$u(x) = \sum_{i=1}^{n} c_i \ddot{}_{\;i}$$

where i is the number of grid points (the edges of our elements) defined at locations x_i. As the basis functions look the same in all elements (apart from some constant) we make life easier by moving to a local coordinate system

$$\P = \frac{x - x_i}{x_{i+1} - x_i}$$

so that the element is defined for x=[0,1].

There is not much choice for the shape of a (straight) 1-D element! Notably the length can vary across the domain.

We require that our function u(ξ) be approximated locally by the linear function

$$u(\xi) = c_1 + c_2 \xi$$

Our node points are defined at $\xi_{1,2}$=0,1 and we require that

$$u_1 = c_1 \quad \Rightarrow \quad c_1 = u_1$$
$$u_2 = c_1 + c_2 \quad \Rightarrow c_2 = -u_1 + u_2$$

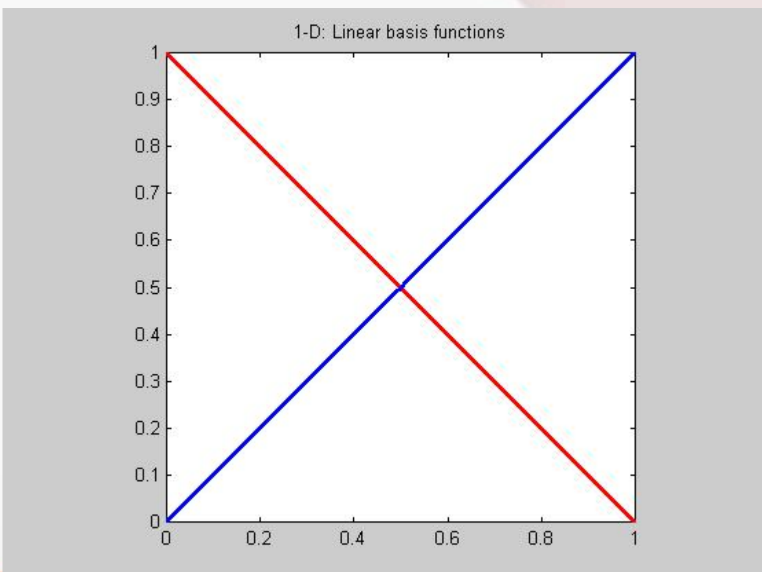
$$\mathbf{c} = \mathbf{A}\mathbf{u}$$

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

1-D elements – linear basis functions

As we have expressed the coefficients c_i as a function of the function values at node points $\xi_{1,2}$ we can now express the approximate function using the node values

$$u(\P) = u_1 + (-u_1 + u_2)\P$$
$$= u_1(1-\P) + u_2\P$$
$$= u_1 N_1(\P) + N_2(\P)\P$$



.. and $N_{1,2}(x)$ are the linear basis functions for 1-D elements.

Now we require that our function u(x) be approximated locally by the quadratic function

$$u(\P) = c_1 + c_2 \P + c_3 \P^2$$

Our node points are defined at $\xi_{1,2,3}$=0,1/2,1 and we require that

$$u_1 = c_1$$

$$u_2 = c_1 + 0.5c_2 + 0.25c_3$$

$$u_3 = c_1 + c_2 + c_3$$

→ $\mathbf{c} = \mathbf{A}\mathbf{u}$

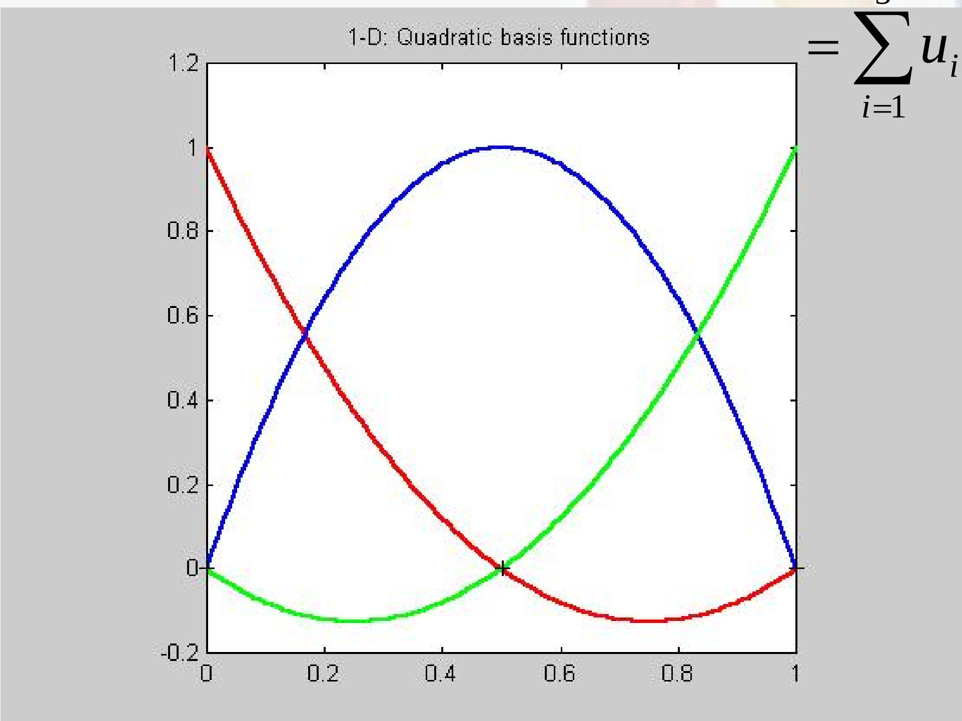
→ $$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 4 & -1 \\ 2 & -4 & 2 \end{bmatrix}$$

