Chapter 1

Introduction

1.1 What is a Plasma?

1.1.1 An ionized gas

A plasma is a gas in which an important fraction of the atoms is *ionized*, so that the electrons and ions are separately free.

When does this ionization occur? When the temperature is hot enough.

Balance between collisional ionization and recombination:

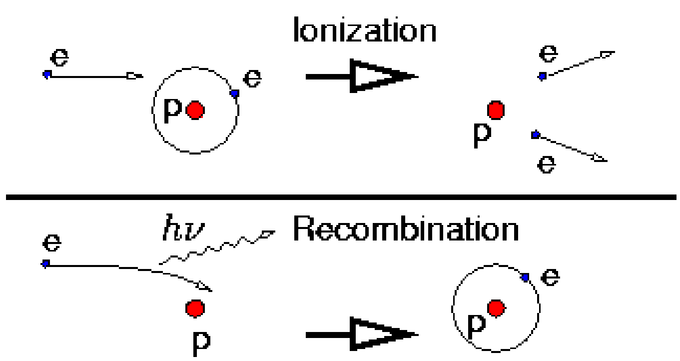

Figure 1.1: Ionization and Recombination

Ionization has a threshold energy. Recombination has not but is much less probable.

Threshold is *ionization energy* (13.6eV, H). χ_i

Integral over Maxwellian distribution gives rate coefficients (reaction rates). Because of the tail of the Maxwellian distribution, the ionization rate extends below $T = \chi_i$. And in equilibrium, when

$$\frac{n_{\text{ions}}}{n_{\text{neutrals}}} = \frac{< \sigma_i v >}{< \sigma_r v >} \quad , \tag{1.1}$$

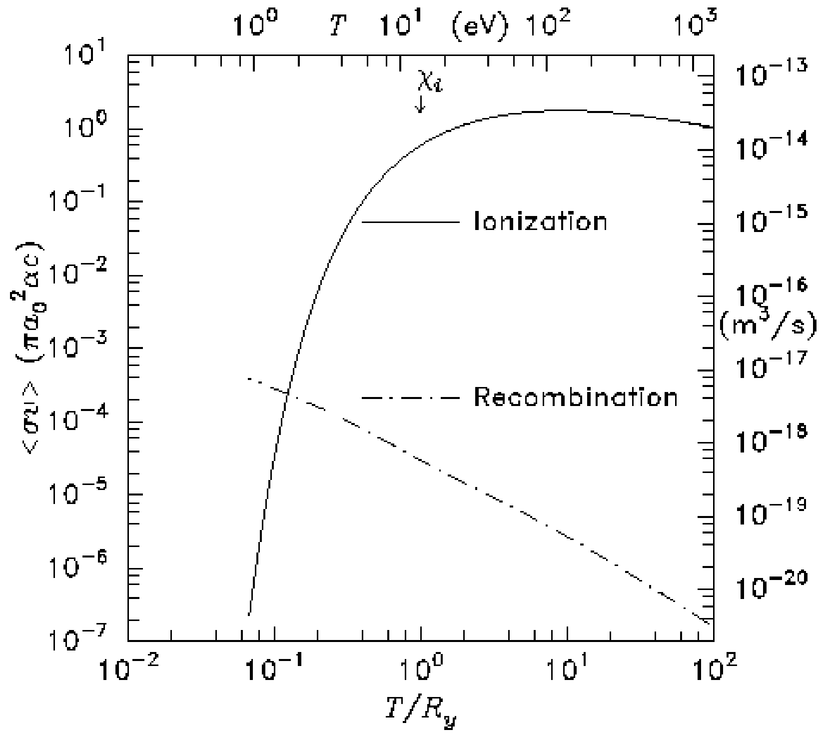

Figure 1.2: Ionization and radiative recombination rate coefficients for atomic hydrogen

the percentage of ions is large ($\sim 100\%$) if electron temperature: $T_e \gtrsim \chi_i/10$. e.g. Hydrogen is ionized for $T_e \gtrsim 1eV$ (11,600°k). At room temp **r** ionization is negligible.

For dissociation and ionization balance figure see e.g. Delcroix *Plasma Physics* Wiley (1965) figure 1A.5, page 25.

1.1.2 Plasmas are Quasi-Neutral

If a gas of electrons and ions (singly charged) has unequal numbers, there will be a net charge density, ρ.

$$\rho = n_e(-e) + n_i(+e) = e(n_i - n_e) \tag{1.2}$$

This will give rise to an electric field via

$$\nabla.\mathbf{E} = \frac{\rho}{\epsilon_0} = \frac{e}{\epsilon_0}(n_i - n_e) \tag{1.3}$$

Example: Slab.

$$\frac{dE}{dx} = \frac{\rho}{\epsilon_0} \tag{1.4}$$

$$\rightarrow E = \rho\frac{x}{\epsilon_0} \tag{1.5}$$

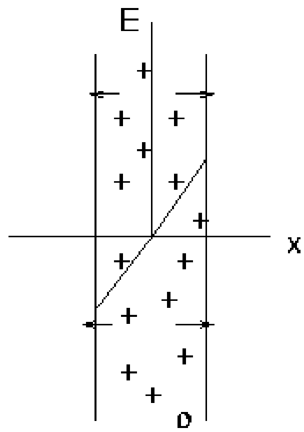

Figure 1.3: Charged slab

This results in a force on the charges tending to expel whichever species is in excess. That is, if $n_i > n_e$, the E field causes n_i to decrease, n_e to increase tending to reduce the charge. This restoring *force is enormous*!

<u>Example</u>

Consider $T_e = 1eV$, $n_e = 10^{19}m^{-3}$ (a modest plasma; c.f. density of atmosphere $n_{\text{molecules}} \sim 3 \times 10^{25}m^{-3}$). Suppose there is a small difference in ion and electron densities $\Delta n = (n_i - n_e)$

$$\text{so} \quad \rho = \Delta n \; e \tag{1.6}$$

Then the force per unit volume at distance x is

$$F_e = \rho E = \rho^2 \frac{x}{\epsilon_0} = (\Delta n \; e)^2 \; \frac{x}{\epsilon_0} \tag{1.7}$$

Take $\Delta n / n_e = 1\%$, $x = 0.10m$.

$$F_e = (10^{17} \times 1.6 \times 10^{-19})^2 \; 0.1/8.8 \times 10^{-12} = 3 \times 10^6 N.m^{-3} \tag{1.8}$$

Compare with this the pressure force per unit volume $\sim p/x : p \sim n_e T_e (+n_i T_i)$

$$F_p \sim 10^{19} \times 1.6 \times 10^{-19}/0.1 = 16 N m^{-3} \tag{1.9}$$

Electrostatic force >> Kinetic Pressure Force.

This is one aspect of the fact that, because of being ionized, plasmas exhibit all sorts of collective behavior, different from neutral gases, mediated by the long distance electromagnetic forces E, B.

Another example (related) is that of longitudinal waves. In a normal gas, sound waves are propagated via the intermolecular action of collisions. In a plasma, waves can propagate when collisions are negligible because of the coulomb interaction of the particles.

1.2 Plasma Shielding

1.2.1 Elementary Derivation of the Boltzmann Distribution

Basic principle of Statistical Mechanics:

Thermal Equilibrium $\leftrightarrow$ *Most Probable State* i.e. State with large number of possible arrangements of micro-states.

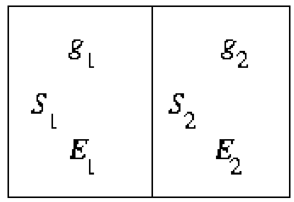

Figure 1.4: Statistical Systems in Thermal Contact

Consider two weakly coupled systems S_1, S_2 with energies E_1, E_2. Let g_1, g_2 be the number of microscopic states which give rise to these energies, for each system. Then the total number of micro-states of the *combined* system is (assuming states are independent)

$$g = g_1 g_2 \tag{1.10}$$

If the total energy of combined system is fixed $E_1 + E_2 = E_t$ then this can be written as a function of E_1:

$$g = g_1(E_1)g_2(E_t - E_1) \tag{1.11}$$

$$\text{and} \quad \frac{dg}{dE_1} = \frac{dg_1}{dE}g_2 - g_1\frac{dg_2}{dE} \quad . \tag{1.12}$$

The most probable state is that for which $\frac{dg}{dE_1} = 0$ i.e.

$$\frac{1}{g_1}\frac{dg_1}{dE} = \frac{1}{g_2}\frac{dg_2}{dE} \quad \text{or} \quad \frac{d}{dE}\ln g_1 = \frac{d}{dE}\ln g_2 \tag{1.13}$$

Thus, in equilibrium, states in thermal contact have equal values of $\frac{d}{dE}\ln g$.

One defines $\sigma \equiv \ln g$ as the *Entropy.*

And $[\frac{d}{dE}\ln g]^{-1} = T$ the *Temperature.*

Now suppose that we want to know the relative probability of 2 *micro*-states of system 1 in equilibrium. There are, in all, g_1 of these states, for each specific E_1 but we want to know how many states of the *combined* system correspond to a *single* microstate of S_1.

Obviously that is just equal to the number of states of system 2. So, denoting the two values of the energies of S_1 for the two microstates we are comparing by E_A, E_B the ratio of the number of combined system states for S_{1A} and S_{1B} is

$$\frac{g_2(E_t - E_A)}{g_2(E_t - E_B)} = \exp[\sigma(E_t - E_A) - \sigma(E_t - E_B)] \tag{1.14}$$

Now we suppose that system S_2 is large compared with S_1 so that E_A and E_B represent very small changes in S_2's energy, and we can Taylor expand

$$\frac{g_2(E_t - E_A)}{g_2(E_t - E_A)} \simeq \exp\left[-E_A\frac{d\sigma}{dE} + E_B\frac{d\sigma}{dE}\right] \tag{1.15}$$

Thus we have shown that the ratio of the probability of a system (S_1) being in any two micro-states A, B is simply

$$\exp\left[\frac{-(E_A - E_B)}{T}\right] \ , \tag{1.16}$$

when in equilibrium with a (large) thermal "reservoir". This is the well-known "Boltzmann factor".

You may notice that Boltzmann's constant is absent from this formula. That is because of using natural thermodynamic units for entropy (dimensionless) and temperature (energy).

Boltzmann's constant is simply a conversion factor between the *natural units of temperature* (energy, e.g. Joules) and (e.g.) degrees Kelvin. Kelvins are based on °C which arbitrarily choose melting and boiling points of water and divide into 100.

Plasma physics is done almost always using energy units for temperature. Because Joules are very large, usually electron-volts (eV) are used.

$$1eV = 11600K = 1.6 \times 10^{-19}\text{Joules}. \tag{1.17}$$

One consequence of our Botzmann factor is that a gas of moving particles whose energy is $\frac{1}{2}mv^2$ adopts the Maxwell-Boltzmann (Maxwellian) distribution of velocities $\propto \exp[-\frac{mv^2}{2T}]$.

1.2.2 Plasma Density in Electrostatic Potential

When there is a varying potential, ϕ, the densities of electrons (and ions) is affected by it.

If electrons are in thermal equilibrium, they will adopt a Boltzmann distribution of density

$$n_e \propto \exp(\frac{e\phi}{T_e}) \quad . \tag{1.18}$$

This is because each electron, regardless of velocity possesses a potential energy $-e\phi$.

Consequence is that (fig 1.5) a self-consistent loop of dependencies occurs.

This is one elementary example of the general principle of plasmas requiring a self-consistent solution of Maxwell's equations of electrodynamics *plus* the particle dynamics of the plasma.

1.2.3 Debye Shielding

A slightly different approach to discussing quasi-neutrality leads to the important quantity called the Debye Length.

Suppose we put a plane grid into a plasma, held at a certain potential, ϕ_g.

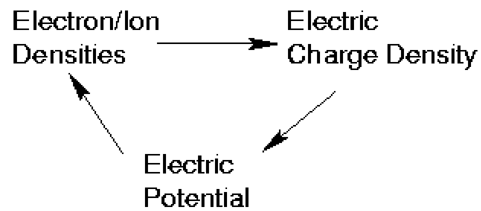

Figure 1.5: Self-consistent loop of dependencies

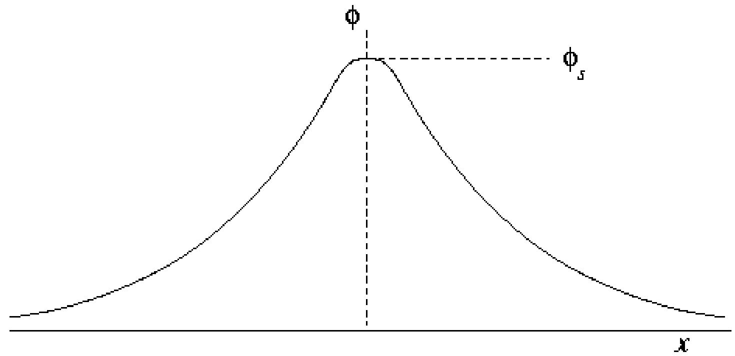

Figure 1.6: Shielding of fields from a 1-D grid.

Then, unlike the vacuum case, the perturbation to the potential falls off rather rapidly into the plasma. We can show this as follows. The important equations are:

$$\text{Poisson's Equation} \qquad \nabla^2\phi = \frac{d^2\phi}{dx^2} = -\frac{e}{\epsilon_0}\,(n_i - n_e) \tag{1.19}$$

$$\text{Electron Density} \qquad n_e = n_\infty \exp(e\phi/T_e). \tag{1.20}$$

[This is a Boltzmann factor; it assumes that electrons are in thermal equilibrium. n_∞ is density far from the grid (where we take $\phi = 0$).]

$$\text{Ion Density} \qquad n_i = n_\infty\ . \tag{1.21}$$

[Applies far from grid by quasineutrality; we just *assume*, for the sake of this illustrative calculation that ion density is not perturbed by ϕ-perturbation.]

Substitute:

$$\frac{d^2\phi}{dx^2} = \frac{en_\infty}{\epsilon_0}\left[\exp\left(\frac{e\phi}{T_e}\right) - 1\right]\ . \tag{1.22}$$

This is a nasty nonlinear equation, but *far from the grid* $|e\phi/T_e| << 1$ so we can use a Taylor expression: $\exp\frac{e\phi}{T_e} \simeq 1 + \frac{e\phi}{T_e}$. So

$$\frac{d^2\phi}{dx^2} = \frac{en_\infty}{\epsilon_0}\frac{e}{T_e}\phi = \frac{e^2 n_\infty}{\epsilon_0 T_e}\phi \tag{1.23}$$

Solutions: $\phi = \phi_0 \exp(-|x|/\lambda_D)$ where

$$\lambda_D \equiv \left(\frac{\epsilon_0 T_e}{e^2 n_\infty}\right)^{\frac{1}{2}} \tag{1.24}$$

This is called the *Debye Length*

Perturbations to the charge density and potential in a plasma tend to fall off with characteristic length λ_D.

In Fusion plasmas λ_D is typically small. [e.g. $n_e = 10^{20} m^{-3} T_e = 1 keV \quad \lambda_D = 2 \times 10^{-5} m = 20 \mu m$]

Usually we include as part of the *definition* of a plasma that $\lambda_D <<$ the size of plasma. This ensures that collective effects, quasi-neutrality etc. are important. Otherwise they probably aren't.

1.2.4 Plasma-Solid Boundaries (Elementary)

When a plasma is in contact with a solid, the solid acts as a "sink" draining away the plasma. *Recombination* of electrons and ions occur at surface. Then:

1. Plasma is normally charged positively with respect to the solid.

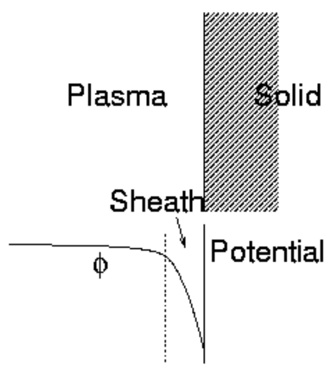

Figure 1.7: Plasma-Solid interface: Sheath

2. There is a relatively thin region called the "sheath", at the boundary of the plasma, where the main potential variation occurs.

Reason for potential drop:
Different velocities of electrons and ions.

If there were no potential variation (**E**= 0) the electrons and ions would hit the surface at the random rate

$$\frac{1}{4} n \bar{v} \qquad \text{per unit area} \tag{1.25}$$

[This equation comes from elementary gas-kinetic theory. See problems if not familiar.]

The mean speed $\bar{v} = \sqrt{\frac{8T}{\pi m}} \sim \sqrt{\frac{T}{m}}$.

Because of mass difference electrons move $\sim \sqrt{\frac{m_i}{m_e}}$ faster and hence would drain out of plasma faster. Hence, plasma charges up enough that an electric field opposes electron escape and reduces total electric current to zero.

Estimate of potential:

Ion escape flux $\quad \frac{1}{4} n_i' \bar{v}_i$

Electron escape flux $\quad \frac{1}{4} n_e' \bar{v}_i$

Prime denotes values at solid surface.

Boltzmann factor applied to electrons:

$$n_e' = n_\infty \exp[e\phi_s/T_e] \tag{1.26}$$

where ϕ_s is solid potential relative to distant (∞) plasma.

