Theorem(Archimedean property): Let x and y be any two positive real numbers with $y < x$. Then, there exists a positive integer n (or natural number) such that $ny > x$.

if

………0………………y (real number)…………..x(real number)………………………………………………………..

Then

…………………………0…………………………………………x…………………………….ny………………………………..

Proof: Suppose not i.e., $ny \le x \; \forall \, n \in \boldsymbol{N}$.

Let $A = \{ny : n \in \boldsymbol{N}\}$

Then, **A** is non empty (since it true for $y = 1$) and $ny \le x \; \forall \, n \in \boldsymbol{N}$ this implies that x is an upper bound of set **A**. This implies **A** is bounded above.

By completeness property in **R,** every bounded above subset has a supremum.

Sup **A**=α (say or assume)

This implies α is an upper bound ($ny \leq \alpha \; \forall \; n \in \boldsymbol{N} \; \boldsymbol{and} \; ny \in \boldsymbol{A}$) as well as lowest of all upper bound.

If $n \in \boldsymbol{N}$ then $n+1 \in \boldsymbol{N}.$

Therefore $(\boldsymbol{n}+\boldsymbol{1})\boldsymbol{y} \in \boldsymbol{A}$ this implies that $(n+1)y \leq \alpha \; \forall \; n \in \boldsymbol{N} \; \boldsymbol{and} \; ny \in \boldsymbol{A}.$

Then, $ny + y \leq \alpha$ this implies $ny \leq \alpha - y \; \forall \; n \in \boldsymbol{N}.$

This implies $\alpha - y$ is an upper bound but $\alpha - y < \alpha.$

This contradict that α is lowest of all upper bounds.

This implies that $ny \leq x \ \forall \ n \in \boldsymbol{N}$ **is wrong.**

Thus, $ny > x \ \forall \ n \in \boldsymbol{N}$. **Proved.**

Theorem: Suppose x and y be any two rational numbers. Then, there exists at least one rational number between x and y and hence infinitely many rational numbers.

Theorem: Suppose x and y be any two rational numbers. Then, there exists at least one rational number between x and y and hence infinitely many rational numbers.

..................x(rational)........................r1.........r2............r3.................................y(rational)...

Proof: Given, x and y be any two rational numbers. Then, $\frac{x+y}{2}$ is also a rational number.

Also , $x < \frac{x+y}{2} = r1 < y$. Proved

Similarly we can prove that there exists infinitely many rational numbers.

Theorem: Suppose x and y be any two real numbers. Then, there exists at least one rational number and hence infinitely rational numbers.

Proof: Given, x and y be any two real numbers.

Let us assume that $x < y \quad or \quad y < x$ this implies $y - x > 0 \quad or \quad x - y > 0.$

(since $x - x < y - x \quad or \quad y - y < x - y$ or
$0 < y - x \quad or \quad 0 < x - y$)

Also **n=1** is a real number.

We have now two positive real numbers **n=1** and $y - x > 0$.

By Achimedean property (if we take x=1, Y=y-x) there exists an integer m

Such that $m(y - x) > 1.$

This implies $my - mx > 1.$

This implies that difference of two real numbers is strictly greater than 1.

......0...0.2.........1...1.200000009...........2...2.20000000000009...........3..

Then, my *and* mx definitely contains an integer, say, n .

This implies $mx < n < my$.

Now divide by m on the both sides, we get

$x < \frac{n}{m} = r(rational\ number) < y$. Proved

Similarly we can prove that there are infinitely many rational numbers.

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Theorem: Suppose x and y be any two real numbers. Then, there exists at least one irrational number and hence infinitely rational numbers.

Proof: Given x and y be any two real numbers then $\sqrt{2}$ x and $\sqrt{2}$ y also be real numbers.

Then, by above theorem there exists a rational number r between $\sqrt{2}$ x and $\sqrt{2}$ y such that

$\sqrt{2}$ x<r<$\sqrt{2}$ y

Divide by $\sqrt{2}$ both the sides

$x < \frac{r}{\sqrt{2}} = irrational\ numvber <\ y$. Proved

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Corollaries of Archimedean properties:

Corollary 1. Let y be any positive real number and x be any real number. Then, there exists a positive integer n (or natural number) such that $ny > x$.

Corollary 2. For any real number x there exists an integer n such that $n > x$

Corollary 3. For any real number x there exists two integers m and n such that $n < x < m$.

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Reference book: Bansi Lal and Sanjay Arora; Introduction to Real Analysis, Satya Prakashan, 1st Vol (1991)