Introduction to Supergravity

Lectures presented by

Henning Samtleben

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typed by Martin Ammon and Cornelius Schmidt-Colinet

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1 Introduction

There are several reasons to consider the combination of supersymmetry and gravitation. The foremost is that if supersymmetry turns out to be realized at all in nature, then it must eventually appear in the context of gravity. As is characteristic for supersymmetry, its presence is likely to improve the quantum behavior of the theory, particularly interesting in the context of gravity, a notoriously non-renormalizable theory. Indeed, in supergravity divergences are typically delayed to higher loop orders, and to date it is still not ruled out that the maximally supersymmetric extension of four-dimensional Einstein gravity might eventually be a finite theory of quantum gravity — only recently very tempting indications in this direction have been unvealed. On the other hand, an underlying supersymmetric extension of general relativity provides already many interesting aspects for the theory itself, given by the intimate interplay of the two underlying symmetries. For example, the possibility of defining and satisfying BPS bounds opens a link to the treatment of solitons in general backgrounds. Last but not least of course, supergravity theories arise as the low-energy limit of string theory compactifications.

These lecture notes can only cover some very selected aspects of the vast field of supergravity theories. We shall begin with "minimal" supergravity, i.e. $N = 1$ supergravity in four space-time dimensions, which is minimal in the sense that it is the smallest possible supersymmetric extension of Einstein's theory of general relativity. Next, we will consider extended supersymmetry ($N > 1$) in higher dimensions ($D > 4$), and try to understand the restrictions which supersymmetry imposes on the possible construction of consistent supergravity theories. The third part is devoted to studying Kaluza-Klein compactification of supergravity from higher space-time dimensions. On the one hand this turns out to be a useful method in itself to construct extended supergravity theories in four dimensions, on the other hand it is an indispensable aspect in the study of supergravities descending from string theories that typically live in higher-dimensional spaces. Finally, we shall have a closer look at gauged supergravities, which are generically obtained in the Kaluza-Klein compactification procedure (unfortunately we will not make it to this last point ...).

There exist many introductions and reviews on the subject, of which we mention the following articles:

- P. van Nieuwenhuizen [1] for a general introduction, and in particular for the $N = 1$ case,
- Y. Tanii [2], especially for theories with extended supersymmetry,
- M. Duff, B. Nilsson and C. Pope [3] for aspects of the Kaluza-Klein compactification,
- B. de Wit [4], especially for the gauged supergravities.

2 $N = 1$ supergravity in $D = 4$ dimensions

2.1 General aspects

Let us recollect a few facts about the structure of globally supersymmetric theories. A supersymmetry transformation changes bosonic into fermionic particles and vice versa. Denoting bosonic particles by B and fermionic particles by F, the global transformation is schematically

$$\delta B = \bar{\epsilon} F \, , \qquad \delta F = \epsilon \partial B \, , \tag{1}$$

where ϵ is the infinitesimal supersymmetry parameter, carrying spinor indices, and ∂ stands for a space-time derivative. The commutator of two supersymmetry transformations hence amounts to an operator which is proportional to the space-time derivative,

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}] B \propto (\bar{\epsilon}_1 \gamma^\mu \epsilon_2) \, \partial_\mu B \, ,$$

where μ labels space-time coordinates. From this we can immediately deduce the important fact that a supersymmetric theory will necessarily be invariant under translations. This link between supersymmetry and the Lorentz group is recorded in the algebra in form of the anticommutator of two supersymmetry charges, which is schematically

$$\{Q, \bar{Q}\} \quad \propto \quad P \, . \tag{2}$$

The first step towards a generalisation is now to allow the supersymmetry parameter to depend on space-time coordinates, i.e. to consider local supersymmetry instead of the former global one. Since this means that the system will be invariant under a larger class of symmetries, there will be different restrictions on the form the theory can take. Making the symmetry local will again lead to a transformation law of the form (1), as well as to a commutator involving a space-time derivative, with the difference that this time both will have local parameters, schematically

$$\delta_\epsilon B = \bar{\epsilon}(x) \, F \, , \quad \delta_\epsilon F = \epsilon(x) \, \partial B \quad \Rightarrow \quad [\delta_{\epsilon_1}, \delta_{\epsilon_2}] B = (\bar{\epsilon}_1 \gamma^\mu \epsilon_2)(x) \, \partial_\mu B \, . \tag{3}$$

The right-hand side is written in a form to stress the difference to the global case: The commutator of two infinitesimal supersymmetry transformations yields a space-time dependent vector field $(\bar{\epsilon}\gamma^\mu \epsilon)(x)$, which is an element of the infinitesimal version of the group of local diffeomorphisms on space-time. A locally supersymmetric theory will hence necessarily be diffeomorphism invariant and thus requires to treat the space-time metric as a dynamical object. The simplest theory which achieves this is Einstein's general relativity which we shall consider in the following — although higher order curvature corrections may equally give rise to supersymmetric extensions.

The candidate for the quantized particle of gravity, the graviton, is provided by the spin-2 state appearing in the supergravity multiplets. Consider e.g. the supergravity multiplet in $N = 2$ supersymmetry, which contains a spin-2 state $g_{\mu\nu}$ giving rise to the graviton, together with a vector-spinor of spin $\frac{3}{2}$ denoted ψ_μ^α, and a spin-1 gauge field A_μ. If supersymmetry is unbroken, these states will be degenerate in mass. The fact that under extended supersymmetry bosonic fields of different spin join in a single multiplet is quite remarkable. It shows that from a "supercovariant" point of view their respective interactions are truly unified: we may thus be able to recover the way gravity works from our knowledge of gauge theory by exploiting the underlying supersymmetric structure.

2.2 Gauging a global symmetry

In order to construct a theory of supergravity, we shall first address the general way to construct a local symmetry from a global one. One way to do this is provided by the Noether method, which we will briefly discuss.
Consider as an example a massless complex scalar field with Lagrangian density

$$\mathcal{L} = \partial_\mu \phi \, \partial^\mu \bar{\phi} \, . \tag{4}$$

This theory has a global $U(1)$ symmetry, acting as a constant phase shift on the field ϕ :

$$\phi(x) \mapsto e^{i\Lambda} \phi(x) \, . \tag{5}$$

To construct a theory with local symmetry from this, we allow the phase to depend on the space-time coordinate, $\Lambda = \Lambda(x)$. Since an operation of type (5) is then no longer a symmetry for the Lagrangian of type (4), we need to change the Lagrangian by introducing the covariant derivative on the fiber bundle of a gauge field vector space over the space-time manifold,

$$D_\mu \phi(x) \;\equiv\; \partial_\mu \phi(x) - i A_\mu(x) \, \phi(x) \, . \tag{6}$$

We then substitute this for the ordinary derivatives in (4). The resulting Lagrangian

$$\mathcal{L} \;=\; D_\mu \phi \, D^\mu \bar{\phi} \, , \tag{7}$$

is locally $U(1)$ invariant. Adding a standard Maxwell term $F_{\mu\nu} F^{\mu\nu}$ to capture the dynamics of the vector fields gives rise to a consistent $U(1)$-invariant theory.
We could proceed the very same way, starting from a model with global supersymmetry, e.g. the WZ model, and turn the global symmetry into a local one. In the $U(1)$ example, the compensating gauge field introduced to correct the derivative was a helicity (or spin) 1 field A_μ. In the supersymmetric case where the transformation parameter is itself a spinor, we hence naively expect a spin-$\frac{3}{2}$ field ψ_μ^α, carrying both a spinor and a vector index. This field is called the "gravitino" and naturally arises in the supergravity multiplets. Gauging global supersymmetry we would hence introduce supercovariant derivatives and a kinetic term for the gravitino field. However, unlike the $U(1)$ example, the construction does not stop here. We have seen in (3) above, that a locally supersymmetric theory requires a dynamical metric, i.e. the Noether method will furthermore give rise to coupling the entire model to gravity. In other words, the analogue of "adding a standard Maxwell term $F_{\mu\nu} F^{\mu\nu}$" in the example above, here corresponds to adding a kinetic term for the gravitino as well as for the gravity field, as only together they can be supersymmetric.

Rather than going through this lengthy procedure (which is possible!) we will restrict here to a construction of the minimal supergravity theory in four dimensions, i.e. to the theory describing only the graviton and the associated gravitino which together form the smallest $N = 1$ supergravity multiplet. Our task shall be in the following to construct a theory with this field content, such that by setting $\psi_\mu^\alpha \equiv 0$ we find Einstein's general relativity again. I.e. we will have to work out the Lagrangian

$$\mathcal{L} = \mathcal{L}_{\text{EH}} + \mathcal{L}(\psi)\,, \tag{8}$$

which consists of a part describing the graviton by the Einstein-Hilbert action, and another part describing its coupling to the gravitino.

2.3 The vielbein formalism

We shall work in units where $\hbar = G = c = 1$. The standard formulation of general relativity includes the metric tensor $g_{\mu\nu}$ together with the Levi-Civita connection ∇_μ, defining covariant derivatives of tensors, as e.g. in the case of a vector field

$$\nabla_\mu X_\nu = \partial_\mu X_\nu - \Gamma^\lambda_{\mu\nu} X_\lambda\,, \tag{9}$$

implying that $\nabla_\mu X_\nu$ transforms as a tensor under diffeomorphisms. Further requiring that the metric is covariantly constant

$$\nabla_\lambda\, g_{\mu\nu} = 0\,, \tag{10}$$

defines the Christoffel symbols $\Gamma^\lambda_{\mu\nu}$ as

$$\Gamma^\lambda_{\mu\nu} = \tfrac{1}{2} g^{\lambda\rho}\left(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu}\right) + K^\lambda{}_{\mu\nu}\,, \tag{11}$$

where $K^\lambda{}_{\mu\nu}$ is the so-called contorsion, defined in terms of a torsion tensor $T^\lambda{}_{\mu\nu} = T^\lambda{}_{[\mu\nu]}$ as $K^\lambda{}_{\mu\nu} = \frac{1}{2}(T_\nu{}^\lambda{}_\mu + T_\mu{}^\lambda{}_\nu + T^\lambda{}_{\mu\nu})$. The torsion tensor is not fixed by the metricity condition (10) and usually set to zero, which amounts to having symmetric Christoffel symbols $\Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{(\mu\nu)}$. Supergravity in its simplest formulation however exhibits a nontrivial torsion bilinear in the fermionic fields, as we shall see.

