Methods of Separation of Variables

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Introduction

An equation involving derivatives of one or more dependent variables with respect to one or more independent variables is called a differential equation.

- **Ordinary Differential Equation:**

Function has 1 independent variable.

- **Partial Differential Equation:**

At least 2 independent variables.

Methods of Separation of Variables

Method of separation of variables is one of the most widely used techniques to solve PDE. It is based on the assumption that the solution of the equation is separable, that is, the final solution can be represented as a product of several functions, each of which is only dependent upon a single independent variable.

Ex. String displacement function

$u(x,t)= X(x)T(t)$, is a product of two functions $X(x)$ & $T(t)$, where $X(x)$ is a function of only x, not t. On the other hand, $T(t)$ is a function of t, not x.
By substituting the new product solution form into the original PDE one can obtain a set of ordinary differential equations (hopefully), each of which involves only one independent variable.

Example: String Vibrations

The string vibration problem can be modeled by the one-dimensional wave equation: $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$, $c^2 = \frac{T}{\rho}$.

The variables T and ρ represent tension and linear density of the string, respectively. The solution, u(x,t), is the deflection of the string. Since the string is fixed at both ends, the boundary conditions are:

u(x=0,t)=u(x=L,t)=0 for all t.

To solve the wave equation, method of the separation of variables is used by assuming the solution can be written in this form:

$$u(x,t) = X(x)T(t)$$

where X is a function of x only, and T is a function of t only.

Exercises

Example 1. *Applying the method of separation of variables techniques, find the solution to the P.D.E.*

$$3u_x + 2u_y = 0 \ldots, \text{ where } u_x = \frac{\partial u}{\partial x}, u_y = \frac{\partial u}{\partial y}.$$

Solution : Here we have

$$\frac{3\,\partial u}{\partial x} + \frac{2\,\partial u}{\partial y} = 0 \qquad \ldots(1)$$

Let $u = X(x)\,Y(y)$ ...(2)

Where X is a function of x only and Y is a function of y only.

On differentiating (2) partially w.r.t. x, we get

$$\frac{\partial u}{\partial x} = \frac{\partial X}{\partial x}.\,Y \qquad \ldots(3)$$

On differentiating (2) partially w.r.t. y, we get

$$\frac{\partial u}{\partial y} = X.\frac{\partial Y}{\partial y} \qquad \ldots(4)$$

Putting the values of $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ from (3) and (4) in (1), we get

$$3\frac{\partial X}{\partial x}.\,Y + 2\,X\frac{\partial Y}{\partial y} = 0 \qquad \ldots(5)$$

Cond...

Dividing (5) by XY, we get

$$\frac{3}{X}\frac{\partial X}{\partial x}+\frac{2}{Y}\frac{\partial Y}{\partial y}=0$$

$\left[\begin{array}{l}\text{R.H.S is constant for L.H.S,}\\ \text{So we take both equations}\\ \text{are equal to } k \text{ (constant)}\end{array}\right.$

$$\Rightarrow \quad \frac{3}{X}\frac{\partial X}{\partial x}=-\frac{2}{Y}\frac{\partial Y}{\partial y}=k \quad \Rightarrow \quad \frac{3}{X}\frac{\partial X}{\partial x}=k \text{ and } -\frac{2}{Y}\frac{\partial Y}{\partial y}=k$$

$$\Rightarrow \quad \frac{\partial X}{X}=\frac{k}{3}\partial x \text{ and } \frac{\partial Y}{Y}=-\frac{k}{2}\partial y \quad \Rightarrow \quad \log X=\frac{k}{3}x+c_1 \text{ and } \log Y=\frac{-k}{2}y+c_2$$

$$\Rightarrow \quad X=e^{\frac{k}{3}x+c_1} \text{ and } Y=e^{-\frac{k}{2}y+c_2}$$

Putting the values of X and Y in (2), we get

$$u=e^{\frac{k}{3}x+c_1}\,e^{\frac{-k}{2}y+c_2}=e^{k\left(\frac{x}{3}-\frac{y}{2}\right)+c_1+c_2}=e^{k\left(\frac{x}{3}-\frac{y}{2}\right)}.\,e^{c_1+c_2}$$

