QUOTIENT GROUP & HOMOMORPHISM

Quotient Group

DEFINITION: If G is a group and N is its normal subgroup then the set G/N of all cosets of N in G is a group under operation $(aN)(bN) = (ab)N$ of cosets. It is called **Quotient group (Factor group)** of G by N.

Theorem: If G is a group, N is a normal subgroup of G, then G/N is also a group.

Proof: Let N be a normal subgroup of a group G.
N is normal in G ⇒ every right coset of N in G is the corresponding left coset of N in G.
Therefore we shall call them simply as cosets.
Let G/N be the collection of all cosets of N in G, i.e., $G/N = \{Na \mid a \in G\}$.
We define a product of elements of G/N as follows:

$$(Na)\,(Nb) = N(ab)\ \forall\, a, b \in G$$

1) Closure Property

Let $Na, Nb \in G/N$ for some $a, b \in G$.

$\therefore (Na)(Nb) = N(aN)b$
$= N(Na)b$ (since N is normal $aN = Na$)
$= N\,N\,a\,b$
$= N(ab)$ (since $NN = N$)

Since $a, b \in G$ and G is a group, we have $ab \in G$ and hence Nab is a coset of N in G, i.e., $Nab \in G/N$

$\therefore Na, Nb \in G/N \Rightarrow Nab \in G/N \Rightarrow G/N$ is closed with respect to coset multiplication.

2) Associative

Let $Na, Nb, Nc \in G/N$ for some a, b, c ∈ G.

$\therefore Na\,(Nb\,Nc) = Na\,N(bc)$

$= Na(bc)$

$= N(ab)c$ (since, $a(bc) = (ab)c$, by associativity in G)

$= (Nab)Nc$

$= (Na\,Nb)\,Nc$

The multiplication of cosets is associative in G/N.

3) Existence of an identity element

We have $e \in G$ and hence $Ne = N \in G/N$.

Let $Na \in G/N$ for some $a \in G$.

$\therefore Na\,Ne = Nae$

$= Na$

And $Ne\,Na = Nea$

$= Na$

$\therefore Na\,Ne = Ne\,Na = Na \;\forall\, Na \in G/N$

$\therefore N = Ne$ is an identity element of G/N.

4) Existence of Inverse

Let $Na = G/N$ for some $a \in G$.

Now, $a \in G \Rightarrow a^{-1} \in G \Rightarrow Na^{-1} \in G/N$.

We have $(Na)(Na^{-1}) = N(aa^{-1}) = Ne = N$

And $(Na^{-1})(Na) = N(a^{-1}a) = Ne = N$

$\therefore$ Na^{-1} is an inverse of Na in G/N

$\therefore$ Every element of G/N possesses an inverse in G/N.

Thus, all the group axioms are satisfied in G/N. G/N, the set of all cosets of N in G is a group with respect to the product of cosets.