1-D quadratic basis functions

... again we can now express our approximated function as a sum over our basis functions weighted by the values at three node points

$$u(\P) = c_1 + c_2\P + c_3\P^2 = u_1(1 - 3\P + 2\P^2) + u_2(4\P - 4\P^2) + u_3(-\P + 2\P^2)$$

$$= \sum_{i=1}^{3} u_i N_i(\P)$$

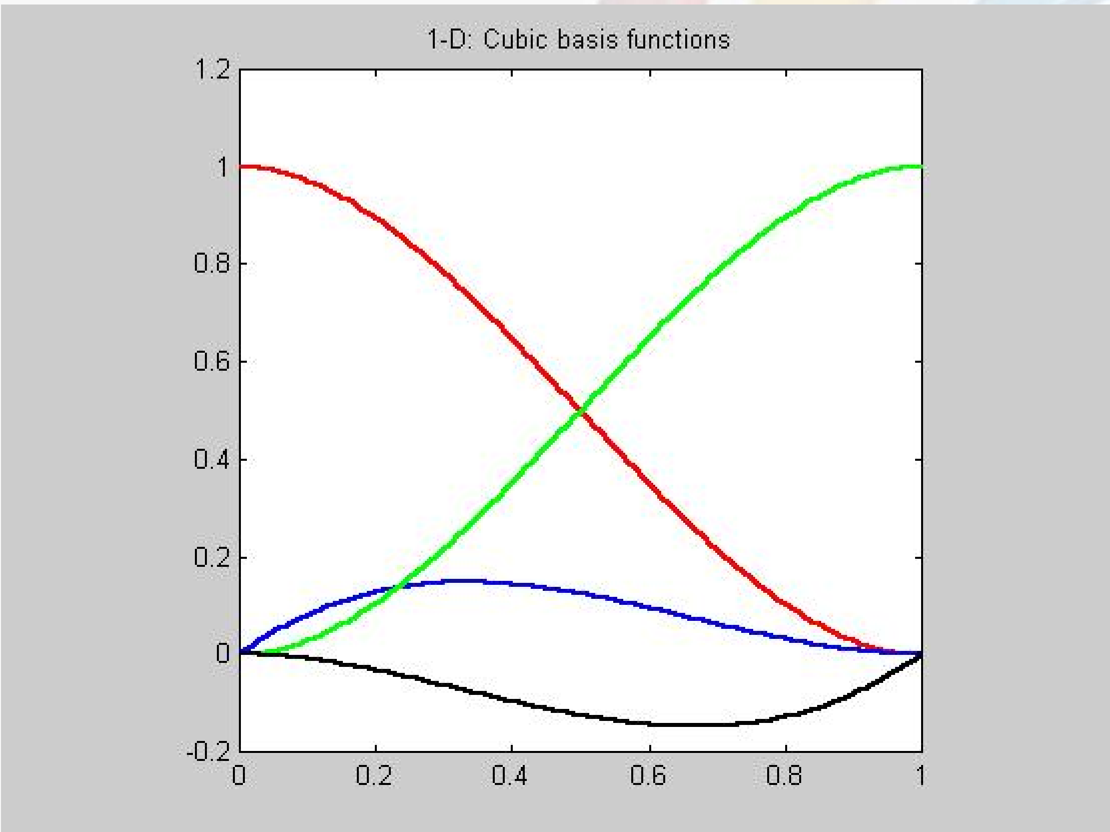


... note that now we re using three grid points per element ...