The Riemann tensor can be expressed in terms of the Christoffel symbols; it is given by

$$R_{\mu\nu}{}^\rho{}_\sigma = 2\partial_{[\mu}\Gamma^\rho_{\nu]\sigma} + 2\Gamma^\rho_{[\mu|\lambda}\Gamma^\lambda_{|\nu]\sigma}\,, \tag{12}$$

where $A_{[\mu|\nu}B_{|\sigma]} := \frac{1}{2}(A_{\mu\nu}B_\sigma - A_{\sigma\nu}B_\nu)$. From this one computes the Ricci tensor

$$R_{\mu\nu} = R_{\rho\mu}{}^\rho{}_\nu\,,$$

and the Ricci scalar

$$R = g^{\mu\nu} R_{\mu\nu}\,.$$

The Einstein-Hilbert action is given by the Lagrangian density

$$\mathcal{L}_{\text{EH}} = -\tfrac{1}{4}\sqrt{|g|}R\,, \tag{13}$$

where g denotes the determinant of the metric, and the field equations are the vacuum Einstein equations

$$R_{\mu\nu} - \tfrac{1}{2} R g_{\mu\nu} = 0\,. \tag{14}$$

In order to deal with spinors, whose transformation rules are difficult to generalise to curved backgrounds, it is helpful to reformulate this theory in the so-called vielbein formalism. The idea behind this is to consider a set of coordinates that is locally inertial, so that one can apply the usual Lorentz behaviour of spinors, and to find a way to translate back to the original coordinate frame. To be a bit more precise, let $y^a(x_0; x)$, $a = 0, \ldots, 3$, denote a coordinate frame that is inertial at the space-time point x_0. We shall call these the "Lorentz" coordinates. Then

$$e_\mu{}^a(x_0) := \left.\frac{\partial y^a(x_0; x)}{\partial x^\mu}\right|_{x=x_0} ,$$

compose the so-called vielbein, or tetrad in our case where a takes four different values. Under general coordinate transformations $x \mapsto x'$, the vielbein transforms covariantly, i.e.

$$e'_\mu{}^a(x') = \frac{\partial x^\nu}{\partial x'^\mu} e_\nu{}^a(x)\,, \tag{15}$$

while a Lorentz transformation $y^a \mapsto y'^a = y^b \Lambda_b{}^a$ leads to

$$e'_\mu{}^a(x) = e_\mu{}^b(x) \Lambda_b{}^a\,. \tag{16}$$

In particular, the space-time metric can be expressed as

$$g_{\mu\nu}(x) = e_\mu{}^a(x) e_\nu{}^b(x)\, \eta_{ab}\,,$$

in terms of the Minkowski metric $\eta_{ab} = \mathrm{diag}(1, -1, -1, -1)$. The vielbein $e_\mu{}^a$ thus takes lower Lorentz (or "flat") indices a, b to lower indices in the coordinate basis (or "curved") indices μ, ν. Its inverse $e_a{}^\mu$ performs the transformation in the other direction. As an example consider a 1-form with coordinates X_μ, for which

$$X_\mu = e_\mu{}^a\, X_a\,, \quad X_a = e_a{}^\mu X_\mu\,.$$

On the other hand, contravariant indices are transformed as

$$X^\mu = X^a e_a{}^\mu\,, \quad X^a = X^\mu e_\mu{}^a\,.$$

We can now introduce spinors $\psi_\alpha(x)$ in our theory, which transform as scalars under the general space-time coordinate transformations, but in a spinor representation $\mathcal{R}$ of the local Lorentz group:

$$\psi'_\alpha(x) = \mathcal{R}(\Lambda(x))_\alpha{}^\beta\, \psi_\beta(x)\,.$$

In addition, we can convert the constant γ matrices of the inertial frame into γ matrices in the curved frame by the action of the vielbein:

$$\gamma_\mu(x) = e_\mu{}^a(x) \gamma_a \qquad \Longrightarrow \qquad \{\gamma_\mu, \gamma_\nu\} = 2 g_{\mu\nu}\,.$$

The choice of the locally inertial frame y^a is of course only unique up to Lorentz transformations, i.e. the tetrad is only defined up to a rotation in the Lorentz indices $e_\mu{}^a \mapsto e_\mu{}^b \Lambda_b{}^a$. The algebra $\mathfrak{so}(1,3)$ of the local Lorentz transformation group $\mathrm{SO}(1,3)$ is given by the generators M_{ab}, which are antisymmetric in their indices and satisfy the commutation relations

$$[M_{ab}, M_{cd}] = 2\eta_{a[c}\, M_{d]b} - 2\eta_{b[c}\, M_{d]a}\,. \tag{17}$$

The action of M_{ab} on vectors is given by

$$M_{ab} X^c = 2\delta^c_{[a} X_{b]}\,,$$

while on spinors it is

$$M_{ab}\psi = \tfrac{1}{2}\gamma_{ab}\psi\,,$$

where γ_{ab} stands for $\gamma_{[a}\gamma_{b]}$ and we have dropped the explicit spinor indices α, as we will usually do from now on. On the other hand, the space-time metric $g_{\mu\nu}$ stays invariant under $\mathfrak{so}(1,3)$ transformations, so that these will automatically preserve the Einstein-Hilbert action.
In order to couple matter to our theory, we will also need the Lorentz covariant derivatives. They are defined by

$$D_\mu \;\equiv\; \partial_\mu + \tfrac{1}{2}\omega_\mu{}^{ab}\, M_{ab}\,.$$

The $\omega_\mu{}^{ab}$ are a set of gauge fields; there is one for each generator of $\mathfrak{so}(1,3)$, correspondingly they are antisymmetric in the Lorentz indices a, b. Together, they define an object called the spin connection ω. It is determined by requiring that the tetrad is covariantly constant (the so-called vielbein postulate):[1]

$$0 \;=\; D_\mu e_\nu{}^a - \Gamma^\lambda_{\mu\nu} e_\lambda{}^a \;=\; \partial_\mu e_\nu{}^a + \omega_\mu{}^a{}_b\, e_\nu{}^b - \Gamma^\lambda_{\mu\nu} e_\lambda{}^a\,.$$

An equivalent way to write this is

$$D_{[\mu} e_{\nu]}{}^a = \frac{1}{2} T^a{}_{\mu\nu}\,. \tag{18}$$

This constitutes a set of algebraic equations for ω which has a unique solution expressing the spin connection in terms of the vielbein and the torsion. For vanishing torsion, the equations reduce to

$$D_{[\mu} e_{\nu]}{}^a = 0\,, \tag{19}$$

of which the solution is given by

$$\omega_\mu{}^{ab} = \omega_\mu{}^{ab}[e] = \frac{1}{2} e_{c\mu}\left(\Omega^{abc} - \Omega^{bca} - \Omega^{cab}\right)\,, \tag{20}$$

where $\Omega_{abc} = e_a{}^\mu e_b{}^\nu (\partial_\mu e_{\nu c} - \partial_\nu e_{\mu c})$ are the so-called objects of anholonomity. In the presence of torsion, the solution of (18) is given by $\omega_\mu{}^{ab} = \omega_\mu{}^{ab}[e] + K^a{}_\mu{}^b$, where the contorsion $K^a{}_\mu{}^b$ has been expressed in terms of $T^a{}_{\mu\nu}$ after equation (11) above. The spin connection ω also defines a curvature tensor with coefficients

$$R_{\mu\nu}{}^{ab}[\omega] = 2\partial_{[\mu}\omega_{\nu]}{}^{ab} + 2\omega_{[\mu}{}^{ac}\omega_{\nu]c}{}^b\,. \tag{21}$$

[1] In our notation, here and in the following the covariant derivative $D_\mu = D(\omega)_\mu$ always includes the spin connection, but not the Christoffel symbols.

It can be shown that this curvature is in fact nothing but the ordinary space-time curvature where two space-time indices have been converted to Lorentz indices, i.e.

$$R_{\mu\nu}{}^{ab} = R_{\mu\nu}{}^{\rho}{}_{\sigma}\, e_{\rho}{}^{a}\, e^{b\sigma}\,.$$

2.4 The Palatini action

We shall now apply the vielbein formalism to the formulation of general relativity. This is done by rewriting the Einstein-Hilbert Lagrangian (13) directly in terms of vielbein quantities:

$$\mathcal{L}_{\mathrm{EH}}[e] = -\tfrac{1}{4}\sqrt{|g|}R = -\tfrac{1}{4}\,|e|\, e_{a}{}^{\mu} e_{b}{}^{\nu} R_{\mu\nu}{}^{ab}\,. \tag{22}$$

Note that $|e|$ stands for the determinant of the tetrad on the right hand side, while on the left hand side the argument e is a short hand notation to denote dependence of $\mathcal{L}$ on the full tetrad $e_{\mu}{}^{a}$. The equations of motion for this Lagrangian are the Einstein equations (14). In the vielbein formalism, these second order equations can be expressed quite nicely in terms of a set of two equations of first order. This goes under the name of Palatini (or first order) formulation. It is achieved by considering the spin connection ω to be a priori independent of the tetrad, described by the Palatini Lagrangian

$$\mathcal{L}_{\mathrm{P}}[e,\omega] = -\tfrac{1}{4}|e|\, e_{a}{}^{\mu} e_{b}{}^{\nu} R_{\mu\nu}{}^{ab}[\omega]\,. \tag{23}$$

There are now two field equations: The first one comes from the variation with respect to the vielbein, $\delta\mathcal{L}_{\mathrm{P}}/\delta e$, and the second from the variation with respect to the spin connection, $\delta\mathcal{L}_{\mathrm{P}}/\delta\omega$. Explicitly, the variation yields

$$\delta\mathcal{L}_{\mathrm{P}} = -\frac{1}{2}|e|\,(R_{\mu}{}^{a} - \tfrac{1}{2}e_{\mu}{}^{a}R)\,\delta e_{a}{}^{\mu} - \frac{3}{2}|e|\,(D_{\mu}e_{\nu}{}^{a})\, e_{[a}{}^{\mu} e_{b}{}^{\nu} e_{c]}{}^{\rho}\,\delta\omega_{\rho}{}^{bc}\,, \tag{24}$$

as we show in appendix A. The second equation thus gives precisely (19) and is solved by by (20). Upon inserting this solution $\omega = \omega[e]$ into the first equation, we recover the Einstein equations (14). In the classical theory, the Palatini formulation is hence equivalent to the standard second order formulation of general relativity.
It is instructive to note that the relation $\mathcal{L}_{\mathrm{EH}}[e] = \mathcal{L}_{\mathrm{P}}[e,\omega]|_{\omega=\omega[e]}$ between the Einstein-Hilbert and the Palatini action provides a very simple way to also compute the variation of the former. Namely

$$\delta\mathcal{L}_{\mathrm{EH}}[e] \;=\; \left(\frac{\delta\mathcal{L}_{\mathrm{P}}}{\delta e_{\mu}{}^{a}}\Big|_{\omega=\omega[e]} \;+\; \frac{\delta\mathcal{L}_{\mathrm{P}}}{\delta\omega_{\nu}{}^{bc}}\Big|_{\omega=\omega[e]}\,\frac{\delta\omega_{\nu}{}^{bc}}{\delta e_{\mu}{}^{a}}\right)\delta e_{\mu}{}^{a}\,. \tag{25}$$

However, the second term vanishes identically, because $\delta\mathcal{L}_{\mathrm{P}}/\delta\omega$ is proportional to (19) and thus zero for $\omega = \omega[e]$. I.e. the variation of $\mathcal{L}_{\mathrm{EH}}$ is simply given by the first term of (24) and thus by Einstein's equations (14). A similar observation will greatly facilitate the computation of a supersymmetric extension of Einstein gravity in the following.

2.5 The supersymmetric action

We now come back to the reason why we introduced the vielbein formalism, which was the inclusion of spinor fields and in particular of the gravitino in our theory. While $\mathcal{L}_{\mathrm{EH}}$ (or $\mathcal{L}_{\mathrm{P}}$) can be considered as the "kinetic term" for gravity, we should now add a kinetic term for the "gauge field" ψ_μ^α. Such a term has in fact been known since the 1940ies, when Rarita and Schwinger published a proposition for the kinetic term of a spin-$\frac{3}{2}$ field [5]:

$$\mathcal{L}_{\rm RS} = \tfrac{1}{2}\epsilon^{\mu\nu\rho\sigma}\bar{\psi}_\mu\gamma_\nu\gamma_5 D_\rho\psi_\sigma \,. \qquad (26)$$

In general background, the Lorentz-covariant derivative already provides a non-trivial coupling to gravity. Just adding $\mathcal{L}_{\rm RS}$ to the Einstein-Hilbert Lagrangian will however not yet yield a supersymmetric theory. In order to guarantee supersymmetry, we must also allow further interaction terms which contain higher powers of ψ. Our ansatz should thus be a Lagrangian of the form

$$\mathcal{L}[e,\psi] \;=\; \mathcal{L}_{\rm EH}[e] + \mathcal{L}_{\rm RS}[e,\psi] + \mathcal{L}_{\psi^4}[e,\psi]\,, \qquad (27)$$

where the last part only contains terms of order at least 4 in the gravitino. However, to determine the actual form of $\mathcal{L}$ we must first understand how the fields transform under a supersymmetry variation.

