

**School of Basic and Applied Sciences**

**Course Code : MSCM301**

**Course Name: Functional Analysis**

# **Space of Bounded linear transformations**

**By**

**Dr. Alok Tripathi**  
**Department of Mathematics**

**Name of the Faculty: Dr. Alok Tripathi**

**Program Name: M. Sc**

Let us denote by  $\beta(N, N')$  the set of all bounded linear transformations of a normed linear space  $N$  into  $N'$ . With the notation, we establish the following theorem.

Theorem:  $\beta(N, N')$  is a normed linear space with respect to pointwise operations

$$(T_1 + T_2)(x) = T_1(x) + T_2(x), (\alpha T)(x) = \alpha T(x)$$

and the norm defined by

$$\|T\| = \sup \left\{ \frac{\|T(x)\|}{\|x\|} : x \in N \text{ and } x \neq 0 \right\}$$

## School of Basic and Applied Sciences

Course Code : MSCM301

Course Name: Functional Analysis

Let  $T_1, T_2 \in \beta(N, N')$  and  $\alpha$  be a scalar. Then  $T_1$  and  $T_2$  are bounded. So there exist real numbers  $M_1$  and  $M_2$  such that

$$\|T_1(x)\| \leq M_1 \|x\| \text{ and } \|T_2(x)\| \leq M_2 \|x\| \text{ for all } x \in N. \quad \dots(2)$$

Now 
$$\|(T_1 + T_2)(x)\| = \|T_1(x) + T_2(x)\| \leq \|T_1(x)\| + \|T_2(x)\|.$$

Using (2), we get  $\|(T_1 + T_2)x\| \leq M_1 \|x\| + M_2 \|x\| = M \|x\|$  where  $M = M_1 + M_2$ .

Hence  $T_1 + T_2 \in \beta(N, N')$  for all  $T_1, T_2 \in \beta(N, N')$ . In a similar manner  $\alpha T \in \beta(N, N')$  for all scalars  $\alpha$  and  $T \in \beta(N, N')$ . So  $\beta(N, N')$  is a linear space.

To prove that  $\beta(N, N')$  is a normed space, we show that (1) satisfies (N1), (N2) and (N3) of the definition of a norm.

## School of Basic and Applied Sciences

Course Code : MSCM301

Course Name: Functional Analysis

(N1) Since  $\|T(x)\| \geq 0$ ,  $\|T\| \geq 0$  from the definition of  $\|T\|$ .

If  $T = 0$ , then  $\|T\| = 0$ . Conversely we have to show that  $\|T\| = 0$  implies  $T = 0$ . If  $\|T\| = 0$ , we get from  $\|T(x)\| \leq \|T\| \|x\|$ ,  $T(x) = 0$  for all  $x \in N$ . That is  $T = 0$ . Hence  $\|T\| \geq 0$  and  $\|T\| = 0$  if and only if  $T = 0$ .

(N2) Let  $T_1, T_2 \in \beta(N, N')$ . Then we have

$$\|(T_1 + T_2)(x)\| = \|T_1(x) + T_2(x)\| \leq \|T_1(x)\| + \|T_2(x)\| \leq (\|T_1\| + \|T_2\|) \|x\|.$$

Hence we have from the above for  $x \neq 0$ ,  $\sup_{x \in N} \left\{ \frac{\|(T_1 + T_2)(x)\|}{\|x\|} \right\} \leq (\|T_1\| + \|T_2\|)$ .

## School of Basic and Applied Sciences

Course Code : MSCM301

Course Name: Functional Analysis

Using the definition of the norm of a bounded linear transformation,  
we get

$$\|T_1 + T_2\| \leq \|T_1\| + \|T_2\|.$$

(N3) If  $\alpha$  is a scalar, and  $x \neq 0$ , we get  $\|\alpha T(x)\| = |\alpha| \|T(x)\|$

$$\sup_{x \in N} \frac{\|\alpha T(x)\|}{\|x\|} = |\alpha| \sup_{x \in N} \frac{\|T(x)\|}{\|x\|}.$$

This proves that  $\|\alpha T\| = |\alpha| \|T\|$ .

UNIVERSITY

## School of Basic and Applied Sciences

Course Code : MSCM301

Course Name: Functional Analysis

Video Links:

1. <https://youtu.be/l7sx9kAXjzg>

The logo of Galgotias University is a stylized 'G' composed of three curved, overlapping bands in shades of yellow, blue, and red. Below the logo, the text 'GALGOTIAS UNIVERSITY' is displayed in a large, light grey, serif font.

GALGOTIAS  
UNIVERSITY

Name of the Faculty: Dr. Alok Tripathi

Program Name: M. Sc

**Reference**

- ❑ A first course in functional Analysis by D. Somasundaram

**GALGOTIAS**  
**UNIVERSITY**

# School of Basic and Applied Sciences

Course Code :

Course Name: Calculus



# Thank You

GALGOTIAS  
UNIVERSITY

Name of the Faculty: Dr. Alok Tripathi

Program Name: B.Tech