Since ions are being dragged out by potential assume $n_i' \sim n_\infty$ ($Z_i = 1$). [This is only approximately correct.]

Hence total current density out of plasma is

$$\begin{aligned} j &= q_i \frac{1}{4} n_i' \bar{v}_i + q_e \frac{1}{4}\, n_e' \bar{v}_e & (1.27) \\ &= \frac{e n_\infty}{4} \{ \bar{v}_i - \exp\left[\frac{e\phi_s}{T_e}\right] \bar{v}_e \} & (1.28) \end{aligned}$$

This must be *zero* so

$$\begin{aligned} \phi_s &= \frac{T_e}{e} \ln |\frac{\bar{v}_i}{\bar{v}_e}| = \frac{T_e}{e} \frac{1}{2} \ln \left(\frac{T_i}{T_e} \frac{m_e}{m_i} \right) & (1.29) \\ &= \frac{T_e}{e} \frac{1}{2} \ln \left(\frac{m_e}{m_i} \right) \quad \text{[if } T_e = T_i.\text{]} & (1.30) \end{aligned}$$

For hydrogen $\frac{m_i}{m_e} = 1800$ so $\frac{1}{2} \ln \frac{m_e}{m_i} = -3.75$.

The potential of the surface relative to plasma is approximately $-4\ \frac{T_e}{e}$.

[Note $\frac{T_e}{e}$ is just the electron temp **r** in electron-volts expressed as a voltage.]

1.2.5 Thickness of the sheath

Crude estimates of sheath thickness can be obtained by assuming that ion density is uniform. Then equation of potential is, as before,

$$\frac{d^2\phi}{dx^2} = \frac{e n_\infty}{\epsilon_0} \left[\exp\left(\frac{e\phi}{T_e}\right) - 1 \right] \tag{1.31}$$

We know the rough scale-length of solutions of this equation is

$$\lambda_D = \left(\frac{\epsilon_0 T_e}{e^2 n_\infty}\right)^{\frac{1}{2}} \qquad \text{the Debye Length.} \tag{1.32}$$

Actually our previous solution was valid only for $|e\phi/T_e| << 1$ which is no longer valid.

When $-e\phi/T_e > 1$ (as will be the case in the sheath). We can practically ignore the electron density, in which case the solution will continue only quadratically. One might expect, therefore, that the sheath thickness is roughly given by an electric potential gradient

$$-\frac{T}{e}\frac{1}{\lambda_D} \tag{1.33}$$

extending sufficient distance to reach $\phi_S = -4\frac{T_e}{e}$ i.e.

$$\text{distance } x \sim 4\lambda_D$$

This is correct for the typical sheath thickness but not at all rigorous.

1.3 The 'Plasma Parameter'

Notice that in our development of Debye shielding we used $n_e e$ as the charge density and supposed that it could be taken as smooth and continuous. However if the density were so low that there were less than approximately one electron in the Debye shielding region this approach would not be valid. Actually we have to address this problem in 3-d by defining the 'Plasma Parameter', N_D, as

$$N_D = \text{Number of particles in the 'Debye Sphere'.}$$

$$= n.\frac{4}{3}\pi\lambda_D^3 \qquad \left(\propto \frac{T^{\frac{3}{2}}}{n^{\frac{1}{2}}}\right) \quad . \tag{1.34}$$

If $N_D \lesssim 1$ then the individual particles cannot be treated as a smooth continuum. It will be seen later that this means that collisions dominate the behaviour: i.e. short range correlation is just as important as the long range collective effects.

Often, therefore we add a further qualification of plasma:

$$N_D >> 1 \quad \text{(Collective effects dominate over collisions)} \tag{1.35}$$

1.4 Summary

Plasma is an *ionized* gas in which collective effects dominate over collisions.

$$[\lambda_D << \text{size} \quad , \quad N_D >> 1 \quad .] \tag{1.36}$$

1.5 Occurrence of Plasmas

Gas Discharges: Fluorescent Lights, Spark gaps, arcs, welding, lighting

Controlled Fusion

Ionosphere: Ionized belt surrounding earth

Interplanetary Medium: Magnetospheres of planets and starts. Solar Wind.

Stellar Astrophysics: Stars. Pulsars. Radiation-processes.

Ion Propulsion: Advanced space drives, etc.

& Space Technology Interaction of Spacecraft with environment

Gas Lasers: Plasma discharge pumped lasers: CO_2, He, Ne, HCN.

Materials Processing: Surface treatment for hardening. Crystal Growing.

Semiconductor Processing: Ion beam doping, plasma etching & sputtering.

Solid State Plasmas: Behavior of semiconductors.

For a figure locating different types of plasma in the plane of density versus temperature see for example Goldston and Rutherford *Introduction to Plasma Physics* IOP Publishing, 1995, figure 1.3 page 9. Another is at `http://www.plasmas.org/basics.htm`

1.6 Different Descriptions of Plasma

1. Single Particle Approach. (Incomplete in itself). Eq. of Motion.

2. Kinetic Theory. Boltzmann Equation.

$$\left[\frac{\partial}{\partial t} + \mathbf{v}.\frac{\partial}{\partial \mathbf{x}} + \mathbf{a}.\frac{\partial}{\partial \mathbf{v}}\right] f = \left.\frac{\partial f}{\partial t}\right)_{\text{col.}} \tag{1.37}$$

3. Fluid Description. Moments, Velocity, Pressure, Currents, etc.

Uses of these.

Single Particle Solutions → Orbits
→ Kinetic Theory Solutions → Transport Coefs.
→ Fluid Theory → Macroscopic Description

All descriptions should be consistent. Sometimes they are different ways of looking at the same thing.

1.6.1 Equations of Plasma Physics

$$\begin{aligned} \nabla.\mathbf{E} &= \frac{\rho}{\epsilon_0} & \nabla.\mathbf{B} &= 0 \\ \nabla\wedge\mathbf{E} &= -\frac{\partial\mathbf{B}}{\partial t} & \nabla\wedge\mathbf{B} &= \mu_0\mathbf{j} + \frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t} \end{aligned} \tag{1.38}$$

$$F = q(\mathbf{E} + \mathbf{v}\wedge\mathbf{B})$$

1.6.2 Self Consistency

In solving plasma problems one usually has a 'circular' system:

The problem is solved only when we have a model in which all parts are self consistent. We need a 'bootstrap' procedure.

Generally we have to do it in stages:

- Calculate Plasma Response (to given E,B)
- Get currents & charge densities
- Calculate **E** & **B** for **j**, p.

Then put it all together.

Chapter 2

Motion of Charged Particles in Fields

Plasmas are complicated because motions of electrons and ions are determined by the electric and magnetic fields but *also change* the fields by the currents they carry.

For now we shall ignore the second part of the problem and assume that *Fields are Prescribed.*

Even so, calculating the motion of a charged particle can be quite hard.

Equation of motion:

$$\underbrace{m\frac{d\mathbf{v}}{dt}}_{\text{Rate of change of momentum}} = \underbrace{\underset{\text{charge}}{q}\,(\underset{\text{E-field}}{\mathbf{E}} + \underset{\text{velocity}}{\mathbf{v}} \wedge \underset{\text{B-field}}{\mathbf{B}})}_{\text{Lorentz Force}} \tag{2.1}$$

Have to solve this differential equation, to get position $\mathbf{r}$ and velocity ($\mathbf{v}= \dot{\mathbf{r}}$) given $\mathbf{E}(\mathbf{r}, t), \mathbf{B}(\mathbf{r}, t)$.

Approach: Start simple, gradually generalize.

2.1 Uniform B field, $\mathbf{E} = 0$.

$$m\dot{\mathbf{v}} = q\mathbf{v} \wedge \mathbf{B} \tag{2.2}$$

2.1.1 Qualitatively

in the plane perpendicular to B: Accel. is perp to $\mathbf{v}$ so particle moves in a circle whose radius r_L is such as to satisfy

$$m r_L \Omega^2 = m\frac{v_\perp^2}{r_L} = |q| v_\perp B \tag{2.3}$$

Ω is the angular (velocity) frequency

1st equality shows $\Omega^2 = v_\perp^2/r_L^2$ ($r_L = v_\perp/\Omega$)

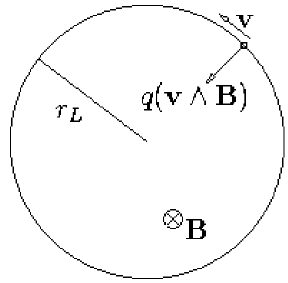

Figure 2.1: Circular orbit in uniform magnetic field.

Hence second gives $m\frac{v_\perp}{\Omega}\Omega^2 = |q|v_\perp B$

$$\text{i.e.} \quad \Omega = \frac{|\mathrm{q}|\mathrm{B}}{\mathrm{m}} \quad . \tag{2.4}$$

Particle moves in a circular orbit with

$$\text{angular velocity} \quad \Omega = \frac{|q|B}{m} \quad \text{the “Cyclotron Frequency”} \tag{2.5}$$

$$\text{and radius} \quad r_l = \frac{v_\perp}{\Omega} \quad \text{the “Larmor Radius.} \tag{2.6}$$

2.1.2 By Vector Algebra

- Particle Energy is constant. *proof*: take $\mathbf{v}$. Eq. of motion then

$$m\mathbf{v}.\dot{\mathbf{v}} = \frac{d}{dt}\left(\frac{1}{2}mv^2\right) = q\mathbf{v}.(\mathbf{v} \wedge \mathbf{B}) = 0. \tag{2.7}$$

- Parallel and Perpendicular motions separate. $v_\parallel$ = constant because accel ($\propto \mathbf{v} \wedge \mathbf{B}$) is perpendicular to $\mathbf{B}$.

Perpendicular Dynamics:

Take B in $\hat{z}$ direction and write components

$$m\dot{v}_x = qv_yB \quad , \quad m\dot{v}_y = -qv_xB \tag{2.8}$$

Hence

$$\ddot{v}_x = \frac{qB}{m}\dot{v}_y = -\left(\frac{qB}{m}\right)^2 v_x = -\Omega^2 v_x \tag{2.9}$$

Solution: $v_x = v_\perp \cos \Omega t$ (choose zero of time)

Substitute back:

$$v_y = \frac{m}{qB}\dot{v}_x = -\frac{|q|}{q}v_\perp \sin \Omega t \tag{2.10}$$

Integrate:

$$x = x_0 + \frac{v_\perp}{\Omega} \sin \Omega t \quad , \quad y \;=\; y_0 + \frac{q}{|q|} \frac{v_\perp}{\Omega} \;\cos \Omega t \tag{2.11}$$

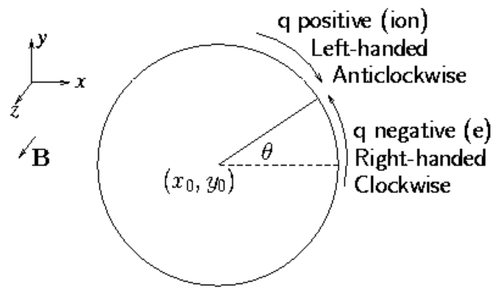

Figure 2.2: Gyro center (x_0, y_0) and orbit

This is the equation of a circle with center $\mathbf{r}_0 = (x_0, y_0)$ and radius $r_L = v_\perp/\Omega$: Gyro Radius. [Angle is $\theta = \Omega t$]

Direction of rotation is as indicated opposite for opposite sign of charge:

Ions rotate anticlockwise. Electrons clockwise about the magnetic field.

The current carried by the plasma always is in such a direction as to *reduce* the magnetic field.

This is the property of a magnetic material which is *"Diagmagnetic"*.

When $v_\parallel$ is non-zero the total motion is along a helix.

2.2 Uniform B and non-zero E

$$m\dot{\mathbf{v}} = q(\mathbf{E} + \mathbf{v} \wedge \mathbf{B}) \tag{2.12}$$

Parallel motion: Before, when $\mathbf{E} = 0$ this was $v_\parallel$ = const. Now it is clearly

$$\dot{v}_\parallel = \frac{qE_\parallel}{m} \tag{2.13}$$

Constant acceleration along the field.

Perpendicular Motion

Qualitatively:

Speed of positive particle is greater at top than bottom so radius of curvature is greater. Result is that guiding center moves perpendicular to both $\mathbf{E}$ and $\mathbf{B}$. It 'drifts' across the field.

Algebraically: It is clear that if we can find a constant velocity $\mathbf{v}_d$ that satisfies

$$\mathbf{E} + \mathbf{v}_d \wedge \mathbf{B} = 0 \tag{2.14}$$

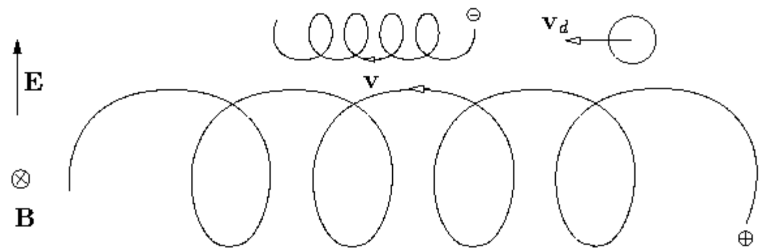

Figure 2.3: $\mathbf{E} \wedge \mathbf{B}$ drift orbit

then the sum of this drift velocity plus the velocity

$$\mathbf{v}_L = \frac{d}{dt}[\mathbf{r}_L e^{i\Omega(t-t_0)}] \tag{2.15}$$

which we calculated for the $\mathbf{E} = 0$ gyration will satisfy the equation of motion.
Take $\wedge\mathbf{B}$ the above equation:

$$0 = \mathbf{E} \wedge \mathbf{B} + (\mathbf{v}_d \wedge \mathbf{B}) \wedge \mathbf{B} = \mathbf{E} \wedge \mathbf{B} + (v_d.\mathbf{B})\mathbf{B} - B^2\mathbf{v}_d \tag{2.16}$$

so that

$$\mathbf{v}_d = \frac{\mathbf{E} \wedge \mathbf{B}}{B^2} \tag{2.17}$$

does satisfy it.
Hence the full solution is

$$\mathbf{v} = \underset{\text{parallel}}{\mathbf{v}_\parallel} + \underset{\text{cross-field drift}}{\mathbf{v}_d} + \underset{\text{Gyration}}{\mathbf{v}_L} \tag{2.18}$$

where

$$\dot{v}_\parallel = \frac{qE_\parallel}{m} \tag{2.19}$$

and
$\mathbf{v}_d$ (eq 2.17) is the "E × B drift" of the gyrocenter.

Comments on E × B drift:

1. It is *independent* of the properties of the drifting particle (q, m, v, whatever).

2. Hence it is in the *same* direction for electrons and ions.

3. Underlying physics for this is that in the frame moving at the E × B drift E = 0. We have 'transformed away' the electric field.

4. Formula given above is exact except for the fact that relativistic effects have been ignored. They would be important if $v_d \sim c$.

2.2.1 Drift due to Gravity or other Forces

Suppose particle is subject to some other force, such as gravity. Write it **F** so that

$$m\dot{\mathbf{v}} = \mathbf{F} + q\ \mathbf{v} \wedge \mathbf{B} = q(\frac{1}{q}\mathbf{F} + \mathbf{v} \wedge \mathbf{B}) \tag{2.20}$$

This is just like the Electric field case except with $\mathbf{F}/q$ replacing **E**.

The drift is therefore

$$\mathbf{v}_d = \frac{1}{q}\frac{\mathbf{F} \wedge \mathbf{B}}{B^2} \tag{2.21}$$

In this case, if force on electrons and ions is same, they drift in *opposite* directions.

This general formula can be used to get the drift velocity in some other cases of interest (see later).

2.3 Non-Uniform B Field

If B-lines are straight but the magnitude of B varies in space we get orbits that look qualitatively similar to the E ⊥ B case:

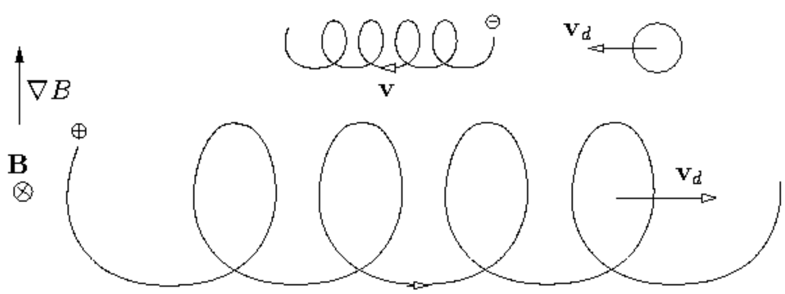

Figure 2.4: ∇B drift orbit

Curvature of orbit is greater where B is greater causing loop to be small on that side. Result is a drift perpendicular to both **B** and ∇B. Notice, though, that electrons and ions go in *opposite* directions (unlike $\mathbf{E} \wedge \mathbf{B}$).

Algebra

We try to find a decomposition of the velocity as before into $\mathbf{v} = \mathbf{v}_d + \mathbf{v}_L$ where $\mathbf{v}_d$ is constant.

