Lecture-01: Introduction to Numerical Methods and Error Analysis

Many engineering problems take too much time for solving or may not possibly solved analytically. In these situations, numerical methods are usually employed. Numerical methods are techniques designed to solve a problem using numerical approximations.

Example: An application of numerical methods is trying to determine the velocity of a falling object. If you know the exact function that determines the position of your object, then you could potentially differentiate the function to obtain an expression for the velocity. More often, you will use a machine to record readings of times and positions that you can then use to numerically solve for velocity:

$$f'(t) \cong \frac{f(t+h) - f(t)}{h}$$

where f is your function, t is the time of the reading, and h is the distance to the next time step.

Introduction of Error

Because our answer is an approximation of the analytical solution, there is an inherent error between the approximated answer and the exact solution. Errors can result prior to computation in the form of measurement errors or assumptions in modeling. The focus of this lecture will be on understanding two types of errors that can occur during computation:

- Roundoff errors
- Truncation errors.

Roundoff errors

Roundoff errors occur because our memory/computers have a limited ability to represent numbers. For example, π has infinite digits, but due to precision limitations, only 16 digits may be stored in MATLAB/SciLab. While this roundoff error may seem insignificant, if your process involves multiple iterations that are dependent on one another, these small errors may accumulate over time and result in a significant deviation from the expected value. Furthermore, if a manipulation involves adding a large and small number, the effect of the smaller number may be lost if rounding is utilized. Thus, it is advised to sum numbers of similar magnitudes first so that smaller numbers are not "lost" in the calculation.

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One interesting example that we covered in our Numerical Methods class, that can be used to illustrate this point, involves the quadratic formula. The quadratic formula is represented as follows:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Using a = 0.2, b = – 47.91, c = 6 and if we carry out rounding to two decimal places at every intermediate step:

$$x = \frac{-47.91 \pm \sqrt{(-47.91)^2 - 4(0.2)(6)}}{2(0.2)} = \frac{-47.91 \pm \sqrt{2295.36 - 4.8}}{0.4} = \frac{-47.91 \pm 47.86}{0.4}$$

$$x_1 = 239.425$$

$$x_2 = 0.125$$

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The error between our approximations and true values can be found as follo

$$Absolute\ Error_{x1} = \left|\frac{actual - approximation}{actual}\right| * 100 = \frac{239.4246996 - 239.425}{239.4246996} * 100$$
$$= 1.25 \times 10^{-4}\ \%$$

$$Absolute\ Error_{x2} = \left|\frac{0.1253003556 - 0.125}{0.1253003556}\right| * 100 = 24\%$$

As can be seen, the smaller root has a larger error associated with it because deviations will be more apparent with smaller numbers than larger numbers. If you have the insight to see that your computation will involve operations with numbers of differing magnitudes, the equations can sometimes be cleverly manipulated to reduce roundoff error.

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In our example, if the quadratic formula equation is rationalized, the resulting absolute error is much smaller because fewer operations are required and numbers of similar magnitudes are being multiplied and added together:

$$x = \frac{2c}{-b \pm \sqrt{b^2 - 4ac}} = \frac{2(6)}{47.91 \pm \sqrt{(-47.91)^2 - 4(0.2)(6)}} = \frac{12}{47.91 \pm 47.86}$$

$$x_1 = 240$$

$$x_2 = 0.1253001984$$

$$Absolute\ Error_{x1} = \left|\frac{actual - approximation}{actual}\right| * 100 = \frac{239.4246996 - 240}{239.4246996} * 100$$
$$= 0.24\%$$

$$Absolute\ Error_{x2} = \left|\frac{0.1253003556 - 0.1253001984}{0.1253003556}\right| * 100 = 1.25 \times 10^{-4}\ \%$$

Truncation Error

Truncation errors are introduced when exact mathematical formulas are represented by approximations. An effective way to understand truncation error is through a Taylor Series approximation. Let's say that we want to approximate some function, f(x) at the point x_{i+1}, which is some distance, h, away from the basepoint x_i, whose true value is shown in black in Figure 1. The Taylor series approximation starts with a single zero order term and as additional terms are added to the series, the approximation begins to approach the true value. However, an infinite number of terms would be needed to reach this true value.

