School of Mechanical Engineering

Course Code : BAUT4001 Course Name: CAD/CAM

Unit-2
Geometric Transformations

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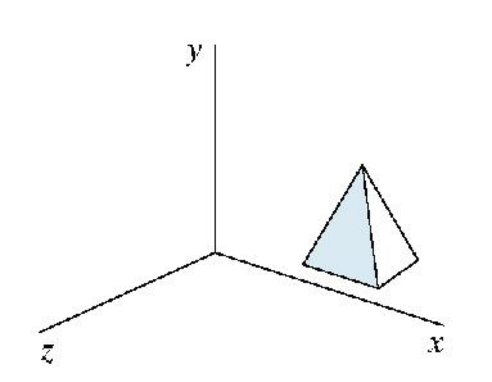
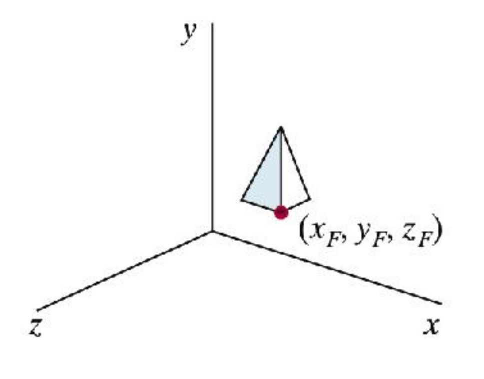
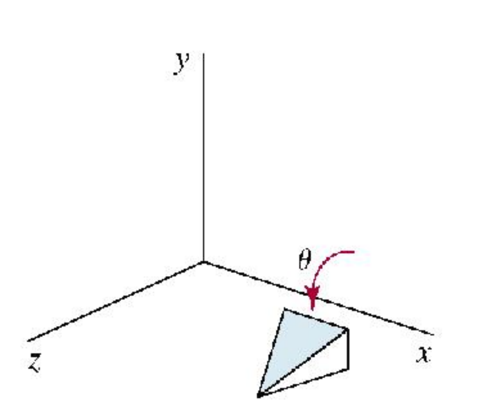
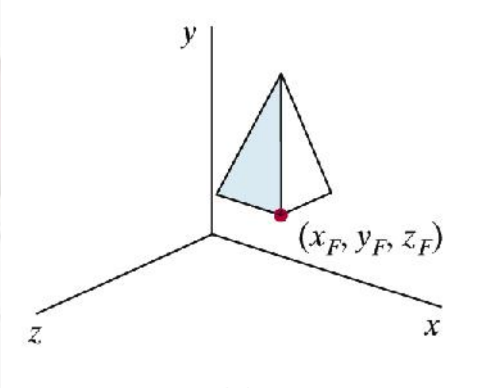
Geometric Transformations

- **Basic transformations:**
  - **Translation**
  - **Scaling**
  - **Rotation**
- **Purposes:**
  - **To move the position of objects**
  - **To alter the shape / size of objects**
  - **To change the orientation of objects**









Basic two-dimensional geometric transformations

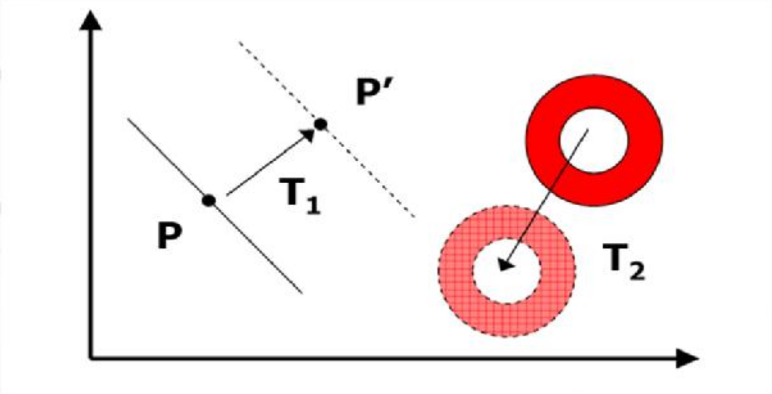
- **Two-Dimensional translation**
  - **One of rigid-body transformation, which move objects without deformation**
  - **Translate an object by Adding offsets to coordinates to generate new coordinates positions**
  - **Set t_x,t_y be the translation distance, we have**

$$x' = x + t_x \qquad y' = y + t_y$$

  - **In matrix format, where T is the translation vector**

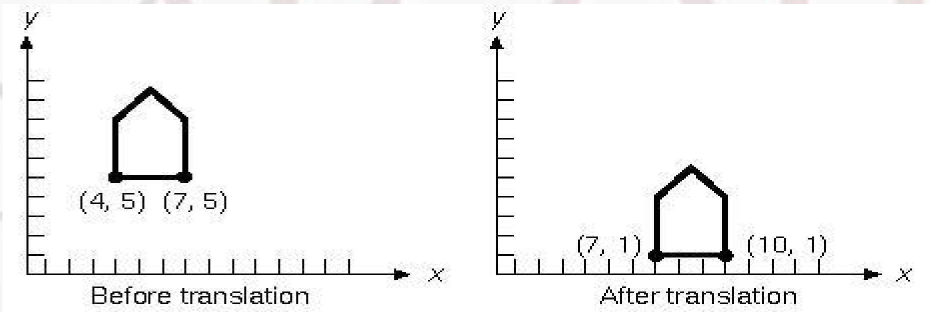
$$P' = \begin{bmatrix} x' \\ y' \end{bmatrix} \quad P = \begin{bmatrix} x \\ y \end{bmatrix} \quad T = \begin{bmatrix} t_x \\ t_y \end{bmatrix}$$

