School of Computing Science and Engineering

Course Code : MCAS2140 Course Name: Algorithm Analysis and Design

SINGLE SOURCE SHORTEST PATHS

Name of the Faculty: Unnikrishnan Program Name: MCA

Single Source Shortest Paths

- Properties of shortest paths
- Dijkstra's algorithm
- Correctness
- Analysis
- Breadth-first search

Paths in graphs

Consider a digraph $G = (V, E)$ with edge-weight function $w : E \to \mathbb{R}$. The ***weight*** of path $p = v_1 \to v_2 \to \cdots \to v_k$ is defined to be

$$w(p) = \sum_{i=1}^{k-1} w(v_i, v_{i+1}).$$

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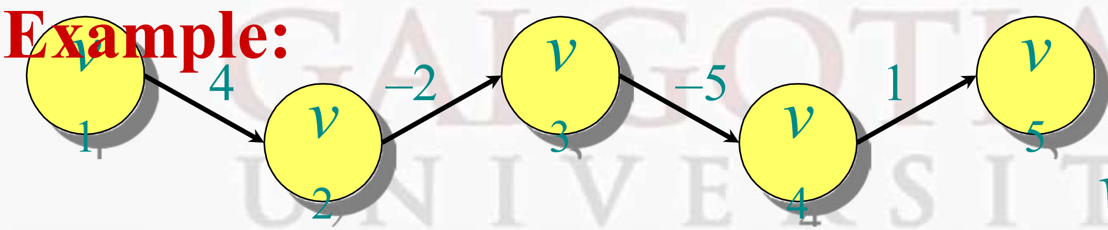
Example:



$w(p) = -2$

Shortest paths

A ***shortest path*** from u to v is a path of minimum weight from u to v. The ***shortest-path weight*** from u to v is defined as

$$\delta(u, v) = \min\{w(p) : p \text{ is a path from } u \text{ to } v\}.$$

Note: $\delta(u, v) = \infty$ if no path from u to v exists.

Well-definedness of shortest paths

If a graph *G* contains a negative-weight cycle, then some shortest paths do not exist.

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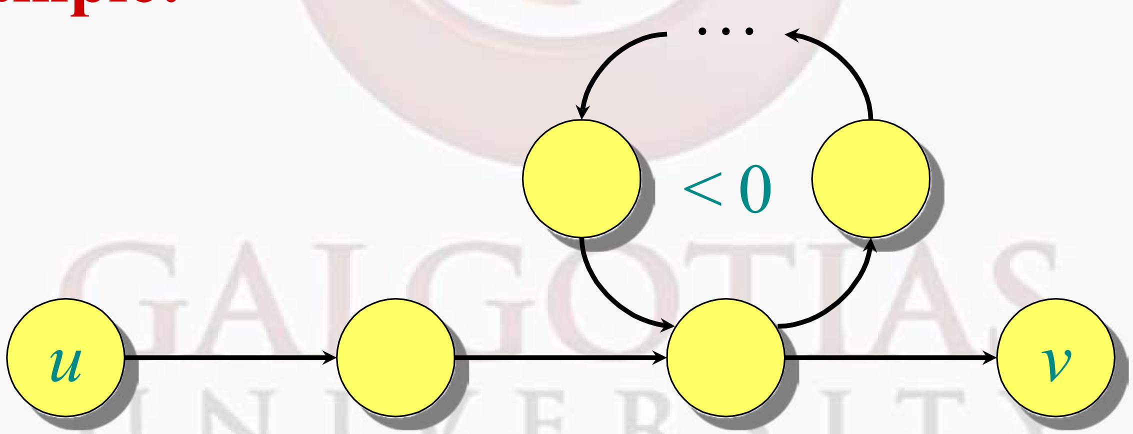

Optimal substructure

Theorem. A subpath of a shortest path is a shortest path.