Our strategy shall be the following. We will first present the transformation laws for the fields under a supersymmetry transformation, and verify them by checking the supersymmetry algebra to lowest order in the fermions. After that, we shall come back to the Lagrangian (27) and determine its precise form such that it is supersymmetric.

To lowest order in the fermions, the supersymmetry transformation laws of the vielbein and the gravitino are given by

$$\begin{aligned} \delta_\epsilon e_\mu{}^a &= -i\bar{\epsilon}\gamma^a\psi_\mu\,, \\ \delta_\epsilon\psi_\mu &= D_\mu\epsilon\,. \end{aligned} \qquad (28)$$

At least heuristically, they seem of a good form, as the bosonic vielbein transforms into its presumed superpartner according to (3), and as ψ_μ which is supposed to play the role of a "gauge field of local supersymmetry" indeed transforms as $D_\mu\epsilon$, the derivative of the local parameter. On the other hand, there seem to be other possibilities (like a contribution $\bar{\epsilon}\gamma_\mu\psi^a$ in $\delta_\epsilon e_\mu{}^a$) whose absence we should justify.

To establish the proposed transformations we will have to check that they satisfy the supersymmetry algebra. In particular, the commutator of two such supersymmetry transformations should give rise to a suitable combination of local symmetry transformations of the theory, in a way similar to (3). Consider the commutator of two supersymmetry transformations on the vielbein,

$$\begin{aligned} [\delta_{\epsilon_1},\delta_{\epsilon_2}]e_\mu{}^a &= 2\delta_{[\epsilon_1}\delta_{\epsilon_2]}e_\mu{}^a \\ &= \delta_{\epsilon_2}(i\bar{\epsilon}_1\gamma^a\psi_\mu) - \delta_{\epsilon_1}(i\bar{\epsilon}_2\gamma^a\psi_\mu) \\ &= i\bar{\epsilon}_1\gamma^a D_\mu\epsilon_2 - i\bar{\epsilon}_2\gamma^a D_\mu\epsilon_1 \\ &= \tfrac{i}{2}D_\mu(\bar{\epsilon}_2\gamma^a\epsilon_1 - \bar{\epsilon}_1\gamma^a\epsilon_2) \\ &= iD_\mu(\bar{\epsilon}_2\gamma^a\epsilon_1)\,, \end{aligned}$$

where we have used the property $\bar{\chi}\gamma^a\xi = -\bar{\xi}\gamma^a\chi$ of Majorana spinors χ, ψ in the last line. In order to understand this result, define the space-time vector field $\xi^\mu := i\bar{\epsilon}_2\gamma^\mu\epsilon_1$, such that

$$\begin{aligned} [\delta_{\epsilon_1}, \delta_{\epsilon_2}]e_\mu{}^a &= D_\mu(\xi^\nu e_\nu{}^a) \\ &= \xi^\nu D_\mu e_\nu{}^a + e_\nu{}^a\partial_\mu\xi^\nu \, . \end{aligned} \tag{29}$$

In the last equation we have used that D_μ is the covariant derivative with respect to only the Lorentz indices, which means that it operates as a mere space-time derivative on ξ^ν. Taking into account (19), we can exchange the lower indices in the first term of the last line in (29), such that by writing out the covariant derivative we are finally left with

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}]e_\mu{}^a = \underbrace{\xi^\nu\partial_\nu e_\mu{}^a + e_\nu{}^a\partial_\mu\xi^\nu}_{I} + \underbrace{\xi^\nu\omega_\nu{}^a{}_b e_\mu{}^b}_{II} \, . \tag{30}$$

We recognise part I as being the infinitesimal action of a diffeomorphism on $e_\mu{}^a$, and part II as an infinitesimal internal Lorentz transformation $\delta_\Lambda e_\mu{}^a$ with $\Lambda^a{}_b = \xi^\nu\omega_\nu{}^a{}_b$. We have thus shown that the transformations (28) close onto the vielbein as

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}]e_\mu{}^a = \delta_\xi + \delta_\Lambda \, , \tag{31}$$

into a combination of local symmetries: diffeomorphism and Lorentz transformation.
Proceeding, we should next apply the commutator of two supersymmetries to the gravitino. This however involves already fermionic terms of higher order (e.g. variation of the connection term in $D_\mu\epsilon_2$ in $\delta_2\psi_\mu$ gives rise to terms in $(\bar{\epsilon}_1\epsilon_2\psi)$ which are of the same fermion order as the variation of possible higher order terms $\psi\psi\epsilon$ in $\delta_2\psi_\mu$ that we are neglecting for the moment). Furthermore, it turns out that closure of the supersymmetry algebra on the fermionic fields will require these fields to obey their first order equations of motion — meaning that the supersymmetry algebra only closes on-shell. This involves the exact form of the Lagrangian, so let us first turn to the Lagrangian; we shall come back to this point in a short comment in section 2.6.
We hence go back to the Lagrangian (27), and try to determine its constituents such that it becomes supersymmetric. To do this, we will make use of a trick and formally include the higher powers in the fermionic fields $\mathcal{L}_{\psi^4}$ into the first two parts, without changing their general form. This can be achieved by properly adapting the spin connection. Our starting point is the Lagrangian

$$\mathcal{L}_0 \equiv \mathcal{L}_{\mathrm{EH}} + \mathcal{L}_{\mathrm{RS}} \, , \tag{32}$$

which depends on the vielbein and the gravitino field. As we have done in the case of the Palatini action, we may write this in the explicit form

$$\mathcal{L}_0[e, \psi, \omega[e]] = -\tfrac{1}{4}|e|\, e_a{}^\mu e_b{}^\nu R_{\mu\nu}{}^{ab}[\omega[e]] + \tfrac{1}{2}\epsilon^{\mu\nu\rho\sigma}\bar{\psi}_\mu\gamma_\nu\gamma_5 D_\rho\psi_\sigma \, , \tag{33}$$

with $\omega[e]$ from (20). Computing the variation of this Lagrangian under supersymmetry, we will thus find schematically

$$\delta\mathcal{L}_0 = \frac{\delta\mathcal{L}_0}{\delta e}\Big|_{\omega=\omega[e]}\delta e + \frac{\delta\mathcal{L}_0}{\delta\psi}\Big|_{\omega=\omega[e]}\delta\psi + \frac{\delta\mathcal{L}_0}{\delta\omega}\Big|_{\omega=\omega[e]}\frac{\delta\omega[e]}{\delta e}\,\delta e \, . \tag{34}$$

It would greatly simplify the calculation, if we could drop the last term by the same argument used at the end of section 2.4, using the fact that $\delta\mathcal{L}_0/\delta\omega$ was identically zero for $\omega = \omega[e]$. Unfortunately, this is no longer true. As $\delta\mathcal{L}_0/\delta\omega$ receives additional contributions from the Rarita-Schwinger term, it follows that variation of $\mathcal{L}_0$ w.r.t. the spin connection gives rise to the equation

$$D_{[\mu}e_{\nu]}{}^a = -\tfrac{i}{2}\bar{\psi}_\mu\gamma^a\psi_\nu \,, \tag{35}$$

instead of (19). As we have seen for (18) above, the solution of (35) is given by the modified spin connection

$$\hat{\omega}_\mu{}^{ab} = \hat{\omega}_\mu{}^{ab}[e,\psi] = \omega_\mu{}^{ab}[e] + K^a{}_\mu{}^b \,, \qquad \text{with} \quad K^a{}_\mu{}^b = -i(\bar{\psi}^{[a}\gamma^{b]}\psi_\mu + \tfrac{1}{2}\,\bar{\psi}^a\gamma_\mu\psi^b) \,, \tag{36}$$

which depends quadratically on the gravitino ψ_μ. Our ansatz for the supergravity Lagrangian will thus be

$$\mathcal{L} \;=\; \mathcal{L}_0[e,\psi,\hat{\omega}[e,\psi]] \;=\; -\tfrac{1}{4}|e|\, e_a{}^\mu e_b{}^\nu R_{\mu\nu}{}^{ab}[\hat{\omega}] + \tfrac{1}{2}\epsilon^{\mu\nu\rho\sigma}\bar{\psi}_\mu\gamma_\nu\gamma_5\hat{D}_\rho\psi_\sigma \,, \tag{37}$$

where $\hat{\omega}$ is given by (36). We also use the notation $\hat{D} = D(\hat{\omega})$ to indicate that the Lorentz connection in this covariant derivative is built by $\hat{\omega}$. The Lagrangian (37) thus differs from (33) by quartic terms in ψ. Correspondingly, we modify the supersymmetry transformation rules (28) to

$$\begin{aligned} \delta_\epsilon e_\mu{}^a &= -i\bar{\epsilon}\gamma^a\psi_\mu \,, \\ \delta_\epsilon\psi_\mu &= \hat{D}_\mu\epsilon \,, \end{aligned} \tag{38}$$

including additional cubic terms in the variation of the gravitino. We will now show that (37) already gives the full supersymmetric Lagrangian to all orders in the fermions.

Schematically, the variation of (37) is given by

$$\delta\mathcal{L} \;=\; \frac{\delta\mathcal{L}_0}{\delta e}\Big|_{\omega=\hat{\omega}[e,\psi]}\delta e + \frac{\delta\mathcal{L}_0}{\delta\psi}\Big|_{\omega=\hat{\omega}[e,\psi]}\delta\psi + \frac{\delta\mathcal{L}_0}{\delta\omega}\Big|_{\omega=\hat{\omega}[e,\psi]}\Big(\frac{\delta\hat{\omega}[e,\psi]}{\delta e}\,\delta e + \frac{\delta\hat{\omega}[e,\psi]}{\delta\psi}\,\delta\psi\Big) \,, \tag{39}$$

and now the entire last term vanishes, as $\delta\mathcal{L}_0/\delta\omega$ is proportional to (35) and thus identically zero for $\omega = \hat{\omega}[e,\psi]$. We can thus in the following computation consistently neglect all contributions due to explicit variation w.r.t. the spin connection.

Using (24), we find that variation of the Einstein-Hilbert term then gives rise to

$$\begin{aligned} \delta_\epsilon\mathcal{L}_{\rm EH} &= -\tfrac{1}{2}\,|e|\left(\hat{R}_\mu{}^a - \tfrac{1}{2}e_\mu{}^a\hat{R}\right)(\delta_\epsilon e_a{}^\mu) \\ &= -\tfrac{i}{2}\,|e|\left(\hat{R}_\mu{}^a - \tfrac{1}{2}e_\mu{}^a\hat{R}\right)\bar{\epsilon}\gamma^\mu\psi_a \,, \end{aligned} \tag{40}$$

where $\hat{R}_\mu{}^a = R[\hat{\omega}]_\mu{}^a$ denotes the Ricci tensor obtained by contracting the curvature (21) of the modified spin connection $\hat{\omega}$.