$$P' = P + T$$

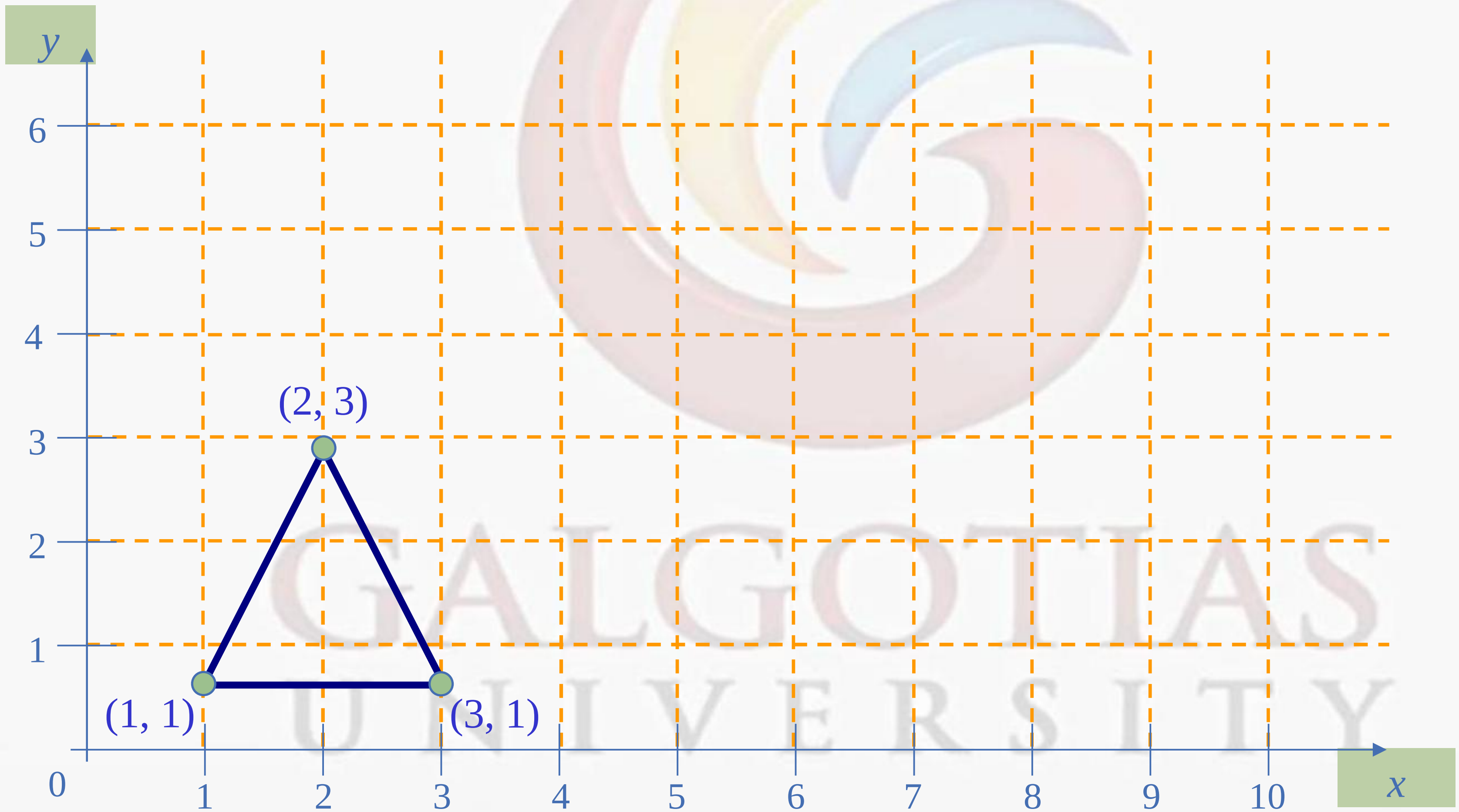


Basic Two-dimensional geometric transformations

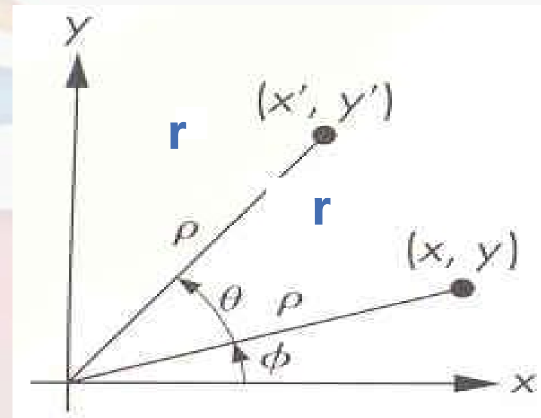
- **We could translate an object by applying the equation to every point of an object.**
  - **Because each line in an object is made up of an infinite set of points, however, this process would take an infinitely long time.**
  - **Fortunately we can translate all the points on a line by translating only the line's endpoints and drawing a new line between the endpoints.**
  - **This figure translates the "house" by (3, -4)**



Translation Example



Basic two-dimensional geometric transformations



- **Two-Dimensional rotation**
  - **Rotation axis and angle are specified for rotation**
  - **Convert coordinates into polar form for calculation**

    $x = r\cos\phi \qquad y = y\sin\phi$
  - **Example, to rotation an object with angle a**
    - **The new position coordinates**

$$x' = r\cos(\phi+\theta) = r\cos\phi\cos\theta - r\sin\phi\sin\theta = x\cos\theta - y\sin\theta$$

$$y' = r\sin(\phi+\theta) = r\cos\phi\sin\theta + r\sin\phi\sin\theta = x\sin\theta + y\cos\theta$$

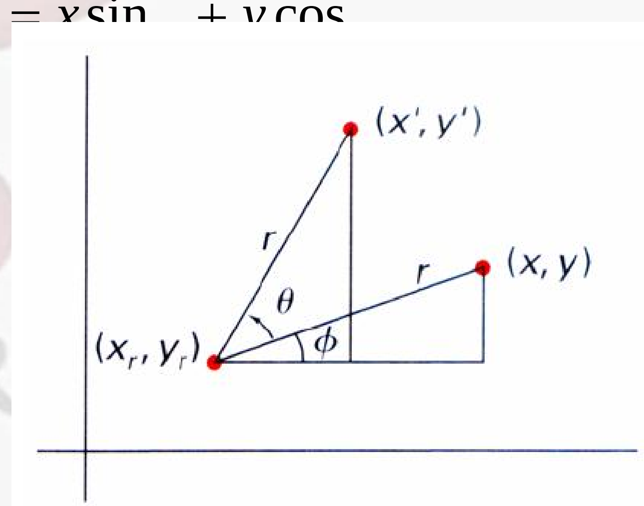
- **In matrix format**

$$R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \qquad P' = R \cdot P$$

- **Rotation about a point (x_r, y_r)**

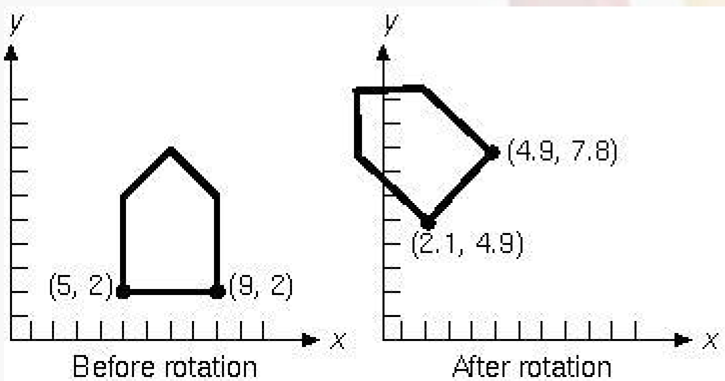
$$x' = x_r + (x - x_r)\cos\theta - (y - y_r)\sin\theta$$

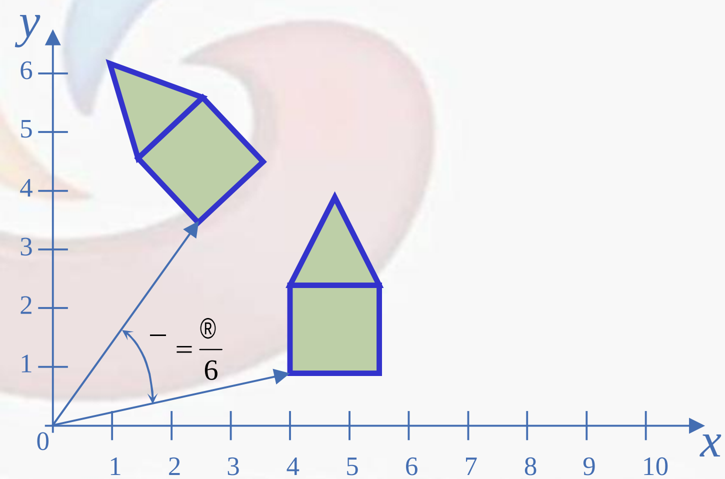
$$y' = y_r + (x - x_r)\sin\theta + (y - y_r)\cos\theta$$