Can we approximate a constant function?

1-D cubic basis functions

... using similar arguments the cubic basis functions can be derived as



$$u(¶) = c_1 + c_2 ¶ + c_3 ¶^2 + c_4 ¶^3$$

$$N_1(¶) = 1 - 3¶^2 + 2¶^3$$

$$N_2(¶) = ¶ - 2¶^2 + ¶^3$$

$$N_3(¶) = 3¶^2 - 2¶^3$$

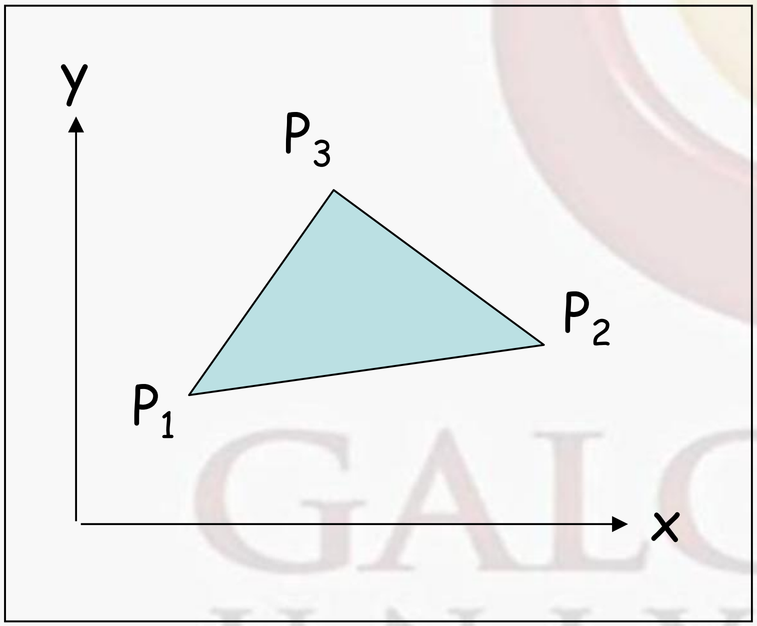
$$N_4(¶) = -¸^2 + ¶^3$$

... note that here we need derivative information at the boundaries ...

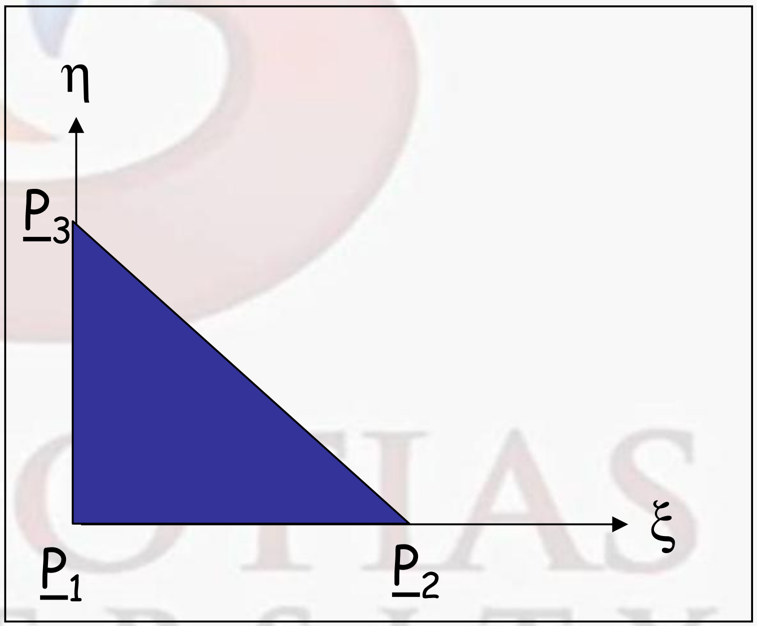
How can we approximate a constant function?