We shall find that this can be done only approximately. Also we must have a simple expression for B. This we get by assuming that the Larmor radius is much smaller than the scale length of B variation i.e.,

$$r_L << B/|\nabla B| \tag{2.22}$$

in which case we can express the field approximately as the first two terms in a Taylor expression:

$$\mathbf{B} \simeq \mathbf{B}_0 + (\mathbf{r}.\nabla)\mathbf{B} \tag{2.23}$$

Then substituting the decomposed velocity we get:

$$m\frac{d\mathbf{v}}{dt} = m\dot{\mathbf{v}}_L = q(\mathbf{v} \wedge \mathbf{B}) = q[\mathbf{v}_L \wedge \mathbf{B}_0 + \mathbf{v}_d \wedge \mathbf{B}_0 + (\mathbf{v}_L + \mathbf{v}_d) \wedge (\mathbf{r}.\nabla)\mathbf{B}] \tag{2.24}$$

$$\text{or} \quad 0 = \mathbf{v}_d \wedge \mathbf{B}_0 + \mathbf{v}_L \wedge (\mathbf{r}.\nabla)\mathbf{B} + \mathbf{v}_d \wedge (\mathbf{r}.\nabla)\mathbf{B} \tag{2.25}$$

Now we shall find that v_d/v_L is also small, like $r|\nabla B|/B$. Therefore the last term here is second order but the first two are first order. So we drop the last term.

Now the awkward part is that $\mathbf{v}_L$ and $\mathbf{r}_L$ are periodic. Substitute for $\mathbf{r} = \mathbf{r}_0 + \mathbf{r}_L$ so

$$0 = \mathbf{v}_d \wedge \mathbf{B}_0 + \mathbf{v}_L \wedge (\mathbf{r}_L.\nabla)\mathbf{B} + \mathbf{v}_L \wedge (\mathbf{r}_0.\nabla)\mathbf{B} \tag{2.26}$$

We now average over a cyclotron period. The last term is $\propto e^{-i\Omega t}$ so it averages to zero:

$$0 = \mathbf{v}_d \wedge \mathbf{B} + \langle \mathbf{v}_L \wedge (\mathbf{r}_L.\nabla)\mathbf{B} \rangle \ . \tag{2.27}$$

To perform the average use

$$\mathbf{r}_L = (x_L, y_L) = \frac{v_\perp}{\Omega}\left(\sin \Omega t, \frac{q}{|q|}\cos \Omega t\right) \tag{2.28}$$

$$\mathbf{v}_L = (\dot{x}_L, \dot{y}_L) = v_\perp \left(\cos \Omega t, \frac{-q}{|q|}\sin \Omega t\right) \tag{2.29}$$

$$\text{So} \quad [v_L \wedge (\mathbf{r}.\nabla)\mathbf{B}]_x = v_y y \frac{d}{dy} B \tag{2.30}$$

$$[\mathbf{v}_L \wedge (\mathbf{r}.\nabla)\mathbf{B}]_y = -v_x y \frac{d}{dy} B \tag{2.31}$$

(Taking ∇B to be in the y-direction).

Then

$$\langle v_y y \rangle = -\langle \cos \Omega t \sin \Omega t \rangle \frac{v_\perp^2}{\Omega} = 0 \tag{2.32}$$

$$\langle v_x y \rangle = \frac{q}{|q|}\langle \cos \Omega t \cos \Omega t \rangle \frac{v_\perp^2}{\Omega} = \frac{1}{2}\frac{v_\perp^2}{\Omega}\frac{q}{|q|} \tag{2.33}$$

So

$$\langle \mathbf{v}_L \wedge (\mathbf{r}.\nabla)\mathbf{B} \rangle = -\frac{q}{|q|}\,\frac{1}{2}\frac{v_\perp^2}{\Omega}\nabla B \tag{2.34}$$

Substitute in:

$$0 = \mathbf{v}_d \wedge \mathbf{B} - \frac{q}{|q|}\frac{v_\perp^2}{2\Omega}\nabla B \tag{2.35}$$

and solve as before to get

$$\mathbf{v}_d = \frac{\left(\frac{-1}{|q|}\frac{v_\perp^2}{2\Omega}\ \nabla B\right) \wedge \mathbf{B}}{B^2} = \frac{q}{|q|}\frac{v_\perp^2}{2\Omega}\frac{\mathbf{B} \wedge \nabla B}{B^2} \tag{2.36}$$

or equivalently

$$\mathbf{v}_d = \frac{1}{q}\frac{mv_\perp^2}{2B}\frac{\mathbf{B} \wedge \nabla B}{B^2} \tag{2.37}$$

This is called the 'Grad B drift'.

2.4 Curvature Drift

When the B-field lines are curved and the particle has a velocity $v_\parallel$ along the field, another drift occurs.

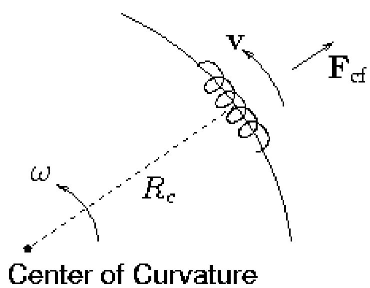

Figure 2.5: Curvature and Centrifugal Force

Take $|B|$ constant; radius of curvature R_e.

To 1st order the particle just spirals along the field.

In the frame of the guiding center a force appears because the plasma is rotating about the center of curvature.

This centrifugal force is F_{cf}

$$F_{cf} = m\frac{v_\parallel^2}{R_c} \text{ pointing outward} \tag{2.38}$$

as a vector

$$\mathbf{F}_{cf} = mv_\parallel^2\frac{\mathbf{R}_c}{R_c^2} \tag{2.39}$$

[There is also a coriolis force $2m(\omega \wedge \mathbf{v})$ but this averages to zero over a gyroperiod.]

Use the previous formula for a force

$$\mathbf{v}_d = \frac{1}{q}\frac{\mathbf{F}_{cf} \wedge \mathbf{B}}{B^2} = \frac{mv_\parallel^2}{qB^2}\frac{\mathbf{R}_c \wedge \mathbf{B}}{R_c^2} \tag{2.40}$$

This is the "Curvature Drift".

It is often convenient to have this expressed in terms of the field gradients. So we relate $\mathbf{R}_c$ to ∇B etc. as follows:

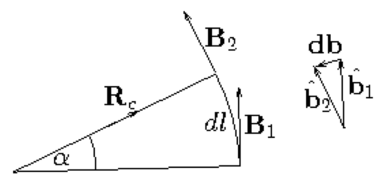

Figure 2.6: Differential expression of curvature

(Carets denote unit vectors)

From the diagram

$$\mathbf{db} = \hat{\mathbf{b}}_2 - \hat{\mathbf{b}}_1 = -\hat{\mathbf{R}}_c \alpha \tag{2.41}$$

and

$$d\ell = \propto R_c \tag{2.42}$$

So

$$\frac{d\mathbf{b}}{dl} = -\frac{\hat{\mathbf{R}}_c}{R_c} = -\frac{\mathbf{R}_c}{R_c^2} \tag{2.43}$$

But (by definition)

$$\frac{d\mathbf{b}}{dl} = (\hat{\mathbf{B}}.\nabla)\hat{\mathbf{b}} \tag{2.44}$$

So the curvature drift can be written

$$\mathbf{v}_d = \frac{mv_\parallel^2}{q}\frac{\mathbf{R}_c}{R_c^2} \wedge \frac{\mathbf{B}}{B^2} = \frac{mv_\parallel^2}{q}\frac{\mathbf{B} \wedge (\hat{\mathbf{b}}.\nabla)\hat{\mathbf{b}}}{B^2} \tag{2.45}$$

2.4.1 Vacuum Fields

Relation between ∇B & $\mathbf{R}_c$ drifts

The curvature and ∇B are related because of Maxwell's equations, their relation depends on the current density $\mathbf{j}$. A particular case of interest is $\mathbf{j} = 0$: vacuum fields.

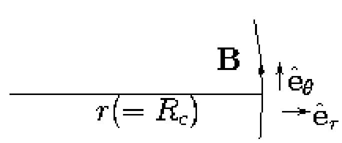

Figure 2.7: Local polar coordinates in a vacuum field

$$\nabla \wedge \mathbf{B} = 0 \quad \text{(static case)} \tag{2.46}$$

Consider the z-component

$$
\begin{aligned}
0 = (\nabla \wedge \mathbf{B})_z &= \frac{1}{r}\frac{\partial}{\partial r}(rB_\theta) \quad (B_r = 0 \text{ by choice}). && (2.47)\\
&= \frac{\partial B_\theta}{\partial r} + \frac{B_\theta}{r} && (2.48)
\end{aligned}
$$

or, in other words,

$$
(\nabla B)_r = -\frac{B}{R_c} \qquad (2.49)
$$

[Note also $0 = (\nabla \wedge \mathbf{B})_\theta = \partial B_\theta/\partial z \ : (\nabla B)_z = 0$]
and hence $(\nabla B)_{\text{perp}} = -B\ \mathbf{R}_c/R_c^2$.
Thus the grad B drift can be written:

$$
\mathbf{v}_{\nabla B} = \frac{mv_\perp^2}{2q}\frac{\mathbf{B} \wedge \nabla B}{B^3} = \frac{mV_\perp^2}{2q}\frac{\mathbf{R}_c \wedge \mathbf{B}}{R_c^2 B^2} \qquad (2.50)
$$

and the total drift across a vacuum field becomes

$$
\mathbf{v}_R + \mathbf{v}_{\nabla B} = \frac{1}{q}\left(mv_\parallel^2 + \frac{1}{2}mv_\perp^2\right)\frac{\mathbf{R}_c \wedge \mathbf{B}}{R_c^2 B^2}\ . \qquad (2.51)
$$

Notice the following:

1. R_c & ∇B drifts are in the *same* direction.
2. They are in *opposite* directions for opposite charges.
3. They are proportional to particle *energies*
4. Curvature $\leftrightarrow$ Parallel Energy ($\times$ 2)
   $\nabla B \leftrightarrow$ Perpendicular Energy
5. As a result one can very quickly calculate the average drift over a thermal distribution of particles because

$$
\begin{aligned}
\langle \frac{1}{2}mv_\parallel^2 \rangle &= \frac{T}{2} && (2.52)\\
\langle \frac{1}{2}mv_\perp^2 \rangle &= T \qquad \text{2 degrees of freedom} && (2.53)
\end{aligned}
$$

Therefore

$$
\langle \mathbf{v}_R + \mathbf{v}_{\nabla B} \rangle = \frac{2T}{q}\frac{\mathbf{R}_c \wedge \mathbf{B}}{R_c^2 B^2}\left(= \frac{2T}{q}\frac{\mathbf{B} \wedge \left(\hat{\mathbf{b}}.\nabla\right)\hat{\mathbf{b}}}{B^2}\right) \qquad (2.54)
$$

2.5 Interlude: Toroidal Confinement of Single Particles

Since particles can move freely along a magnetic field even if not across it, we cannot obviously confine the particles in a straight magnetic field. Obvious idea: bend the field lines into circles so that they have no ends.

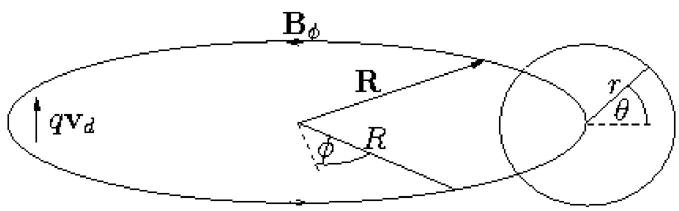

Figure 2.8: Toroidal field geometry

<u>Problem</u>

Curvature & ∇B drifts

$$\mathbf{v}_d = \frac{1}{q}\left(mv_\parallel^2 + \frac{1}{2}mv_\perp^2\right)\frac{\mathbf{R}\wedge\mathbf{B}}{R^2B^2} \tag{2.55}$$

$$|\mathbf{v}_d| = \frac{1}{q}\left(mv_\parallel^2 + \frac{1}{2}mv_\perp^2\right)\frac{1}{BR} \tag{2.56}$$

Ions drift *up*. Electrons down. There is no confinement. When there is finite density things

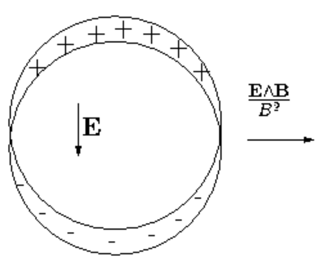

Figure 2.9: Charge separation due to vertical drift

are even worse because charge separation occurs → E → $E \wedge B$ → Outward Motion.

2.5.1 How to solve this problem?

Consider a beam of electrons $v_\parallel \neq 0$ $v_\perp = 0$. Drift is

$$v_d = \frac{mv_\parallel^2}{q}\frac{1}{B_T R} \tag{2.57}$$

What B_z is required to cancel this?

Adding B_z gives a compensating vertical velocity

$$v = v_\parallel \frac{B_z}{B_T} \qquad \text{for } \mathrm{B_z} \ << \ \mathrm{B_T} \tag{2.58}$$

We want total

$$v_z = 0 = v_\parallel \frac{B_z}{B_T} + \frac{m v_\parallel^2}{q} \frac{q}{B_T R} \tag{2.59}$$

So $B_z = -mv_\parallel / Rq$ is the right amount of field.

Note that this is such as to make

$$r_L(B_z) = \frac{|mv_\parallel|}{|qB_z|} \ = R \ \ . \tag{2.60}$$

But B_z required depends on $v_\parallel$ and q so we can't compensate for all particles simultaneously. Vertical field along cannot do it.

2.5.2 The Solution: Rotational Transform

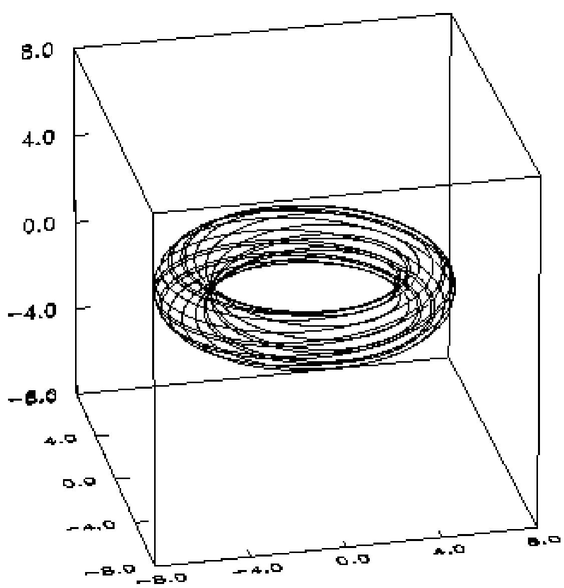

Figure 2.10: Tokamak field lines with rotational transform

Toroidal Coordinate system (r, θ, ϕ) (minor radius, poloidal angle, toroidal angle), see figure 2.8.

Suppose we have a *poloidal field* B_θ

Field Lines become helical and wind around the torus: figure 2.10.

In the poloidal cross-section the field describes a circle as it goes round in ϕ.
Equation of motion of a particle *exactly* following the field is:

$$r\frac{d\theta}{dt} = \frac{B_\theta}{B_\phi}v_\phi = \frac{B_\theta}{B_\phi}\frac{B_\phi}{B}v_\parallel \; = \frac{B_\theta}{B}\; v_\parallel \tag{2.61}$$

and

$$r = \text{constant.} \tag{2.62}$$

Now add on to this motion the cross field drift in the $\hat{\mathbf{z}}$ direction.

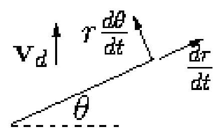

Figure 2.11: Components of velocity

$$r\frac{d\theta}{dt} \;=\; \frac{B_\theta}{B}v_\parallel + v_d\cos\theta \tag{2.63}$$

$$\frac{dr}{dt} \;=\; v_d\sin\theta \tag{2.64}$$

Take ratio, to eliminate time:

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{u_d\sin\theta}{\frac{B_\theta}{B}v_\parallel + v_d\cos\theta} \tag{2.65}$$

Take $B_\theta, B, v_\parallel, v_d$ to be constants, then we can integrate this orbit equation:

$$[\ln r] = [-\ln|\frac{B_\theta v_\parallel}{B} + v_d\cos\theta|]\ . \tag{2.66}$$

Take $r = r_0$ when $\cos\theta = 0$ ($\theta = \frac{\pi}{2}$) then

$$r = r_0/\left[1 + \frac{Bv_d}{b_\theta v_\parallel}\cos\theta\right] \tag{2.67}$$

If $\frac{Bv_d}{B_\theta v_\parallel} << 1$ this is approximately

$$r = r_0 - \Delta\cos\theta \tag{2.68}$$

where $\Delta = \frac{Bv_d}{B_\theta v_\parallel}r_0$
This is approximately a circular orbit shifted by a distance Δ:

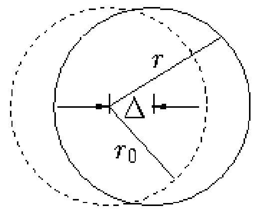

Figure 2.12: Shifted, approximately circular orbit

Substitute for v_d

$$\Delta \simeq r_0 \frac{B}{B_\theta} \frac{1}{q} \frac{(mv_\parallel^2 + \frac{1}{2} mv_\perp^2)}{v_\parallel} \frac{1}{B_\phi R} \tag{2.69}$$

$$\simeq \frac{1}{qB_\theta} \frac{mv_\parallel^2 + \frac{1}{2} mv_\perp^2}{v_\parallel} \frac{r_p}{R} \tag{2.70}$$

$$\text{If } v_\perp = 0 \qquad \Delta = \frac{mv_\parallel}{qB_\theta} \frac{r_0}{R} = r_{L\theta} \frac{r_0}{R}, \tag{2.71}$$

where $r_{L\theta}$ is the Larmor Radius in a field $B_\theta \times r/R$.