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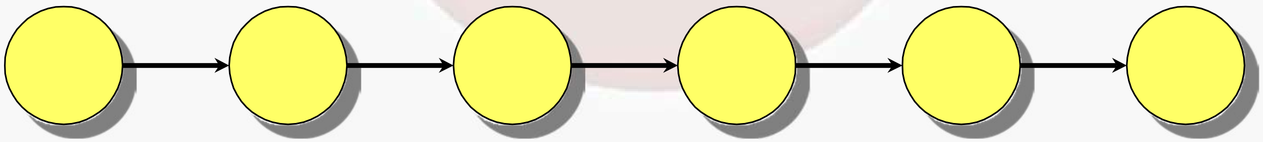
Proof. Cut and paste:



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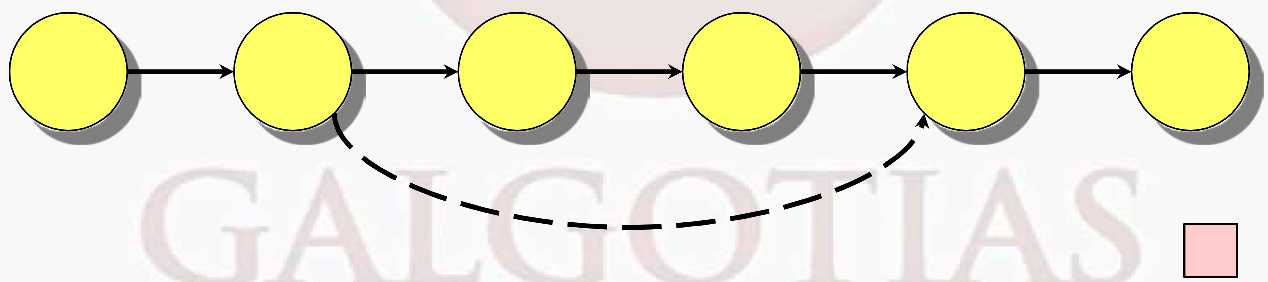

Triangle inequality

Theorem. For all $u, v, x \in V$, we have

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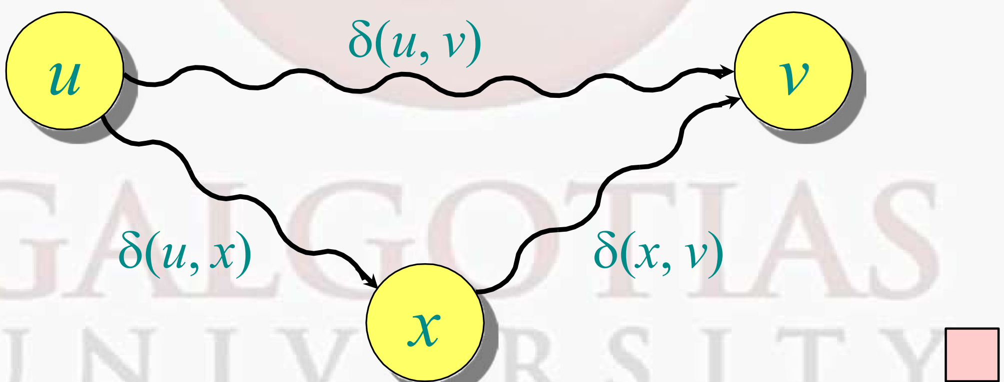

Single-source shortest paths (nonnegative edge weights)

Problem. Assume that $w(u, v) \geq 0$ for all $(u, v) \in E$. (Hence, all shortest-path weights must exist.) From a given source vertex $s \in V$, find the shortest-path weights $\delta(s, v)$ for all $v \in V$.

IDEA: Greedy.

1. Maintain a set S of vertices whose shortest-path distances from s are known.
2. At each step, add to S the vertex $v \in V - S$ whose distance estimate from s is minimum.
3. Update the distance estimates of vertices adjacent to v.

Dijkstra's algorithm

$d[s] \leftarrow 0$
for each $v \in V - \{s\}$
  do $d[v] \leftarrow \infty$
$S \leftarrow \varnothing$
$Q \leftarrow V$ ⊳ Q is a priority queue maintaining $V - S$, keyed on $d[v]$

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while $Q \neq \varnothing$
    do $u \leftarrow$ EXTRACT-MIN(Q)
        $S \leftarrow S \cup \{u\}$
        for each $v \in Adj[u]$
            do if $d[v] > d[u] + w(u, v)$
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relaxation step

Implicit DECREASE-KEY

Thank You