2-D elements: coordinate transformation

Let us now discuss the geometry and basis functions of 2-D elements, again we want to consider the problems in a local coordinate system, first we look at triangles

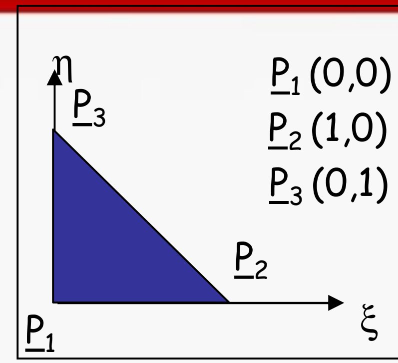


before



after

2-D elements: coordinate transformation



Any triangle with corners $P_i(x_i,y_i)$, i=1,2,3 can be transformed into a rectangular, equilateral triangle with

$$x = x_1 + (x_2 - x_1)\xi + (x_3 - x_1)\eta$$

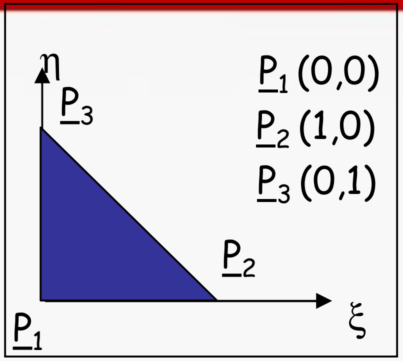
$$y = y_1 + (y_2 - y_1)\xi + (y_3 - y_1)\eta$$

using counterclockwise numbering. Note that if η=0, then these equations are equivalent to the 1-D tranformations. We seek to approximate a function by the linear form

$$u(\xi, \eta) = c_1 + c_2\xi + c_3\eta$$

we proceed in the same way as in the 1-D case

2-D elements: coefficients



... and we obtain

$$u_1 = u(0,0) = c_1$$
$$u_2 = u(1,0) = c_1 + c_2$$
$$u_3 = u(0,1) = c_1 + c_3$$

... and we obtain the coefficients as a function of the values at the grid nodes by matrix inversion

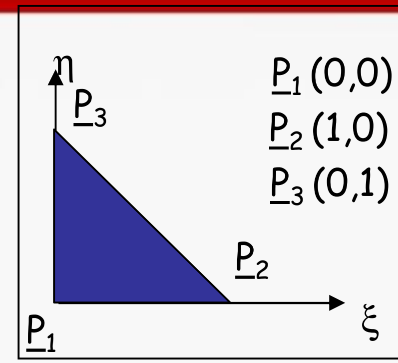
$$\Rightarrow \quad \mathbf{c} = \mathbf{A}\mathbf{u}$$

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

containing the 1-D case

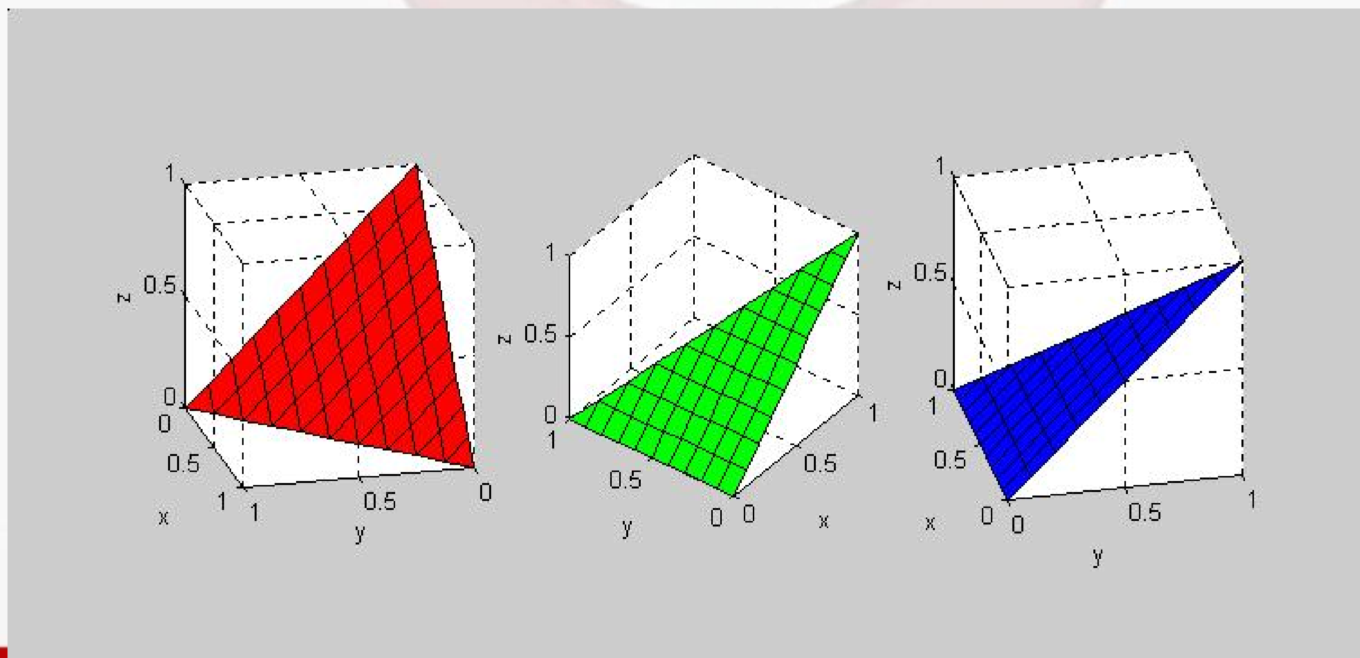
$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