Basic two-dimensional geometric transformations

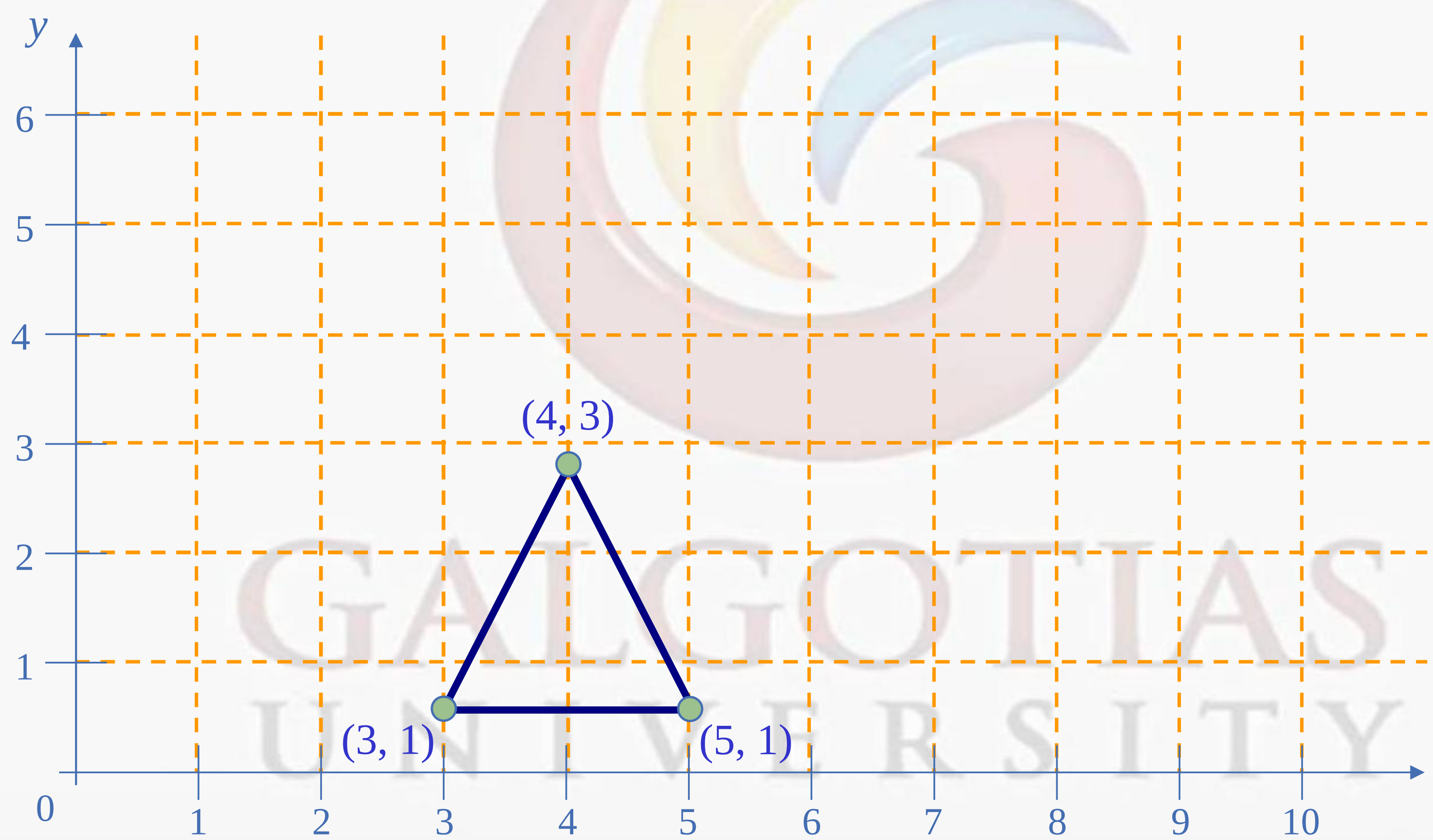
– **This figure shows the rotation of the house by 45 degrees.**





- **Positive angles are measured counterclockwise (from x towards y)**
- **For negative angles, you can use the identities:**
  – **$\cos(-\theta) = \cos(\theta)$ and $\sin(-\theta) = -\sin(\theta)$**

Rotation Example



Basic two-dimensional geometric transformations

- **Two-Dimensional scaling**
  - **To alter the size of an object by multiplying the coordinates with scaling factor s_x and s_y**

    $$x' = x \cdot s_x \qquad y = y \cdot s_y$$

  - **In matrix format, where S is a 2by2 scaling matrix**

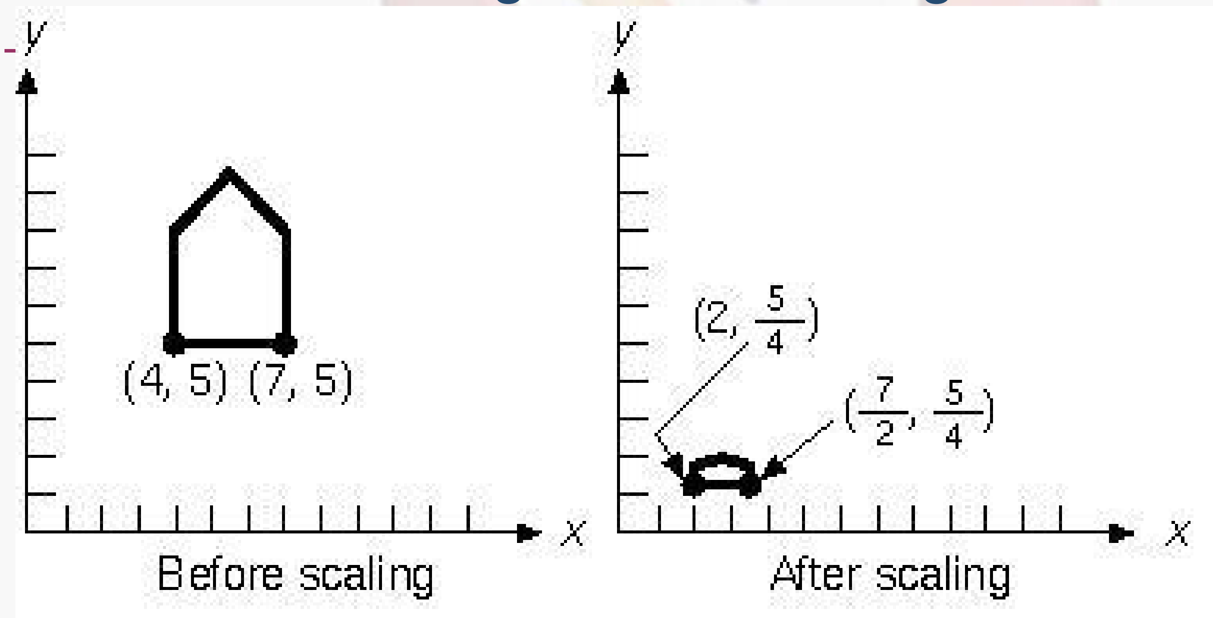
    $$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} \qquad P' = S \cdot P$$