On the other hand, varying $\hat{\mathcal{L}}_{\rm RS}$ while again neglecting the terms from variation w.r.t. the spin connection yields

$$\begin{aligned} \delta_\epsilon\mathcal{L}_{\rm RS} &= \delta_\epsilon(\tfrac{1}{2}\epsilon^{\mu\nu\rho\sigma}\bar{\psi}_\mu\gamma_\nu\gamma_5\hat{D}_\rho\psi_\sigma) \\ &= \tfrac{1}{2}\epsilon^{\mu\nu\rho\sigma}\left(\overline{\hat{D}_\mu\epsilon}\gamma_\nu\gamma_5\hat{D}_\rho\psi_\sigma + \bar{\psi}_\mu\gamma_\nu\gamma_5\hat{D}_\rho\hat{D}_\sigma\epsilon + \bar{\psi}_\mu\,(\delta_\epsilon\gamma_\nu)\gamma_5\hat{D}_\rho\psi_\sigma\right) . \end{aligned}$$

Since the Lagrangian has to stay invariant only up to a total derivative, we can shift the derivative $\hat{D}_\mu$ in the first term to the right by partial integration. Hence

$$\begin{aligned}\delta_\epsilon \mathcal{L}_{\rm RS} &= \tfrac{1}{2}\epsilon^{\mu\nu\rho\sigma}\Big(\bar{\psi}_\mu\gamma_\nu\gamma_5\hat{D}_{[\rho}\hat{D}_{\sigma]}\epsilon - \bar{\epsilon}\gamma_\nu\gamma_5\hat{D}_{[\mu}\hat{D}_{\rho]}\psi_\sigma - \bar{\epsilon}(\hat{D}_{[\mu}\gamma_{\nu]})\gamma_5\hat{D}_\rho\psi_\sigma + \bar{\psi}_\mu(\delta_\epsilon\gamma_\nu)\gamma_5\hat{D}_\rho\psi_\sigma\Big)\\ &\quad + (\text{total derivatives})\,, \end{aligned} \tag{41}$$

where we have used the possibility to antisymmetrise in various indices in the bracket, as they are contracted with the totally antisymmetric ϵ tensor density. In the third term in the bracket, the covariant derivative of the gamma matrix $\hat{D}_{[\mu}\gamma_{\nu]} = \gamma_a\,\hat{D}_{[\mu}e_{\nu]}{}^a$ is quadratic in the fermions due to (35). Together with the two fermionic fields $\bar{\epsilon}$ and ψ, this term is thus at least quartic in the fermions. Likewise, the last term in (41) is at least quartic in the fermions. Denoting these two quartic terms by $\delta_\epsilon^{(4)}\mathcal{L}_{\rm RS}$ and making use of $[\hat{D}_\mu, \hat{D}_\nu]\psi = \frac{1}{4}\hat{R}_{\mu\nu}{}^{ab}\gamma_{ab}\psi$, we find

$$\delta_\epsilon\mathcal{L}_{\rm RS} = -\tfrac{1}{16}\epsilon^{\mu\nu\rho\sigma}\left(-\bar{\epsilon}\gamma_\nu\gamma_5\gamma_{ab}\psi_\mu + \bar{\psi}_\mu\gamma_\nu\gamma_5\gamma_{ab}\epsilon\right)\hat{R}_{\rho\sigma}{}^{ab} + \delta_\epsilon^{(4)}\mathcal{L}_{\rm RS}\,, \tag{42}$$

where we have relabeled a few of the indices in the first terms. We observe that

$$\begin{aligned}\gamma_\nu\gamma_5\gamma_{ab} &= \gamma_\nu\gamma_{ab}\gamma_5\\ &= \gamma_{\nu ab}\gamma_5 + 2e_{\nu[a}\gamma_{b]}\gamma_5\,,\end{aligned}$$

and since $\gamma_{\nu ab} = ie_\nu{}^d\,\epsilon_{dabc}\gamma^c\gamma_5$ and $(\gamma_5)^2 = \mathbb{1}$, we have

$$\gamma_\nu\gamma_5\gamma_{ab} = ie_\nu{}^d\,\epsilon_{dabc}\gamma^c + 2e_{\nu[a}\gamma_{b]}\gamma_5\,.$$

Plugging this back into the variation (42) and keeping in mind that $\bar{\epsilon}\gamma^c\psi = -\bar{\psi}\gamma^c\epsilon$ for Majorana spinors, we are thus led to

$$\delta_\epsilon\mathcal{L}_{\rm RS} = -\tfrac{i}{8}e_\nu{}^d\epsilon^{\nu\mu\rho\sigma}\epsilon_{dabc}\,\bar{\epsilon}\gamma^c\psi_\mu\,\hat{R}_{\rho\sigma}{}^{ab} + \delta_\epsilon^{(4)}\mathcal{L}_{\rm RS} - \tfrac{1}{8}\epsilon^{\mu\nu\rho\sigma}(-\bar{\epsilon}e_{\nu[a}\gamma_{b]}\gamma_5\psi_\mu + \bar{\psi}_\mu e_{\nu[a}\gamma_{b]}\gamma_5\epsilon)\hat{R}_{\rho\sigma}{}^{ab}\,.$$

Since we also have $\bar{\epsilon}\gamma_5\gamma_a\psi = \bar{\psi}\gamma_5\gamma_a\epsilon$ for Majorana spinors, the last two terms in this expression mutually cancel. If we furthermore make use of the identity $e_\nu{}^d\epsilon^{\nu\mu\rho\sigma}\epsilon_{dabc} = -6|e|e_{[a}{}^\mu e_b{}^\rho e_{c]}{}^\sigma$, we finally obtain

$$\begin{aligned}\delta_\epsilon\mathcal{L}_{\rm RS} = -\tfrac{i}{8}e_\nu{}^d\epsilon^{\nu\mu\rho\sigma}\epsilon_{dabc}\,\bar{\epsilon}\gamma^c\psi_\mu\,\hat{R}_{\rho\sigma}{}^{ab} &= \tfrac{6i}{8}\,|e|e_{[a}{}^\mu e_b{}^\rho e_{c]}{}^\sigma(\bar{\epsilon}\gamma^c\psi_\mu)\hat{R}_{\rho\sigma}{}^{ab} + \delta_\epsilon^{(4)}\mathcal{L}_{\rm RS}\\ &= \tfrac{3i}{4}\,|e|(\bar{\epsilon}\gamma^{[\sigma}\psi_\mu)\hat{R}_{\rho\sigma}{}^{\mu\rho]} + \delta_\epsilon^{(4)}\mathcal{L}_{\rm RS}\\ &= \tfrac{i}{2}\,|e|(\bar{\epsilon}\gamma^\mu\psi_a)\left(\hat{R}_\mu{}^a - \tfrac{1}{2}e_\mu{}^a\hat{R}\right) + \delta_\epsilon^{(4)}\mathcal{L}_{\rm RS}\,.\end{aligned}$$

The first term precisely cancels the variation of the Einstein-Hilbert part (40).
We have thus shown that up to total derivatives the variation of the full Lagrangian $\mathcal{L}$ from (27) reduces to the last two terms of (41)

$$\delta_\epsilon\mathcal{L} \;=\; \delta_\epsilon^{(4)}\mathcal{L}_{\rm RS} \;=\; \tfrac{1}{2}\epsilon^{\mu\nu\rho\sigma}\left(-\bar{\epsilon}(\hat{D}_{[\mu}\gamma_{\nu]})\gamma_5\hat{D}_\rho\psi_\sigma + \bar{\psi}_\mu(\delta_\epsilon\gamma_\nu)\gamma_5\hat{D}_\rho\psi_\sigma\right)\,, \tag{43}$$

quartic in the fermions. With (35) we find

$$\delta_\epsilon^{(4)}\mathcal{L}_{\rm RS} \;=\; \tfrac{i}{2}\epsilon^{\mu\nu\rho\sigma}\left(\tfrac{1}{2}(\bar{\psi}_\mu\gamma^a\psi_\nu)(\bar{\epsilon}\gamma_a\gamma_5\hat{D}_\rho\psi_\sigma) - (\bar{\psi}_\mu\,\gamma_a\gamma_5\hat{D}_\rho\psi_\sigma)(\bar{\epsilon}\gamma^a\psi_\nu)\right)\,. \tag{44}$$

A priori, the form of the two terms is rather different. In order to bring them into the same form, we will need some so-called Fierz identities:

$$\chi)(\bar{\lambda} \;=\; -\tfrac{1}{4}(\bar{\lambda}\chi)I - \tfrac{1}{4}(\bar{\lambda}\gamma_5\chi)\gamma_5 - \tfrac{1}{4}(\bar{\lambda}\gamma_a\chi)\gamma^a + \tfrac{1}{4}(\bar{\lambda}\gamma_a\gamma_5\chi)\gamma^a\gamma_5 + \tfrac{1}{8}(\bar{\lambda}\gamma_{ab}\chi)\gamma^{ab}\;, \qquad (45)$$

where the l.h.s. is to be understood as a matrix to be contracted with further spinors from the left and the right. (The easiest way to prove this identity is to note that the 16 matrices $\{I, \gamma_5, \gamma^a, \gamma^a\gamma_5, \gamma^{ab}\} \equiv \mathcal{B}$ form a basis of the 4×4 matrices which is orthogonal with respect to $\mathrm{Tr}(AB)$.) Applying (45) to the second term of (44) (with $\chi = \hat{D}_\rho\psi_\sigma$ and $\bar{\lambda} = \bar{\epsilon}$), and noting that due to the symmetry properties of Majorana spinors the bilinear $\bar{\psi}_{[\mu}\Gamma\psi_{\nu]}$ for $\Gamma \in \mathcal{B}$ is non-vanishing only for $\Gamma = \gamma_a$ and $\Gamma = \gamma_{ab}$, we find

$$\begin{aligned} -\tfrac{i}{2}\epsilon^{\mu\nu\rho\sigma}\,(\bar{\psi}_\mu\,\gamma_a\gamma_5\hat{D}_\rho\psi_\sigma)(\bar{\epsilon}\gamma^a\psi_\nu) \;&=\; -\tfrac{i}{4}\epsilon^{\mu\nu\rho\sigma}\,(\bar{\psi}_\mu\,\gamma_a\gamma_5\gamma_b\gamma_5\gamma^a\psi_\nu)(\bar{\epsilon}\gamma^b\gamma_5\hat{D}_\rho\psi_\sigma) \\ &\quad - \tfrac{i}{8}\epsilon^{\mu\nu\rho\sigma}\,(\bar{\psi}_\mu\,\gamma_a\gamma_5\gamma_{bc}\gamma_5\gamma^a\psi_\nu)(\bar{\epsilon}\gamma^{bc}\gamma_5\hat{D}_\rho\psi_\sigma)\,. \end{aligned}$$

Using that $\gamma_a\gamma_b\gamma^a = -2\gamma_b$ and $\gamma_a\gamma_{bc}\gamma^a = 0$, the second term vanishes while the first term precisely cancels the first term of (44). Together, we have thus shown that the Lagrangian (37) is invariant up to total derivatives under the supersymetry transformations (38). It constitutes the sought-after minimal supersymmetric extension of Einstein gravity in four dimensions.

