UNIT 1: WAVE-PARTICLE DUALITY

Phase & Group Velocity

WAVES: PHASE AND GROUP VELOCITIES OF A WAVE PACKET

The velocity of a wave can be defined in many different ways, partly because there are different kinds of waves, and partly because we can focus on different aspects or components of any given wave.

The wave function depends on both time, t, and position, x, i.e.:

$$A = A(x,t) \quad ,$$

where A is the amplitude.

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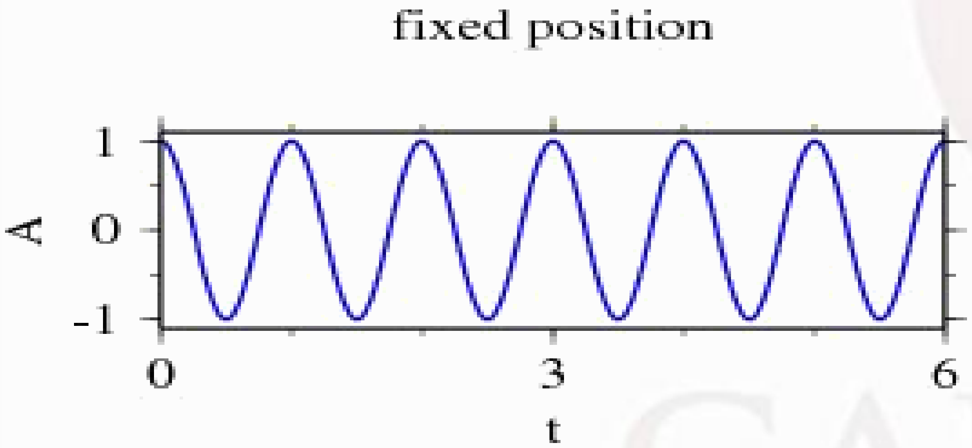
At any fixed location on the x axis the function varies sinusoidally with time.



The angular frequency, ω, of a wave is the number of radians (or cycles) per unit of time at a fixed position.

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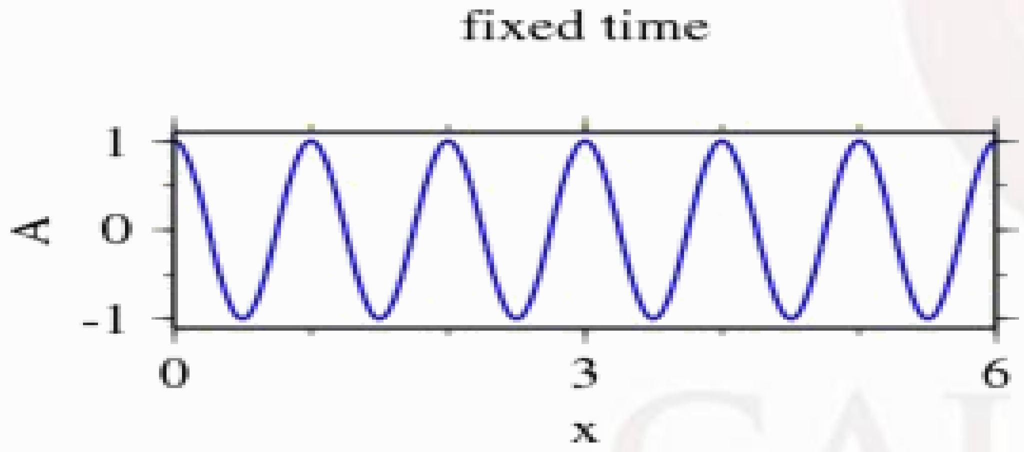
Similarly, at any fixed instant of time, the function varies sinusoidally along the horizontal axis.



The wave number, k, of a wave is the number of radians (or cycles) per unit of distance at a fixed time.

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A pure traveling wave is a function of w and k as follows:

$$A(t,x) = A_0 \sin(\omega t - kx) \quad ,$$

where A_0 is the maximum amplitude.