  - **Choosing a fix point (x_f, y_f) as its centroid to perform scaling**

    $$x' = x \cdot s_x + x_f(1 - s_x)$$
    $$y' = y \cdot s_y + y_f(1 - s_y)$$

Basic two-dimensional geometric transformations

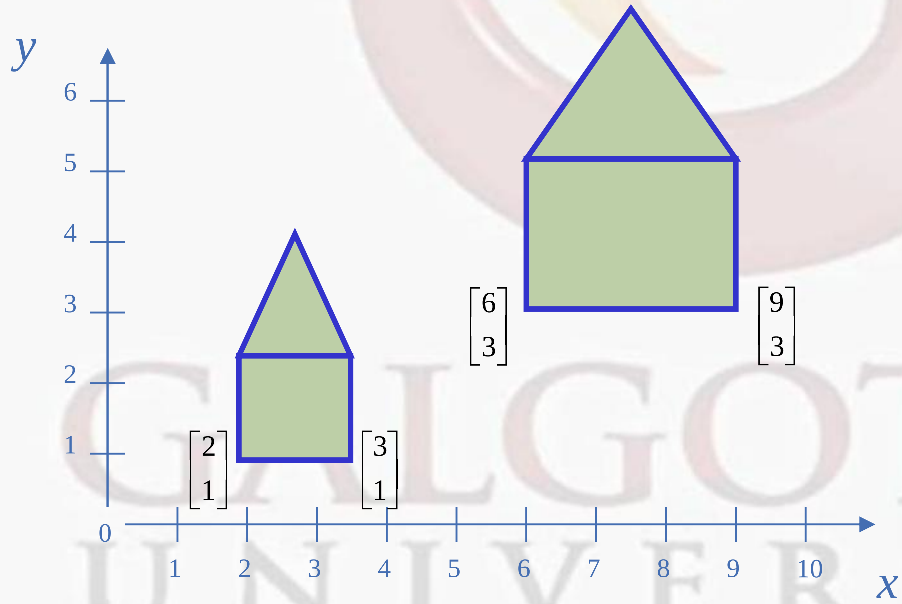
- In this figure, the house is scaled by 1/2 in x and 1/4 in y
  - Notice that the scaling is about the origin:



Scaling

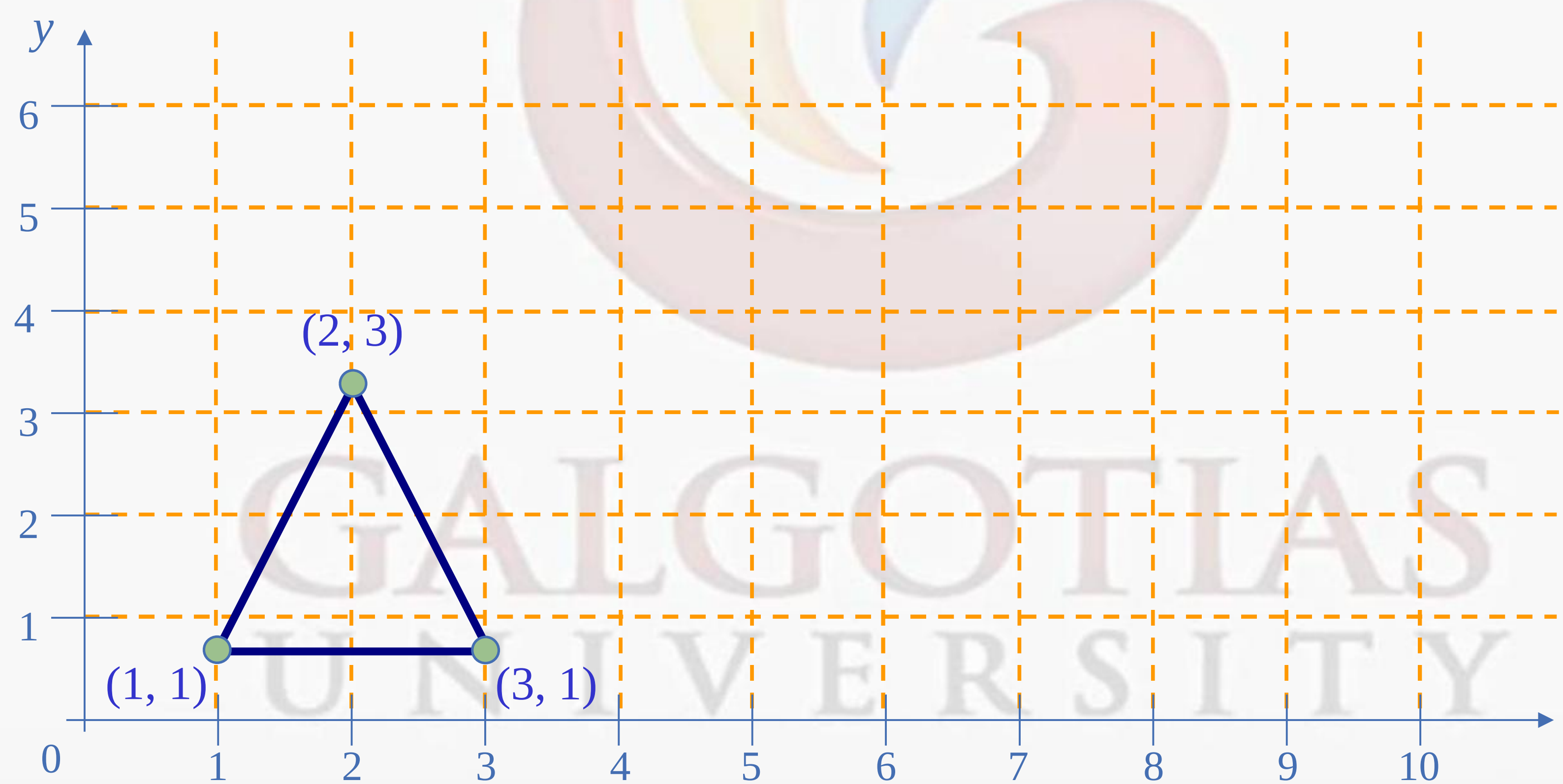
– **If the scale factor had been greater than 1, it would be larger and farther away.**

WATCH OUT: **Objects grow and move!**



Note: House shifts position relative to origin

Scaling Example



Homogeneous Coordinates

- **A point** (x, y) **can be re-written in** homogeneous coordinates **as** (x_h, y_h, h)
- **The** homogeneous parameter h **is a non-zero value such that:**

$$x = \frac{x_h}{h} \qquad y = \frac{y_h}{h}$$

- **We can then write any point** (x, y) **as** (hx, hy, h)
- **We can conveniently choose** $h = 1$ **so that** (x, y) **becomes** $(x, y, 1)$

Why Homogeneous Coordinates?

