Corollaries of Archimedean properties:

Corollary 1. Let y be any positive real number and x be any real number. Then, there exists a positive integer n (or natural number) such that $ny > x$.

Proof: Case 1. If y<x.

……………………………………0……………y(given that y is positive)……………x……………………………………………

Then, x is also positive.

Thus, x and y both are positive and y<x. Proof done by Archimedean property.

Case 2. If $0<x<y$

............0..................x............... y(given that y is positive)..

In this case we have x and y both are positive. But the proof of Archimedean depends on y not x.

Similar proof works.

Case 3. If $x<0<y$

...............x...0............................ y(given that y is positive)..................

Proof of **Archimedean property** depends only on y not x.

Similar proof works.

Corollary 2. For any real number x there exists an integer n such that $n > x$.

Proof: Put y=1 in Corollary 1. We will get the result.

Corollary 3. For any real number x there exists two integers m and n such that $n < x < m$.

Proof: For $x < m$ we use corollary 2. ....(1)

If x is a real number (-1.2 is real number) **then $-x$ is also a real number** (-(-1.2)=1.2 is also a real number).

By using Corollary 2, there exists an integer "p" such that $p > -x$.

Operate "-" sign both the sides, we get

$-p < x$ **if p is an integer then** $-p = n$**(assume)** is also an integer.

Thus, we get $n < x$**........................(2)**

From (1) and (2)

$n < x < m$**. Proved.**

Characterization of supremum and infimum:

Characterization of supremum and infimum:

Theorem 1. Let α be a supremum of a subset **S** of **R if and only if**

(i) $x \le \alpha \ \forall \ x \in S$ **i.e.,** α is an upper bound for set **S**

(ii) $For\ any\ given\ \in > 0$ there exists some $x \in S$ such that $x > \alpha - \in$.

Proof: **If part is sufficient** and only if is necessary part.

Only if (necessary pat) i.e.,

Assume α is a supremum of a subset **S** of **R.**

Aim: Our aim is to prove that

a. $x \le \alpha \ \forall \ x \in S$ **i.e.,** α is an upper bound for set **S**

b. $For\ any\ given\ \in > 0$ there exists some $x \in S$ such that $x > \alpha - \in$.

BY assumptio α a supremum of a subset **S** of **R.** Then, by the definition of supremum

i. α is an upper bound of **S**

ii. α is the lowest(or smallest) of all upper bounds of **S.**

(i) condition implies that $x \leq \alpha \; \forall \, x \in S$

(ii)condition implies α *is the smallest of all upper bounds* so that **for given** $\varepsilon > 0$,

$\alpha - \varepsilon < \alpha.$

There $\alpha - \varepsilon$ will not be an upper bound for **S.**

This implies that **there exists some** $x \in S$ **such that** $x > \alpha - \varepsilon.$

Proved

If part(sufficient)

Assume

a. $x \leq \alpha \; \forall \; x \in S$ **i.e.,** α is an upper bound for set **S**

b. *For any given* $\in > 0$ there exists some $x \in S$ such that $x > \alpha - \in$.

Aim: α is the supremum of **S.**

(a) Condition implies that α is an upper bound
(b) Condition implies that α is smallest
BY def α is the supremum of **S.**

Proved.

School of basic and applied sciences
Galgotias University

Course Code : BSCM301 **Course Name: Real Analysis-I**

Theorem 1. Let β be a infimum of a subset **S** of **R if and only if**

(i) $x \geq \beta \ \forall \ x \in S$ **i.e.,** β is a lower bound for set **S**

(ii) *For any given* $\epsilon > 0$ there exists some $x \in S$ such that $x < \beta + \epsilon$.

Proof: Similar proof as above

Reference book: Bansi Lal and Sanjay Arora; Introduction to Real Analysis, Satya Prakashan, 1st Vol (1991)