Provided Δ is small, particles *will* be confined. Obviously the important thing is the poloidal rotation of the field lines: Rotational Transform.

Rotational Transform

$$\text{rotational transform} \equiv \frac{\text{poloidal angle}}{\text{1 toroidal rotation}} \tag{2.72}$$

$$(\text{transform}/2\pi =) \quad \iota \equiv \frac{\text{poloidal angle}}{\text{toroidal angle}} \quad . \tag{2.73}$$

(Originally, ι was used to denote the transform. Since about 1990 it has been used to denote the transform divided by 2π which is the inverse of the safety factor.)

'Safety Factor'

$$`q_s' = \frac{1}{\iota} = \frac{\text{toroidal angle}}{\text{poloidal angle}} \quad . \tag{2.74}$$

Actually the value of these ratios may vary as one moves around the magnetic field. Definition strictly requires one should take the limit of a large no. of rotations.

q_s is a topological number: number of rotations the long way per rotation the short way.

Cylindrical approx.:

$$q_s = \frac{rB_\phi}{RB_\theta} \tag{2.75}$$

In terms of safety factor the orbit shift can be written

$$|\Delta| = r_{L\theta}\ \frac{r}{R} = r_{L\phi}\frac{B_\phi r}{B_\theta R} = r_L q_s \tag{2.76}$$

(assuming $B_\phi >> B_\theta$).

2.6 The Mirror Effect of Parallel Field Gradients: $\mathbf{E} = 0$, $\nabla B \parallel \mathbf{B}$

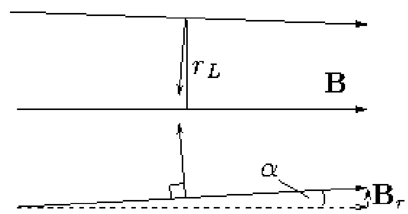

Figure 2.13: Basis of parallel mirror force

In the above situation there is a net force along **B**.
Force is

$$\begin{aligned} < F_\parallel > &= -|q\mathbf{v}\wedge\mathbf{B}|\ \sin\alpha = \ -|q|v_\perp B\sin\alpha & (2.77)\\ \sin\alpha &= \frac{-B_r}{B} & (2.78)\end{aligned}$$

Calculate B_r as function of B_z from $\nabla.\mathbf{B} = 0$.

$$\nabla.\mathbf{B} = \frac{1}{r}\frac{\partial}{\partial r}(rB_r) + \frac{\partial}{\partial z}B_z = 0\ . \tag{2.79}$$

Hence

$$rB_r = -\int r\frac{\partial B_z}{\partial z}dr \tag{2.80}$$

Suppose r_L is small enough that $\frac{\partial B_z}{\partial z} \simeq$ const.

$$[rB_r]_0^{r_L} \simeq \int_0^{r_L} rdr\ \frac{\partial B_z}{\partial z} = -\frac{1}{2}\ r_L^2\frac{\partial B_z}{\partial z} \tag{2.81}$$

So

$$B_r(r_L) = -\frac{1}{2}r_L\frac{\partial B_z}{\partial z} \tag{2.82}$$

$$\sin\alpha = -\frac{B_r}{B} = +\frac{r_L}{2}\frac{1}{2}\frac{\partial B_z}{\partial z} \tag{2.83}$$

Hence

$$< F_{\|} >= -|q| \; \frac{v_\perp r_L}{2}\frac{\partial B_z}{\partial z} = -\frac{\frac{1}{2}mv_\perp^2}{B}\frac{\partial B_z}{\partial z}. \quad (2.84)$$

As particle enters increasing field region it experiences a net parallel *retarding* force.

Define *Magnetic Moment*

$$\mu \equiv \frac{1}{2}mv_\perp^2/B \ . \quad (2.85)$$

Note this is consistent with loop current definition

$$\mu = AI = \pi r_L^2 . \frac{|q|v_\perp}{2\pi r_L} = \frac{|q|r_L v_\perp}{2} \quad (2.86)$$

Force is $F_{\|} = \mu.\nabla_{\|}\mathbf{B}$

This is force on a 'magnetic dipole' of moment μ.

$$F_{\|} = \mu.\nabla_{\|}\mathbf{B} \quad (2.87)$$

Our μ always points along $\mathbf{B}$ but in opposite direction.

2.6.1 Force on an Elementary Magnetic Moment Circuit

Consider a plane rectangular circuit carrying current I having elementary area $dxdy = dA$. Regard this as a vector pointing in the $\mathbf{z}$ direction $\mathbf{dA}$. The force on this circuit in a field $\mathbf{B}(\mathbf{r})$ is $\mathbf{F}$ such that

$$F_x = Idy[B_z(x+dx) - B_z(x)] = Idydx\frac{\partial B_z}{\partial x} \quad (2.88)$$

$$F_y = -Idx[B_z(y+dy) - B_z(y)] = Idydx\frac{\partial B_z}{\partial y} \quad (2.89)$$

$$F_z = -Idx[B_y(y+dy) - B_y(y)] - Idy[B_x(x+dx) - B_x(x)] \quad (2.90)$$

$$= -Idxdy\left[\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y}\right] = Idydx\frac{\partial B_z}{\partial z} \quad (2.91)$$

(Using $\nabla.\mathbf{B} = 0$).

Hence, summarizing: $\mathbf{F} = Idydx\nabla B_z$. Now define $\mu = I\mathbf{dA} = Idydx\hat{\mathbf{z}}$ and take it constant. Then clearly the force can be written

$$\mathbf{F} = \nabla(\mathbf{B}.\mu) \quad [\text{Strictly} = (\nabla\mathbf{B}).\mu] \quad (2.92)$$

μ is the (vector) magnetic moment of the circuit.

The shape of the circuit does not matter since any circuit can be considered to be composed of the sum of many rectangular circuits. So in general

$$\mu = I\mathbf{dA} \quad (2.93)$$

and force is

$$\mathbf{F} = \nabla(\mathbf{B}.\mu) \qquad (\mu \text{ constant}), \tag{2.94}$$

We shall show in a moment that $|\mu|$ is constant for a circulating particle, regard as an elementary circuit. Also, μ for a particle always points in the -**B** direction. [Note that this means that the effect of particles on the field is to *decrease* it.] Hence the force may be written

$$\mathbf{F} = -\mu \nabla B \tag{2.95}$$

This gives us both:

- *Magnetic Mirror Force:*

$$F_{\parallel} = -\mu \nabla_{\parallel} B \tag{2.96}$$

  and

- *Grad B Drift:*

$$\mathbf{v}_{\nabla B} = \frac{1}{q}\frac{\mathbf{F} \wedge \mathbf{B}}{B^2} = \frac{\mu}{q}\frac{\mathbf{B} \wedge \nabla B}{B^2}. \tag{2.97}$$

2.6.2 μ is a constant of the motion

'Adiabatic Invariant'

Proof from $F_{\parallel}$

Parallel equation of motion

$$m\frac{dv_{\parallel}}{dt} = F_{\parallel} = -\mu\frac{dB}{dz} \tag{2.98}$$

So

$$mv_{\parallel}\frac{dv_{\parallel}}{dt} = \frac{d}{dt}(\frac{1}{2}mv_{\parallel}^2) = -\mu v_z\frac{dB}{dz} = -\mu\frac{dB}{dt} \tag{2.99}$$

or

$$\frac{d}{dt}(\frac{1}{2}mv_{\parallel}^2) + \mu\frac{dB}{dt} = 0 \quad . \tag{2.100}$$

Conservation of Total KE

$$\frac{d}{dt}(\frac{1}{2}mv_{\parallel}^2 + \frac{1}{2}mv_{\perp}^2) = 0 \tag{2.101}$$

$$= \quad \frac{d}{dt}(\frac{1}{2}mv_{\parallel}^2 + \mu B) = 0 \tag{2.102}$$

Combine

$$\frac{d}{dt}(\mu B) - \mu\frac{dB}{dt} = 0 \tag{2.103}$$

$$= \frac{d\mu}{dt} = 0 \qquad \text{As required} \tag{2.104}$$

Angular Momentum

of particle about the guiding center is

$$r_L m v_\perp = \frac{m v_\perp}{|q| B} m v_\perp = \frac{2m}{|q|} \frac{\frac{1}{2} m v_\perp^2}{B} \quad (2.105)$$

$$= \frac{2m}{|q|} \mu \quad . \quad (2.106)$$

Conservation of magnetic moment is basically conservation of angular momentum about the guiding center.

Proof direct from Angular Momentum

Consider angular momentum about G.C. Because θ is ignorable (locally) Canonical angular momentum is conserved.

$$p = [\mathbf{r} \wedge (m\mathbf{v} + q\mathbf{A})]_z \qquad \text{conserved.} \quad (2.107)$$

Here $\mathbf{A}$ is the vector potential such that $\mathbf{B} = \nabla \wedge \mathbf{A}$

the definition of the vector potential means that

$$B_z = \frac{1}{r} \frac{\partial (r A_\theta))}{\partial r} \quad (2.108)$$

$$\Rightarrow r_L A_\theta(r_L) = \int_0^{r_L} r.B_z dr = \frac{r_L^2}{2} B_z = \frac{\mu m}{|q|} \quad (2.109)$$

Hence

$$p = \frac{-q}{|q|} r_L v_\perp m + q \frac{m\mu}{|q|} \quad (2.110)$$

$$= -\frac{q}{|q|} m\mu. \quad (2.111)$$

So p = const $\leftrightarrow$ μ = constant.

Conservation of μ is basically conservation of angular momentum of particle about G.C.

2.6.3 Mirror Trapping

$F_\parallel$ may be enough to reflect particles back. But may not!

Let's calculate whether it will:

Suppose reflection occurs.

At reflection point $v_{\parallel r} = 0$.

Energy conservation

$$\frac{1}{2} m (v_{\perp 0}^2 + v_{\parallel 0}^2) = \frac{1}{2} m v_{\perp r}^2 \quad (2.112)$$

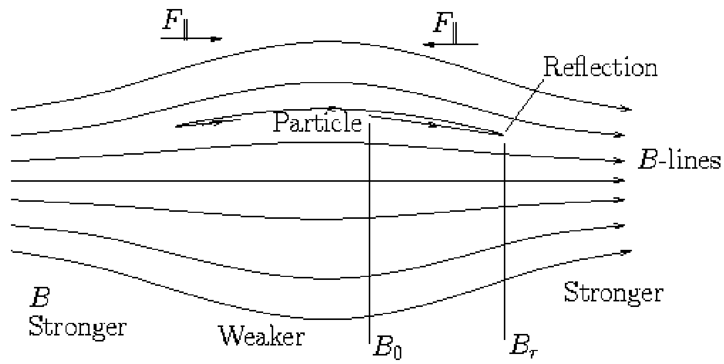

Figure 2.14: Magnetic Mirror

μ conservation

$$\frac{\frac{1}{2}mv_{\perp 0}^2}{B_0} = \frac{\frac{1}{2}mv_{\perp r}^2}{B_r} \tag{2.113}$$

Hence

$$v_{\perp 0}^2 + v_{\parallel 0}^2 = \frac{B_r}{B_0} v_{\perp 0}^2 \tag{2.114}$$

$$\frac{B_0}{B_r} = \frac{v_{\perp 0}^2}{v_{\perp 0}^2 + v_{\parallel o}^2} \tag{2.115}$$

2.6.4 Pitch Angle θ

$$\tan\theta = \frac{v_\perp}{v_\parallel} \tag{2.116}$$

$$\frac{B_0}{B_r} = \frac{v_{\perp 0}^2}{v_{\perp 0}^2 + v_{\parallel 0}^2} = \sin^2\theta_0 \tag{2.117}$$

So, given a pitch angle θ_0, reflection takes place where $B_0/B_r = \sin^2\theta_0$.
If θ_0 is too small no reflection can occur.
Critical angle θ_c is obviously

$$\theta_c = \sin^{-1}(B_0/B_1)^{\frac{1}{2}} \tag{2.118}$$

Loss Cone is all $\theta < \theta_c$.
Importance of Mirror Ratio: $R_m = B_1/B_0$.

2.6.5 Other Features of Mirror Motions

Flux enclosed by gyro orbit is constant.

$$\Phi = \pi r_L^2 B = \frac{\pi m^2 v_\perp^2}{q^2 B^2} B \tag{2.119}$$

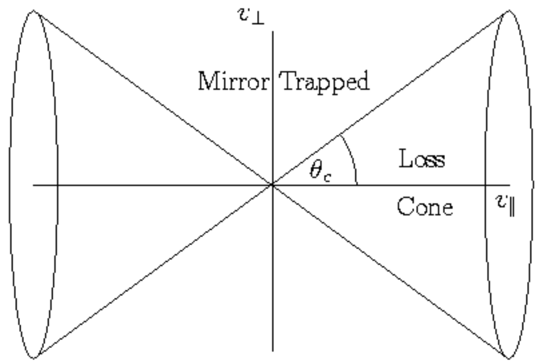

Figure 2.15: Critical angle θ_c divides velocity space into a loss-cone and a region of mirror-trapping

$$= \frac{2\pi m}{q^2}\frac{\frac{1}{2}mv_\perp^2}{B} \tag{2.120}$$

$$= \frac{2\pi m}{q^2}\mu = \text{constant.} \tag{2.121}$$

Note that if B changes 'suddenly' μ might not be conserved.

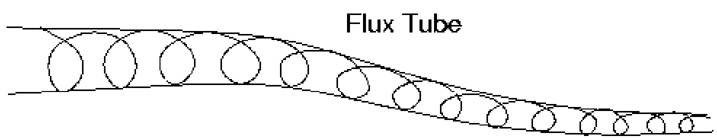

Figure 2.16: Flux tube described by orbit

Basic requirement

$$r_L << B/|\nabla B| \tag{2.122}$$

Slow variation of B (relative to r_L).

2.7 Time Varying B Field (E inductive)

Particle can gain energy from the inductive $\mathbf{E}$ field

$$\nabla \wedge \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{2.123}$$

$$\text{or} \oint \mathbf{E}.\mathbf{dl} = -\int_s \dot{\mathbf{B}}.\mathbf{ds} = -\frac{d\Phi}{dt} \tag{2.124}$$

Hence work done on particle in 1 revolution is

$$\delta w = -\oint |q|\mathbf{E}.\mathbf{d}\ell = +|q|\int_s \dot{\mathbf{B}}.\mathbf{ds} = +|q|\frac{d\Phi}{dt} = |q|\dot{B}\pi r_L^2 \tag{2.125}$$

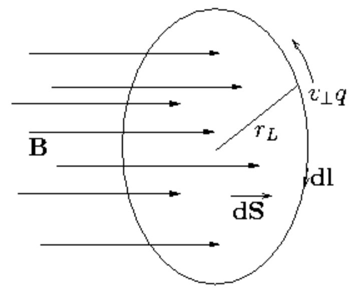

Figure 2.17: Particle orbits round **B** so as to perform a line integral of the Electric field

($\mathbf{d}\ell$ and $\mathbf{v}_\perp q$ are in opposition directions).

$$\delta\left(\frac{1}{2}mv_\perp^2\right) = |q|\dot{B}\pi r_L^2 = \frac{2\pi\dot{B}m}{|q|B}\frac{\frac{1}{2}mv_\perp^2}{B} \tag{2.126}$$

$$= \frac{2\pi\dot{B}}{|\Omega|}\mu. \tag{2.127}$$

Hence

$$\frac{d}{dt}\left(\frac{1}{2}mv_\perp^2\right) = \frac{|\Omega|}{2\pi}\delta\left(\frac{1}{2}mv_\perp^2\right) = \mu\frac{db}{dt} \tag{2.128}$$

but also

$$\frac{d}{dt}\left(\frac{1}{2}mv_\perp^2\right) = \frac{d}{dt}\left(\mu B\right) \; . \tag{2.129}$$

Hence

$$\frac{d\mu}{dt} = 0. \tag{2.130}$$

Notice that since $\Phi = \frac{2\pi m}{q^2}\mu$, this is just another way of saying that the flux through the gyro orbit is conserved.

Notice also *energy increase*. Method of 'heating'. Adiabatic Compression.