- **Mathematicians commonly use homogeneous coordinates as they allow scaling factors to be removed from equations**
- **We will see in a moment that all of the transformations we discussed previously can be represented as 3*3 matrices**
- **Using homogeneous coordinates allows us use matrix multiplication to calculate transformations – extremely efficient!**

Homogenous Coordinates

- **Combine the geometric transformation into a single matrix with 3by3 matrices**
- **Expand each 2D coordinate to 3D coordinate with homogenous parameter**
- **Two-Dimensional translation matrix**

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

- **Two-Dimensional rotation matrix**

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\cos\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

- **Two-Dimensional scaling matrix**

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Inverse transformations

- **Inverse translation matrix**

$$T^{-1} = \begin{bmatrix} 1 & 0 & -t_x \\ 0 & 1 & -t_y \\ 0 & 0 & 1 \end{bmatrix}$$

- **Two-Dimensional translation matrix**

$$R^{-1} = \begin{bmatrix} \cos{}^{-} & \sin{}^{-} & 0 \\ -\sin{}^{-} & \cos{}^{-} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- **Two-Dimensional translation matrix**

$$S^{-1} = \begin{bmatrix} \frac{1}{s_x} & 0 & 0 \\ 0 & \frac{1}{s_x} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Basic 2D Transformations

- Basic 2D transformations as 3x3 Matrices

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & tx \\ 0 & 1 & ty \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Translate

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} sx & 0 & 0 \\ 0 & sy & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Scale

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Rotate

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & shx & 0 \\ shy & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Shear

Geometric transformations in three-dimensional space

- **Extend from two-dimensional transformation by including considerations for the z coordinates**
- **Translation and scaling are similar to two-dimension, include the three Cartesian coordinates**
- **Rotation method is less straight forward**
- **Representation**
  - **Four-element column vector for homogenous coordinates**
  - **Geometric transformation described 4by4 matrix**

Geometric transformations in three-dimensional space

- **Three-dimensional translation**
  - **A point P (x,y,z) in three-dimensional space translate to new location with the translation distance T (tx, ty, tz)**

$$x' = x + t_x \qquad y' = y + t_y \qquad z' = z + t_z$$

  - **In matrix format**

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \qquad P' = T \cdot P$$

Geometric transformations in three-dimensional space

- **Three-dimensional scaling**
  - **Relative to the coordinate origin, just include the parameter for z coordinate scaling in the transformation matrix**

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \qquad P' = S \cdot P$$

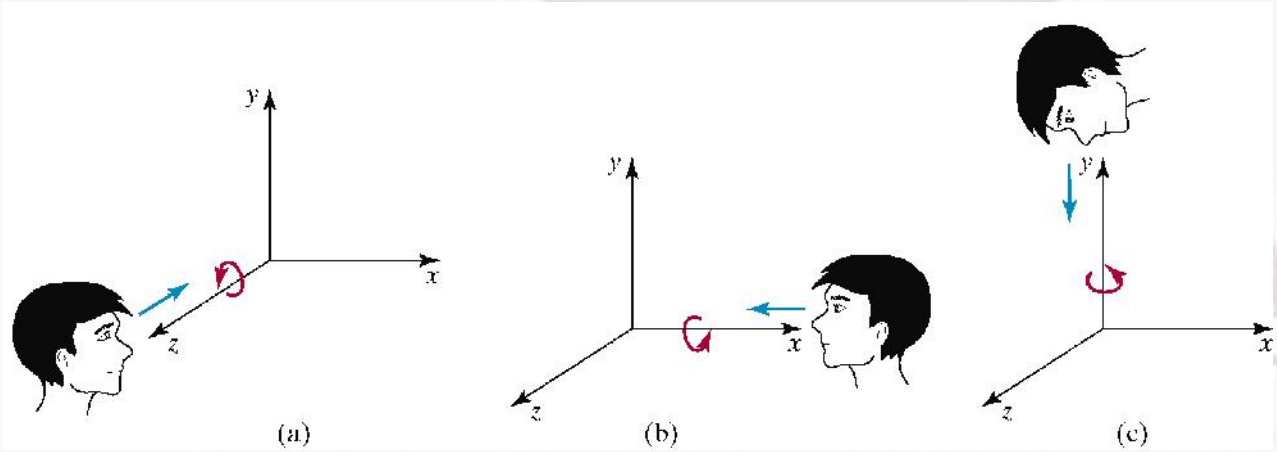
  - **Relative to a fixed point (x_f, y_f z_f)**
    - **Perform a translate-scaling-translate composite transformation**

$$t(x_f, y_f, z_f) \cdot S(s_x, s_y, s_z) \cdot T(-x_f, -y_f, -z_f)$$

$$= \begin{bmatrix} s_x & 0 & 0 & (1-s_x)x_f \\ 0 & s_y & 0 & (1-s_y)y_f \\ 0 & 0 & s_z & (1-s_z)z_f \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Geometric transformations in three-dimensional space

- **Three-dimensional rotation definition**
  - **Assume looking in the negative direction along the axis**
  - **Positive angle rotation produce counterclockwise rotations about a coordinate axis**



Geometric transformations in three-dimensional space

- **Three-dimensional coordinate-axis rotation**
  - **Z-axis rotation equations**

$$x' = x\cos\theta - y\sin\theta$$
$$y' = x\sin\theta + y\cos\theta$$
$$z' = z$$

R_z

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

  - **Transformation equations for rotation about the other two coordinate axes can be obtained by a cyclic permutation**

x → y → z → x

  - **X-axis rotation equations**

$$y' = y\cos\theta - z\sin\theta$$
$$z' = y\sin\theta + z\cos\theta$$
$$x' = x$$

$$R_x = R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Geometric transformations in three-dimensional space

- **Three-dimensional coordinate-axis rotation**
  - **Y-axis rotation equations**

$$z' = z\cos\theta - x\sin\theta$$
$$x' = z\sin\theta + x\cos\theta$$
$$y' = y$$

$$R_y = R_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

  - **General Three-dimensional rotations**
    - **Translate object so that the rotation axis coincides with the parallel coordinate axis**
    - **Perform the specified rotation about that axis**
    - **Translate object back to the original position**

$$P' = T^{-1} \cdot R_x(\theta) \cdot T \cdot P$$
$$R(\theta) = T^{-1} \cdot R_x(\theta) \cdot T$$

Geometric transformations in three-dimensional space (7)

- **Inverse of a rotation matrix**

$$R^{-1}(\theta) = R(-\theta)$$

$$\cos(-\theta) = \cos\theta, \quad \sin(-\theta) = -\sin\theta$$

$$R_z^{-1}(\theta) = R_z(-\theta) = \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) & 0 & 0 \\ \sin(-\theta) & \cos(-\theta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R^{-1} = R^T$$: *orthogonal matrix*

Concatenation of Transformations

- **Concatenating**
  - **affine transformations by multiplying together**
  - **sequences of the basic transformations**
    - ➔ **define an arbitrary transformation directly**
  - **ex) three successive transformations**

