CORRECTNESS OF DIJKSTRA'S ALGORITHM

Correctness — Part I

Lemma. Initializing $d[s] \leftarrow 0$ and $d[v] \leftarrow \infty$ for all $v \in V - \{s\}$ establishes $d[v] \geq \delta(s, v)$ for all $v \in V$, and this invariant is maintained over any sequence of relaxation steps.

Correctness — Part I

Lemma. Initializing $d[s] \leftarrow 0$ and $d[v] \leftarrow \infty$ for all $v \in V - \{s\}$ establishes $d[v] \geq \delta(s, v)$ for all $v \in V$, and this invariant is maintained over any sequence of relaxation steps.

Proof. Suppose not. Let v be the first vertex for which $d[v] < \delta(s, v)$, and let u be the vertex that caused $d[v]$ to change: $d[v] = d[u] + w(u, v)$. Then,

$d[v] < \delta(s,v)$	supposition
$\leq \delta(s, u) + \delta(u, v)$	triangle inequality
$\leq \delta(s,u) + w(u, v)$	sh. path $\leq$ specific path
$\leq d[u] + w(u, v)$	v is first violation

Contradiction. □

Correctness — Part II

Lemma. Let u be v's predecessor on a shortest path from s to v. Then, if $d[u] = \delta(s, u)$ and edge (u, v) is relaxed, we have $d[v] = \delta(s, v)$ after the relaxation.

Correctness — Part II

Lemma. Let u be v's predecessor on a shortest path from s to v. Then, if $d[u] = \delta(s, u)$ and edge (u, v) is relaxed, we have $d[v] = \delta(s, v)$ after the relaxation.

Proof. Observe that $\delta(s, v) = \delta(s, u) + w(u, v)$. Suppose that $d[v] > \delta(s, v)$ before the relaxation. (Otherwise, we're done.) Then, the test $d[v] > d[u] + w(u, v)$ succeeds, because $d[v] > \delta(s, v) = \delta(s, u) + w(u, v) = d[u] + w(u, v)$, and the algorithm sets $d[v] = d[u] + w(u, v) = \delta(s, v)$. □

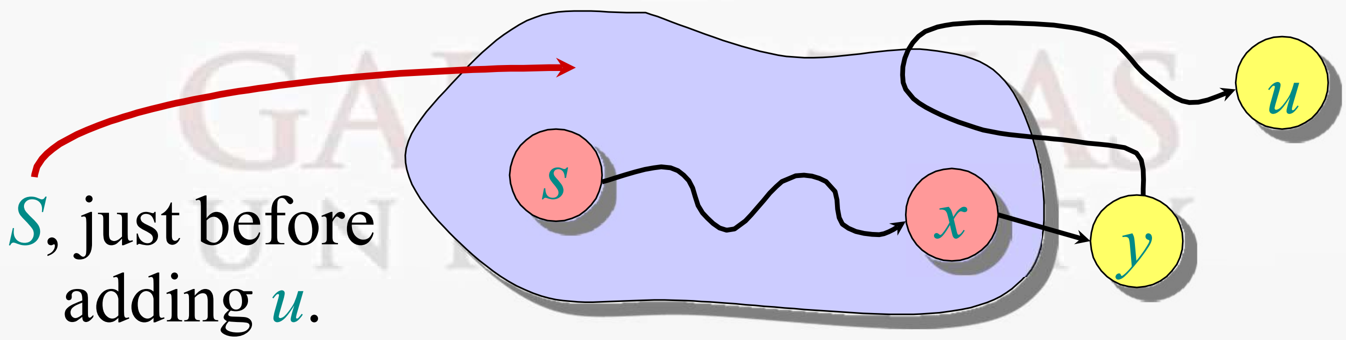
Correctness — Part III

Theorem. Dijkstra's algorithm terminates with $d[v] = \delta(s, v)$ for all $v \in V$.

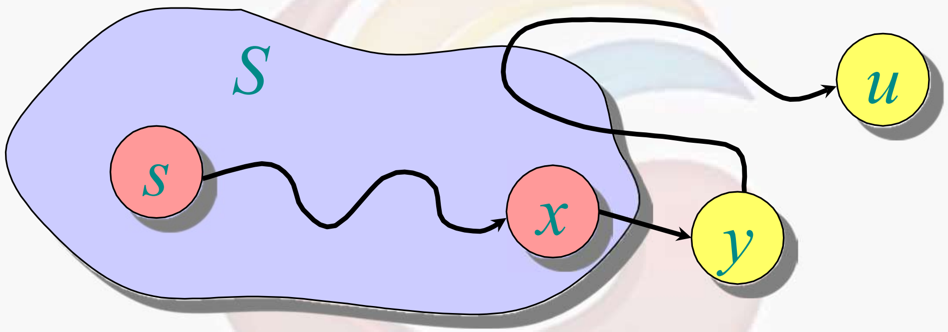
Correctness — Part III

Theorem. Dijkstra's algorithm terminates with $d[v] = \delta(s, v)$ for all $v \in V$.