Hence $\quad u=A\,e^{k\left(\frac{x}{3}-\frac{y}{2}\right)} \quad$ [where $A=e^{c_1+c_2}$] **Ans.**

Example 2. *Solve the following equation* $\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0$ *by the method of separation of variables.*

Solution. Given equation is

$$\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0 \qquad \text{... (1)}$$

Let $z = X(x)\, Y(y)$ ... (2)

where X is a function of x only and Y is a function of y only.

$$\frac{\partial z}{\partial x} = Y\frac{dX}{dx}, \qquad \frac{\partial^2 z}{\partial x^2} = Y\frac{d^2 X}{dx^2}$$

$$\frac{\partial z}{\partial y} = X\frac{dY}{dy}$$

Putting all values in equation (1), we get $Y\frac{d^2X}{dx^2} - 2Y\frac{dX}{dx} + X\frac{dY}{dy} = 0$

Dividing by XY, we have $\frac{1}{X}\frac{d^2X}{dx^2} - \frac{2}{X}\frac{dX}{dx} + \frac{1}{Y}\frac{dY}{dy} = 0$

Separating the variables, we have $\frac{1}{X}\frac{d^2X}{dx^2} - \frac{2}{X}\frac{dX}{dx} = -\frac{1}{Y}\frac{dY}{dy} = K \ (let)$

Cond...

where K is a constant.

$$\frac{1}{X}\frac{d^2X}{dx^2}-\frac{2}{X}\frac{dX}{dx}=K \qquad\qquad -\frac{1}{Y}\frac{dY}{dy}=K$$

$$\frac{d^2X}{dx^2}-2\frac{dX}{dx}=KX \qquad\qquad \frac{dY}{dy}+KY=0$$

$$\Rightarrow \quad (D^2-2D-K)X=0 \qquad\qquad (D+K)Y=0$$

A. E. is $m^2-2m-K=0$ | A.E. is $m+K=0 \Rightarrow m=-K$

$$\Rightarrow \quad m=\frac{2\pm\sqrt{4+4K}}{2} \qquad\qquad \Rightarrow \quad Y=C_3e^{-Ky} \quad \ldots(3)$$

$$\Rightarrow \quad m=1\pm\sqrt{1+K}$$

Thus $X=C_1e^{(1+\sqrt{1+K})x}+C_2e^{(1-\sqrt{1+K})x}$ ...(4)

Putting the values of X and Y from (3) and (4) in (2), we get

$$z=\left\{C_1e^{(1+\sqrt{1+K})x}+C_2e^{(1-\sqrt{1+K})x}\right\}C_3e^{-Ky}$$

Ans.

Example 3. *Using the method of separation of variables, solve*

$$\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u \qquad \text{where} \qquad u(x, 0) = 6e^{-3x}$$

Solution. $$\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u \qquad \dots (1)$$

Let $u = X(x).T(t)$ ... (2)

where X is a function of x only and T is a function of t only.

Putting the value of u in (1), we get

$$\frac{\partial (X.T)}{\partial x} = 2\frac{\partial}{\partial t}(X.T) + X.T, \qquad T\frac{dX}{dx} = 2X\frac{dT}{dt} + X.T$$

On separating the variables, we get

$$\frac{1}{X}\frac{dX}{dx} = \frac{2}{T}\frac{dT}{dt} + 1 = C \qquad \text{[On dividing by } XT\text{]}$$

Cond...

$$\frac{1}{X}\frac{dX}{dx} = C \qquad\qquad \frac{2}{T}\frac{dT}{dt} + 1 = C$$

$$\Rightarrow \frac{dX}{dx} = CX \qquad\qquad \Rightarrow \frac{dT}{dt} + \frac{T}{2} = \frac{CT}{2}$$

$$\Rightarrow DX - CX = 0 \qquad\qquad \Rightarrow DT - \left(\frac{C}{2} - \frac{1}{2}\right)T = 0$$

$$\Rightarrow (D - C)X = 0 \qquad\qquad \text{A.E. is } m - \left(\frac{C}{2} - \frac{1}{2}\right) = 0 \Rightarrow m = \frac{1}{2}(C - 1)$$