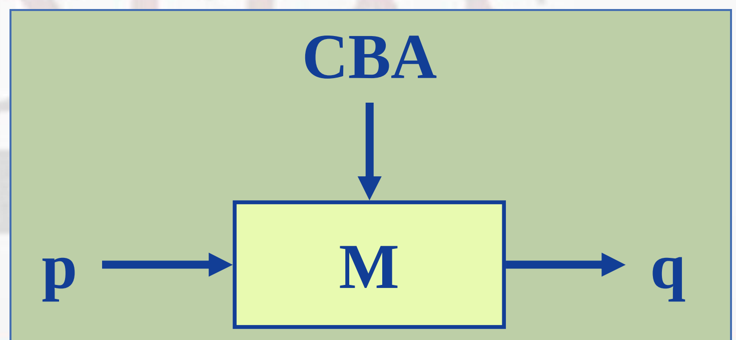
p1 = Ap
p2 = Bp1
q = Cp2

q = CBp1
q = CBAp

$$\Rightarrow q = (C(B(Ap))) = CBAp$$

$$M = CBA$$

$$q = Mp$$

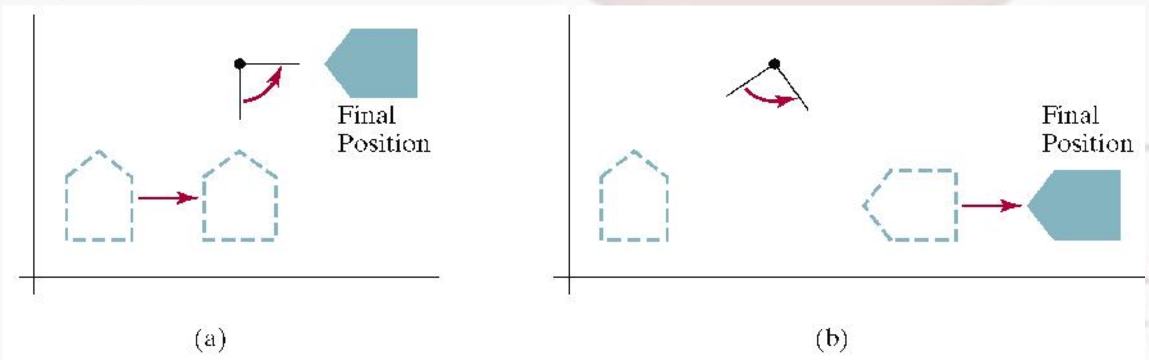


Matrix Concatenation Properties

- **Associative properties**

$$M_1 \cdot M_2 \cdot M_3 = M_1 \cdot (M_2 \cdot M_3) = (M_1 \cdot M_2) \cdot M_3$$

- **Transformation is not commutative (CopyCD!)**
  - **Order of transformation may affect transformation position**



Two-dimensional composite transformation

- **Composite transformation**
  - **A sequence of transformations**
  - **Calculate composite transformation matrix rather than applying individual transformations**

$$P' = M_2 \cdot M_1 \cdot P$$

$$P' = M \cdot P$$

- **Composite two-dimensional translations**
  - **Apply two successive translations, T_1 and T_2**

$$T_1 = T(t_{1x}, t_{1y})$$

$$T_2 = T(t_{2x}, t_{2y})$$

$$P' = T_2 \cdot (T_1 \cdot P) = (T_2 \cdot T_1) \cdot P$$

  - **Composite transformation matrix in coordinate form**

$$T(t_{2x}, t_{2y}) \cdot T(t_{1x}, t_{1y}) = T(t_{1x} + t_{2x}, t_{1y} + t_{2y})$$

Two-dimensional composite transformation

- **Composite two-dimensional rotations**
  - **Two successive rotations, R_1 and R_2 into a point P**

$$P' = R(\theta_1) \cdot \{R(\theta_2) \cdot P\}$$

$$P' = \{R(\theta_1) \cdot R(\theta_2)\} \cdot P$$

  - **Multiply two rotation matrices to get composite transformation matrix**

$$R(\theta_1) \cdot R(\theta_2) = R(\theta_1 + \theta_2)$$

$$P' = R(\theta_1 + \theta_2) \cdot P$$

- **Composite two-dimensional scaling**

$$S(s_{1x}, s_{1y}) \cdot S(s_{2x}, s_{2y}) = S(s_{1x} \cdot s_{2x}, s_{1y} \cdot s_{2y})$$

$$P' = S(s_{1x} \cdot s_{2x}, s_{1y} \cdot s_{2y}) \cdot P$$

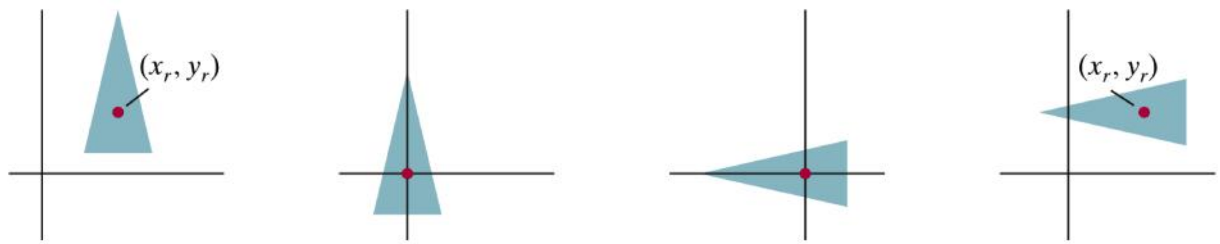
Two-dimensional composite transformation

- **General two-dimensional Pivot-point rotation**
  - **Graphics package provide only origin rotation**
  - **Perform a translate-rotate-translate sequence**
    - **Translate the object to move pivot-point position to origin**
    - **Rotate the object**
    - **Translate the object back to the original position**
  - **Composite matrix in coordinates form**

$$T(x_r, y_r)\cdot R(\theta)\cdot T(-x_r, -y_r) = R(x_r, y_r, \theta)$$

Two-dimensional composite transformation

- **Example of pivot-point rotation**



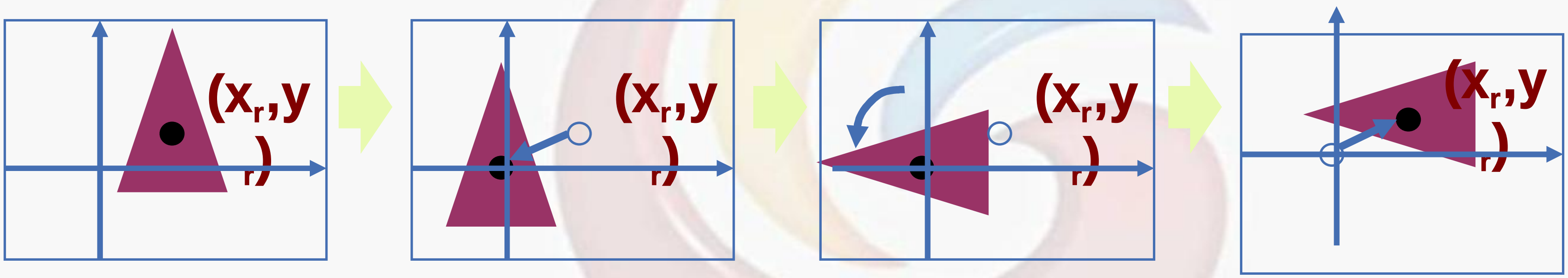
(a) Original Position of Object and Pivot Point

(b) Translation of Object so that Pivot Point (x_r, y_r) is at Origin

(c) Rotation about Origin

(d) Translation of Object so that the Pivot Point is Returned to Position (x_r, y_r)

