Space of Bounded linear transformations

By

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Let us denote by $\beta(N, N')$ the set of all bounded linear transformations of a normed linear space N into N'. With the notation, we establish the following theorem.

Theorem: $\beta(N, N')$ is a normed linear space with respect to pointwise operations

$$(T_1 + T_2)(x) = T_1(x) + T_2(x), (\alpha T)(x) = \alpha T(x)$$

and the norm defined by

$$||T|| = \sup \left\{ \frac{||T(x)||}{||x||} : x \in N \; and \; x \neq 0 \right\}$$

Let $T_1, T_2 \in \beta(N, N')$ and α be a scalar. Then T_1 and T_2 are bounded. So tnere exist real numbers M_1 and M_2 such that

$$||T_1(x)|| \leq M_1 ||x|| \text{ and } ||T_2(x)|| \leq M_2 ||x|| \text{ for all } x \in N. \quad ...(2)$$

Now $$||(T_1 + T_2)(x)|| = ||T_1(x) + T_2(x)|| \leq ||T_1(x)|| + ||T_2(x)||.$$

Using (2), we get $||(T_1 + T_2)x|| \leq M_1 ||x|| + M_2 ||x|| = M ||x||$ where $M = M_1 + M_2$.

Hence $T_1 + T_2 \in \beta(N, N')$ for all $T_1, T_2 \in \beta(N, N')$. In a similar manner $\alpha T \in \beta(N, N')$ for all scalars α and $T \in \beta(N, N')$. So $\beta(N, N')$ is a linear space.

To prove that $\beta(N, N')$ is a normed space, we show that (1) satisfies (N1), (N2) and (N3) of the definition of a norm.

(N1) Since $||T(x)|| \geq 0$, $||T|| \geq 0$ from the definition of $||T||$.

If $T = 0$, then $||T|| = 0$. Conversely we have to show that $||T|| = 0$ implies $T = 0$. If $||T|| = 0$, we get from $||T(x)|| \leq ||T||\ ||x||$, $T(x) = 0$ for all $x \in N$. That is $T = 0$. Hence $||T|| \geq 0$ and $||T|| = 0$ if and only if $T = 0$.

(N2) Let $T_1, T_2 \in \beta(N, N')$. Then we have

$$||(T_1 + T_2)(x)|| = ||T_1(x) + T_2(x)|| \leq ||T_1(x)|| + ||T_2(x)|| \leq (||T_1|| + ||T_2||)\ ||x||.$$

Hence we have from the above for $x \neq 0$, $\sup\limits_{x \in N} \left\{ \dfrac{||(T_1 + T_2)(x)||}{||x||} \right\} \leq (||T_1|| + ||T_2||)$.

Using the definition of the norm of a bounded linear transformation,

we get $$||T_1 + T_2|| \leq ||T_1|| + ||T_2||.$$

(N3) If α is a scalar, and $x \neq 0$, we get $||\alpha T(x)|| = |\alpha| \; ||T(x)||$

$$\sup_{x \in N} \frac{||\alpha T(x)||}{||x||} = |\alpha| \sup_{x \in N} \frac{||T(x)||}{||x||}.$$

This proves that $||\alpha T|| = |\alpha| \; ||T||$.

Video Links:

1. https://youtu.be/l7sx9kAXjzg

Reference

- A first course in functional Analysis by D. Somasundaram

Thank You