triangles: linear basis functions



from matrix A we can calculate the linear basis functions for triangles

$$N_1(\xi,\eta) = 1 - \xi - \eta$$
$$N_2(\xi,\eta) = \xi$$
$$N_3(\xi,\eta) = \eta$$



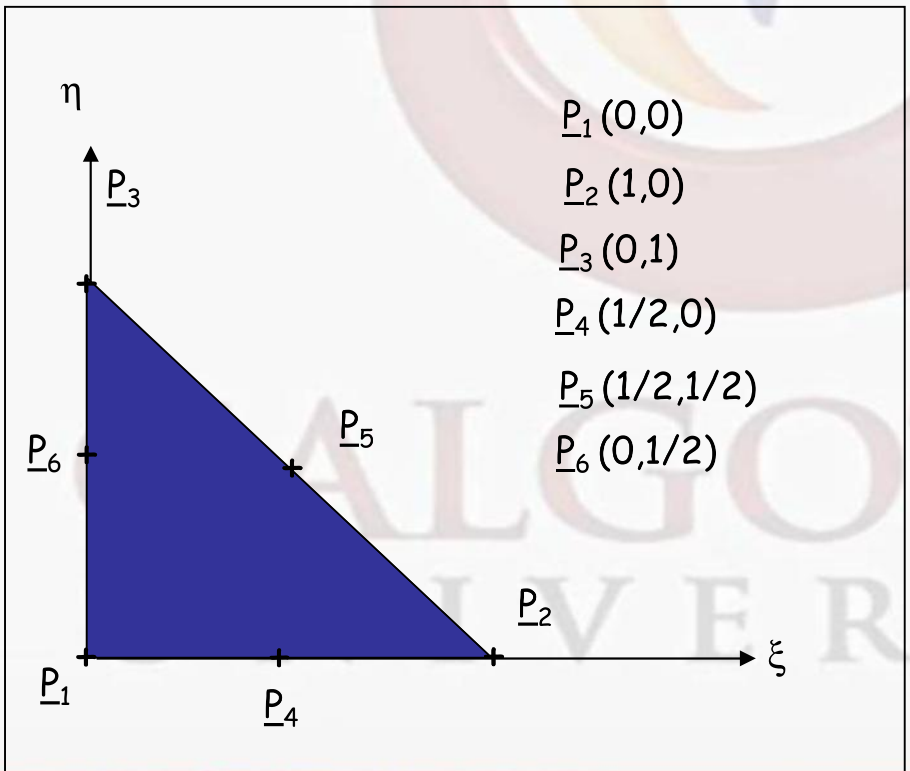
triangles: quadratic elements

Any function defined on a triangle can be approximated by the quadratic function

$$u(x,y) = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 x^2 + \alpha_5 xy + \alpha_6 y^2$$

and in the transformed system we obtain

$$u(\xi,\eta) = c_1 + c_2\xi + c_3\eta + c_4\xi^2 + c_5\xi\eta + c_6\eta^2$$



as in the 1-D case we need additional points on the element.

Questions-

1. Using polynomial functions (generalized coordinates) determine shape functions for a two noded beam element.

2. Using generalized coordinate approach, find shape functions for two noded bar/truss element.

3. Using polynomial functions (generalized coordinates) determine shape functions for a two noded beam element

Text Book-

1. Finite Element Analysis by S.S bhavikatti six multicolour edition,2018.New age International publisher. ISBN: 678-26-74589-23-4.

2. A Textbook of Finite Element Analysis Formulation and Programming by D.K.mahraj,Edition 2019. Publisher Willey India ISBN : 978-93-88425-93-3.

Reference Book-

1. Finite element analysis ,Theory and application with Ansys by Moaveni ,2nd edition 2015 ,publisher Pearson, ISBN- 528-43-88435-9.

2. Finite element Analysis By David V. Hutton ,Publisher Elizabeth A. Jomes ,4th edition 2017. ISBN: 0-07-23-9536-2

THANK YOU