Lecture-10

Solution of Volterra integral equation of second kind by Resolvent kernels

Consider a Volterra integral equation of second kind

$$y(x) = f(x) + \lambda \int_a^x K(x,t) y(t) dt. \quad ...(1)$$

where $K(x,t)$ is a continuous function in $a \leq x \leq b$, $a \leq t \leq x$ and $f(x)$ is continuous function in $a \leq x \leq b$.

We seek the solution of integral equation (1) in the form of an infinite power in series λ.

we may write equation (1) as,

$$y(x)=f(x)+\lambda\int_a^x K(x,t_1)y(t_1)dt_1. \qquad ...(2)$$

Replace x by t in (2), we get

$$y(t)=f(t)+\lambda\int_a^t K(t,t_1)y(t_1)dt_1. \qquad ...(3)$$

Substituting the value of $y(t)$ in equation (1), we get

$$y(x)=f(x)+\lambda\int_a^x K(x,t)\left(f(t)+\lambda\int_a^t K(t,t_1)y(t_1)dt_1\right)dt. \qquad ...(4)$$

or

$$y(x)=f(x)+\lambda\int_a^x K(x,t)f(t)dt+\lambda^2\int_a^x K(x,t)\int_a^t K(t,t_1)y(t_1)dt_1dt. \quad ...(5)$$

we may write equation (3) as

$$y(t)=f(t)+\lambda\int_a^t K(t,t_2)y(t_2)dt_2. \quad ...(6)$$

Replacing t by t_1 in (6), we have

$$y(t_1)=f(t_1)+\lambda\int_a^{t_1} K(t_1,t_2)y(t_2)dt_2. \quad ...(7)$$

Substituting the above value of $y(t_1)$ in (5), we have

$$y(x) = f(x) + \lambda \int_a^x K(x,t) f(t) dt$$
$$+ \lambda^2 \int_a^x K(x,t) \int_a^t K(t,t_1) \left(f(t_1) + \lambda \int_a^{t_1} K(t_1,t_2) y(t_2) dt_2 \right) dt_1 dt,$$

or

$$y(x) = f(x) + \lambda \int_a^x K(x,t) f(t) dt + \lambda^2 \int_a^x K(x,t) \int_a^t K(t,t_1) f(t_1) dt_1 dt$$
$$+ \lambda^3 \int_a^x K(x,t) \int_a^t K(t,t_1) \int_a^{t_1} K(t_1,t_2) y(t_2) dt_2 dt_1 dt. \quad ...(8)$$

Proceeding likewise, we have

$$y(x) = f(x) + \lambda\int_a^x K(x,t)f(t)dt + \lambda^2\int_a^x K(x,t)\int_a^t K(t,t_1)f(t_1)dt_1dt + \ldots$$

$$+\lambda^n\int_a^x K(x,t)\int_a^t K(t,t_1)\ldots\int_a^{t_{n-2}} K(t_{n-2},t_{n-1})f(t_{n-1})dt_{n-1}\ldots dt_1dt + R_{n+1}(x), \quad \ldots(9)$$

where

$$R_{n+1}(x) = \lambda^{n+1}\int_a^x K(x,t)\int_a^t K(t,t_1)\ldots\int_a^{t_{n-1}} K(t_{n-1},t_n)y(t_n)dt_n\ldots dt_1dt. \quad \ldots(10)$$

Now, let us consider the following infinite series

$$f(x) + \lambda\int_a^x K(x,t)f(t)dt + \lambda^2\int_a^x K(x,t)\int_a^t K(t,t_1)f(t_1)dt_1 + \ldots \quad \ldots(11)$$

In view of the assumptions, it follows that the series (11) converges uniformly and absolutely.

Further, $$\lim_{n \to \infty} R_{n+1}(x) = 0,$$

and hence the function satisfying (9) is the continuous function given by the series (11) and it is a unique solution of (1).

Reference:
https://nptel.ac.in/courses/111/107/111107103/