2.6 Results

Let us collect and complete our results, and simplify the notation. The full supersymmetric Lagrangian is given by $\mathcal{L} = \mathcal{L}_0[e, \psi, \hat{\omega}[e, \psi]]$ from (37). By properly adapting the spin connection $\hat{\omega}$ we got rid of all explicit quartic fermion interaction terms. Equation (35) corresponds precisely to the equation $\delta\mathcal{L}_0/\delta\hat{\omega} = 0$ and is identically solved by $\hat{\omega}$ given in equation (36) as (schematically)

$$\hat{\omega} = \hat{\omega}[e, \psi] = \omega[e] + K\,,$$

depending on the tetrad and the gravitino. Equivalently, the full Lagrangian $\mathcal{L}$ can be brought into the form (27) in which the first terms depend on the standard spin connection $\omega[e]$, and the quartic interaction terms are explicit. They can in fact be retrieved rather conveniently by formally expanding $\mathcal{L}_0[e, \psi, \omega]$ around $\omega = \hat{\omega}$. Since $\mathcal{L}_0$ is no more than quadratic in ω, this expansion stops after two terms; we find (again schematically)

$$\mathcal{L}_0[e, \psi, \omega] = \mathcal{L}_0[e, \psi, \omega]\Big|_{\omega=\hat{\omega}} - K\,\frac{\delta\mathcal{L}_0[e, \omega, \psi]}{\delta\omega}\Big|_{\omega=\hat{\omega}} + \frac{1}{2}\,K^2\,\frac{\delta}{\delta\omega}\frac{\delta}{\delta\omega}\mathcal{L}_0[e, \omega, \psi]\Big|_{\omega=\hat{\omega}}\,.$$

Now the term linear in K vanishes, because $\delta\mathcal{L}_0/\delta\omega = 0$ at $\omega = \hat{\omega}$, as we have already used above. Moreover, the first term on the right-hand side is precisely the full supersymmetric Lagrangian (37). The last term finally has only contributions from the $\omega\omega$ term in the curvature $R_{\mu\nu}{}^{ab}[\omega]$ in the Einstein-Hilbert term. Together, this yields for the full supersymmetric Lagrangian

$$\mathcal{L} = \mathcal{L}_0[e, \psi, \omega[e]] - \tfrac{1}{4}|e|\left(K_a{}^{ac}K_b{}^b{}_c + K^{abc}K_{cab}\right)\,, \qquad (46)$$

where K_{abc} given in (36) explicitly describes the quartic fermion interactions. One can see that the fermions are subject to an interaction of current-current type.
Let us mention two more things before we conclude this chapter. The first one is that with the full supersymmetry rules (38)

$$\begin{aligned} \delta_\epsilon e_\mu{}^a &= -i\bar{\epsilon}\gamma^a\psi_\mu \,, \\ \delta_\epsilon \psi_\mu &= \hat{D}_\mu \epsilon \,, \end{aligned} \tag{47}$$

containing cubic fermion terms in the variation of the gravitino, the computation of the supersymmetry algebra leading to (31) needs to be revisited. Extending our previous calculation, the commutator of two supersymmetry transformations on the tetrad can be shown to be

$$\begin{aligned} [\delta_{\epsilon_1}, \delta_{\epsilon_2}] e_\mu{}^a &= \hat{D}_\mu(\xi^\nu e_\nu{}^a) \\ &= \xi^\nu \partial \nu e_\mu{}^a + e_\nu{}^a \partial_\mu \xi^\nu + \xi^\nu \hat{\omega}_\nu{}^a{}_b e_\mu{}^b + i\xi^\nu \bar{\psi}_\nu \gamma^a \psi_\mu \,, \end{aligned}$$

for $\xi^\mu = i\bar{\epsilon}_2\gamma^\mu\epsilon_1$ as in (30), where the last term is due to the non-vanishing torsion (35). Again, the result is given by a combination of local transformations; the first two terms being an infinitesimal diffeomorphism, the third term an internal Lorentz transformation (now with $\Lambda^a{}_b = \xi^\nu \hat{\omega}_\nu{}^a{}_b$), while the last term represents an additional supersymmetry transformation with parameter $\epsilon = -\xi^\nu\psi_\nu$. If on the other hand one applies the commutator of supersymmetry transformations on the gravitino, one will find that the only way to interpret the result as a combination of local transformations involves the application of the equations of motion of the gravitino, i.e. the supersymmetry algebra closes only on-shell (some details for the computation can be found in [2]). This is a generic feature of supersymmetric theories.
Finally, a remark concerning nomenclature: the straightforward although very tedious way to prove supersymmetry of the full Lagrangian would amount to starting from its explicit form (46) and check its invariance under (47) including lengthy terms up to order ψ^5 due to the explicit dependence of the spin connection on the tetrad. Historically, this is how four-dimensional supergravity was first constructed by Freedman, van Nieuwenhuizen and Ferrara in 1976 [6]. Nowadays, this is referred to as the "second order" formalism. Soon after, Deser and Zumino proved supersymmetry by using an analogue of the Palatini formalism, treating the spin connection as an independent field. This "first order" formalism simplifies the computation but requires to find the correct supersymmetry transformation law for the independent spin connection. The method outlined in these lectures is the most common used these days and goes under the name of the "1.5th order formalism". It combines features of both of the older approaches. Morally, we stay second order in the sense that only tetrad and gravitino are independent fields in the Lagrangian. From the first order formalism we borrow however the observation that the equation obtained by formal variation of the Lagrangian $\delta\mathcal{L}/\delta\hat{\omega}$ vanishes identically for $\omega = \hat{\omega}$. As we have seen, this greatly simplifies the computation of the higher order fermion terms.

3 Extended supergravity in $D = 4$ dimensions

3.1 Matter couplings in $N = 1$ supergravity

In the last chapter, we have discussed how to construct minimal $N = 1$ supergravity in $D = 4$ dimensions, i.e. how to construct the supersymmetric couplings for the $N = 1$ supergravity multiplet $(g_{\mu\nu}, \psi_\mu^\alpha)$. The aim of this section is to sketch how we can implement further matter in $N = 1$ supergravity (for details see [8, 9]). Besides the supergravity multiplet, we may consider chiral multiplets (ϕ^i, χ^i), consisting of two spin-1/2 fermions χ^i and a complex spin-0 scalar ϕ^i for each i, and vector multiplets (χ^M, A_μ^M) containing one spin-1 gauge field A_μ^M and two spin-1/2 fermions χ^M for each M. Both are representations of the $N = 1$ supersymmetry algebra and contain four degrees of freedom.

In the following we just discuss the couplings of the bosonic sector, i.e. the interactions of the metric $g_{\mu\nu}$, the scalar fields ϕ^i and the (abelian) gauge fields A_μ^M. Restricting the dynamics to two-derivative terms, these couplings are rather constrained by diffeomorphism and gauge invariance which allows for the following terms:

- the gravitational sector, described by the Einstein-Hilbert term

$$\mathcal{L}_{\text{EH}} = -\frac{1}{4}|e|R \, , \tag{48}$$

  which was extensively discussed in the last chapter.

- the scalar sector

$$\mathcal{L}_{\text{scalar}} = -\frac{1}{2}|e|\, g^{\mu\nu}\, \partial_\mu\phi^j \partial_\nu\phi^k\, G_{jk}(\phi^i) - |e|\, \mathrm{V}(\phi^i) \, , \tag{49}$$

  where $G_{jk}(\phi^i)$ is the metric on the scalar target space and $\mathrm{V}(\phi^i)$ is a scalar potential.

- the vector sector

$$\mathcal{L}_{\text{vector}} = -\frac{1}{4}\, |e|\, g^{\mu\rho} g^{\nu\sigma}\, F_{\mu\nu}^M F_{\rho\sigma}^N\, \mathcal{I}_{MN}(\phi^i) - \frac{1}{4}\, \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu}^M F_{\rho\sigma}^N\, \mathcal{R}_{MN}(\phi^i) \, , \tag{50}$$

  described by a standard Maxwell term and a topological term (which does not depend on the metric) in terms of the abelian field strength $F_{\mu\nu}^M \equiv \partial_\mu A_\nu^M - \partial_\nu A_\mu^M$. The matrices $\mathcal{I}_{MN}(\phi^i)$ and $\mathcal{R}_{MN}(\phi^i)$ may depend on the scalar fields. If $\mathcal{R}_{MN}$ is constant, the second term is a total derivative and can be considered as a generalisation of the usual instanton term.

The "data" that need to be specified in order to fix the bosonic Lagrangian thus are the scalar dependent matrices $G_{jk}(\phi^i)$, $\mathcal{I}_{MN}(\phi^i)$, and $\mathcal{R}_{MN}(\phi^i)$, and the potential $\mathrm{V}(\phi^i)$. The fermionic couplings of the Lagrangian are then entirely determined by supersymmetry, without any freedom left. In turn, the bosonic data cannot be chosen arbitrarily, but are constrained by supersymmetry, i.e. by demanding that the bosonic Lagrangian $\mathcal{L}_{\text{EH}} + \mathcal{L}_{\text{scalar}} + \mathcal{L}_{\text{vector}}$ can be completed by fermionic couplings to a Lagrangian invariant under local supersymmetry. Without going into any details, we just state that this requires the scalar target space to be a Kähler

λ	-2	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	
$N=2$							1	2	1	sugra
					1	2	1			vector
				1	2	1				hyper
$N=4$					1	4	6	4	1	sugra
			1	4	6	4	1			vector
$N=8$	1	8	28	56	70	56	28	8	1	sugra

Table 1: The various multiplets for $N = 2, 4, 8$ supersymmetry. λ denotes the helicity of the states, i.e. their charge under the SO(2) little group in $D = 4$ dimensions.

manifold, and the combination $\mathcal{N}_{MN} \equiv \mathcal{R}_{MN} + i\,\mathcal{I}_{MN}$ to be a holomorphic function of the complex scalar fields ϕ^i. We refer to [8, 9] for details. There are more complicated versions of the theory, in which the gauge group is non-abelian and part of the scalar fields may be charged under the gauge group, this can be accommodated by introducing proper covariant derivatives in (49) and replacing (50) by the corresponding Yang-Mills terms [10]. These theories, in which the scalar potential is generically non-vanishing are referred to as *gauged supergravities.*
Within extended supergravity, which will be briefly discussed in the next section, the generic structure of the Lagrangian remains the same. However, the conditions on $G_{jk}(\phi^i)$, $\mathcal{I}_{MN}(\phi^i)$, and $\mathcal{R}_{MN}(\phi^i)$ are much more restrictive.

3.2 Extended supergravity in $D = 4$ dimensions

In the last section we have discussed the couplings of chiral and vector multiplets to minimal $N = 1$ supergravity. In principle, there is another $N = 1$ multiplet that one may consider, the so-called gravitino multiplet which consists of a spin-$\frac{3}{2}$ field ψ_μ and a spin-1 field $\mathcal{A}_\mu$. Its coupling is slightly more subtle due to a simple reason: just as the two degrees of freedom of the massless vector are described by a standard U(1) gauge theory, the additional massless gravitino must be described by introducing yet another local supersymmetry. In other words, coupling the gravitino multiplet to the $N = 1$ supergravity multiplet gives rise to a theory, which has two local supersymmetries and is thus referred to as $N = 2$ supergravity [11]. Accordingly, multiplets now organize under the $N = 2$ supersymmetry algebra in $D = 4$ dimensions. The $N = 2$ supergravity multiplet comprises the two $N = 1$ multiplets mentioned above and thus includes one spin-2 field, two spin-$\frac{3}{2}$ fields and a spin-1 field $\mathcal{A}_\mu$. Analogously to the discussion of the last section, this theory may be enlarged by further coupling matter in $N = 2$ multiplets. Upon coupling more gravitino multiplets one arrives at theories with larger extended supersymmetry, whose structures and field contents are more and more constrained. The particle content of the most common multiplets with extended supersymmetry ($N = 2, 4, 8$) are summarized in table 1
In addition there are multiplets for other values of N, for example involving $N = 5$ or $N = 6$ supercharges. We restrict the discussion to $N \leq 8$ multiplets since representations of super-

symmetry algebras with $N > 8$ supercharges contain states with helicities larger than two. No consistent interacting theories involving a finite number of such fields are known.
As mentioned above, the constraints on supergravity theories with extended supersymmetry are very severe. For example, the scalar target space of $N = 4$ supergravity has to be of the form

$$\frac{\mathrm{SO}(6,n)}{\mathrm{SO}(6)\times\mathrm{SO}(n)} \times \frac{\mathrm{SL}(2)}{\mathrm{SO}(2)} \,, \tag{51}$$

with some positive integer n. The dimension of this manifold is $6n + 2$ and thus coincides with the number of scalar fields expected for n vector multiplets and one supergravity multiplet (including the CPT conjugated multiplet) with $N = 4$ supercharges.
The (ungauged) $N = 8$ supergravity in $D = 4$ dimensions is unique and its field content can only consist of a single supergravity multiplet. The corresponding scalar target space is given by the coset-space[2]

$$\frac{\mathrm{E}_{7(7)}}{\mathrm{SU}(8)} \,. \tag{52}$$

Because of $\dim \mathrm{E}_{7(7)} = 133$ and $\dim \mathrm{SU}(8) = 63$, the target space dimension is 70 in agreement with the number of scalars in the theory. The construction of the $N = 8$ supergravity is not easy with the toolkit provided up to now. However, it can rather straightforwardly be derived from eleven-dimensional supergravity [12] compactified on a seven-dimensional torus T^7, as was originally done by Cremmer and Julia [13]. In order to follow this route, it is important to understand the construction of supergravity theories in higher dimensions and their dimensional reduction which we shall discuss in the following.

