Properties of Bounded Linear Transformation

By

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If $T_1, T_2 \in \beta(N)$, then $\|T_1 . T_2\| \leq \|T_1\| \, \|T_2\|$and if $T_n \to T$ and $T_n' \to T'$, then $T_n T_n' \to TT'$ as $n \to \infty$ which implies that the multiplication is jointly continuous.

Proof:

(1) $$||T_1 \cdot T_2|| = \sup \left\{ \frac{||(T_1 \cdot T_2)(x)||}{||x||}, x \in N \text{ and } x \neq 0 \right\}$$

$$= \sup \left\{ \frac{||T_1(T_2(x))||}{||x||}, x \in N \text{ and } x \neq 0 \right\} \leq \sup \left\{ \frac{||T_1|| \, ||T_2(x)||}{||x||}, x \in N \text{ and } x \neq 0 \right\}$$

Hence $||T_1 \cdot T_2|| \leq ||T_1|| \; ||T_2||$ from the definition of the norm of an operator.

(2) To prove that the multiplication is jointly continuous, let $T_n \to T$ and $T_n' \to T'$ as $n \to \infty$.

$$||T_n T_n' - TT'|| = ||T_n(T_n' - T') + (T_n - T)T'||$$

$$\leq ||T_n|| \, ||T_n' - T'|| + ||T_n - T|| \, ||T'|| \to 0.$$

Hence $T_n T_n' \to TT'$ as $n \to \infty$.

NOTE If $N \neq 0$, then the identity operator I is the identity element of $\beta(N)$ and $|||I||| = 1$.

$|||I||| = \sup \left\{ \frac{||I(x)||}{||x||}, x \in N \text{ and } x \neq 0. \right\}$ Since $||I(x)|| = ||x||$, we get $|||I||| = 1$.

Let N and N' be normed linear spaces. An isometric isomorphism of N into N' is a one-to-one linear transformation T of N into N' such that $||T(x)|| = ||x||$ for every $x \in N$. Further for any $x, y \in N$, we have

$$||T(x) - T(y)|| = ||T(x-y)|| = ||x-y||.$$

Thus an isometry preserves the distances. If there is an isometric isomorphism of N onto N', then N is said to be isometrically isomorphic to N' or N and N' are said to be congruent. If N and N' are congruent, it is necessary and sufficient that there exists a linear operator T with domain N and range N' such that T^{-1} exists and $||Tx|| = ||x||$ for every $x \in N$.

Two normed linear spaces N and N' are said to be topologically isomorphic, if

(i) there exists a linear operator $T : N \to N'$ having the inverse T^{-1}.

(ii) T establishes the isomorphism of N and N'.

(iii) T and T^{-1} are continuous in their respective domains.

This means that N and N' are topologically isomorphic if there is a homeomorphism T of N onto N' where T is also a linear operator. On account of this reason, N and N' are said to be linearly homeomorphic.

NOTE. Topologically isomorphic spaces need not be isometrically isomorphic. There exist examples of pairs of spaces which are topologically isomorphic but not isometrically isomorphic.

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Theorem: Let N and N' be normed linear spaces and let T be linear transformation of N into N'. If T(N) is the range of T, then the inverse T^{-1} exists and is bounded(continuous) in its domain of definition iff there exists a constant $m > 0$ such that

$$m\|x\| \leq \|T(x)\|, \forall\, x \in N.$$

Let us assume (1) and show that T^{-1} exists and it is continuous. If the condition (1) is true and if $Tx = 0$, then $x = 0$. Therefore T is one-to-one onto $T(N)$. So T^{-1} exists on $T(N)$. Therefore to each $y \in T(N)$, there exists x in N such that $T(x) = y$ and $T^{-1}(y) = x$. ...(2)

Using (2) in (1), we get $m\,||T^{-1}(y)|| \leq ||y||$ which implies,

$$||T^{-1}(y)|| \leq \frac{1}{m}||y|| \text{ for all } y \in T(N).$$

Hence T^{-1} is bounded and consequently T^{-1} is continuous.

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Conversely, let T^{-1} exists and continuous on $T(N)$. Let $x \in N$. Since T^{-1} exists, there is $y \in T(N)$ such that $T^{-1}(y) = x$ which implies and is implied by $T(x) = y$. Since T^{-1} is continuous, it is bounded so that there exists a positive constant M such that $||T^{-1}(y)|| \leq M||y||$ for all $y \in T(N)$. Hence we get $||x|| \leq M||T(x)||$.

If we take $\frac{1}{M} = m$, then we get $m||x|| \leq ||T(x)||$. This completes the proof of the theorem.

Let N and N′ be normed linear spaces. Then N and N′ are topologically isomorphic if and only if there exist a linear operator T on N onto N′ and positive constants m and M such that

$$m\,||x|| \leq ||T(x)|| \leq M\,||x|| \text{ for every } x \in N.$$

PROOF

N is topologically isomorphic to N' if and only if there exists a linear transformation T of N onto N' such that T and T^{-1} are continuous. But T is continuous if and only if there exists a positive constant M such that

$$||T(x)|| \leq M||x|| \text{ for all } x \in N \qquad ...(1)$$

by Theorem 2 of 1.9 By the previous theorem T^{-1} is continuous if and only if there exists a positive constant m such that

$$m||x|| \leq ||T(x)|| \text{ for all } x \in N \qquad ...(2)$$

If (1) and (2), it follows that N and N’ are topologically isomorphic iff $m\|x\| \leq \|T(x)\| \leq M\|x\|, \forall\, x \in N$.

Video Links:

1. https://youtu.be/TJcnVWj6jVg

Reference

- A first course in functional Analysis by D. Somasundaram

Thank You