Integral Equation:

An integral equation is an equation in which unknown function appear under integral sign.

Applications of Integral Equations:

1. Mechanics
2. Mathematical physics
3. Applied Mathematics
4. Computational Electromagnetics

Abel's Problem:

Let us consider a smooth curve in a vertical plane. Let a particle starts from rest at any point P with co-ordinate x under the influence of gravity along the curve.

Let t=f(x) where f(x) is the given curve. The absolute velocity of the particle at any point Q whose ordinate is η is given by

$$v=\sqrt{2g(x-\eta)}$$

Let β be the angle of inclination of the tangent to the curve at Q with ξ axis.

then $$\frac{d\eta}{dt} = -\sqrt{2g(x-\eta)}\sin\beta$$

or $$dt = -\frac{d\eta}{\sin\beta\sqrt{2g(x-\eta)}} .$$

Now, let $\frac{1}{\sin\beta} = \phi(\eta)$ then

$$\sqrt{2g}\,dt = -\frac{\phi(\eta)d\eta}{\sqrt{(x-\eta)}} .$$

Hence

$$t\sqrt{2g} = -\int_{P}^{R} \frac{\phi(\eta)d\eta}{\sqrt{x-\eta}}$$

$$\sqrt{2g}\, f_1(x) = \int_{0}^{x} \frac{\phi(\eta)d\eta}{\sqrt{x-\eta}}$$

$$f(x) = \int_{0}^{x} \frac{\phi(\eta)d\eta}{\sqrt{x-\eta}}.$$

where $f(x)=f_1(x)\sqrt{2g}$ is known and $\phi(\eta)$ is the unknown function.

After finding $\phi(\eta)$, We obtain the equation of the curve as

$$\phi(\eta)=\frac{1}{\sin\beta}$$

$$\eta=\Phi(\beta)\,.$$

$$\frac{d\eta}{d\xi}=\tan\beta$$

or $$\frac{\Phi'(\beta)d\beta}{d\xi} = \tan\beta$$

$$\Rightarrow \quad \xi = \int \frac{\Phi'(\beta)}{\tan\beta} d\beta = \Phi_1(\beta) \, .$$

Thus the required curve is given in parametric form as

$$\xi = \Phi_1(\beta), \eta = \Phi(\beta) \, .$$

Hence the Abel's Problem reduces to a solution of Volterra Integral equation of First kind.

$$f(x)=\int_0^x K(x,t)\phi(t)dt,$$

where $\phi(x)$ is an unknown function, $K(x,t)=\frac{1}{\sqrt{x-t}}$ and $f(x)$ are given functions.

Reference:

https://nptel.ac.in/courses/111/107/111107103/