4 Extended supergravity in higher Dimensions

The motivation to study supergravity theories in dimensions $D > 4$, and their dimensional reduction is two-fold. On the one hand supergravity appears as the low energy effective action for fundamental theories such as string theories, which generically live in higher dimensions. In this *top-down* approach one has to consider the dimensional reduction of these theories in order to derive from the fundamental theories a meaningful four-dimensional theories supposed to describe our universe. On the other hand from a purely four-dimensional point of view, the dimensional reduction of higher-dimensional supergravities can be considered as a powerful technique in order to construct extended $D = 4$ supergravity theories. This is usually referred to as a *bottom-up* approach.

4.1 Spinors in higher dimensions

In order to discuss extended supergravities we have to know the possible types of spinors in higher space-time dimensions, see e.g. [14] for a reference. In this section we consider D-dimensional Minkowski space-time with signature $\eta = \mathrm{diag}(+1, -1, \ldots, -1)$. The spinorial representation of

[2] By $\mathrm{E}_{7(7)}$ we denote here a particular real form of the complex exceptional group E_7, whose compact part is given by SU(8).

the Lorentz group can be obtained from a representation of the D-dimensional Clifford algebra associated with the metric η_{ab}. The generators of this algebra, called Gamma matrices Γ_a, satisfy the anticommutation relations

$$\{\Gamma_a, \Gamma_b\} = 2\,\eta_{ab}\mathbb{1}\ . \tag{53}$$

The spinorial representation of the D-dimensional Lorentz algebra SO$(1, D-1)$ can then be constructed using antisymmetric products of gamma matrices; it is straightforward to verify that the Clifford algebra (53) implies that the generators

$$M_{ab} \equiv \frac{1}{2}\Gamma_{ab}, \qquad \Gamma_{ab} = \Gamma_{[a}\Gamma_{b]}\ , \tag{54}$$

satisy the Lorentz algebra (17). Finite Lorentz transformations are given by exponentiation

$$\mathcal{R}(\Lambda) = \exp\left(\frac{1}{2}\Lambda^{ab}\,M_{ab}\right)\ .$$

It can be shown that in even dimensions, the Clifford algebra has a unique (up to similarity transformations) irreducible representation $\mathcal{C}_D$ which has $2^{[D/2]}$ complex dimensions. Its elements are referred to as *Dirac spinors*. An explicit construction can be given recursively: denoting by γ_j the Gamma-matrices in $(D-2)$-dimensional space-time, the ensemble

$$i\begin{pmatrix} 0 & \gamma_j \\ -\gamma_j & 0 \end{pmatrix}\ ,\quad i\begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}\ ,\quad i\begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}\ , \tag{55}$$

satisfies the algebra (53) in D-dimensional space-time.
An important object is the following combination of Gamma matrices

$$\bar{\Gamma} = i^{\frac{D}{2}+1}\Gamma_0\Gamma_1\cdots\Gamma_{D-1}\ , \tag{56}$$

which generalizes the γ_5 in $D = 4$ dimensions. The Clifford algebra implies that it satisfies relations

$$\{\bar{\Gamma}, \Gamma_a\} \;=\; 0\,, \qquad \bar{\Gamma}^2 \;=\; -\,\mathbb{1}\ . \tag{57}$$

This has two important consequences:

- The ensemble $(\bar{\Gamma}, \Gamma_a)$ of matrices defines a representation of the Clifford algebra in $D+1$ dimensions. This shows that in also odd dimensions, the Clifford algebra (53) has an irreducible representation $\mathcal{C}_{D+1} \equiv \mathcal{C}_D$ of $2^{[D/2]}$ complex dimensions. In fact, in odd dimensions there are two inequivalent representations $\mathcal{C}_{D+1}$, $\mathcal{C}'_{D+1}$ of the Clifford algebra, related by $\bar{\Gamma} \to -\bar{\Gamma}$, $\Gamma_a \to -\Gamma_a$, which can not be achieved by a similarity transformation. Via (54) this defines a (unique and irreducible) $2^{[D/2]}$-dimensional representation of the Lorentz group in odd dimensions.

- In even dimensions, the operator $\bar{\Gamma}$ commutes with all generators of the Lorentz algebra: $[\bar{\Gamma}, M_{ab}] = 0$. This implies that as a representation of the Lorentz algebra, the representation $\mathcal{C}_D$ is reducible and may be decomposed into its irreducible parts by virtue of the projectors

$$\mathbb{P}_{\pm} \;\equiv\; \frac{1}{2}\left(\mathbb{1} \pm i\,\bar{\Gamma}\right)\ . \tag{58}$$

The corresponding $2^{[D/2]-1}$-dimensional chiral spinors are called *Weyl spinors.*

Let us now consider the reality properties of the spinors. It follows from (53) that also the complex conjugated Gamma matrices $\Gamma_a^\star$ fulfill the same Clifford Algebra

$$\{\Gamma_a^\star, \Gamma_b^\star\} = 2\eta_{ab}\mathbb{1} \ .$$

Because of the uniqueness of the representations of the Clifford algebra discussed above, this implies that $\Gamma_a^\star$ has to be related to Γ_a (up to a sign) by a similarity transformation

$$\pm\Gamma_a^\star = \mathcal{B}_\pm \, \Gamma_a \mathcal{B}_\pm^{-1} \ .$$

This suggests to impose on the spinors on of the following reality conditions

$$\psi^\star = \mathcal{B}_- \psi; \qquad \text{and} \qquad \psi^\star = \mathcal{B}_+ \psi \ ,$$

which are referred to as *Majorana* and *pseudo-Majorana*, respectively. Obviously, these conditions only make sense provided that

$$\mathcal{B}_\pm \mathcal{B}_\pm^\star = 1 \ ,$$

which only holds in certain space-time dimensions. A closer analysis shows that (pseudo) Majorana spinors exits in dimensions $D \equiv 0, 1, 2, 3, 4 \mod 8$. They possess half of the degrees of freedom of a Dirac spinor.
In even dimensions, we may further ask if these reality conditions are compatible with the chiral split (58) into Weyl spinors. It turns out that this is only the case in dimensions $D \equiv 2 \mod 8$. All possible types of spinors in $D \leq 11$ dimensional Minkowski space-time are given in table 2. Based on these structures, supergravity theories can be constructed in dimensions $D \leq 11$. The highest-dimensional supergravity theory is the unique eleven-dimensional theory, constructed 1978 by Cremmer, Julia and Scherk [12], which we shall discuss in some more detail now.

4.2 Eleven-dimensional supergravity

The highest-dimensional supergravity theory lives in eleven space-time dimensions and was constructed by Cremmer, Julia and Scherk [12]. Its field content is given by the lowest massless representation of the supersymmetry algebra (2). For massless states which fulfill $P_\mu P^\mu = 0$ we can set without loss of generality $P_0 = P_{10}$ and all other components to zero. The supersymmetry algebra then turns into

$$\{Q_\alpha, Q_\beta^\dagger\} \ = \ 2P_\mu \left(\Gamma^\mu \Gamma^0\right)_{\alpha\beta} \ = \ 2P_0 \left(1 + \Gamma^{10}\Gamma^0\right)_{\alpha\beta} , \tag{59}$$

where the Q_α are the 32 independent real supercharges in $D = 11$ dimensional Minkowski space-time. As the r.h.s. of (59) describes a projector of half-maximal rank in spinor space, it follows that only 16 out of the 32 supercharges act non-trivially on massless states and satisfy the Clifford algebra (53). From the discussion of the previous section, we then know that there is a $2^8 = 256$-dimensional irreducible representation $\mathcal{C}_{16}$ of this algebra which then gives the field content of the eleven-dimensional theory. Under SO(16), this representation decomposes into $128_s + 128_c$

D	W	M	pM	MW	dimension (real)
2	×	×	×	×	1
3		×			2
4	×	×			4
5					8
6	×				8
7					16
8	×		×		16
9			×		16
10	×	×	×	×	16
11		×			32

Table 2: Possible types of spinors in D-dimensional Minkowski space-time. Weyl, Majorana, pseudo-Majorana and Majorana-Weyl are denoted by W, M, pM and MW respectively.

chiral spinors (58). It is important to stress that in contrast to the discussion of the last section, the 16 here has no meaning as a number of space-time dimensions, but just denotes the number of supercharges; we are just exploiting the fact, that the same abstract algebra (53) appears in a rather different context again. In order to find the space-time interpretation of the 256 physical states, we note that the 16 supercharges form the physical degrees of freedom of a space-time spinor in eleven dimensions, i.e. they build an irreducible 16-dimensional representation of the little group SO(9) under which massless states are organized in eleven dimensions.
We thus consider the embedding SO(9) $\subset$ SO(16) of the little group into SO(16), under which the relevant representations decompose as [17]

$$\begin{aligned} 16 &\rightarrow 16\,, \\ 128_s &\rightarrow 44 \oplus 84\,, \\ 128_c &\rightarrow 128\,. \end{aligned} \tag{60}$$

What is the meaning of these SO(9) representations? Recall, that a massless vector in eleven dimensions has 9 degrees of freedom and transforms in the fundamental representation of SO(9). The 44 corresponds to the symmetric traceless product of two vectors: these are the degrees of freedom of a massless spin-2 field, the graviton. The 84 on the other hand describes the three-fold completely antisymmetric tensor product of vectors: $\binom{9}{3} = 84$. The bosonic field content of eleven-dimensional supergravity thus is given by the metric $g_{\mu\nu}$ and a totally anti-symmetric three-form field $A_{\mu\nu\rho}$. The fermionic field content of the theory is the irreducible 128 of SO(9) which is the Gamma-traceless vector-spinor $9 * 16 - 16 = 128$. Accordingly, this describes the degrees of freedom of a massless spin-3/2 field, the gravitino. The full eleven-dimensional supergravity multiplet thus is given by $(g_{\mu\nu}, \psi_\mu, A_{\mu\nu\rho})$. A similar argument shows why supergravity theories do not exist beyond eleven dimensions: repeating the same analysis for say a twelve-dimensional space-time yields a minimal field content that includes fields with spin larger than two. No consistent interacting theory for such fields can be constructed.

The action of eleven-dimensional supergravity is given by the eleven-dimensional analogue of (46)

$$\mathcal{L}_0[e,\psi] \;=\; -\frac{1}{4}|e|\,R - \frac{1}{4}|e|\bar{\psi}_\mu \Gamma^{\mu\nu\rho} D_\nu \psi_\rho\;, \tag{61}$$

supplemented by a kinetic, a topological term and a fermionic interaction term for the three-form field $A_{\mu\nu\rho}$

$$\begin{aligned}\mathcal{L}_A[A,e,\psi] \;=\;& -\frac{1}{96}|e|F_{\mu\nu\rho\sigma}F^{\mu\nu\rho\sigma} - \frac{\sqrt{2}}{6912}\,\varepsilon^{\kappa\lambda\mu\nu\rho\sigma\tau\upsilon\xi\zeta\omega}\,F_{\kappa\lambda\mu\nu}\,F_{\rho\sigma\tau\upsilon}\,A_{\xi\zeta\omega}\\ & - \frac{\sqrt{2}}{384}|e|\Big(\bar{\psi}_\rho \Gamma^{\kappa\lambda\mu\nu\rho\sigma}\psi_\sigma + 12\,\bar{\psi}^\kappa \Gamma^{\lambda\mu}\psi^\nu\Big)F_{\kappa\lambda\mu\nu}\;,\end{aligned} \tag{62}$$

with the abelian field strength $F_{\mu\nu\rho\sigma} = 4\partial_{[\mu}A_{\nu\rho\sigma]}$, and quartic fermion terms. Most other supergravities can be obtained by Kaluza-Klein reduction from this theory.