A wave packet is formed from the superposition of several such waves, with different A, ω, and k:

$$A(t,x) = \sum_n A_n \sin(\omega_n t - k_n x) \quad .$$

WAVE PACKET

Wave packet: As the particle velocity V is less than c and the wave velocity of de Broglie wave comes out to be greater than c. This means that the de Broglie wave associated with the particle would travel much faster than the particle itself and would leave the particle far behind. This is physically inconsistent and hence it is clear that a single wave cannot be used to describe a moving particle.

Instead a moving particle can be represented by superposition of a group of waves slightly differing in velocity and wavelength, with phases and amplitude such that they interfere constructively over a small region of space where the particle can be located and outside this space they interfere destructively so that the amplitude reduces to zero. Such a group of waves is called a wave packet.

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Consider two wavs having same amplitude A but slightly different angular frequencies (ω and $\omega+d\omega$) and wave numbers (k and $k+dk$). These waves can be represented as

$$y_1 = A\cos(\omega t - kx)$$

and $$y_2 = A\cos[(\omega + d\omega)t - (k + dk)x]$$

When these two waves interfere the resultant is given by

$$y = y_1 + y_2 = A\cos(\omega t - kx) + A\cos[(\omega + d\omega)t - (k + dk)x]$$

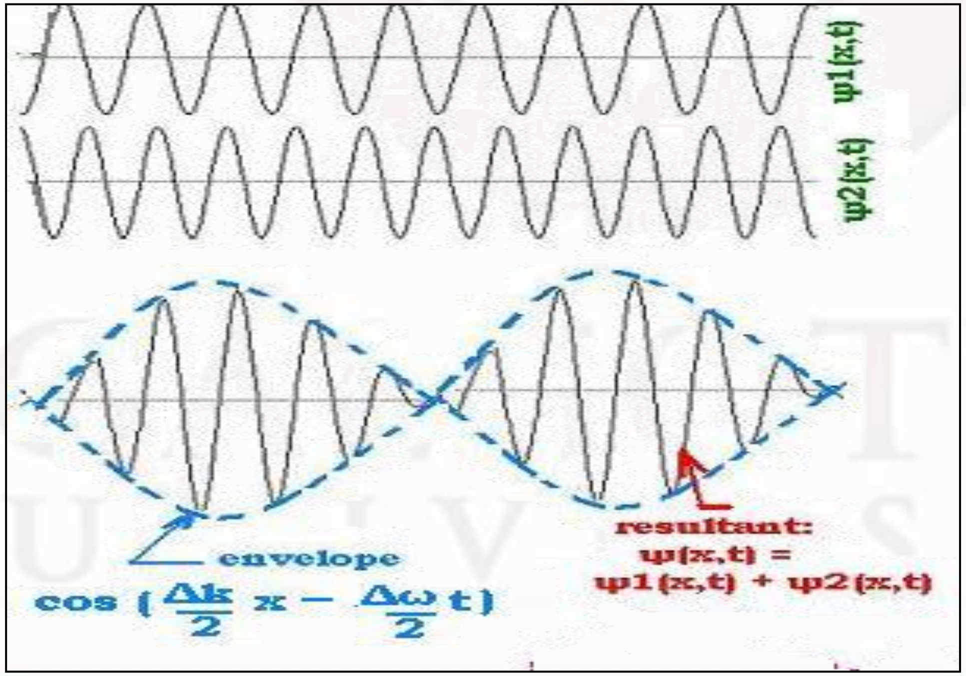
$$= 2A\cos[\frac{\omega t - kx + (\omega + d\omega)t - (k + dk)x}{2}]\cos[\frac{\omega t - kx - [(\omega + d\omega)t - (k + dk)x]}{2}]$$

$$= 2A\cos[\frac{(2\omega + d\omega)t - (2k + dk)x}{2}]\cos[\frac{d\omega\ t - dk\ x]}{2}]$$

since $d\omega$ *and* dk are very small, so $2\omega + d\omega \sim 2\omega$ *and* $2k + dk \sim 2k$.

$$Thus, y = 2A\cos[\frac{(2\omega t - 2kx)}{2}]\cos[\frac{d\omega\ t - dk\ x]}{2}] = 2A\cos(\omega t - kx)\cos(\frac{d\omega\ t - dk\ x}{2})$$

The above equation represents a wave of $2A\cos(\omega t - kx)$ with a modulation (envelope) of $\cos(\frac{d\omega\ t - dk\ x}{2})$. The velocity of first part is the wave velocity or phase velocity (v_p) while that of the envelop is called the group velocity (v_g). Thus $v_p = \frac{\omega}{k}$ *and* $v_g = \frac{d\omega}{dk}$. The figure below shows the wave packet formed by superposition of two waves ψ_1 *and* ψ_2.