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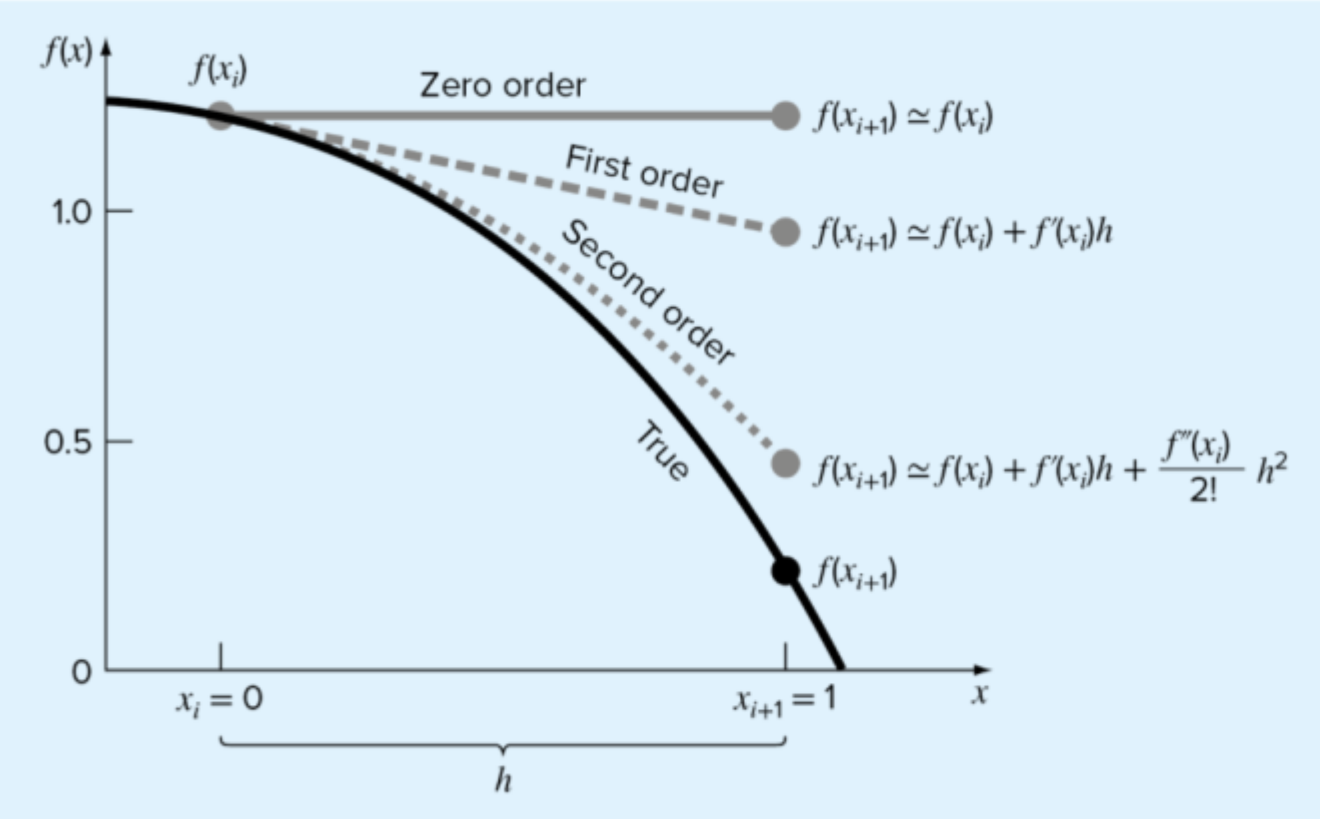

Figure 1: Graphical representation of a Taylor Series approximation (Chapra, 2017)

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The Taylor Series can be written as follows:

$$f(x_{i+1}) \cong f(x_i) + f'(x_i)h + \frac{f''(x_i)}{2!}h^2 + \cdots + \frac{f^{(n)}(x_i)}{n!}h^n + R_n$$

where R_n is a remainder term used to account for all of the terms that were not included in the series and is therefore a representation of the truncation error. The remainder term is generally expressed as $R_n = O(h^{n+1})$ which shows that truncation error is proportional to the step size, h, raised to the n+1 where n is the number of terms included in the expansion. It is clear that as the step size decreases, so does the truncation error.

References

- https://waterprogramming.wordpress.com/2018/02/21/types-of-errors-in-numerical-methods/
- Chapra, Steven C. Applied Numerical Methods with MATLAB for Engineers and Scientists. McGraw-Hill, 2017.
- Class Notes from ENGRD 3200: Engineering Computation taught by Professor Peter Diamessis at Cornell University