4.3 Kaluza-Klein supergravity

Kaluza's and Klein's observation that electromagnetism can be considered as part of a five-dimensional gravity theory, where the fifth dimension is curled up, is one of most striking ideas of modern physics. Later investigations generalized this idea to a space-time with more than five dimensions and to non-abelian gauge theories.[3] Kaluza-Klein theories do not only form an interesting approach to unify gravity and (non-)abelian gauge theories. For example, they are also useful to generate consistent supergravity theories by dimensional reduction of $D = 11$ dimensional supergravity (for a review see [3]).

Reduction of pure gravity

Dimensional reduction of gravity can be performed most conveniently by using the vielbein formalism, see e.g. [16]. We will apply this procedure to reduce pure gravity in an $(D+n)$-dimensional space-time on an n-dimensional torus T^n down to D dimensions. In the following, capital Latin letters will be used for the $(D+n)$-dimensional space-time whereas the indices of the D-dimensional space-time and of the n circles will be denoted by small Greek and Latin letters respectively. For curved (flat) indices letters of the middle (beginning) of the corresponding alphabet will be taken. The notation is summarized in table 3. We will denote the coordinates of the D-dimensional space-time and of the n-torus by x^μ and y^m, respectively.
In ordinary dimensional reduction, all fields are taken to be independent of the coordinates y^m of the n-torus. More precisely, this corresponds to a normal mode expansion of the fields

$$\Phi(x,y) \;=\; \sum_{k_1,\ldots,k_n\in\mathbb{Z}} \varphi_{k_1k_2\ldots k_n}(x)\,e^{2\pi i k_1 y^1/R_1}\,e^{2\pi i k_2 y^2/R_2}\ldots e^{2\pi i k_n y^n/R_n}\;, \tag{63}$$

on an n-torus with radii $R_1, \ldots R_n$, and the dropping of all fields $\varphi_{k_1k_2\ldots k_n}(x)$ for $(k_1k_2\ldots k_n) \neq (00\ldots0)$, which become infinitely heavy if the size of the n-torus is shrunk to zero.[4]

[3] In fact, O. Klein himself came close to discovering non-abelian Yang-Mills theories by studying higher-dimensional gravity, see [15] for a historical account.

[4] In general, one has to be careful when truncating the action by setting some fields to specific values (say zero).

	curved indices (manifold)	flat indices (tangent space)
$(D+n)$-dim. space-time	$M, N = 0, \ldots, D+n-1$	$A, B = 0, \ldots, D+n-1$
D dim. space-time	$\mu, \nu = 0, \ldots, D-1$	$\alpha, \beta = 0, \ldots, D-1$
n circles (T^n)	$m = 1, \ldots, n$	$a, b = 1, \ldots, n$

Table 3: Use of indices in the reduction of $(D+n)$-dimensional gravity on an n-torus T^n.

For pure gravity, the fields are the various components of the vielbein $E_M{}^A$ of the $(D+n)$-dimensional space-time, which will depend only on the coordinates x^μ of the D-dimensional space-time. The local Lorentz invariance in $(D+n)$ dimensions can be used to bring the vielbein into a triangular form

$$E_M{}^A = \begin{pmatrix} E_\mu{}^\alpha & E_\mu{}^a \\ 0 & E_m{}^a \end{pmatrix} , \tag{64}$$

which breaks the $\mathrm{SO}(1, D+n-1)$ down to $\mathrm{SO}(1, D-1) \times \mathrm{SO}(n)$. It turns out to be convenient to further parametrize (64) as

$$E_M{}^A = \begin{pmatrix} \rho^\kappa \, e_\mu{}^\alpha & \rho^{1/n} \, V_m{}^a \, B_\mu{}^m \\ 0 & \rho^{1/n} \, V_m{}^a \end{pmatrix} , \tag{65}$$

with a constant κ and a matrix $V_m{}^a \in \mathrm{SL}(n)$ of unit determinant, such that $\rho = \det E_m{}^a$. Plugging (65) into the $(D+n)$-dimensional Einstein-Hilbert action, one finds

$$\mathcal{L}_{\mathrm{EH}}^{(D+n)} \quad \longrightarrow \quad -\frac{1}{4} |e| \, \rho^{1+(D-2)\kappa} \, R_{(D)} + \ldots \; , \tag{66}$$

where $R_{(D)}$ is the Ricci scalar computed from the D-dimensional vielbein $e_\mu{}^\alpha$. This shows that upon choosing $\kappa = -\frac{1}{D-2}$, the Einstein-Hilbert term of the the reduced theory takes the standard form.

What about the remaining terms in (66)..? In principle, they could be derived straightforwardly by plugging (65) into the $(D+n)$-dimensional Einstein-Hilbert action and working out the separate terms. Rather than going through that rather lengthy exercise, we will deduce the form of the result by mere symmetry arguments. From a D-dimensional point of view, the components of the $(D+n)$-dimensional vielbein correspond to a vielbein $e_\mu{}^\alpha$ (graviton), to n vector fields $B_\mu{}^n$ and to a set of scalar fields ρ, $V_m{}^a$. In addition, the D-dimensional theory inherits a number of symmetries from its higher-dimensional ancestor. Namely, with the vielbein (65) transforming under infinitesimal diffeomorphisms as (cf. (15))

$$\delta E_M{}^A \;\; = \;\; \xi^N \partial_N E_M{}^A + E_N{}^A \, \partial_M \, \xi^N \; , \tag{67}$$

Some constraints, which arise by truncating the equations of motion of the full theory, might not be reproduced by the truncated action. As a result one might find a Lagrangian which admits solutions that can not be lifted to solutions of the original theory. Truncation to the zero-modes of (63) on the other hand is always consistent on the level of the action.

it is straightforward to verify that x-dependent diffeomorphisms of the type $\xi^\mu(x)$ generate D-dimensional diffeomorphisms on the fields $e_\mu{}^\alpha$, $B_\mu{}^n$, ρ, and $V_m{}^a$. On the other hand, under x-dependent diffeomorphisms of the type $\xi^m(x)$, the fields $B_\mu{}^n$ transform as

$$\delta B_\mu{}^m \;=\; \partial_\mu \zeta^m(x) \;, \tag{68}$$

whereas the graviton and the scalar fields are left inert. This shows that the resulting theory is an abelian $\mathrm{U}(1)^n$ gauge theory with gauge fields $B_\mu{}^m$ and none of the matter charged under the gauge group. Accordingly, the vector fields will only couple with a standard Maxwell term.
Furthermore, diffeomorphisms of the type $\xi^m(y) = g^m{}_n\, y^n$, linear in the compactified coordinates y^m, are also compatible with the truncation (64) and induce a global symmetry $\mathrm{SL}(n)$ acting on the matter fields as

$$\delta V_m{}^a \;=\; g^n{}_m V_n{}^a \;, \qquad \delta B_\mu{}^m \;=\; -\,g^m{}_n\, B_\mu{}^n \;, \tag{69}$$

where we have taken the matrix $g^m{}_n$ to be traceless $g^m{}_m \equiv 0$. The trace part of the transformation is somewhat more subtle, as due to the parametrization (65) it also acts non-trivially on the D-dimensional vielbein and has to be accompanied by a Weyl rescaling in order to yield a proper off-shell symmetry of the theory

$$\delta\rho \;=\; n\,(D-2)\,\rho \;, \qquad \delta B_\mu{}^m \;=\; -\,(D+n-2)\,B_\mu{}^m \;. \tag{70}$$

Finally, local Lorentz invariance is also a symmetry of the theory. As mentioned above, the upper triangular form (64) of the vielbein breaks $\mathrm{SO}(1, D+n-1)$ down to $\mathrm{SO}(1, D-1) \times \mathrm{SO}(n)$, of which the first factor acts as D-dimensional Lorentz transformation on $e_\mu{}^\alpha$, and the second factor acts as an additional local symmetry on $V_m{}^a$

$$\delta V_m{}^a = V_m{}^b \Lambda_b{}^a \;, \qquad \Lambda \in \mathfrak{so}(n) \;. \tag{71}$$

This shows that not all scalar fields are physical. For example we could fix the symmetry (71) by putting $V_m{}^a$ into (upper) triangular form. Consequently the number of physical scalars is given by $\frac{1}{2}n(n+1)$ rather than n^2.
To summarize, the reduction of $(D+n)$-dimensional gravity on an n-torus gives rise to a D-dimensional theory of gravity coupled to n abelian $\mathrm{U}(1)$ vector fields and $\frac{1}{2}n(n+1)$ scalars. In addition, the theory possesses a global $\mathrm{GL}(n)$ symmetry (69), (70). The vector field coupling in (66) is completely determined by all these symmetries up to a global factor and given by

$$\mathcal{L}_{\text{vector}} \;=\; -\frac{1}{16}\,|e|\,\rho^{2/n+2/(D-2)}\,V_m{}^a V_n{}^b\,\delta_{ab}\,F^m_{\mu\nu}\,F^{\mu\nu\,n} \;, \tag{72}$$

with the abelian field strength $F^m_{\mu\nu} = \partial_\mu B^m_\nu - \partial_\nu B^m_\mu$. Indeed this is confirmed by explicit computation.
The coupling of the scalar fields is slightly more complicated. Recall from the above, that the scalar fields parametrize a matrix $V_m{}^a \in \mathrm{SL}(n)$ transforming under global $\mathrm{SL}(n)$ and local $\mathrm{SO}(n)$ as

$$\delta V \;=\; gV + V\Lambda(x) \qquad g \in \mathrm{SL}(n)_{\text{global}} \,, \quad \Lambda \in \mathrm{SO}(n)_{\text{local}} \;. \tag{73}$$

This turns out to be a generic structure in extended supergravity theories: the scalar fields are described by a non-compact coset-space σ-model $\mathrm{G/K} = \mathrm{SL}(n)/\mathrm{SO}(n)$.

Coset-space σ-model G/K

The scalar fields, which appear in extended supergravity theories, are typically described by G/K non-linear coset-space σ-models, where G is a noncompact Lie group and K is its maximal compact subgroup. Explicitly, the scalar fields are most conveniently represented by a G-valued field $V(x)$ subject to the action

$$V(x) \;\rightarrow\; G\,V(x)K(x)\;, \qquad G\in \mathrm{G}\,,\quad K(x)\in \mathrm{K}\;, \tag{74}$$

under global G and local K transformations. Because of the local K invariance, we can get rid of dim K degrees of freedom. The number of degrees of freedom in the scalar sector is therefore given by

$$\dim(\mathrm{G/K}) = \dim \mathrm{G} - \dim \mathrm{K}\;.$$

The action is constructed in terms of the G-invariant currents

$$J_\mu = V(x)^{-1}\partial_\mu V(x) \in \mathfrak{g}\;, \tag{75}$$

where $\mathfrak{g}$ is the Lie algebra of the Lie group G. We decompose $\mathfrak{g}$ into the Lie algebra of K, denoted by $\mathfrak{k}$, and its orthogonal complement $\mathfrak{p}$ (orthogonal w.r.t. the Cartan-Killing form)

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}\;.$$

Accordingly, the current $J_\mu = V(x)^{-1}\partial_\mu V(x)$ can be decomposed into

$$V(x)^{-1}\partial_\mu V(x) \;\equiv\; Q_\mu(x) + P_\mu(x)\;,$$

with $Q_\mu(x)\in\mathfrak{k}$ and $P_\mu(x)\in\mathfrak{p}$. While Q_μ and P_μ are invariant under rigid G transformations, their transformation under local K transformations is given by

$$Q_\mu \;\rightarrow\; K(x)^{-1}Q_\mu(x)K(x) + K(x)^{-1}\partial_\mu K(x)\;, \tag{76}$$

$$P_\mu \;\rightarrow\; K(x)^{-1}P_\mu(x)K(x). \tag{77}$$

Consequently, the simplest action that is invariant under $\mathrm{G}_{\mathrm{global}}\times\mathrm{K}_{\mathrm{local}}$ is given by

$$\mathcal{L}_{\mathrm{scalar}} = -\frac{1}{4}\,|e|\,\mathrm{Tr}\,P_\mu\,P^\mu\;. \tag{78}$$