Relation between particle velocity (V) and group velocity (v_g):

The wavelength of this wave is given by de Broglie's relation i.e.,

$$\lambda = \frac{h}{p} = \frac{h}{\gamma m_0 V}$$

Thus

$$v_p = \lambda \nu$$

$$v_p = \frac{h}{\gamma m_0 V}\nu \Rightarrow \nu = \frac{\gamma m_0 V v_p}{h} = \frac{\gamma m_0 c^2}{h}$$

$$v_p = \frac{c^2}{V}$$

$$\nu = \frac{\gamma m_0 c^2}{h} \Rightarrow \omega = 2\pi\nu = \frac{2\pi\gamma m_0 c^2}{h} \ldots\ldots(1)$$

$$v_g = \frac{d\omega}{dk} = \frac{d\omega/dV}{dk/dV}$$

$$k = \frac{2\pi}{\lambda} = \frac{2\pi p}{h} = \frac{2\pi\gamma m_0 V}{h} \ldots\ldots(2)$$

$$d\omega/dV \text{ and } dk/dV$$

$$\frac{d\omega}{dV} = \frac{d\left(\frac{2\pi\gamma m_0 c^2}{h}\right)}{dV} = \frac{2\pi m_0 c^2}{h}\frac{d\gamma}{dV} \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$\frac{d\gamma}{dV} = \frac{d}{dV}\left(\frac{1}{\sqrt{1 - V^2/c^2}}\right) = \frac{V}{c^2(1 - V^2/c^2)^{3/2}}$$

$$\frac{dk}{dV} = \frac{d\left(\frac{2\pi\gamma m_0 V}{h}\right)}{dV} = \frac{2\pi m_0}{h}\frac{d(\gamma V)}{dV} \qquad \text{.....................}(2)$$

$$\frac{d(\gamma V)}{dV} = \frac{c^3}{(c^2 - V^2)^{3/2}}$$

$$v_g = \frac{d\omega}{dk} = \frac{d\omega/dV}{dk/dV} = \frac{\frac{2\pi m_0 c^2}{h}\frac{d\gamma}{dV}}{\frac{2\pi m_0}{h}\frac{d(\gamma V)}{dV}} = \frac{c^2\frac{d\gamma}{dV}}{\frac{d(\gamma V)}{dV}} = \frac{c^2\frac{V}{c^2(1 - V^2/c^2)^{3/2}}}{\frac{c^3}{(c^2 - V^2)^{3/2}}} = V$$

Thus, $\boxed{v_g = \frac{d\omega}{dk} = V}$, i.e. the group velocity is equal to the particle velocity. Hence a moving particle can be described using a wave packet.

Relation between group velocity and phase velocity:

$$\omega = v_p k \text{ or } d\omega = dv_p k + v_p dk$$

$$\frac{d\omega}{dk} = v_p + k\frac{dv_p}{dk} \text{ or } v_g = v_p + k\frac{dv_p}{dk}$$

Since, $k = 2\pi/\lambda$ hence,

$$v_g = v_p + \frac{2\pi}{\lambda}\frac{dv_p}{d\left(\frac{2\pi}{\lambda}\right)} = v_p + \frac{1}{\lambda}\frac{dv_p}{d\left(\frac{1}{\lambda}\right)}$$

$$\because \quad d\left(\frac{1}{\lambda}\right) = -\frac{1}{\lambda^2}d\lambda$$

$$\boxed{v_g = v_p - \lambda\frac{dv_p}{d\lambda}}$$

REFERENCES

- CONCEPTS OF MODERN PHYSICS, ARTHUR BEISER, MCGRAW-HILL.
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- INTRODUCTION TO QUANTUM MECHANICS, DAVID J. GRIFFITH, PEARSON EDUCATION.