Theorem: S is closed iff R- S is open.

Proof: (Necessary part)

Assume: S is closed.

Aim: our aim is to prove that R-S is open **or** we have to prove that every point of R-S is an interior point of R-S.

Let us assume that $p \in R - S = S^c$ be any arbitrary point of R-S.

Implies $p \notin S$ implies p is not a limit point of S.

This implies $\exists\ \varepsilon > 0$ such that $(N_\varepsilon(p) - \{p\}) \cap S = \emptyset$ implies $N_\varepsilon(p) \cap S = \emptyset$ implies $N_\varepsilon(p) \subseteq S^c$.

This implies p is an interior point of S^c.

S^c is open.

(Sufficient part) S^c is open.

Aim : S is closed.

Suppose S is not closed this implies that there is a point p in S which is not a limit point of S implies that p is an interior point of S. This implies that p is not an interior point of S^c. This implies that S^c is not open. A contradiction that S^c is open.

Our assumption is wrong implies S is closed.

Th: If A^o defined interior set of A.

Then,

(i) $A^o = Int(A) = \bigcup_i G_i$, G_i's are open sets.

(ii) A^o is an open set.

(iii) A^o is the largest open set contained in 'A'.

(iv) A is open iff $A^{o} = A$.

Proof: Let $\mathcal{G} = \{G_i \mid i \in \Lambda\}$ be a class of open subsets of A.

(Λ: index set)

Let $p \in \text{Int}(A) \Rightarrow p$ is a member of ~~some open subset~~ $\underline{G_{i_0}}$ for some i_0.

$\Rightarrow \exists$ an $\epsilon > 0$ such that

$p \in N_\epsilon(p) \subset G_{i_0}$

$\Rightarrow p \in N_\epsilon(p) \subset \cup G_i$

$\Rightarrow p \in \cup G_i$

$\Rightarrow \text{int}(A) \subset \cup G_i$.

let $x \in \cup G'$ $\Rightarrow x \in G_{\omega_0}$ for some G_{ω_0}.

$\Rightarrow x \in \text{int}(A)$ - collection of all interior points.

$\Rightarrow \cup G' \subset \text{int}(A)$.

$\Rightarrow \boxed{\text{int}(A) = \cup G'}$.

Proof:

(ii) simple ~~operational~~ observation.
that arbitrary union of open sets is open.

(iii). Let G is an open set

$\Rightarrow$ ~~$G \subseteq \cup G_i$~~ $G \subseteq \bigcup_i G_i = \text{int}(A)$

$\Rightarrow \underline{G} \subset \underline{\text{int}(A)}$ — (1)

largest open set.

Reference book: Bansi Lal and Sanjay Arora; Introduction to Real Analysis, Satya Prakashan, 1st Vol (1991)

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If $x \in \text{int}(A) \Rightarrow x$ is an interior point of A

$\Rightarrow \exists$ an $\epsilon > 0$ such that

$x \in N_\epsilon(x) \subset A$

$\Rightarrow x \in A$

$\Rightarrow \text{int}(A) \subset A$ —— (2)

$\Rightarrow \text{int}(A) \subset A$ —— (2)

$\Rightarrow \underbrace{G \subset \text{int}(A)}_{\text{largest open set}} \underbrace{\subset A}_{\text{contained in } A}$.

Reference book: Bansi Lal and Sanjay Arora; Introduction to Real Analysis, Satya Prakashan, 1st Vol (1991)

From ① & ②

$\Rightarrow$ $\boxed{A = \text{int}(A)}$.

Sufficient part:

Let $A = \text{int}(A)$

Aim: A is open.

Since $\text{int}(A)$ is open

$\Rightarrow$ A is also an open set.

Reference book: Bansi Lal and Sanjay Arora; Introduction to Real Analysis, Satya Prakashan, 1st Vol (1991)