Pivot-Point Rotation



Translate Rotate Translate

$$T(x_r, y_r)\cdot R(\theta)\cdot T(-x_r, -y_r) = R(x_r, y_r, \theta)$$

$$\begin{bmatrix} 1 & 0 & x_r \\ 0 & 1 & y_r \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & -x_r \\ 0 & 1 & -y_r \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & x_r(1-\cos\theta)+y_r\sin\theta \\ \sin\theta & \cos\theta & y_r(1-\cos\theta)-x_r\sin\theta \\ 0 & 0 & 1 \end{bmatrix}$$

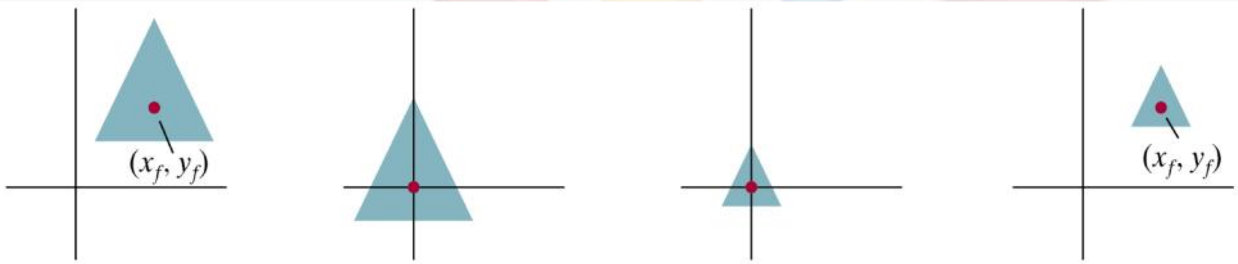
Two-dimensional composite transformation

- **General two-dimensional Fixed-point scaling**
  - **Perform a translate-scaling-translate sequence**
    - **Translate the object to move fixed-point position to origin**
    - **Scale the object wrt. the coordinate origin**
    - **Use inverse of translation in step 1 to return the object back to the original position**
  - **Composite matrix in coordinates form**

$$T(x_f, y_f) \cdot S(s_x, s_y) \cdot T(-x_f, -y_f) = S(x_f, y_f, s_x, s_y)$$

Two-dimensional composite transformation

- **Example of fixed-point scaling**



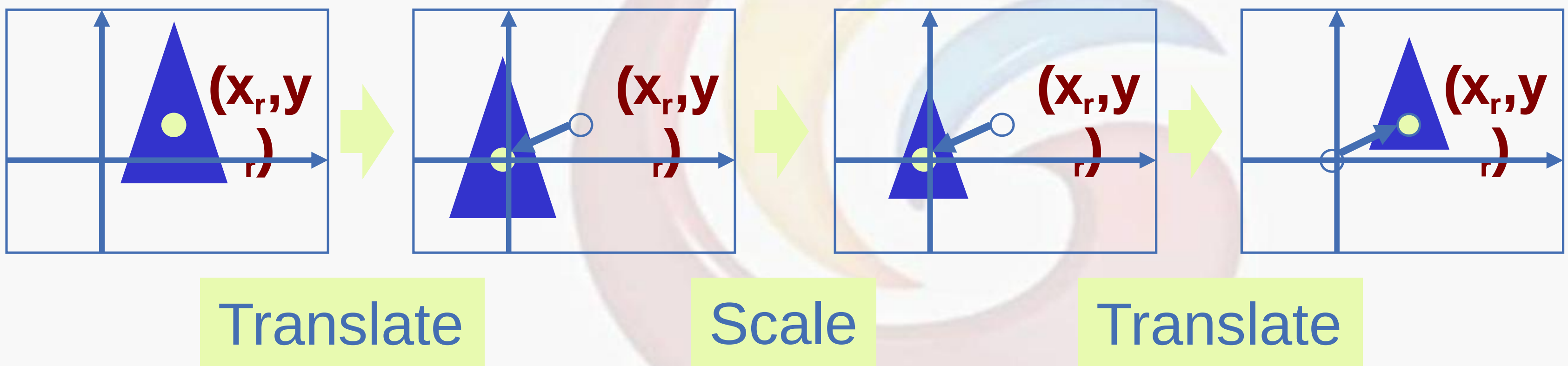
(a) Original Position of Object and Fixed Point

(b) Translate Object so that Fixed Point (x_f, y_f) is at Origin

(c) Scale Object with Respect to Origin

(d) Translate Object so that the Fixed Point is Returned to Position (x_f, y_f)

GeNeral Fixed-Point Scaling



$$T(x_{f,}y_f)\cdot S(s_x,s_y)\cdot T(-x_f,-y_f)=S(x_{f,}y_f,s_x,s_y)$$

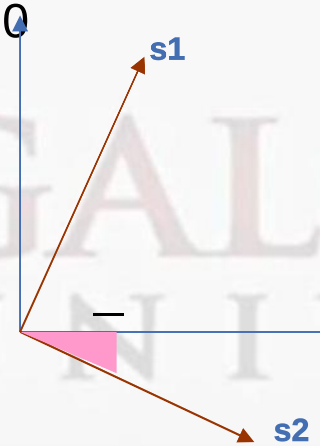
$$\begin{bmatrix}1 & 0 & x_f\\0 & 1 & y_f\\0 & 0 & 1\end{bmatrix}\cdot\begin{bmatrix}s_x & 0 & 0\\0 & s_y & 0\\0 & 0 & 1\end{bmatrix}\cdot\begin{bmatrix}1 & 0 & -x_f\\0 & 1 & -y_f\\0 & 0 & 1\end{bmatrix}=\begin{bmatrix}s_x & 0 & x_f(1-s_x)\\0 & s_y & y_f(1-s_y)\\0 & 0 & 1\end{bmatrix}$$

Two-dimensional composite transformation

- **General two-dimensional scaling directions**
  - **Perform a rotate-scaling-rotate sequence**
  - **Composite matrix in coordinates form**

