# UNIT 2: QUANTUM MECHANICS

# The Free Particle: Continuous State

# THE FREE PARTICLE: CONTINUOUS STATES

This is the simplest one-dimensional problem; it corresponds to $V(x) = 0$ for any value of $x$. In this case the Schrödinger equation is given by

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} = E\psi(x) \qquad \Longrightarrow \qquad \left(\frac{d^2}{dx^2} + k^2\right)\psi(x) = 0, \tag{4.5}$$

where $k^2 = 2mE/\hbar^2$, $k$ being the wave number. The most general solution to (4.5) is a combination of two linearly independent *plane waves* $\psi_+(x) = e^{ikx}$ and $\psi_-(x) = e^{-ikx}$:

$$\psi_k(x) = A_+e^{ikx} + A_-e^{-ikx}, \tag{4.6}$$

where $A_+$ and $A_-$ are two arbitrary constants. The complete wave function is thus given by the stationary state

$$\Psi_k(x,t) = A_+ e^{i(kx-\omega t)} + A_- e^{-i(kx+\omega t)} = A_+ e^{i(kx-\hbar k^2 t/2m)} + A_- e^{-i(kx+\hbar k^2 t/2m)}, \quad (4.7)$$

since $\omega = E/\hbar = \hbar k^2/2m$. The first term, $\Psi_+(x,t) = A_+ e^{i(kx-\omega t)}$, represents a wave traveling to the right, while the second term, $\Psi_-(x,t) = A_- e^{-i(kx+\omega t)}$, represents a wave traveling to the left. The intensities of these waves are given by $|A_+|^2$ and $|A_-|^2$, respectively. We should note that the waves $\Psi_+(x,t)$ and $\Psi_-(x,t)$ are associated, respectively, with a free particle traveling to the right and to the left with *well-defined* momenta and energy: $p_\pm = \pm\hbar k$, $E_\pm = \hbar^2 k^2/2m$. We will comment on the physical implications of this in a moment. Since there are no boundary conditions, there are no restrictions on $k$ or on $E$; all values yield solutions to the equation.

The free particle problem is simple to solve mathematically, yet it presents a number of physical subtleties. Let us discuss briefly three of these subtleties. First, the probability densities corresponding to either solutions

$$P_{\pm}(x, t) = |\Psi_{\pm}(x, t)|^2 = |A_{\pm}|^2 \tag{4.8}$$

are constant, for they depend neither on $x$ nor on $t$. This is due to the complete loss of information about the position and time for a state with definite values of momentum, $p_{\pm} = \pm\hbar k$, and energy, $E_{\pm} = \hbar^2 k^2/2m$. This is a consequence of Heisenberg's uncertainty principle: when the momentum and energy of a particle are known exactly, $\Delta p = 0$ and $\Delta E = 0$, there must be total uncertainty about its position and time: $\Delta x \longrightarrow \infty$ and $\Delta t \longrightarrow \infty$. The second subtlety pertains to an apparent discrepancy between the speed of the wave and the speed of the particle it is supposed to represent. The speed of the plane waves $\Psi_{\pm}(x, t)$ is given by

$$v_{wave} = \frac{\omega}{k} = \frac{E}{\hbar k} = \frac{\hbar^2 k^2/2m}{\hbar k} = \frac{\hbar k}{2m} \tag{4.9}$$

solutions by means of a linear superposition of plane waves. The answer is provided by *wave packets*, which we have seen in Chapter 1:

$$\psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \phi(k) e^{i(kx-\omega t)} dk, \tag{4.12}$$

where $\phi(k)$, the amplitude of the wave packet, is given by the Fourier transform of $\psi(x, 0)$ as

$$\phi(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \psi(x,0) e^{-ikx} dx. \tag{4.13}$$

The wave packet solution cures and avoids all the subtleties raised above. First, the momentum, the position and the energy of the particle are no longer known exactly; only probabilistic outcomes are possible. Second, as shown in Chapter 1, the wave packet (4.12) and the particle travel with the same speed $v_g = p/m$, called the *group* speed or the speed of the whole packet. Third, the wave packet (4.12) is normalizable.

**To summarize, a free particle cannot be represented by a single (monochromatic) plane wave; it has to be represented by a wave packet. The physical solutions of the Schrōdinger equation are thus given by wave packets, not by stationary solutions.**

# REFERENCES


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