2.8 Time Varying E-field (E, B uniform)

Recall the $\mathbf{E}\wedge\mathbf{B}$ drift:

$$\mathbf{v}_{E\wedge B} = \frac{\mathbf{E}\wedge\mathbf{B}}{B^2} \tag{2.131}$$

when E varies so does $\mathbf{v}_{E\wedge B}$. Thus the guiding centre experiences an acceleration

$$\dot{\mathbf{v}}_{E\wedge B} = \frac{d}{dt}\left(\frac{\mathbf{E}\wedge\mathbf{B}}{B^2}\right) \tag{2.132}$$

In the frame of the guiding centre which is accelerating, a force is felt.

$$\mathbf{F}_a = -m\frac{d}{dt}\left(\frac{\mathbf{E}\wedge\mathbf{B}}{B^2}\right) \qquad \text{(Pushed back into seat! – ve.)} \tag{2.133}$$

This force produces another drift

$$\mathbf{v}_D = \frac{1}{q}\frac{\mathbf{F}_a \wedge \mathbf{B}}{B^2} = \frac{m}{qB^2}\frac{d}{dt}\left(\frac{\mathbf{E}\wedge\mathbf{B}}{B^2}\right)\wedge\mathbf{B} \tag{2.134}$$

$$= -\frac{m}{qB}\frac{d}{dt}\left((\mathbf{E}.\mathbf{B})\,\mathbf{B} - B^2\mathbf{E}\right) \tag{2.135}$$

$$= \frac{m}{qB^2}\dot{\mathbf{E}}_\perp \tag{2.136}$$

This is called the *'polarization drift'*.

$$\mathbf{v}_D = \mathbf{v}_{E\wedge B} \quad + \quad \mathbf{v}_p = \frac{\mathbf{E}\wedge\mathbf{B}}{B^2} \quad + \frac{m}{qB^2}\,\dot{\mathbf{E}}_\perp \tag{2.137}$$

$$= \frac{E\wedge B}{B^2} \quad + \quad \frac{1}{\Omega B}\dot{\mathbf{E}}_\perp \tag{2.138}$$

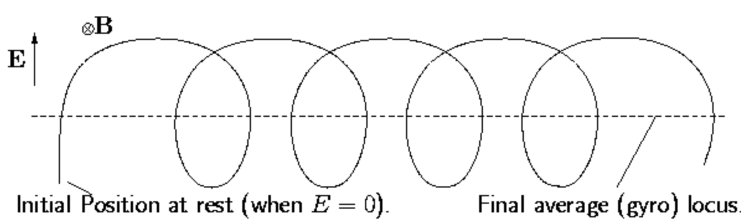

Figure 2.18: Suddenly turning on an electric field causes a shift of the gyrocenter in the direction of force. This is the polarization drift.

Start-up effect: When we 'switch on' an electric field the average position (gyro center) of an initially stationary particle shifts over by $\sim \frac{1}{2}$ the orbit size. The polarization drift is this polarization effect on the medium.
Total *shift* due to $\mathbf{v}_p$ is

$$\Delta\mathbf{r}\int \mathbf{v}_p dt = \frac{m}{qB^2}\int \hat{\mathbf{E}}_\perp dt = \frac{m}{qB^2}\left[\Delta\mathbf{E}_\perp\right] \quad . \tag{2.139}$$

2.8.1 Direct Derivation of $\frac{d\mathbf{E}}{dt}$ effect: 'Polarization Drift'

Consider an oscillatory field $\mathbf{E} = \mathbf{E}e^{-i\omega t}$ $(\perp r_0\mathbf{B})$

$$m\frac{d\mathbf{v}}{dt} = q\left(\mathbf{E} + \mathbf{v}\wedge\mathbf{B}\right) \tag{2.140}$$

$$= q\left(\mathbf{E}e^{-i\omega t} + \mathbf{v}\wedge\mathbf{B}\right) \tag{2.141}$$

Try for a solution in the form

$$\mathbf{v} = \mathbf{v}_D e^{-i\omega t} + \mathbf{v}_L \tag{2.142}$$

where, as usual, $\mathbf{v}_L$ satisfies $m\dot{\mathbf{v}}_L = q\mathbf{v}_L \wedge \mathbf{B}$

Then

$$(1) \qquad m(-i\omega\mathbf{v}_D = q\left(\mathbf{E} + \mathbf{v}_D \wedge \mathbf{B}\right) \qquad x\ell^{-i\omega t} \tag{2.143}$$

Solve for $\mathbf{v}_D$: Take $\wedge\mathbf{B}$ this equation:

$$(2) \qquad -mi\omega\left(\mathbf{v}_D \wedge \mathbf{B}\right) = q\left(\mathbf{E} \wedge \mathbf{B} + \left(\mathbf{B}^2.\mathbf{v}|D\right)\mathbf{B} - B^2\mathbf{v}_D\right) \tag{2.144}$$

add $mi\omega \times (1)$ to $q \times (2)$ to eliminate $\mathbf{v}_D \wedge \mathbf{B}$.

$$m^2\omega^2\mathbf{v}_D + q^2(\mathbf{E} \wedge \mathbf{B} - B^2\mathbf{v}_D) = mi\omega q\mathbf{E} \tag{2.145}$$

$$\text{or :} \qquad \mathbf{v}_D\left[1 - \frac{m^2\omega^2}{q^2B^2}\right] = -\frac{mi\omega}{qB^2}\mathbf{E} + \frac{\mathbf{E} \wedge \mathbf{B}}{B^2} \tag{2.146}$$

$$\text{i.e.} \qquad \mathbf{v}_D\left[1 - \frac{\omega^2}{\Omega^2}\right] = -\frac{i\omega q}{\Omega B|q|}\mathbf{E} + \frac{\mathbf{E} \wedge \mathbf{B}}{B^2} \tag{2.147}$$

Since $-i\omega \leftrightarrow \frac{\partial}{\partial t}$ this is the same formula as we had before: the sum of polarization and $\mathbf{E} \wedge \mathbf{B}$ drifts *except* for the $[1 - \omega^2\Omega^2]$ term.

This term comes from the change in $\mathbf{v}_D$ with time (accel).

Thus our earlier expression was only approximate. A good approx if $\omega << \Omega$.

2.9 Non Uniform E (Finite Larmor Radius)

$$m\frac{d\mathbf{v}}{dt} = q\left(\mathbf{E}(\mathbf{r}) + \mathbf{v} \wedge \mathbf{B}\right) \tag{2.148}$$

Seek the usual soltuion $\mathbf{v} = \mathbf{v}_D + \mathbf{v}_g$.

Then average out over a gyro orbit

$$\begin{aligned} \langle m\frac{dv_D}{dt}\rangle &= 0 = \langle q\left(\mathbf{E}(\mathbf{r}) + \mathbf{v} \wedge \mathbf{B}\right)\rangle && (2.149)\\ &= q\left[\langle\mathbf{E}(\mathbf{r})\rangle + \mathbf{v}_D \wedge \mathbf{B}\right] && (2.150) \end{aligned}$$

Hence drift is obviously

$$\mathbf{v}_D = \frac{\langle\mathbf{E}(\mathbf{r})\rangle \wedge \mathbf{B}}{B^2} \tag{2.151}$$

So we just need to find the *average* E field experienced.

Expand $\mathbf{E}$ as a Taylor series about the G.C.

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}_0 + \left(\mathbf{r}.\nabla\right)\mathbf{E} + \left(\frac{x^2\partial^2}{2!\partial x^2} + \frac{y^2}{2!}\frac{\partial^2}{\partial y^2}\right)\mathbf{E} + \text{cross terms } + . \tag{2.152}$$

(E.g. cross terms are $xy\frac{\partial^2}{\partial x \partial y}\mathbf{E}$).

Average over a gyro orbit: $\mathbf{r} = r_L(\cos\theta, \sin\theta, 0)$.

Average of cross terms = 0.

Then

$$\langle \mathbf{E}(\mathbf{r}) \rangle = \mathbf{E} + (\langle \mathbf{r}_L \rangle . \nabla)\mathbf{E} + \frac{\langle r_L^2 \rangle}{2!} \nabla^2 \mathbf{E}. \tag{2.153}$$

linear term $\langle r_L \rangle = 0$. So

$$\langle \mathbf{E}(\mathbf{r}) \rangle \simeq \mathbf{E} + \frac{r_L^2}{4} \nabla^2 E \tag{2.154}$$

Hence $\mathbf{E} \wedge \mathbf{B}$ with 1st finite-Larmor-radius correction is

$$\mathbf{v}_{E\wedge B} = \left(1 + \frac{r_L^2}{r} \nabla^2 \right) \frac{\mathbf{E} \wedge \mathbf{B}}{B^2}. \tag{2.155}$$

[Note: Grad B drift is a finite Larmor effect already.]

<u>Second and Third Adiabatic Invariants</u>

There are additional approximately conserved quantities like μ in some geometries.

2.10 Summary of Drifts

$$\mathbf{v}_E = \frac{\mathbf{E} \wedge \mathbf{B}}{B^2} \quad \text{Electric Field} \tag{2.156}$$

$$\mathbf{v}_F = \frac{1}{q} \frac{\mathbf{F} \wedge \mathbf{B}}{B^2} \quad \text{General Force} \tag{2.157}$$

$$\mathbf{v}_E = \left(1 + \frac{r_L^2}{4} \nabla^2 \right) \frac{\mathbf{E} \wedge \mathbf{B}}{B^2} \quad \text{Nonuniform E} \tag{2.158}$$

$$\mathbf{v}_{\nabla B} = \frac{m v_\perp^2}{2q} \frac{\mathbf{B} \wedge \nabla B}{B^3} \quad \text{GradB} \tag{2.159}$$

$$\mathbf{v}_R = \frac{m v_\parallel^2}{q} \frac{\mathbf{R}_c \wedge \mathbf{B}}{R_c^2 B^2} \quad \text{Curvature} \tag{2.160}$$

$$\mathbf{v}_R + \mathbf{v}_{\nabla B} = \frac{1}{q} \left(m v_\parallel^2 + \frac{1}{2} m v_\perp^2 \right) \frac{\mathbf{R}_c \wedge \mathbf{B}}{R_c^2 B^2} \quad \text{Vacuum Fields.} \tag{2.161}$$

$$\mathbf{v}_p = \frac{q}{|q|} \frac{\dot{E}_\perp}{|\Omega| B} \quad \text{Polarization} \tag{2.162}$$

<u>Mirror Motion</u>

$$\mu \equiv \frac{m v_\perp^2}{2B} \qquad \text{is constant} \tag{2.163}$$

Force is $\mathbf{F} = -\mu \nabla B$.

Chapter 3

Collisions in Plasmas

3.1 Binary collisions between charged particles

Reduced-mass for binary collisions:

Two particles interacting with each other have forces

$\mathbf{F}_{12}$ force on 1 from 2.

$\mathbf{F}_{21}$ force on 2 from 1.

By Newton's 3rd law, $\mathbf{F}_{12} = -\mathbf{F}_{21}$.

Equations of motion:

$$m_1 \ddot{\mathbf{r}}_1 = \mathbf{F}_{12} \qquad ; \qquad m_2 \ddot{\mathbf{r}}_2 = \mathbf{F}_{21} \tag{3.1}$$

Combine to get

$$\ddot{\mathbf{r}}_1 - \ddot{\mathbf{r}}_2 = \mathbf{F}_{12} \left(\frac{1}{m_1} + \frac{1}{m_2} \right) \tag{3.2}$$

which may be written

$$\frac{m_1 m_2}{m_1 + m_2} \frac{d^2}{dt^2} (\mathbf{r}_1 - \mathbf{r}_2) = \mathbf{F}_{12} \tag{3.3}$$

If F_{12} depends only on the difference vector $\mathbf{r}_1 - \mathbf{r}_2$, then this equation is identical to the equation of a particle of "Reduced Mass" $m_r \equiv \frac{m_1 m_2}{m_1 + m_2}$ moving at position $\mathbf{r} \equiv \mathbf{r}_1 - \mathbf{r}_2$ with respect to a fixed center of force:

$$m_r \ddot{\mathbf{r}} = \mathbf{F}_{12}(\mathbf{r}) \quad . \tag{3.4}$$

This is the equation we analyse, but actually particle 2 *does* move. And we need to recognize that when interpreting mathematics.

If $\mathbf{F}_{21}$ and $\mathbf{r}_1 - \mathbf{r}_2$ are always parallel, then a general form of the trajectory can be written as an integral. To save time we specialize immediately to the *Coulomb force*

$$\mathbf{F}_{12} = \frac{q_1 q_2}{4\pi\epsilon_0} \frac{\mathbf{r}}{r^3} \tag{3.5}$$

Solution of this standard (Newton's) problem:

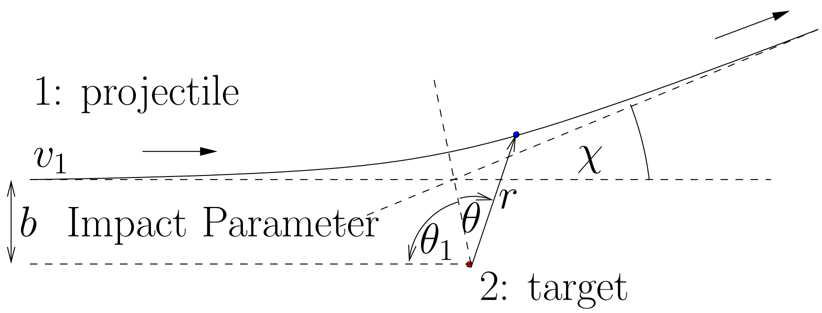

Figure 3.1: Geometry of the collision orbit

Angular momentum is conserved:

$$m_r r^2 \dot{\theta} \;=\; \text{const.} \;=\; m_r b v_1 \qquad (\theta \text{ clockwise from symmetry}) \tag{3.6}$$

Substitute $u \equiv \frac{1}{r}$ then $\dot{\theta} = \frac{bv_1}{r^2} = u^2\ bv_1$
Also

$$\dot{r} \;=\; \frac{d}{dt}\,\frac{1}{u} \;=\; -\frac{1}{u^2}\,\frac{du}{d\theta}\,\dot{\theta} \;=\; -bv_1\,\frac{du}{d\theta} \tag{3.7}$$

$$\ddot{r} \;=\; -bv_1\,\frac{d^2u}{d\theta^2}\,\dot{\theta} \;=\; -(bv_1)^2\ u^2\ \frac{d^2u}{d\theta^2} \tag{3.8}$$

Then radial acceleration is

$$\ddot{r} - r\dot{\theta}^2 = -(bv_1)^2\ u^2 \left(\frac{d^2u}{d\theta^2} + u\right) \;=\; \frac{|F_{12}|}{m_r} \tag{3.9}$$

i.e.

$$\frac{d^2u}{d\theta^2} + u = -\frac{q_1 q_2}{4\pi\epsilon_0}\,\frac{1}{m_r\,(bv_1)^2} \tag{3.10}$$

This orbit equation has the elementary solution

$$u \equiv \frac{1}{r} \;=\; C\cos\theta \;-\; \frac{q_1 q_2}{4\pi\epsilon_0}\,\frac{1}{m_r\,(bv_1)^2} \tag{3.11}$$

The $\sin\theta$ term is absent by symmetry. The other constant of integration, C, must be determined by initial condition. At initial (far distant) angle, θ_1, $u_1 = \frac{1}{\infty} = 0$. So

$$0 = C\cos\theta_1 - \frac{q_1 q_2}{4\pi\epsilon_0}\,\frac{1}{m_r\,(bv_1)^2} \tag{3.12}$$

There:

$$\dot{r}_1 = -v_1 = -bv_1\frac{du}{d\theta}|_1 = +bv_1 C\sin\theta_1 \tag{3.13}$$

Hence

$$\tan\theta_1 = \frac{\sin\theta_1}{\cos\theta_1} = \frac{-1/Cb}{\frac{q_1q_2}{4\pi\epsilon_0}\frac{1}{m_r(bv_1)^2}/C} = -\frac{b}{b_{90}} \tag{3.14}$$

where

$$b_{90} \equiv \frac{q_1 q_2}{4\pi\epsilon_0} \frac{1}{m_r v_1^2} \quad . \tag{3.15}$$

Notice that $\tan\theta_1 = -1$ when $b = b_{90}$. This is when $\theta_1 = -45°$ and $\chi = 90°$. So particle emerges at 90° to initial direction when

$$b \;=\; b_{90} \qquad \text{“90° impact parameter''} \tag{3.16}$$

Finally:

$$C = -\frac{1}{b}\text{cosec}\theta_1 = -\frac{1}{b}\left(1 + \frac{b_{90}^2}{b^2}\right)^{\frac{1}{2}} \tag{3.17}$$

3.1.1 Frames of Reference

Key quantity we want is the scattering angle but we need to be careful about reference frames.

Most “natural” frame of ref is “Center-of-Mass” frame, in which C of M is stationary. C of M has position:

$$\mathbf{R} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2} \tag{3.18}$$

and velocity (in lab frame)

$$\mathbf{V} = \frac{m_1\mathbf{v}_1 + m_2\mathbf{v}_2}{m_1 + m_2} \tag{3.19}$$

Now

$$\mathbf{r}_1 \;=\; \mathbf{R} + \frac{m_2}{m_1 + m_2}\mathbf{r} \tag{3.20}$$

$$\mathbf{r}_2 \;=\; \mathbf{R} - \frac{m_1}{m_1 + m_2}\,\mathbf{r} \tag{3.21}$$

So motion of either particle in C of M frame is a factor times difference vector, $\mathbf{r}$.

