Lecture-4

Conversion of initial value problem into integral equations

Consider the initial value problem

$$\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1} y}{dx^{n-1}} + \ldots + a_n(x) y = F(x) \ ,$$

with the initial conditions

$$y(x_0) = c_0, y'(x_0) = c_1, \ldots, y^{(n-1)}(x_0) = c_{n-1},$$

Where the functions $a_i(i=1,2,\ldots n)$ and F(x) are defined in [a,b].

Now, let $$\frac{d^n y}{dx^n} = \phi(x)$$

then $$\left(\frac{d^{n-1} y}{dx^{n-1}}\right)_{x_0}^{x} = \int_{x_0}^{x} \phi(t)dt$$

or $$\frac{d^{n-1} y}{dx^{n-1}} - c_{n-1} = \int_{x_0}^{x} \phi(t)dt$$

or $$\frac{d^{n-1} y}{dx^{n-1}} = \int_{x_0}^{x} \phi(t)dt + c_{n-1}$$

Again, integrating both sides with respect to x from x_0 to x, we have

$$\left\{\frac{d^{n-2}y}{dx^{n-2}}\right\}_{x_0}^{x} = \int_{x_0}^{x}\phi(x)dx^2 + c_{n-1}\int_{x_0}^{x}dx\ ,$$

$$\Rightarrow \qquad \frac{d^{n-2}y}{dx^{n-2}} - c_{n-2} = \int_{x_0}^{x}\phi(x)dx^2 + c_{n-1}(x-x_0)$$

or

$$\frac{d^{n-2}y}{dx^{n-2}} = \int_{x_0}^{x}\phi(x)dx^2 + c_{n-1}(x-x_0) + c_{n-2}$$

$$\frac{d^{n-2}y}{dx^{n-2}} = \int_{x_0}^{x}(x-t)\phi(t)dt + c_{n-1}(x-x_0) + c_{n-2} \, .$$

Integrating again w. r. to x from x_0 to x, we obtain

$$\frac{d^{n-3}y}{dx^{n-3}} = \int_{x_0}^{x}\phi(x)dx^3 + c_{n-1}\frac{(x-x_0)^2}{2!} + c_{n-2}(x-x_0) + c_{n-3}$$

$$= \int_{x_0}^{x}\frac{(x-t)^2}{2!}\phi(t)dt + c_{n-1}\frac{(x-x_0)^2}{2!} + c_{n-2}(x-x_0) + c_{n-3}$$

and so on.

Finally, we get.

$$y=\int_{x_0}^{x}\phi(x)dx^n+c_{n-1}\frac{(x-x_0)^{n-1}}{(n-1)!}+c_{n-2}\frac{(x-x_0)^{n-2}}{(n-2)!}+\ldots+c_1(x-x_0)+c_0$$

$$=\int_{x_0}^{x}\frac{(x-t)^{n-1}}{(n-1)!}\phi(t)dt+c_{n-1}\frac{(x-x_0)^{n-1}}{(n-1)!}+c_{n-2}\frac{(x-x_0)^{n-2}}{(n-2)!}+\ldots+c_1(x-x_0)+c_0$$

Substituting these values of y and its derivatives in the given differential equation, we obtain

$$\phi(x)+a_1(x)\left(\int_{x_0}^{x}\phi(t)dt+c_{n-1}\right)+a_2(x)\left(\int_{x_0}^{x}\right.$$

$$+a_3(x)\left(\int_{x_0}^{x}\frac{(x-t)^2}{2!}\phi(t)dt+\frac{(x-x_0)^2}{2!}c_{n-}\right.$$

$$\dots+a_n(x)\left(\int_{x_0}^{x}\frac{(x-t)^{n-1}}{(n-1)!}\phi(t)dt+\frac{(x-x_0)}{(n-1)}\right.$$

$$=F(x)$$

$$\Rightarrow \quad F(x)=\phi(x)+\psi(x)-\int_{x_0}^{x}K(x,t)\phi(t)dt\ ,$$

where $\psi(x)=c_{n-1}a_1(x)+\{c_{n-2}+(x-x_0)c_{n-1}\}a_2(x)+\dots$

$$+\left\{c_0+(x-x_0)c_1+\dots+c_{n-1}\frac{(x-x_0)^{n-1}}{(n-1)!}\right\}a_n(x)$$

and

$$K(x,t)=-\left\{a_1(x)+(x-t)a_2(x)+\dots+\frac{(x-t)^{n-1}}{(n-1)!}a_n(x)\right\}.$$

$$\Rightarrow \quad F(x) = \phi(x) + \psi(x) - \int_{x_0}^{x} K(x,t)\phi(t)dt \ ,$$

where $\psi(x) = c_{n-1}a_1(x) + \{c_{n-2} + (x - x_0)c_{n-1}\}a_2(x) + \ldots$

$$+\left\{c_0 + (x - x_0)c_1 + \ldots + c_{n-1}\frac{(x - x_0)^{n-1}}{(n-1)!}\right\}a_n(x)$$

and

$$K(x,t) = -\left\{a_1(x) + (x-t)a_2(x) + \ldots + \frac{(x-t)^{n-1}}{(n-1)!}a_n(x)\right\}.$$

Now, let us assume

$$F(x) - \psi(x) = f(x)$$

then we get

$$\phi(x) = f(x) + \int_{x_0}^{x} K(x,t)\phi(t)dt,$$

which is a Volterra integral equation of the second kind.

Reference:
https://nptel.ac.in/courses/111/107/111107103/