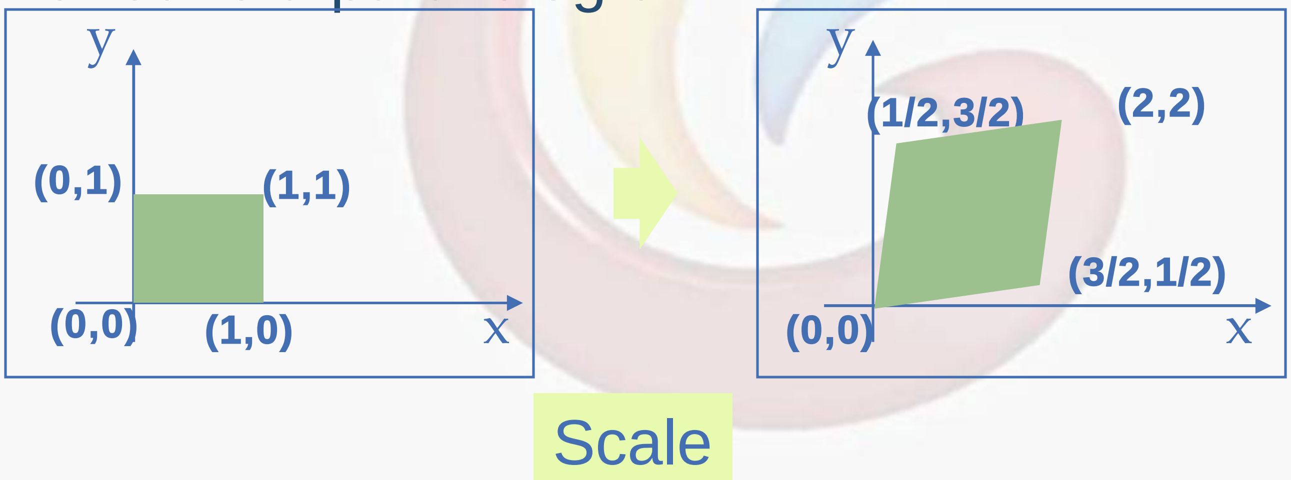
$$R^{-1}(¯)\cdot S(s_1,s_2)\cdot R(¯)$$

$$=\begin{bmatrix} s_1\cos^2¯+s_2\sin^2¯ & (s_2-s_1)\cos¯\ \sin¯ & 0 \\ (s_2-s_1)\cos¯\ \sin¯ & s_1\sin^2¯+s_2\cos^2¯ & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

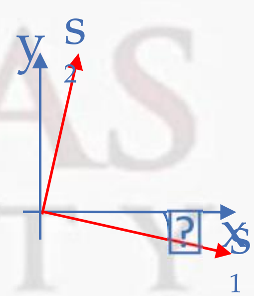


General Scaling Directions

- Converted to a parallelogram

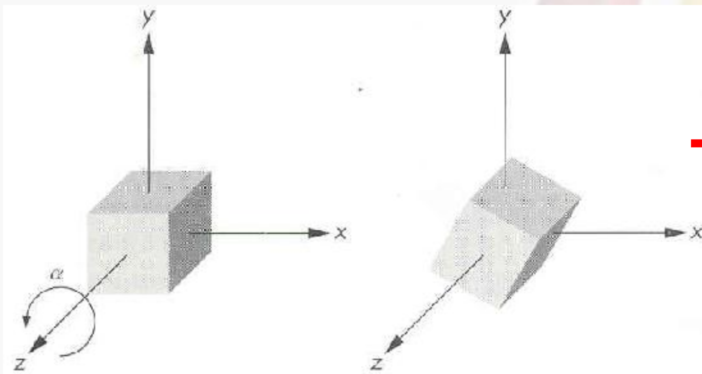


$$R^{-1}(\theta)\cdot S(s_1,s_2)\cdot R(\theta)=\begin{bmatrix} s_1\cos^2\theta+s_2\sin^2\theta & (s_2-s_1)\cos\theta\sin\theta & 0 \\ (s_2-s_1)\cos\theta\sin\theta & s_1\sin^2\theta+s_2\cos^2\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

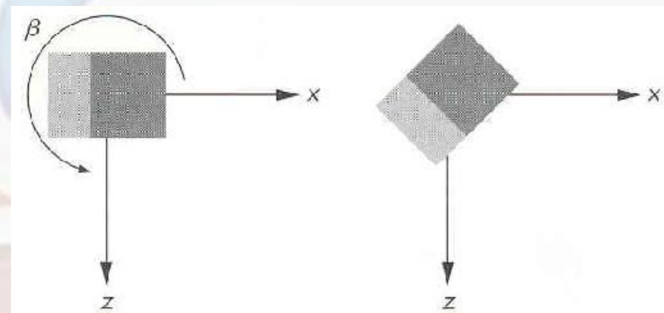


General Rotation

- **Three successive rotations about the three axes**

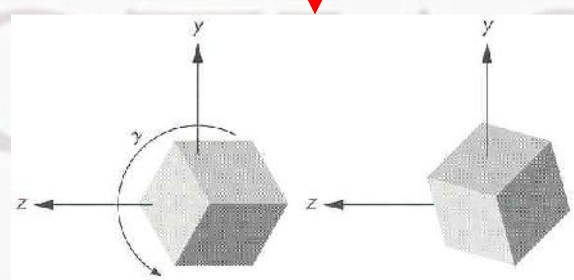


rotation of a cube about the z axis



rotation of a cube about the y axis

?



rotation of a cube about the x axis

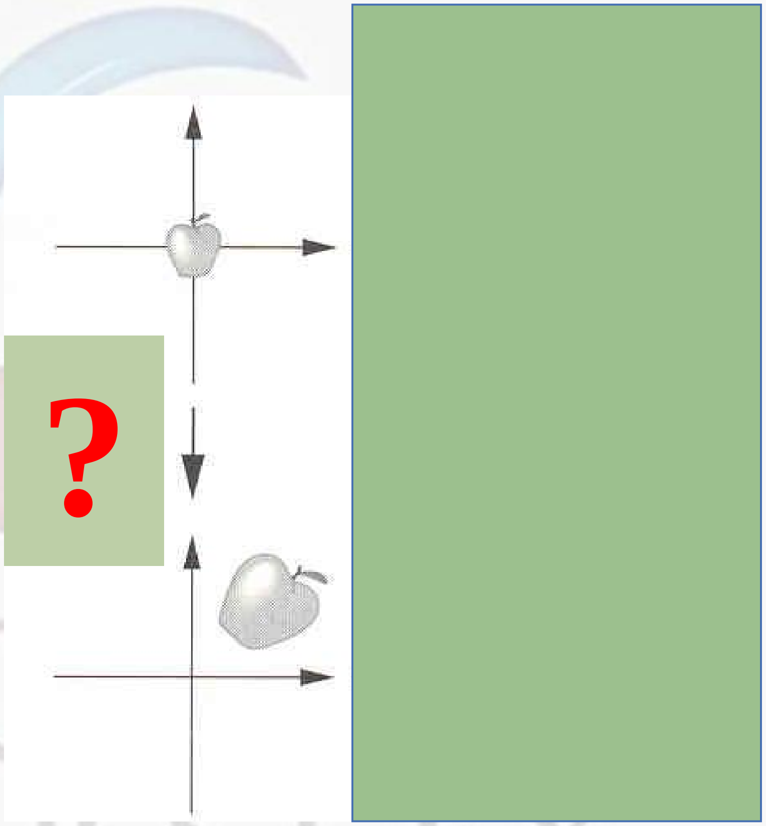
General Rotation

$$\mathbf{R} = R_x R_y R_z$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos¥ & -\sin¥ & 0 \\ 0 & \sin¥ & \cos¥ & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos & 0 & \sin & 0 \\ 0 & 1 & 0 & 0 \\ -\sin & 0 & \cos & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\sim & -\sin\sim & 0 & 0 \\ \sin\sim & \cos\sim & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Instance Transformation

- **Instance of an object's prototype**
  - **occurrence of that object in the scene**
- **Instance transformation**
  - **applying an affine transformation to the prototype to obtain desired size, orientation, and location**



instance transformation

Instance Transformation

M = TRS

$$\begin{bmatrix} 1 & 0 & 0 & \text{¥}_x \\ 0 & 1 & 0 & \text{¥}_y \\ 0 & 0 & 1 & \text{¥}_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos & -\sin & 0 & 0 \\ \sin & \cos & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sim_x & 0 & 0 & 0 \\ 0 & \sim_y & 0 & 0 \\ 0 & 0 & \sim_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$