Proof. It suffices to show that $d[v] = \delta(s, v)$ for every $v \in V$ when v is added to S. Suppose u is the first vertex added to S for which $d[u] > \delta(s, u)$. Let y be the first vertex in $V - S$ along a shortest path from s to u, and let x be its predecessor:



Correctness — Part III (continued)



Since u is the first vertex violating the claimed invariant, we have $d[x] = \delta(s, x)$. When x was added to S, the edge (x, y) was relaxed, which implies that $d[y] = \delta(s, y) \le \delta(s, u) < d[u]$. But, $d[u] \le d[y]$ by our choice of u. Contradiction. □

Analysis of Dijkstra

while $Q \neq \emptyset$
    do $u \leftarrow$ EXTRACT-MIN(Q)
        $S \leftarrow S \cup \{u\}$
        for each $v \in Adj[u]$
            do if $d[v] > d[u] + w(u, v)$
                then $d[v] \leftarrow d[u] + w(u, v)$

Analysis of Dijkstra

$|V|$ times

while $Q \neq \emptyset$
  do $u \leftarrow$ EXTRACT-MIN(Q)
    $S \leftarrow S \cup \{u\}$
    for each $v \in Adj[u]$
      do if $d[v] > d[u] + w(u, v)$
        then $d[v] \leftarrow d[u] + w(u, v)$

Analysis of Dijkstra

$|V|$ times

while $Q \neq \emptyset$
  do $u \leftarrow$ EXTRACT-MIN(Q)
    $S \leftarrow S \cup \{u\}$
    for each $v \in Adj[u]$ — $degree(u)$ times
      do if $d[v] > d[u] + w(u, v)$
        then $d[v] \leftarrow d[u] + w(u, v)$

Analysis of Dijkstra

$|V|$ times

while $Q \neq \emptyset$
  do $u \leftarrow$ EXTRACT-MIN(Q)
    $S \leftarrow S \cup \{u\}$

$degree(u)$ times

    for each $v \in Adj[u]$
      do if $d[v] > d[u] + w(u, v)$
        then $d[v] \leftarrow d[u] + w(u, v)$

Handshaking Lemma $\Rightarrow \Theta(E)$ implicit DECREASE-KEY's.

Analysis of Dijkstra

$|V|$ times:

while $Q \neq \emptyset$
  do $u \leftarrow$ EXTRACT-MIN(Q)
    $S \leftarrow S \cup \{u\}$
    $degree(u)$ times:
    for each $v \in Adj[u]$
      do if $d[v] > d[u] + w(u, v)$
        then $d[v] \leftarrow d[u] + w(u, v)$

Handshaking Lemma $\Rightarrow \Theta(E)$ implicit DECREASE-KEY's.

Time $= \Theta(V \cdot T_{\text{EXTRACT-MIN}} + E \cdot T_{\text{DECREASE-KEY}})$

Note: Same formula as in the analysis of Prim's minimum spanning tree algorithm.

Analysis of Dijkstra (continued)

$$\text{Time} = \Theta(V) \cdot T_{\text{EXTRACT-MIN}} + \Theta(E) \cdot T_{\text{DECREASE-KEY}}$$

Q	$T_{\text{EXTRACT-MIN}}$	$T_{\text{DECREASE-KEY}}$	Total

Analysis of Dijkstra (continued)

$$\text{Time} = \Theta(V) \cdot T_{\text{EXTRACT-MIN}} + \Theta(E) \cdot T_{\text{DECREASE-KEY}}$$

Q	$T_{\text{EXTRACT-MIN}}$	$T_{\text{DECREASE-KEY}}$	Total
array	$O(V)$	$O(1)$	$O(V^2)$

Analysis of Dijkstra (continued)

Time = $\Theta(V) \cdot T_{\text{EXTRACT-MIN}} + \Theta(E) \cdot T_{\text{DECREASE-KEY}}$

Q	$T_{\text{EXTRACT-MIN}}$	$T_{\text{DECREASE-KEY}}$	Total
array	$O(V)$	$O(1)$	$O(V^2)$
binary heap	$O(\lg V)$	$O(\lg V)$	$O(E \lg V)$

Analysis of Dijkstra (continued)

Time = $\Theta(V) \cdot T_{\text{EXTRACT-MIN}} + \Theta(E) \cdot T_{\text{DECREASE-KEY}}$

Q	$T_{\text{EXTRACT-MIN}}$	$T_{\text{DECREASE-KEY}}$	Total
array	$O(V)$	$O(1)$	$O(V^2)$
binary heap	$O(\lg V)$	$O(\lg V)$	$O(E \lg V)$
Fibonacci heap	$O(\lg V)$ amortized	$O(1)$ amortized	$O(E + V \lg V)$ worst case

Thank You