Velocity in lab frame is obtained by adding $\mathbf{V}$ to the C of M velocity, e.g. $\frac{m_2\dot{\mathbf{r}}}{m_1+m_2} + \mathbf{V}$.

Angles of position vectors and velocity *differences* are *same* in all frames.

Angles (i.e. directions) of velocities are *not same*.

3.1.2 Scattering Angle

In *C of M frame* is just the final angle of $\mathbf{r}$.

$$-\,2\theta_1 + \chi = \pi \tag{3.22}$$

(θ_1 is negative)

$$\chi = \pi + 2\theta_1 \quad ; \quad \theta_1 = \frac{\chi - \pi}{2} \quad . \tag{3.23}$$

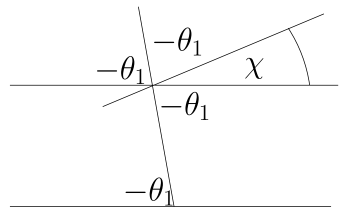

Figure 3.2: Relation between θ_1 and χ.

$$\tan\theta_1 = \tan\left(\frac{\chi}{2} = \frac{\pi}{2}\right) = -\cot\frac{\chi}{2} \tag{3.24}$$

So

$$\cot\frac{\chi}{2} = \frac{b}{b_{90}} \tag{3.25}$$

$$\tan\frac{\chi}{2} = \frac{b_{90}}{b} \tag{3.26}$$

But scattering angle (defined as exit velocity angle relative to initial velocity) in lab frame is *different*.

Final velocity in CM frame

$$\mathbf{v}'_{\mathrm{CM}} = v_{1\mathrm{CM}}(\cos\chi_c, \sin\chi_C) = \frac{m_2}{m_1+m_2}\, v_1(\cos\chi_c, \sin\chi_c) \tag{3.27}$$

[$\chi_c \equiv \chi$ and v_1 is initial relative velocity]. Final velocity in Lab frame

$$\mathbf{v}'_L = \mathbf{v}'_{\mathrm{CM}} + \mathbf{V} = \left(V + \frac{m_2 v_1}{m_1+m_2}\cos\chi_c, \frac{m_2 v_1}{m_1+m_2}\sin\chi_c\right) \tag{3.28}$$

So angle is given by

$$\cot\chi_L = \frac{V + \frac{m_2 v_1}{m_1+m_2}\cos\chi_c}{\frac{m_2 v_1}{m_1+m_2}\sin\chi_x} = \frac{V}{v_1}\,\frac{m_1+m_2}{m_2}\,\mathrm{cosec}\chi_c + \cot\chi_c \tag{3.29}$$

For the specific case when m_2 is initially a *stationary target* in lab frame, then

$$V = \frac{m_1 v_1}{m_1+m_2} \qquad \text{and hence} \tag{3.30}$$

$$\cot\chi_L = \frac{m_1}{m_2}\mathrm{cosec}\chi_c + \cot\chi_c \tag{3.31}$$

This is *exact*.

Small angle approximation ($\cot\chi \to \frac{1}{\chi}, \mathrm{cosec}\chi \to \frac{1}{\chi}$ gives

$$\frac{1}{\chi_L} = \frac{m_1}{m_2}\,\frac{1}{\chi_c} + \frac{1}{\chi_c} \;\Leftrightarrow\; \chi_L = \frac{m_2}{m_1+m_2}\chi_c \tag{3.32}$$

So small angles are proportional, with ratio set by the mass-ratio of particles.

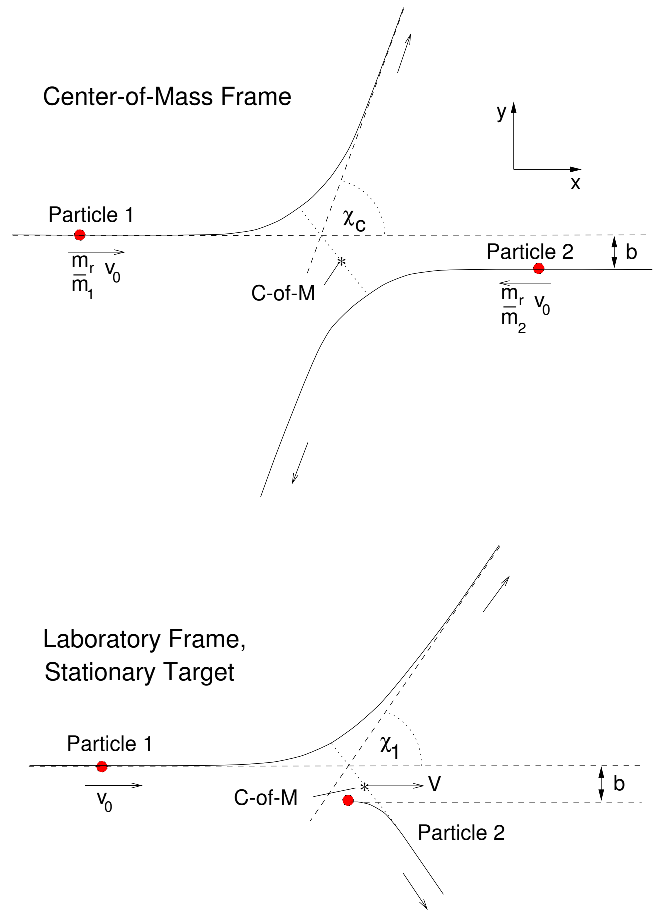

Figure 3.3: Collisions viewed in Center of Mass and Laboratory frame.

3.2 Differential Cross-Section for Scattering by Angle

Rutherford Cross-Section

By definition the cross-section, σ, for any specified collision process when a particle is passing through a density n_2 of targets is such that the number of such collisions per unit path length is $n_2\sigma$.

Sometimes a continuum of types of collision is considered, e.g. we consider collisions at different angles (χ) to be distinct. In that case we usually discuss *differential cross-sections* (e.g $\frac{d\sigma}{d\chi}$) defined such that number of collisions in an (angle) element $d\chi$ per unit path length is $n_2\frac{d\sigma}{d\chi}dx$. [Note that $\frac{d\sigma}{d\chi}$ is just notation for a number. Some authors just write $\sigma(\chi)$, but I find that less clear.]

Normally, for scattering-angle discrimination we discuss the differential cross-section per unit *solid angle*:

$$\frac{d\sigma}{d\Omega_s}. \tag{3.33}$$

This is related to scattering angle integrated over all azimuthal directions of scattering by:

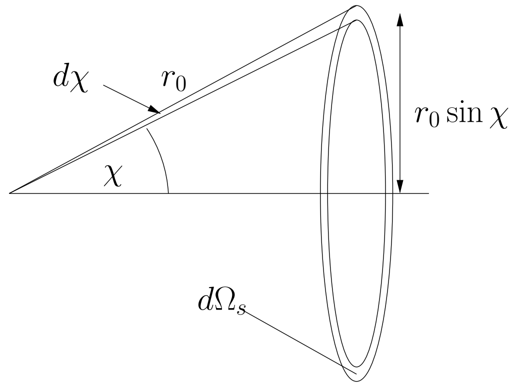

Figure 3.4: Scattering angle and solid angle relationship.

$$d\Omega_s = 2\pi \sin\chi d\chi \tag{3.34}$$

So that since

$$\frac{d\sigma}{d\Omega_s} d\Omega_s = \frac{d\sigma}{d\chi} d\chi \tag{3.35}$$

we have

$$\frac{d\sigma}{d\Omega_s} = \frac{1}{2\pi \sin\chi} \, \frac{d\sigma}{d\chi} \tag{3.36}$$

Now, since χ is a function (only) of the impact parameter, b, we just have to determine the number of collisions per unit length at impact parameter b.

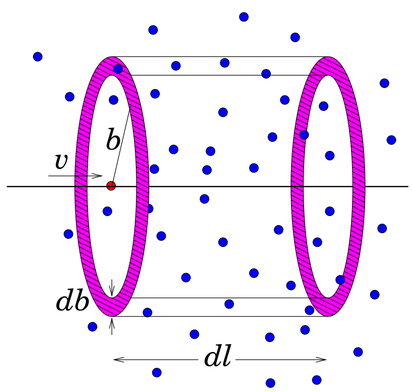

Figure 3.5: Annular volume corresponding to db.

Think of the projectile as dragging along an annulus of radius b and thickness db for an elementary distance along its path, $d\ell$. It thereby drags through a volume:

$$d\ell 2\pi b db \quad . \tag{3.37}$$

Therefore in this distance it has encountered a total number of targets

$$d\ell 2\pi b db \; . \; n_2 \tag{3.38}$$

at impact parameter $b(db)$. By definition this is equal to $d\ell \frac{d\sigma}{db} dbn_2$. Hence the differential cross-section for scattering (encounter) at impact parameter b is

$$\frac{d\sigma}{db} = 2\pi b \quad . \tag{3.39}$$

Again by definition, since χ is a function of b

$$\frac{d\sigma}{d\chi}dx = \frac{d\sigma}{db}db \Rightarrow \frac{d\sigma}{d\chi} = \frac{d\sigma}{db} = \left|\frac{db}{d\chi}\right| \quad . \tag{3.40}$$

[$db/d\chi$ is negative but differential cross-sections are positive.]

Substitute and we get

$$\frac{d\sigma}{d\Omega_s} = \frac{1}{2\pi \sin\chi} \frac{d\sigma}{db}\left|\frac{db}{d\chi}\right| = \frac{b}{\sin}\left|\frac{db}{d\chi}\right| \quad . \tag{3.41}$$

[This is a general result for classical collisions.]

For Coulomb collisions, in C of M frame,

$$\cot\left(\frac{\chi}{2}\right) = \frac{b}{b_{90}} \tag{3.42}$$

$$\Rightarrow \frac{db}{d\chi} = b_{90}\frac{d}{d\chi}\cot\frac{\chi}{2} = -\frac{b_{90}}{2}\text{cosec}^2\frac{\chi}{2} \quad . \tag{3.43}$$

Hence

$$\frac{d\sigma}{d\Omega_s} = \frac{b_{90}\cot\frac{\chi}{2}}{\sin\chi}\frac{b_{90}}{2}\text{cosec}^2\frac{\chi}{2} \tag{3.44}$$

$$= \frac{b_{90}^2}{2}\frac{\cos\frac{\chi}{2}/\sin\frac{\chi}{2}}{2\sin\frac{\chi}{2}\cos\frac{\chi}{2}}\frac{1}{\sin^2\frac{\chi}{2}} \tag{3.45}$$

$$= \frac{b_{90}^2}{4\sin^4\frac{\chi}{2}} \tag{3.46}$$

This is the Rutherford Cross-Section.

$$\frac{d\sigma}{d\Omega_s} = \frac{b_{90}^2}{4\sin^4\frac{\chi}{2}} \tag{3.47}$$

for scattering by Coulomb forces through an angle χ measured in C of M frame.

Notice that $\frac{d\sigma}{d\Omega_s} \to \infty$ as $\chi \to 0$.

This is because of the long-range nature of the Coulomb force. Distant collisions tend to dominate. ($\chi \to 0 \Leftrightarrow b \to \infty$).

3.3 Relaxation Processes

There are 2 (main) different types of collisional relaxation process we need to discuss for a test particle moving through a background of scatterers:

1. Energy Loss (or equilibrium)
2. Momentum Loss (or angular scattering)

The distinction may be illustrated by a large angle (90°) scatter from a heavy (stationary) target.

If the target is fixed, no energy is transferred to it. So the *energy loss* is *zero* (or small if scatterer is just 'heavy'). However, the *momentum* in the x direction is *completely 'lost'* in this 90° scatter.

This shows that the timescales for Energy loss and momentum loss may be very different.

3.3.1 Energy Loss

For an initially stationary target, the final velocity in lab frame of the projectile is

$$v'_L = \left(\frac{m_l v_1}{m_1 + m_2} + \frac{m_2 v_1}{m_1 + m_2} \cos \chi_c \ , \ \frac{m_2 v_1}{m_1 + m_2} \sin \chi_c \right) \tag{3.48}$$

So the final kinetic energy is

$$K' = \frac{1}{2} m_1 v_L'^2 = \frac{1}{2} m_1 v_1^2 \Bigg\{ \left(\frac{m_1}{m_1 + m_2} \right)^2 + \frac{2 m_1 m_2}{(m_1 + m_2)^2} \cos \chi_c \tag{3.49}$$

$$+ \frac{m_2^2}{(m_1 + m_2)^2} \left(\cos^2 \chi_c + \sin^2 \chi_c \right) \Bigg\} \tag{3.50}$$

$$= \frac{1}{2} m_1 v_1^2 \left\{ 1 + \frac{2 m_1 m_2}{(m_1 + m_2)^2} \ (\cos \chi_c - 1) \right\} \tag{3.51}$$

$$= \frac{1}{2} m_1 v_1^2 \left\{ 1 + \frac{2 m_1 m_2}{(m_1 + m_2)^2} \ 2 \sin^2 \frac{\chi_c}{2} \right\} \tag{3.52}$$

Hence the kinetic energy lost is $\Delta K = K - K'$

$$= \frac{1}{2} m_1 v_2 \ \frac{4 m_1 m_2}{(m_1 + m_2)^2} \ \sin^2 \frac{\chi_c}{2} \tag{3.53}$$

$$= \frac{1}{2} m_1 v_1^2 \ \frac{4 m_1 m_2}{(m_1 + m_2)^2} \ \frac{1}{\left(\frac{b}{b_{90}} \right)^2 + 1} \qquad [\text{using } \cot \frac{\chi_c}{2} = \frac{b}{b_{90}}] \tag{3.54}$$

(exact). For small angles $\chi \ll 1$ i.e. $b/b_{90} \gg 1$ this energy lost in a single collision is approximately

$$\left(\frac{1}{2} m_1 v_1^2 \right) \ \frac{4 m_1 m_2}{(m_1 + m_2)^2} \ \left(\frac{b_{90}}{b} \right)^2 \tag{3.55}$$

If what we are asking is: how fast does the projectile lose energy? Then we need add up the effects of all collisions in an elemental length $d\ell$ at all relevant impact parameters.

The contribution from impact parameter range db at b will equal the number of targets encountered times ΔK:

$$\underbrace{n_2 d\ell 2\pi b db}_{\text{encounters}} \quad \underbrace{\frac{1}{2}m_1 v_1^2 \frac{4m_1 m_2}{(m_1+m_2)^2}\left(\frac{b_{90}}{b}\right)^2}_{\text{Loss per encounter } (\Delta K)} \tag{3.56}$$

This must be integrated over all b to get total energy loss.

$$dK = n_2 d\ell K \frac{4m_1 m_2}{(m_1+m_2)^2} \int \left(\frac{b_{90}}{b}\right)^2 \; 2\pi b db \tag{3.57}$$

so

$$\frac{dK}{d\ell} = K \; n_2 \frac{m_1 m_2}{(m_1+m_2)^2} \; 8\pi b_{90}^2 \; \left[\ln|b|\right]_{\min}^{\max} \tag{3.58}$$

We see there is a *problem* both limits of the integral ($b \to 0$, $b \to \infty$) diverge logarithmically. That is because the formulas we are integrating are approximate.

1. We are using small-angle approx for ΔK.

2. We are assuming the Coulomb force applies but this is a plasma so there is screening.

3.3.2 Cut-offs Estimates

1. Small-angle approx breaks down around $b = b_{90}$. Just truncate the integral there; ignore contributions from $b < b_{90}$.

2. Debye Shielding says really the potential varies as

$$\phi \propto \frac{\exp\left(\frac{-r}{\lambda_D}\right)}{r} \qquad \text{instead of } \propto \frac{1}{r} \tag{3.59}$$

   so approximate this by cutting off integral at $b = \lambda_D$ equivalent to

$$b_{\min} = b_{90}. \qquad b_{\max} = \lambda_D. \tag{3.60}$$

$$\frac{dK}{d\ell} = K n_2 \frac{m_1 m_2}{(m_1+m_2)^2} \; 8\pi b_{90}^2 \; \ln|\Lambda| \tag{3.61}$$

$$\Lambda = \frac{\lambda_D}{b_{90}} = \left(\frac{\epsilon_0 T_e}{ne^2}\right)^{\frac{1}{2}} \Big/ \left(\frac{q_1 q_2}{4\pi\epsilon_0 m_r v_1^2}\right) \tag{3.62}$$

So *Coulomb Logarithm* is '$\ln \Lambda$'

$$\Lambda \;=\; \frac{\lambda_D}{b_{90}} \;=\; \left(\frac{\epsilon_0 T_e}{ne^2}\right)^{\frac{1}{2}} \Big/ \left(\frac{q_1 q_2}{4\pi\epsilon_0 m_r v_1^2}\right) \tag{3.63}$$

Because these cut-offs are in ln term result is not sensitive to their exact values.