$$\text{A.E. is } m - C = 0 \Rightarrow m = C \qquad\qquad \Rightarrow T = be^{\frac{1}{2}(c-1)t}$$

$$\Rightarrow X = ae^{cx}$$

Putting the values of X and T in (2), we have

$$u = ae^{cx}.be^{\frac{1}{2}(c-1)t}$$

$$\Rightarrow u = ab\, e^{cx+\frac{1}{2}(c-1)t} \qquad ...(3)$$

On putting $t = 0$ and $u = 6e^{-3x}$ in (3), we get

$$6e^{-3x} = abe^{cx} \Rightarrow ab = 6 \text{ and } c = -3$$

Putting the values of ab and c in (3), we have

$$u = 6e^{-3x+\frac{1}{2}(-3-1)t}$$

$$u = 6e^{-3x-2t} \qquad \textbf{Ans.}$$

which is the required solution.

Example 4. *Solve the following equation by the method of separation of variables*

$$\frac{\partial^2 u}{\partial x\,\partial t} = e^{-t}\cos x \quad \text{given that } u = 0 \text{ when } t = 0 \text{ and } \frac{\partial u}{\partial t} = 0 \text{ when } x = 0.$$

Solution. Let $u = XT$ ...(1)

where X is a function of x only and T is a function of t only.

Then, $$\frac{\partial u}{\partial t} = \frac{\partial}{\partial t}(XT) = X\frac{dT}{dt}$$

$$\therefore \quad \frac{\partial^2 u}{\partial x\partial t} = \frac{\partial}{\partial x}\left(X\frac{dT}{dt}\right) = \frac{dT}{dt}\cdot\frac{dX}{dx} \qquad ...(2)$$

Substituting the value $\frac{\partial^2 u}{\partial x\partial t}$ from (2) in the given equation, we get

$$\frac{dT}{dt}\frac{dX}{dx} = e^{-t}\cos x$$

Separating the variables, we get

$$e^t\frac{dT}{dt} = \frac{\cos x}{\left(\frac{dX}{dx}\right)} = -p^2 \text{ (say)} \qquad ...(3)$$

Now, $$e^t\frac{dT}{dt} = -p^2$$

$$\Rightarrow \quad dT = -p^2 e^{-t}\,dt$$

On integration, we get

$$T = p^2 e^{-t} + c_1 \qquad ...(4)$$

Also, $$\frac{dX}{dx} = -\frac{1}{p^2}\cos x$$

$$dX = -\frac{1}{p^2}\cos x\,dx$$

On integration, we get

$$X = -\frac{1}{p^2}\sin x + c_2 \qquad ...(5)$$

Cond...

Putting the values of X and T from (4) and (5) in (1), we get

$$u = XT = \left(-\frac{1}{p^2}\sin x + c_2\right)(p^2 e^{-t} + c_1) \quad \text{...(6)}$$

On putting $u = 0$ and $t = 0$ in (6), we get

$$0 = \left(-\frac{1}{p^2}\sin x + c_2\right)(p^2 + c_1)$$

$\Rightarrow$ $p^2 + c_1 = 0 \Rightarrow c_1 = -p^2$

Differentiating (6) w.r.t. "t", we get

$$\frac{\partial u}{\partial t} = \left(-\frac{1}{p^2}\sin x + c_2\right)(-p^2 e^{-t}) \quad \text{...(7)}$$

Putting $\frac{\partial u}{\partial t} = 0$ when $x = 0$ in (7), we get

$$0 = c_2(-p^2 e^{-t})$$

$\Rightarrow$ $c_2 = 0$

Substituting the values of $c_1 = -p^2$ and $c_2 = 0$ in (6), we get

$$u = -\frac{1}{p^2}\sin x\,(p^2 e^{-t} - p^2)$$

$$= (1 - e^{-1})\sin x$$ **Ans.**

Applications

Physical systems are often described by coupled Partial Differential Equations (PDEs):

Example:

- Maxwell equations
- Navier-Stokes and Euler equations in fluid dynamics.
- MHD-equations in plasma physics
- Einstein-equations for general relativity etc.

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