

School of Basic and Applied Sciences

Course Code : MSCM303

Course Name: Integral equations and calculus of variation

Lecture-4

Conversion of initial value problem into integral equations

Consider the initial value problem

$$\frac{d^n y}{dx^n} + a_1(x) \frac{d^{n-1} y}{dx^{n-1}} + \dots + a_n(x) y = F(x) ,$$

with the initial conditions

$$y(x_0) = c_0, y'(x_0) = c_1, \dots, y^{(n-1)}(x_0) = c_{n-1},$$

Where the functions $a_i (i=1, 2, \dots, n)$ and $F(x)$ are defined in $[a, b]$.

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Now, let

$$\frac{d^n y}{dx^n} = \phi(x)$$

then

$$\left(\frac{d^{n-1}y}{dx^{n-1}}\right)_{x_0}^x = \int_{x_0}^x \phi(t)dt$$

or

$$\frac{d^{n-1}y}{dx^{n-1}} - c_{n-1} = \int_{x_0}^x \phi(t)dt$$

or

$$\frac{d^{n-1}y}{dx^{n-1}} = \int_{x_0}^x \phi(t)dt + c_{n-1}$$



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Again, integrating both sides with respect to x from x_0 to x , we have

$$\left\{ \frac{d^{n-2}y}{dx^{n-2}} \right\}_{x_0}^x = \int_{x_0}^x \phi(x) dx^2 + c_{n-1} \int_{x_0}^x dx ,$$

$$\Rightarrow \frac{d^{n-2}y}{dx^{n-2}} - c_{n-2} = \int_{x_0}^x \phi(x) dx^2 + c_{n-1}(x-x_0)$$

$$\text{or } \frac{d^{n-2}y}{dx^{n-2}} = \int_{x_0}^x \phi(x) dx^2 + c_{n-1}(x-x_0) + c_{n-2}$$



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$$\frac{d^{n-2}y}{dx^{n-2}} = \int_{x_0}^x (x-t)\phi(t)dt + c_{n-1}(x-x_0) + c_{n-2} .$$

Integrating again w. r. to x from x_0 to x , we obtain

$$\begin{aligned} \frac{d^{n-3}y}{dx^{n-3}} &= \int_{x_0}^x \phi(x)dx^3 + c_{n-1} \frac{(x-x_0)^2}{2!} + c_{n-2}(x-x_0) + c_{n-3} \\ &= \int_{x_0}^x \frac{(x-t)^2}{2!} \phi(t)dt + c_{n-1} \frac{(x-x_0)^2}{2!} + c_{n-2}(x-x_0) + c_{n-3} \end{aligned}$$

and so on.



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Finally, we get.

$$y = \int_{x_0}^x \phi(x) dx^n + c_{n-1} \frac{(x-x_0)^{n-1}}{(n-1)!} + c_{n-2} \frac{(x-x_0)^{n-2}}{(n-2)!} + \dots + c_1(x-x_0) + c_0$$

$$= \int_{x_0}^x \frac{(x-t)^{n-1}}{(n-1)!} \phi(t) dt + c_{n-1} \frac{(x-x_0)^{n-1}}{(n-1)!} + c_{n-2} \frac{(x-x_0)^{n-2}}{(n-2)!} + \dots + c_1(x-x_0) + c_0$$



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Substituting these values of y and its derivatives in the given differential equation, we obtain

$$\begin{aligned} & \phi(x) + a_1(x) \left(\int_{x_0}^x \phi(t) dt + c_{n-1} \right) + a_2(x) \left(\int_{x_0}^x \right. \\ & \left. + a_3(x) \left(\int_{x_0}^x \frac{(x-t)^2}{2!} \phi(t) dt + \frac{(x-x_0)^2}{2!} c_n \right. \right. \\ & \left. \left. \dots + a_n(x) \left(\int_{x_0}^x \frac{(x-t)^{n-1}}{(n-1)!} \phi(t) dt + \frac{(x-x_0)^{n-1}}{(n-1)!} c_n \right) \right) \right) \\ & = F(x) \end{aligned} \Rightarrow F(x) = \phi(x) + \psi(x) - \int_{x_0}^x K(x,t) \phi(t) dt ,$$

where $\psi(x) = c_{n-1} a_1(x) + \{c_{n-2} + (x-x_0)c_{n-1}\} a_2(x) + \dots$

$$+ \left\{ c_0 + (x-x_0)c_1 + \dots + c_{n-1} \frac{(x-x_0)^{n-1}}{(n-1)!} \right\} a_n(x)$$

and

$$K(x,t) = - \left\{ a_1(x) + (x-t)a_2(x) + \dots + \frac{(x-t)^{n-1}}{(n-1)!} a_n(x) \right\} .$$

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$$\Rightarrow F(x) = \phi(x) + \psi(x) - \int_{x_0}^x K(x,t)\phi(t)dt ,$$

where $\psi(x) = c_{n-1}a_1(x) + \{c_{n-2} + (x-x_0)c_{n-1}\}a_2(x) + \dots$
 $+ \left\{c_0 + (x-x_0)c_1 + \dots + c_{n-1} \frac{(x-x_0)^{n-1}}{(n-1)!}\right\}a_n(x)$

and

$$K(x,t) = -\left\{a_1(x) + (x-t)a_2(x) + \dots + \frac{(x-t)^{n-1}}{(n-1)!}a_n(x)\right\}.$$

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Now, let us assume

$$F(x) - \psi(x) = f(x)$$

then we get

$$\phi(x) = f(x) + \int_{x_0}^x K(x,t)\phi(t)dt,$$

which is a Volterra integral equation of the second kind.



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Reference:

<https://nptel.ac.in/courses/111/107/111107103/>

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