One commonly uses *Collision Frequency*. Energy Loss Collision Frequency is

$$\nu_K \equiv v_1 \frac{1}{K}\frac{dK}{dL} = n_2 v_1 \frac{m_1 m_2}{(m_1+m_2)^2}\, 8\pi b_{90}^2 \;\ln|\Lambda| \tag{3.64}$$

Substitute for b_{90} and m_r (in b_{90})

$$\nu_K \;=\; n_2 v_1 \frac{m_1 m_2}{(m_1+m_2)^2}\, 8\pi \left[\frac{q_1 q_2}{4\pi\epsilon_0 \frac{m_1 m_2}{m_1+m_2} v_1^2}\right]^2 \ln\Lambda \tag{3.65}$$

$$\;=\; n_2 \frac{q_1^2 q_2^2}{(4\pi\epsilon_0)^2} \frac{8\pi}{m_1 m_2 v_1^3} \;\ln\Lambda \tag{3.66}$$

Collision time $\tau_K \equiv 1/\nu_K$

Effective (Energy Loss) *Cross-section* $\left[\frac{1}{K}\frac{dK}{d\ell} = \sigma_K n_2\right]$

$$\sigma_K = \nu_K / n_2 v_1 = \frac{q_1^2 q_2^2}{(4\pi\epsilon_0)^2} \frac{8\pi}{m_1 m_2 v_1^4} \;\ln\Lambda \tag{3.67}$$

3.3.3 Momentum Loss

Loss of x-momentum in 1 collision is

$$\Delta p_x \;=\; m_1(v_1 - v'_{Lx}) \tag{3.68}$$

$$\;=\; m_1 v_1 \left\{1 - \left(\frac{m_1}{m_1+m_2} + \frac{m_2}{m_1+m_2}\cos\chi_c\right)\right\} \tag{3.69}$$

$$\;=\; p_x \frac{m_2}{m_1+m_2} \;(1-\cos\chi_c) \tag{3.70}$$

$$\;\simeq\; p_x \frac{m_2}{m_1+m_2} \frac{\chi_c^2}{2} = p_x \frac{m_2}{m_1+m_2} \frac{2b_{90}^2}{b^2} \tag{3.71}$$

(small angle approx). Hence rate of momentum loss can be obtained using an integral identical to the energy loss but with the above parameters:

$$\frac{dp}{d\ell} \;=\; n_2 p \int_{b_{min}}^{b_{max}} \frac{m2}{m1+m_2} \frac{2b_{90}^2}{b^2}\, 2\pi b db \tag{3.72}$$

$$\;=\; n_2 p \frac{m_2}{m_1+m_2}\, 4\pi\, b_{90}^2 \;\ln\Lambda \tag{3.73}$$

Note for the future reference:

$$\frac{dp}{dt} = v\frac{dp}{d\ell} = n_2 v^2 \frac{m_1 m_2}{m_1+m_2}\, 4\pi b_{90}^2 \;\ln\Lambda. \tag{3.74}$$

Therefore *Momentum Loss.*

Collision Frequency

$$\nu_p = v_1 \frac{1}{p}\frac{dp}{d\ell} = n_2 v_1 \ \frac{m_2}{m_1+m_2} \ 4\pi b_{90}^2 \ \ln\Lambda \tag{3.75}$$

$$= n_2 v_1 \ \frac{m_2}{m_1+m_2} \ 4\pi \left[\frac{q_1 q_2}{4\pi\epsilon_0 \frac{m_1 m_2}{m_1+m_2} v_1^2}\right]^2 \ \ln\Lambda \tag{3.76}$$

$$= n_2 \ \frac{q_1^2 q_2^2}{(4\pi\epsilon_0)^2} \ \frac{4\pi\,(m_1+m_2)}{m_2 m_1^2 v_1^3} \ \ln\Lambda \tag{3.77}$$

Collision Time $\tau_p = 1/\nu_p$

Cross-Section (effective) $\sigma = \nu_p / n_2 v_1$

Notice ratio

$$\frac{\text{Energy Loss } \nu_{\rm K}}{\text{Momentum loss } \nu_{\rm p}} = \frac{2}{m_1 m_2} \Bigg/ \frac{m_1+m_2}{m_2 m_1^2} = \frac{2m_1}{m_1+m_2} \tag{3.78}$$

This is

$$\simeq \ 2 \quad \text{if} \quad m_1 >> m_2 \tag{3.79}$$

$$= \ 1 \quad \text{if} \quad m_1 = m_2 \tag{3.80}$$

$$\ll \ 1 \quad \text{if} \quad m_1 \ll m_2. \tag{3.81}$$

Third case, e.g. electrons → shows that mostly the *angle* of velocity scatters. Therefore Momentum 'Scattering' time is often called '90° scattering' time to 'diffuse' through 90° in angle.

3.3.4 'Random Walk' in angle

When $m_1 << m_2$ energy loss << momentum loss. Hence $|\mathbf{v}'_L \simeq v_1$. All that matters is the scattering angle: $\chi_L \simeq \chi_c \simeq 2b_{90}/b$.

Mean angle of deviation in length L is zero because all directions are equally likely.
But:
Mean *square* angle is

$$\overline{\Delta\alpha^2} = n_2 L \int_{b_{\rm min}}^{b_{\rm max}} \chi^2 \ 2\pi b db \tag{3.82}$$

$$= L n_2 \ 8\pi \ b_{90}^2 \ \ln\Lambda \tag{3.83}$$

Spread is 'all round' when $\overline{\Delta\alpha^2} \simeq 1$. This is roughly when a particle has scattered 90° on average. It requires

$$L n_2 \ 8\pi b_{90}^2 \ \ln\Lambda = 1 \quad . \tag{3.84}$$

So can think of a kind of 'cross-section' for 'σ_{90}' 90° scattering as such that

$$n_2 L`\sigma_{90}\text{'} = 1 \text{ when } Ln_2\ 8\pi\ b_{90}^2\ \ln\Lambda = 1 \tag{3.85}$$

$$\text{i.e. }`\sigma_{90}\text{'} = 8\pi\ b_{90}^2\ \ln\Lambda\ \ (= 2\sigma_p) \tag{3.86}$$

This is $8 \ln \Lambda$ larger than cross-section for 90° scattering *in single collision.*

Be Careful! 'σ_{90}' is not a usual type of cross-section because the whole process is really diffusive in angle.

Actually all collision processes due to coulomb force are best treated (in a Mathematical way) as a diffusion in velocity space

$\rightarrow$ *Fokker-Planck equation.*

3.3.5 Summary of different types of collision

The *Energy Loss* collision frequency is to do with slowing down to rest and exchanging energy. It is required for calculating

Equilibration Times (of Temperatures)
Energy Transfer between species.

The *Momentum Loss* frequency is to do with loss of *directed* velocity. It is required for calculating

Mobility: Conductivity/Resistivity
Viscosity
Particle Diffusion
Energy (Thermal) Diffusion

Usually we distinguish between electrons and ions because of their very different mass:

Energy Loss [Stationary Targets] *Momentum Loss*

$$\begin{aligned}
{}^K\nu_{ee} &= n_e \frac{e^4}{(4\pi\epsilon_0)^2} \frac{8\pi}{m_e^2 v_e^3} \ln\Lambda & {}^p\nu_{ee} &= {}^K\ \nu_{ee} \times \left[\frac{m_e + m_e}{2m_e} = 1\right] \\
{}^K\nu_{ei} &= n_i \frac{Z^2 e^4}{(4\pi\epsilon_0)^2} \frac{8\pi}{m_e m_i v_e^3} \ln\Lambda & {}^p\nu_{ei} &= {}^K\ \nu_{ei} \times \left[\frac{m_e + m_i}{2m_e} \simeq \frac{m_i}{2m_e}\right] \\
{}^k\nu_{ii} &= n_i \frac{Z^2 e^4}{(4\pi e_o)^2} \frac{8\pi}{m_i^2 v_i^3} \ln\Lambda & {}^p\nu_{ii} &= {}^K\ \nu_{ii} \times \left[\frac{m_i + m_i}{2m_i} = 1\right] \\
{}^K\nu_{ie} &= n_e \frac{Z_e^2 e^4}{(4\pi\epsilon_0)^2} \frac{8\pi}{m_i m_e v_i^3} \ln\Lambda & {}^p\nu_{ie} &= {}^K\ \nu_{ie} = \times \left[\frac{m_e + m_i}{2mi_i} \simeq \frac{1}{2}\right]
\end{aligned} \tag{3.87}$$

Sometimes one distinguishes between 'transverse diffusion' of velocity and 'momentum loss'. The ratio of these two is

$$\frac{\overline{\Delta p_1^2}}{p^2 \Delta L} \Bigg/ \left|\frac{\Delta p_\parallel}{p\Delta L}\right| = \frac{d\chi_L^2}{dL} \Bigg/ \left|\frac{1}{p}\ \frac{dp}{dL}\right| \tag{3.88}$$

$$= \frac{\left(\frac{m_2}{m_1+m_2}\chi_c\right)^2}{\frac{m_2}{m_1+m_2}\frac{\chi_c^2}{2}} = \frac{2m_2}{m_1 + m_2}. \tag{3.89}$$

So

$$\frac{`\sigma_{90}\text{'}}{`\sigma_p\text{'}} = \frac{2m_2}{m_1+m_2} \qquad = 1 \qquad \text{like particles} \tag{3.90}$$

$$\simeq 2 \qquad m_1 << m_2 \tag{3.91}$$

$$\simeq \frac{2m_2}{m_1} \qquad m_2 << m_1\ . \tag{3.92}$$

Hence

$$^{\perp}\nu_{ee} = {}^{p}\nu_{ee} = {}^{K}\nu_{ee}\,(= `\nu_{ee}\text{'!!}) \tag{3.93}$$

$$^{\perp}\nu_{ei} = 2^{p}\nu_{ei} = {}^{K}\nu_{ee}\frac{n_i}{n_e}Z^2\,(= Z\nu_{ee})\,(= `\nu_{ei}\text{'}) \tag{3.94}$$

$$^{\perp}\nu_{ii} = {}^{p}\nu_{ii} = {}^{K}\nu_{ii}\,(= \nu_{ii}!!)\ \text{(Like Ions)} \tag{3.95}$$

$$^{\perp}\nu_{ie} = \frac{2m_e}{m_i}\,{}^{p}\nu_{ie} = \frac{m_e}{m_i}\,{}^{K}\nu_{ie} = {}^{K}\nu_{ii} = \nu_{ii} \tag{3.96}$$

[But note: ions are slowed down by electrons long before being angle scattered.]

3.4 Thermal Distribution Collisions

So far we have calculated collision frequencies with stationary targets and single-velocity projectiles but generally we shall care about thermal (Maxwellian) distributions (or nearly thermal) of both species. This is harder to calculate and we shall resort to some heuristic calculations.

3.4.1 $e \to i$

Very rare for thermal ion velocity to be ∼ electron. So ignore ion motion.

Average over electron distribution.

Momentum loss to ions from (assumed) drifting Maxwellian electron distribution:

$$f_e(\mathbf{v}) = n_e\left(\frac{m_e}{2\pi T_e}\right)^{\frac{3}{2}}\ \exp\left[-\frac{m\,(\mathbf{v}-\mathbf{v}_d)^2}{2T}\right] \tag{3.97}$$

Each electron in this distribution is losing momentum to the ions at a rate given by the collision frequency

$$\nu_p = n_i\frac{q_e^2 q_i^2}{(4\pi\epsilon_0)^2}\ \frac{4\pi\,(m_e+m_i)}{m_i m_e^2 v^3}\ \ln\Lambda \tag{3.98}$$

so total rate of loss of momentum is given by (per unit volume)

$$-\frac{dp}{dt} = \int f_e(\mathbf{v})\ \nu_p(v)\ m_e\mathbf{v}\ d^3\mathbf{v} \tag{3.99}$$

To evaluate this integral approximately we adopt the following simplifications.

1. Ignore variations of $\ln\Lambda$ with v and just replace a typical thermal value in $\Lambda = \lambda_D/b_{90}(v_1)$.

2. Suppose that drift velocity $\mathbf{v}_d$ is small relative to the typical thermal velocity, written $v_e \equiv \sqrt{t_e/m_e}$ and express f_e in terms of $\mathbf{u} \equiv \frac{\mathbf{v}}{v_e}$ to first order in $\mathbf{u}_d \equiv \frac{\mathbf{v}_d}{v_e}$:

$$f_e = n_e \frac{1}{(2\pi)^{\frac{3}{2}} v_e^3} \exp\left[\frac{-1}{2}(\mathbf{u}-\mathbf{u}_d)^2\right] \quad (3.100)$$

$$\simeq \frac{n_e}{(2\pi)^{\frac{3}{2}} v_e^3}(1+\mathbf{u}.\mathbf{u}_d) \exp\left[\frac{-u^2}{2}\right] = (1+u_x u_d) f_o \quad (3.101)$$

taking x-axis along $\mathbf{u}_d$ and denoting by f_o the unshifted Maxwellian.

Then momentum loss rate per unit volume

$$-\frac{dp_x}{dt} = \int f_e \nu_p m_e v_x d^3 v$$

$$= \nu_p(v_t) m_e \int (1+u_x u_d) f_o \frac{v_e^3}{v^3} v_x \, d^3\,\mathbf{v} \quad (3.102)$$

$$= \nu_p(v_t) m_e v_d \int \frac{u_x^2}{u^3} f_o \, d^3\,\mathbf{v}$$

To evaluate this integral, use the spherical symmetry of f_o to see that:

$$\int \frac{u_x^2}{u^3} f_o d^3\mathbf{v} = \frac{1}{3}\int \frac{u_x^2+u_y^2+u_z^3}{u^3} f_o d^3\mathbf{v} = \frac{1}{3}\int \frac{u^2}{u^3} f_o d^3\mathbf{v}$$

$$= \frac{1}{3}\int_0^\alpha \frac{v_e}{v} f_o 4\pi v^2 dv$$

$$= \frac{2\pi}{3} v_e \int_0^\alpha f_o 2v dv$$

$$= \frac{2\pi}{3} v_e \frac{n_e}{(2\pi)^{\frac{3}{2}} v_e^3} \int_0^\alpha \exp\left(\frac{-v^2}{2v_e^2}\right) dv^2$$

$$= \frac{2\pi}{3} \frac{n_e}{(2\pi)^{\frac{3}{2}}} 2 = \frac{2}{3(2\pi)^{\frac{1}{2}}} n_e \,. \quad (3.103)$$

Thus the Maxwell-averaged momentum-loss frequency is

$$-\frac{1}{p}\frac{dp}{dt} \equiv \overline{\nu}_{ei} = \frac{2}{3(2\pi)^{\frac{1}{2}}} \nu_p(v_t) \quad (3.104)$$

(where $p = m_e v_d n_e$ is the momentum per unit volume attributable to drift).

$$\overline{\nu}_{ei} = \frac{2}{3(2\pi)^{\frac{1}{2}}} n_i \frac{q_e^2 q_i^2}{(4\pi\epsilon_0)^2} \frac{4\pi(m_e+m_i)}{m_i m_e^2 v_e^3} \ln\Lambda_e \quad (3.105)$$

$$= \frac{2}{3(2\pi)^{\frac{1}{2}}} n_i \left(\frac{ze^2}{4\pi\epsilon_0}\right)^2 \frac{4\pi}{m_e^{\frac{1}{2}} T_e^{\frac{3}{2}}} \ln\Lambda_e \quad (3.106)$$

(substituting for thermal electron velocity, v_e, and dropping $\frac{m_e}{m_i}$ order term), where $Ze = q_i$. This is the standard form of electron collision frequency.

3.4.2 $i \rightarrow e$

Ion momentum loss to electrons can be treated by a simple Galilean transformation of the $e \rightarrow i$ case because it is still the electron thermal motions that matter.

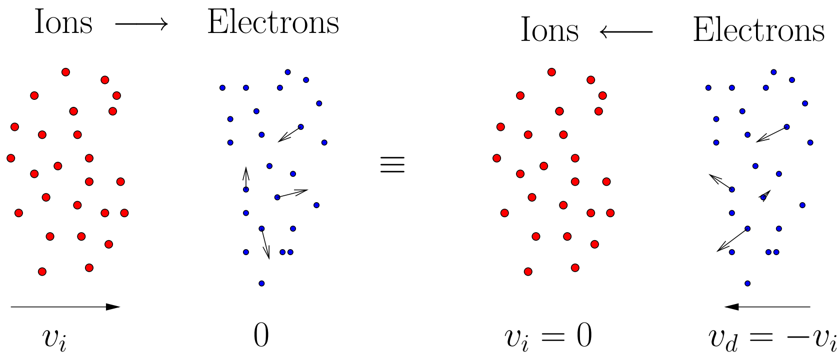

Figure 3.6: Ion-electron collisions are equivalent to electron-ion collisions in a moving reference frame.

Rate of momentum transfer, $\frac{dp}{dt}$, is same in both cases:

$$\frac{dp}{dt} = -p\nu \tag{3.107}$$

Hence $p_e\nu_{ei} = p_i\nu_{ie}$ or

$$\overline{\nu}_{ie} = \frac{p_e}{p_i}\overline{\nu}_{ei} = \frac{n_e m_e}{n_i m_i}\;\overline{\nu}_{ei} \tag{3.108}$$

(since drift velocities are the same).

