BOSE-EINSTEIN STATISTICS

Topics Covered:

- Bose-Einstein Distribution Law

- Consider an isolated system of N identical , indistinguishable particles contained in a volume V at an equilibrium temperature T with total energy U.
- Here we account for the particles whose spin quantum number is an integer.
- These particles are known as Bose particles or bosons. Example: photons with spin 1 and alpha particle, deuteron with spin 0.
- The particles do not follow Pauli's exclusion principle i.e. any number of particles can be in a common level

Suppose there are l states with energies, E_1, E_2, ……E_l, and degeneracies g_1, g_2,….g_l respectively. Let n_1, n_2, ….n_l , respectively, be the number of particles in these levels.

- In the given system the total number of particles is constant

$$i.e., n_1 + n_2 + n_3 + ... + n_l = \sum n_i = N(const.)$$

Hence it's derivative

$$\Sigma \delta n_i = 0.....[1]$$

Also, the total energy of the system(U) is constant

$$i.e., E = E_1 n_1 + E_2 n_2 + E_3 n_3 + ... + E_l n_l = \sum E_i n_i = U(const.)$$

Hence it's derivative

$$\Sigma \delta (E_i n_i) = 0.....[2]$$

Now we want to place n_i indistinguishable particles in g_i levels with the freedom that any number of particles can be placed in any level.

For that let us consider two distinguishable particles to be arranged in two distinguishable Cells i.e. $n_i = 2$ and $g_i = 2$
Number of permutations of 2 particles in 2 cells will be:

Which will be equal to the number of permutations of n_i particles and $(g_i -1)$ partitions.

Considering the particles and partitions to be different objects, the number of permutations of (n_i+g_i-1) objects will be= $(n_i+g_i-1)!$

So, for the given example, $(n_i+g_i-1)! = (2+2-1)! = 3! =$ **6**

Since, the particles among themselves and partitions among themselves are indistinguishable, the number of ways of arranging n_i indistinguishable particles in g_i levels is

$$W_i = \frac{(n_i + g_i - 1)!}{n_i!(g_i - 1)!}$$

Where, i=1,2,3……l

Hence, total number W of ways of arranging the particles among states will be

$$W = W_1 W_2 \ldots\ldots W_l$$

$$W = \frac{(n_1 + g_1 - 1)!}{n_1!(g_1 - 1)!} \frac{(n_2 + g_2 - 1)!}{n_2!(g_2 - 1)!} \ldots\ldots\ldots\ldots \frac{(n_l + g_l - 1)!}{n_l!(g_l - 1)!}$$

$$= \prod_{i=1}^{l} \frac{(n_i + g_i - 1)!}{n_i!(g_i - 1)!}$$

Since, n_i and g_i are large numbers, we can omit 1 in comparison to them :

$$W = \prod_{i=1}^{l} \frac{(n_i + g_i)!}{n_i! g_i!}$$

The most probable distribution of N particles among l states is that for which W is the Maximum!!

For mathematical convenience let us take logarithm of W and consider the maximum of ln(W).

Therefore, **d (ln W)=0** for maxima

$$\frac{\partial}{\partial n_1}(\ln W)dn_1 + \frac{\partial}{\partial n_2}(\ln W)dn_2 + \ldots\ldots\ldots \frac{\partial}{\partial n_l}(\ln W)dn_l = 0$$

$$\sum_{i=1}^{i=l} \frac{\partial}{\partial n_i}(\ln W)dn_i$$

This equation can be written as

$$\sum_{i=1}^{i=l} [\frac{\partial}{\partial n_i}(\ln W) - \alpha - \beta E_i]dn_i = 0$$

$$\Rightarrow \frac{\partial}{\partial n_i}(\ln W) - \alpha - \beta E_i = 0\ldots\ldots\ldots\ldots(1)$$

$$\ln W = \ln[\prod_{i=1}^{l} \frac{(n_i + g_i)!}{n_i!\, g_i!}] = \sum_{i=1}^{i=l} \ln[\frac{(n_i + g_i)!}{n_i!\, g_i!}]$$

$$= \sum_{i=1}^{i=l} [\ln(n_i + g_i)! - \ln n_i\,! - \ln g_i\,!]$$

Applying Stirling approximation

$$\ln x\,! = x \ln x - x,$$

$$\ln W = \sum_{i=1}^{i=l} [(n_i + g_i) \ln(n_i + g_i) - (n_i + g_i) - n_i \ln n_i + n_i - g_i \ln g_i + g_i]$$

$$\ln W = \sum_{i=1}^{i=l} [(n_i + g_i) \ln(n_i + g_i) - n_i \ln n_i - g_i \ln g_i] \ldots\ldots\ldots (2)$$

$$\frac{\partial}{\partial n_i}\ln W = \frac{n_i + g_i}{n_i + g_i} + \ln(n_i + g_i) - \frac{n_i}{n_i} - \ln n_i - 0$$

where all other terms are zero

$$= \ln(n_i + g_i) - \ln n_i = \ln[\frac{n_i + g_i}{n_i}] \ldots\ldots\ldots\ldots(3)$$

Substituting in equation (2) we get

$$\ln[\frac{n_i + g_i}{n_i}] - \alpha - \beta E_i = 0$$

$$\ln[\frac{n_i + g_i}{n_i}] = \alpha + \beta E_i \Rightarrow \frac{n_i + g_i}{n_i} = e^{\alpha + \beta E_i} \Rightarrow \frac{g_i}{n_i} = e^{\alpha + \beta E_i} - 1$$

$$\therefore n_i = \frac{g_i}{e^{\alpha + \beta E_i} - 1}$$

This is the Bose-Einstein distribution of the particles among various states or Bose-Einstein distribution law

$$f_{BE}(E_i) = \frac{n_i}{g_i} = \frac{1}{e^{\alpha+\beta E_i} - 1} = \frac{1}{e^{\alpha+\frac{E_i}{kT}} - 1}$$

where, $\beta = 1/\mathrm{kT}$

This is called Bose-Einstein distribution function

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- Statistical Mechanics, R.K. Pathria, Butterworth Heinemann: 2nd Ed., 1996, Oxford University Press
- Statistical and Thermal Physics, S. Lokanathan and R.S. Gambhir. 1991, Prentice Hall

THANK YOU!