This is the coset-space σ-model action and appears as a very generic structure in supergravity. Comparing (73) to (74) one deduces that dimensional reduction of pure gravity yields a scalar sector described by a G/K = SL(n)/SO(n) coset-space σ-model (78). More precisely, the dilaton ρ completes the target space to the coset-space σ-model G/K = GL(n)/SO(n). The final result for the full D-dimensional action (66) is

$$\mathcal{L} \;=\; -\frac{1}{4}|e|R_{(D)} - \frac{1}{16}\,|e|\,\rho^{2/n+2/(D-2)}\,V_m{}^a V_n{}^b\,\delta_{ab}\,F^m_{\mu\nu}\,F^{\mu\nu\,n} - \frac{1}{4}|e|\,\mathrm{Tr}\,P_\mu P^\mu - \frac{D+n-2}{4n\,(D-2)}\,|e|\,\rho^{-2}\,\partial_\mu\rho\,\partial^\mu\rho\;. \tag{79}$$

Reduction of other fields

So far we only have considered the dimensional reduction of pure gravity. But as we have seen above, eleven-dimensional supergravity also contains an antisymmetric 3-form field A_{KMN}. Dimensional reduction of this field is straightforward: in $D = 11 - n$ dimensions it gives rise to one 3-form field $A_{\mu\nu\rho}$, to n 2-form fields $A_{\mu\nu m}$, $\binom{n}{2}$ vectors $A_{\mu mn}$, and to $\binom{n}{3}$ scalars A_{mnp}. By an analysis similar to the above, one can derive the resulting D-dimensional action which extends (79). In particular, the full scalar sector is in general given by a larger coset space $\mathrm{G/K} \supset \mathrm{GL}(n)/\mathrm{SO}(n)$. A detailed discussion of the full reduction and the appearance of these larger coset spaces in maximal supergravity can e.g. be found in [18].

4.4 $N = 8$ supergravity in $D = 4$ dimensions

Let us finally sketch (some elements of) the construction of the maximally supersymmetric supergravity theory in $D = 4$ dimensions. Its field content is the $N = 8$ supergravity multiplet given in table 1. Historically, this theory was constructed by dimensional reduction of the eleven-dimensional supergravity on a serven-torus T^7 [13].
As discussed in section 4.3, the reduction of pure gravity from eleven dimensions down to $D = 4$ dimensions yields one graviton $g_{\mu\nu}$, seven abelian vector fields ${B_\mu}^n$, $n = 1, \ldots, 7$, and $1 + 27$ scalar fields, parametrizing the coset space GL(7)/SO(7). The dimensional reduction of the antisymmetric 3-form to $D = 4$ dimensions give rise to one 3-form field, seven 2-form fields, $\binom{7}{2} = 21$ vectors and additional $\binom{7}{3} = 35$ scalar fields. A priori, the field content thus looks quite different from the supergravity multiplet of table 1. This is where the last ingredient enters the game: in D dimensions, massless antisymmetric p-form fields have a dual description as massless $(D - p - 2)$-forms

$$\text{massless } p - \text{form field} \quad \overset{\text{onshell}}{\longleftrightarrow} \quad \text{massless } (D-p-2)\text{-form field} . \tag{80}$$

This is a consequence of the fact that both are described by the same irreducible representation $\binom{D-2}{p} = \binom{D-2}{D-p-2}$ under the little group $\mathrm{SO}(1, D-1)$ and can be seen explicitly on the level of the linearized equations of motion: Let $\mathcal{A}$ be the p-form with abelian field strength $\mathcal{F} = d\mathcal{A}$. The linearized equations of motion and the Bianchi-identity are given by

$$d \star \mathcal{F} = 0 \;, \qquad d\mathcal{F} = 0 \;, \tag{81}$$

respectively. In terms of the dual field strength, $\mathcal{G} \equiv \star\mathcal{F}$ these two equations formally exchange their role. The new Bianchi-identity $d\mathcal{G} = 0$ states that $\mathcal{G}$ can be written locally as $\mathcal{G} = d\mathcal{B}$, where $\mathcal{B}$ is the dual $(D - p - 2)$ form. The dynamics of $\mathcal{A}$ can thus equivalently be described in terms of the field $\mathcal{B}$. It turns out that this duality extends to the full nonlinear equations of motion. Applying this duality to $D = 4$ dimensions, we deduce that the the seven two-forms can be dualized into zero-forms (scalar fields), while the three-form is non-propagating and can be set to zero. Together, we thus obtain the field content

$$\begin{aligned} 1 \quad & \text{graviton,} \\ 7{+}21 = 28 \quad & \text{vectors,} \\ 1{+}27{+}35{+}7 = 70 \quad & \text{scalars.} \end{aligned}$$

which reproduces the $N = 8$ supergravity multiplet. The full action can be found by performing the dimensional reduction of the eleven-dimensional action (61), (62), we refer to [13, 18] for details. In particular, the full scalar target space is described by the 70-dimensional coset space

$$\mathrm{G/K} \;=\; \frac{\mathrm{E}_{7(7)}}{\mathrm{SU}(8)} \;\supset\; \frac{\mathrm{GL}_7}{\mathrm{SO}(7)}\;. \tag{82}$$

In fact, the story continues: having set the four-dimensional three-form field to zero is a consistent truncation but strictly speaking not necessary. Its field equations only imply that the field strength is constant and may be set to an arbitrary value, thereby giving a one-parameter deformation of the maximally supersymmetric theory [19]. A closer analysis shows that this deformation parameter in fact is not a singlet under the global symmetry group $\mathrm{E}_{7(7)}$ but only one component of an irreducible 912-dimensional representation [20]. Switching on other parameters within this representation leads to different maximally supersymmetric theories which generically have non-abelian gauge groups and matter charged under the gauge group; these deformations are the so-called *gauged supergravities* and may correspond to more complicated compactifications in the presence of background fluxes and/or geometric fluxes (see [21] for an introduction). In particular, these theories include the compactification of eleven-dimensional supergravity on a seven-sphere S^7 which gives rise to a four-dimensional theory with compact non-abelian gauge group SO(8) [22].

Appendix

A Variation of the Palatini action

Consider the Palatini action (23),

$$\mathcal{L}_{\mathrm{P}}[e,\omega] = -\tfrac{1}{4}|e|\, e_a{}^{\mu} e_b{}^{\nu} R_{\mu\nu}{}^{ab}[\omega]\,, \tag{83}$$

where $|e|$ on the right-hand side denotes the determinant of the tetrad. It can be recast in the form

$$\mathcal{L}_P[e,\omega] = \tfrac{1}{16}\epsilon^{\mu\nu\rho\sigma}\epsilon_{abcd}e_{\rho}{}^{c}e_{\sigma}{}^{d}R_{\mu\nu}{}^{ab}[\omega] \tag{84}$$

To see this, first note that

$$\epsilon^{abcd}\epsilon_{efcd} = -4\delta^{[a}{}_{[e}\,\delta^{b]}{}_{f]}\,, \tag{85}$$

where one can determine the numerical factor by tracing over the index pairs (a, e) and (b, f), on both sides. [5] The tensor density $e^{\mu\nu\rho\sigma}$ is obtained from e^{abcd} by

$$\epsilon^{\mu\nu\rho\sigma} = \epsilon^{abcd}e_a{}^{\mu}e_b{}^{\nu}e_c{}^{\rho}e_d{}^{\sigma}\,|e|\,, \tag{86}$$

where the determinant shows up since we are dealing with a tensor *density* rather than with a tensor (as a consequence $\epsilon^{\mu\nu\rho\sigma}$ is just a number). Plugging (86) into (84) and making use of

[5] Our conventions are $\epsilon_{0123} = -1 = -\epsilon^{0123}$, $\eta = \mathrm{diag}(+1,-1,-1,-1)$.

(85) readily yields (83).

We shall now vary the Palatini action in the form (84) with respect to the spin connection. Since ω only shows up in the expression for the Riemann tensor, we are interested in $\delta R_{\mu\nu}{}^{ab}$, for which we have

$$\delta R_{\mu\nu}{}^{ab} = 2D_{[\mu}\delta\omega_{\nu]}{}^{ab} \, .$$

The covariant derivative must actually arise in this expression, since according to (21), $\delta R_{\mu\nu}{}^{ab}$ starts with the non-covariant term $2\partial_{[\mu}\delta\omega_{\nu]}{}^{ab}$, followed by a term of the form " $\omega \cdot \delta\omega$ ", which must complete the non-covariant first part in order to yield a covariant quantity in total. We therefore find

$$\delta\mathcal{L}_{\mathrm{P}} = \tfrac{1}{16}\epsilon^{\mu\nu\rho\sigma}\epsilon_{abcd}e_\rho{}^c e_\sigma{}^d \, 2D_\mu\delta\omega_\nu{}^{ab} \, ,$$

where we dropped the antisymmetric bracket since $\epsilon^{\mu\nu\rho\sigma}$ already takes care of the antisymmetrisation. Integration by parts leads us to

$$\delta\mathcal{L}_{\mathrm{P}} = -\tfrac{1}{4}\epsilon^{\mu\nu\rho\sigma}\epsilon_{abcd}e_\rho{}^c \left(D_\mu e_\sigma{}^d\right)\delta\omega_\nu{}^{ab} \, ,$$

where we made use of the antisymmetrisation by the ϵ tensor densities again. Using the identity $\epsilon^{\mu\nu\rho\sigma}\epsilon_{abcd}e_\rho{}^c = -6|e| e_{[a}{}^\mu e_b{}^\nu e_{d]}{}^\sigma$ (which follows in complete analogy to (86)), we find

$$\delta\mathcal{L}_{\mathrm{P}} = -\tfrac{3}{2}|e| \left(D_\mu e_\nu{}^d\right) e_{[a}{}^\mu e_b{}^\nu e_{d]}{}^\sigma \, \delta\omega_\sigma{}^{ab} \, .$$

On the other hand, variation of the Palatini action w.r.t. the tetrad gives rise to

$$\begin{aligned}
\delta\mathcal{L}_{\mathrm{P}} &= -\tfrac{1}{4}R_{\mu\nu}{}^{ab}[\omega]\,\delta(|e|\, e_a{}^\mu e_b{}^\nu) \\
&= -\tfrac{1}{4}|e|\, R_{\mu\nu}{}^{ab}[\omega]\,(2e_a{}^\mu\,\delta e_b{}^\nu - e_a{}^\mu e_b{}^\nu(e_\rho{}^c\,\delta e_c{}^\rho)) \\
&= -\tfrac{1}{2}|e|\, R_\nu{}^b[\omega]\,(\delta^\nu_\mu\delta^a_b - \tfrac{1}{2}e_b{}^\nu e_\mu{}^a)\,\delta e_a{}^\mu \;=\; -\tfrac{1}{2}|e|\,(R_\mu{}^a[\omega] - \tfrac{1}{2}e_\mu{}^a R)\,\delta e_a{}^\mu \, , \qquad (87)
\end{aligned}$$

where we have used that $\delta(\det A) = (\det A)\,\mathrm{Tr}[A^{-1}\delta A]$ for any matrix A. Together, we arrive at (24) for the variation of the Palatini action.

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Université de Lyon, Laboratoire de Physique,
Ecole Normale Supérieure de Lyon,
46 allée d'Italie, F-69364 Lyon cedex 07,
FRANCE
henning.samtleben@ens-lyon.fr