Example 1: $K_4 = \{e = a^4, a, a^2, a^3\}$

$H = \{e, a^2\}$ is an normal subgroup of K_4

Coset $M = aH = a^2 H = \{a, a^3\}$

Quotient group $K_4/H = \{H, M\}$

H	M
M	H

Example 2: $S_3 = \{I, (123), (132), (23), (13), (12)\}$

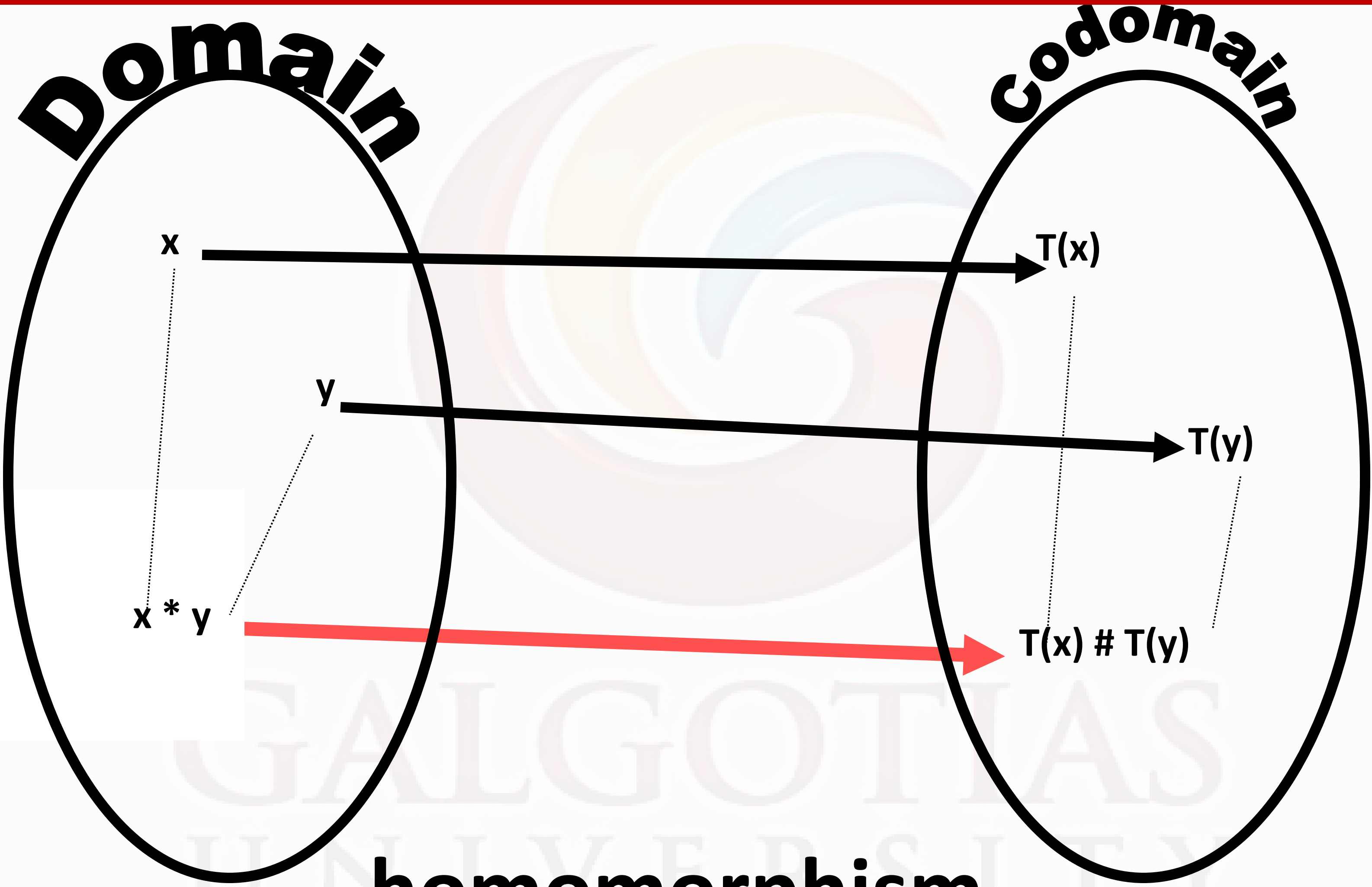
$H = \{e, (123), (132)\}$ is normal subgroup of S_3

Coset $M = \{(23), (13), (12)\}$

Quotient group $S_3/H = \{H, M\}$

I	(123)	(132)	(23)	(13)	(12)
(123)	(132)	I	(12)	(23)	(13)
(132)	I	(123)	(13)	(12)	(23)
(23)	(13)	(12)	I	(123)	(132)
(13)	(12)	(23)	(132)	I	(123)
(12)	(23)	(13)	(123)	(132)	I

Assignment: Show that Quotient group of an Abelian group is Abelian.



Operation: * **homomorphism** Operation: #

$$T(x*y) = T(x) \# T(y)$$

Homomorphism

Definition:

A function f from a group $(G,*)$ to a group $(G',\#)$ is a homomorphism if f satisfies

$$f(a * b) = f(a) \# f(b) \text{ for all } a, b \in G.$$

Example:

Let $r \in \mathbb{Z}$ and let $f_r : \mathbb{Z} \rightarrow \mathbb{Z}$ be defined by $f_r(n) = rn$ for all $n \in \mathbb{Z}$. Is f_r a homomorphism?

Solution: For all $m, n \in \mathbb{Z}$, we have $f_r(m + n) = r(m + n) = rm + rn = f_r(m) + f_r(n)$.
So f_r is a homomorphism.

Note: There always exists a homomorphism between two groups in which every element of domain go to identity element of codomain. And that homomorphism is called **trivial homomorphism**.

Properties of homomorphism: If f from a group $(G,*)$ to a group $(G',\#)$ is a homomorphism then

1. $f(e) = e'$, where $e \in G$ & $e' \in G'$
2. $|f(a)|$ divides $|a|$, for all $a \in G$
3. $f(x^{-1}) = f(x)^{-1}$, for all $x \in G$
4. Kernel of $f = \{a \in G; f(a) = e'\}$ is the normal subgroup of G
5. $f(G)$ is the subgroup of G'

Assignment: Show that inverse of a bijective homomorphism from $G \rightarrow \bar{G}$, is an homomorphism from $\bar{G} \rightarrow G$.

Refernces:

http://faculty.bucks.edu/leutwyle/Linear/Mappings/homomorphism.ppt

https://www.slideshare.net/ayushagrawal106902/quotient-groupsgroup-theory

http://ckw.phys.ncku.edu.tw/public/pub/Notes/Mathematics/GroupTheory/Tung/Powerpoint/2._BasicGroupTheory.ppt