Ion momentum loss to electrons is much lower collision frequency than $e \rightarrow i$ because ions possess so much more momentum for the same velocity.

3.4.3 $i \rightarrow i$

Ion-ion collisions can be treated somewhat like $e \rightarrow i$ collisions except that we have to account for *moving targets* i.e. their thermal motion.

Consider two different ion species moving relative to each other with drift velocity v_d; the targets' thermal motion affects the momentum transfer cross-section.

Using our previous expression for momentum transfer, we can write the average rate of transfer per unit volume as: [see 3.74 "note for future reference"]

$$-\frac{d\mathbf{p}}{dt} = \int\int \mathbf{v}_r\;\frac{m_1 m_2}{m_1+m_2}\;v_r 4\pi\;b_{90}^2\;\;\ln\Lambda\;\;f_1 f_2 d^2 v_1 d^3 v_2 \tag{3.109}$$

where $\mathbf{v}_r$ is the relative velocity $(\mathbf{v}_1 - \mathbf{v}_2)$ and b_{90} is expressed

$$b_{90} = \frac{q_1 q_2}{4\pi\epsilon_0}\;\frac{1}{m_r v_r^2} \tag{3.110}$$

and m_r is the reduced mass $\frac{m_1 m_2}{m_1+m_2}$.
Since everything in the integral apart from $f_1 f_2$ depends only on the relative velocity, we proceed by transforming the velocity coordinates from $\mathbf{v}_1, \mathbf{v}_2$ to being expressed in terms of relative ($\mathbf{v}_r$) and average ($\mathbf{V}$ say)

$$\mathbf{v}_r \equiv \mathbf{v}_1 - \mathbf{v}_2 \quad ; \quad \mathbf{V} \equiv \frac{m_1 \mathbf{v}_1 + m_2 \mathbf{v}_2}{m_1 + m_2} \quad . \tag{3.111}$$

Take f_1 and f_2 to be shifted Maxwellians in the overall C of M frame:

$$f_j = n_j \left(\frac{m_j}{2\pi T}\right)^{\frac{3}{2}} \exp\left[-\frac{m_j \left(\mathbf{v}_j - \mathbf{v}_{dj}\right)^2}{2T}\right] \quad (j = 1, 2) \tag{3.112}$$

where $m_1 \mathbf{v}_{d1} + m_2 \mathbf{v}_{d2} = 0$. Then

$$\begin{aligned} f_1 f_2 &= n_1 n_2 \left(\frac{m_1}{2\pi T}\right)^{\frac{3}{2}} \left(\frac{m_2}{2\pi T}\right)^{\frac{3}{2}} \exp\left[-\frac{m_1 v_1^2}{2T} - \frac{m_2 v_2^2}{2T}\right] \\ &\quad \times \left\{1 + \frac{\mathbf{v}_1 . m_1 \mathbf{v}_{d1}}{T} + \frac{\mathbf{v}_2 . m_2 \mathbf{v}_{d2}}{T}\right\} \end{aligned} \tag{3.113}$$

to first order in $\mathbf{v}_d$. Convert CM coordinates and find (after algebra)

$$\begin{aligned} f_1 f_2 &= n_1 n_2 \left(\frac{M}{2\pi T}\right)^{\frac{3}{2}} \left(\frac{m_r}{2\pi T}\right)^{\frac{3}{2}} \exp\left[-\frac{M V^2}{2T} - \frac{m_r v_r^2}{2T}\right] \\ &\quad \times \left\{1 + \frac{m_r}{T} \mathbf{v}_d . \mathbf{v}_r\right\} \end{aligned} \tag{3.114}$$

where $M = m_1 + m_2$. Note also that (it can be shown) $d^3 v_1 d^3 v_2 = d^3 v_r d^3 V$. Hence

$$\begin{aligned} -\frac{d\mathbf{p}}{dt} = \int\int \quad & \mathbf{v}_r m_r v_r 4\pi\ b_{90}^2\ \ln \Lambda n_1 n_2 \left(\frac{M}{2\pi T}\right)^{\frac{3}{2}} \left(\frac{m_r}{2\pi T}\right)^{\frac{3}{2}} \\ & \exp\left(-\frac{M V^2}{2T}\right) \exp\left(-\frac{m_r v_r^2}{2T}\right) \left\{1 + \frac{m_r}{T} \mathbf{v}_d . \mathbf{v}_r\right\} d^3 v_r d^3 V \end{aligned} \tag{3.115}$$

and since nothing except the exponential depends on V, that integral can be done:

$$-\frac{d\mathbf{p}}{dt} = \int \mathbf{v}_r\ m_r v_r 4\pi\ \ln \Lambda\ n_1 n_2 \left(\frac{m_r}{2\pi T}\right)^{\frac{3}{2}} \exp\left(\frac{-m_r v_r^2}{2\pi}\right) \left\{1 + \frac{m_r}{T} \mathbf{v}_d . \mathbf{v}_r\right\} d^3 v_r \tag{3.116}$$

This integral is of just the same type as for $e - i$ collisions, i.e.

$$\begin{aligned} -\frac{dp}{dt} &= v_d v_{rt} m_r\ 4\pi\ b_{90}^2(v_{rt})\ \ln \Lambda_t\ n_1 n_2 \int \frac{u_x^2}{u_3} \hat{f}_o(v_r) d^3 \mathbf{v}_r \\ &= v_d v_{rt} m_r 4\ \pi\ b_{90}^2(v_{rt})\ \ln\ \Lambda_t\ n_1 n_2\ \frac{2}{3\left(2\pi\right)^{\frac{2}{2}}} \end{aligned} \tag{3.117}$$

where $v_{rt} \equiv \sqrt{\frac{T}{m_r}}$, $b_{90}^2(v_{rt})$ is the ninety degree impact parameter evaluated at velocity v_{tr}, and $\hat{f}_o$ is the normalized Maxwellian.

$$-\frac{dp}{dt} = \frac{2}{3\,(2\pi)^{\frac{1}{2}}} \left(\frac{q_1 q_2}{4\pi\epsilon_0}\right)^2 \frac{4\pi}{m_r^2 v_{rt}^3} \;\ln \Lambda_t \; n_1 n_2 \; m_r v_d \tag{3.118}$$

This is the general result for momentum exchange rate between two Maxwellians drifting at small relative velocity v_d.

To get a collision frequency is a matter of deciding which species is stationary and so what the momentum density of the moving species is. Suppose we regard 2 as targets then momentum density is $n_1 m_1 v_d$ so

$$\overline{\nu}_{12} = \frac{1}{n_1 m_1 v_d} \frac{dp}{dt} = \frac{2}{3\,(2\pi)^{\frac{1}{2}}} n_2 \left(\frac{q_1 q_2}{4\pi\epsilon_0}\right)^2 \frac{4\pi}{m_r v_{rt}^3} \frac{\ln \Lambda_t}{m_1} \quad . \tag{3.119}$$

This expression works immediately for electron-ion collisions substituting $m_r \simeq m_e$, recovering previous.

For equal-mass ions $m_r = \frac{m_i^2}{m_i + m_i} = \frac{1}{2} m_i$ and $v_{rt} = \sqrt{\frac{T}{m_r}} = \sqrt{\frac{2T}{m_i}}$.
Substituting, we get

$$\overline{\nu}_{ii} = \frac{1}{3\pi^{\frac{1}{2}}} \; n_i \; \left(\frac{q_1 q_2}{4\pi\epsilon_0}\right)^2 \frac{4\pi}{m_i^{\frac{1}{2}} T_i^{\frac{3}{2}}} \;\ln \Lambda \tag{3.120}$$

that is, $\frac{1}{\sqrt{2}}$ times the $e - i$ expression but with ion parameters substituted. [Note, however, that we have considered the ion species to be different.]

3.4.4 $e \to e$

Electron-electron collisions are covered by the same formalism, so

$$\overline{\nu}_{ee} = \frac{1}{3\pi^{\frac{1}{2}}} \; n_e \left(\frac{e^2}{4\pi\epsilon_0}\right)^2 \frac{4\pi}{m_e^{\frac{1}{2}} T_e^{\frac{3}{2}}} \;\ln \Lambda \quad . \tag{3.121}$$

However, the physical case under discussion is not so obvious; since electrons are indistiguishable how do we define two different "drifting maxwellian" electron populations? A more specific discussion would be needed to make this rigorous.

Generally $\nu_{ee} \sim \nu_{ei}/\sqrt{2}$: electron-electron collision frequency $\sim$ electron-ion (for momentum loss).

3.4.5 Summary of Thermal Collision Frequencies

For *momentum loss*:

$$\overline{\nu}_{ei} \quad = \quad \frac{\sqrt{2}}{3\sqrt{\pi}} \; n_i \left(\frac{Ze^2}{4\pi\epsilon_0}\right)^2 \frac{4\pi}{m_e^{\frac{1}{2}} T_e^{\frac{3}{2}}} \;\ln \Lambda_e \quad . \tag{3.122}$$

$$\overline{\nu}_{ee} \simeq \frac{1}{\sqrt{2}}\,\overline{\nu}_{ei} \quad . \quad \text{(electron parameters)} \tag{3.123}$$

$$\overline{\nu}_{ie} = \frac{n_e m_e}{n_i m_i}\,\overline{\nu}_{ei} \quad . \tag{3.124}$$

$$\overline{\nu}_{ii\prime} = \frac{\sqrt{2}}{3\sqrt{\pi}}\,n_{i\prime}\left(\frac{q_i q_{i\prime}}{4\pi\epsilon_0}\right)^2 \frac{4\pi}{m_i^{\frac{1}{2}} T_i^{\frac{3}{2}}}\left(\frac{m_{i\prime}}{m_i + m_{i\prime}}\right)^{\frac{1}{2}} \ln\Lambda_i \tag{3.125}$$

Energy loss ${}^K\nu$ related to the above (${}^p\nu$) by

$$^K\nu = \frac{2m_i}{m_1 + m_2}\;{}^p\nu \quad . \tag{3.126}$$

Transverse 'diffusion' of momentum ${}^\perp\nu$, related to the above by:

$$^\perp\nu = \frac{2m_2}{m_1 + m_2}\;{}^p\nu \quad . \tag{3.127}$$

3.5 Applications of Collision Analysis

3.5.1 Energetic ('Runaway') Electrons

Consider an energetic ($\frac{1}{2}m_e v_1^2 >> T$) electron travelling through a plasma. It is slowed down (loses momentum) by collisions with electrons and ions (Z), with collision frequency:

$$^p\nu_{ee} = \nu_{ee} = n_e \frac{e^4}{(4\pi\epsilon_0)^2}\,\frac{8\pi}{m_e^2 v_1^3}\,\ln\Lambda \tag{3.128}$$

$$^p\nu_{ei} = \frac{1}{2}\,Z\,\nu_{ee} \tag{3.129}$$

Hence (in the absence of other forces)

$$\frac{d}{dt}(m_e v) = -\left({}^p\nu_{ee} + {}^p\nu_{ei}\right) m_p v \tag{3.130}$$

$$= -\left(1 + \frac{Z}{2}\right)\nu_{ee}\; m_e v \tag{3.131}$$

This is equivalent to saying that the electron experiences an effective 'Frictional' force

$$F_f = \frac{d}{dt}(m_e v) = -\left(1 + \frac{Z}{2}\right)\nu_{ee}\; m_e v \tag{3.132}$$

$$F_f = -\left(1 + \frac{Z}{2}\right) n_e\,\frac{e^4}{(4\pi\epsilon_0)^2}\,\frac{8\pi\;\ln\Lambda}{m_e v^2} \tag{3.133}$$

Notice

1. for $Z = 1$ slowing down is $\frac{2}{3}$ on electrons $\frac{1}{3}$ ions
2. F_f decreases with v increasing.

Suppose now there is an electric field, E. The electron experiences an accelerating Force.
Total force

$$F = \frac{d}{dt}(mv) = -eE + F_f = -eE - \left(1 + \frac{Z}{2}\right) n_e \frac{e^4}{(4\pi\epsilon_0)^2} \frac{8\pi \ln \Lambda}{m_e v^2} \tag{3.134}$$

Two Cases (When E is accelerating)

1. $|eE| < |F_f|$: Electron Slows Down
2. $|eE| > |F_f|$: Electron Speeds Up!

Once the electron energy exceeds a certain value its velocity increases continuously and the friction force becomes less and less effective. The electron is then said to ahve become a *'runaway'*.

Condition:

$$\frac{1}{2} m_e v^2 > \left(1 + \frac{Z}{2}\right) n_e \frac{e^4}{(4\pi\epsilon_0)^2} \frac{8\pi \ln \Lambda}{2eE} \tag{3.135}$$

3.5.2 Plasma Resistivity (DC)

Consider a bulk distribution of electrons in an electric field. They tend to be accelerated by E and decelerated by collisions.

In this case, considering the electrons as a whole, no loss of total electron momentum by $e - e$ collisions. Hence the friction force we need is just that due to $\overline{\nu}_{ei}$.

If the electrons have a mean drift velocity $v_d (<< v_{the})$ then

$$\frac{d}{dt}(m_e v_d) = -eE - \overline{\nu}_{ei} m_e v_d \tag{3.136}$$

Hence in steady state

$$v_d = \frac{-eE}{m_e \overline{\nu}_{ei}} \quad . \tag{3.137}$$

The current is then

$$j = -n_e e v_d = \frac{n_e e^2 E}{m_e \overline{\nu}_{ei}} \tag{3.138}$$

Now generally, for a conducting medium we define the conductivity, σ, or resistivity, η, by

$$j = \sigma E \quad ; \quad \eta j = E \qquad \left(\sigma = \frac{1}{\eta}\right) \tag{3.139}$$

Therefore, for a plasma,

$$\sigma = \frac{1}{\eta} = \frac{n_e e^2}{m_e \overline{\nu}_{ei}} \tag{3.140}$$

Substitute the value of $\overline{\nu}_{ei}$ and we get

$$\eta \simeq \frac{n_i Z^2}{n_e} \cdot \frac{e^2 m_e^{\frac{1}{2}}\ 8\pi\ \ln\Lambda}{(4\pi\epsilon_0)^2\ 3\sqrt{2\pi}\ T_e^{\frac{3}{2}}} \tag{3.141}$$

$$= \frac{Z e^2 m_e^{\frac{1}{2}}\ 8\pi\ \ln\Lambda}{(4\pi\epsilon_0)^2\ 3\sqrt{2\pi}\ T_e^{\frac{3}{2}}} \quad \text{(for a single ion species).} \tag{3.142}$$

Notice

1. Density cancels out because more electrons means (a) more carriers but (b) more collisions.
2. Main dependence is $\eta \propto T_e^{-3/2}$. High electron temperature implies low resistivity (high conductivity).
3. This expression is only approximate because the current tends to be carried by the more energetic electrons, which have smaller ν_{ei}; thus if we had done a proper average over $f(v_e)$ we expect a lower numerical value. Detailed calculations give

   $$\eta = 5.2 \times 10^{-5} \frac{\ln\Lambda}{(T_e/eV)^{\frac{3}{2}}}\ \Omega m \tag{3.143}$$

   for $Z = 1$ (vs. $\simeq 10^{-4}$ in our expression). This is 'Spitzer' resistivity. The detailed calculation value is roughly a factor of two smaller than our calculation, which is not a negligible correction!

3.5.3 Diffusion

For motion *parallel* to a magnetic field if we take a typical electron, with velocity $v_\parallel \simeq v_{te}$ it will travel a distance approximately

$$\ell_e = v_{te}/\overline{\nu}_{ei} \tag{3.144}$$

before being pitch-angle scattered enough to have its velocity randomised. [This is an order-of-magnitude calculation so we ignore $\overline{\nu}_{ee}$.] ℓ is the mean free path.

Roughly speaking, any electron does a random walk along the field with step size ℓ and step frequency $\overline{\nu}_{ei}$. Thus the diffusion coefficient of this process is

$$D_{e\parallel} \simeq \ell_e^2\ \overline{\nu}_{ei} \simeq \frac{v_{te}^2}{\overline{\nu}_{ei}} \quad . \tag{3.145}$$

Similarly for ions

$$D_{i\parallel} \simeq \ell_i^2 \overline{\nu}_{ii} \simeq \frac{v_{ti}^2}{\overline{\nu}_{ii}} \tag{3.146}$$

Notice

$$\overline{\nu}_{ii}/\overline{\nu}_{ei} \simeq \left(\frac{m_e}{m_i}\right)^{\frac{1}{2}} \simeq \frac{v_{ti}}{v_{te}} \qquad (\text{if } T_e \simeq T_i) \tag{3.147}$$

Hence $\ell_e \simeq \ell_i$

Mean free paths for electrons and ions are $\sim$ same.

The diffusion coefficients are in the ratio

$$\frac{D_i}{D_e} \simeq \left(\frac{m_e}{m_i}\right)^{\frac{1}{2}} \quad : \quad \text{Ions diffuse slower in parallel direction.} \tag{3.148}$$

Diffusion Perpendicular to Mag. Field is different

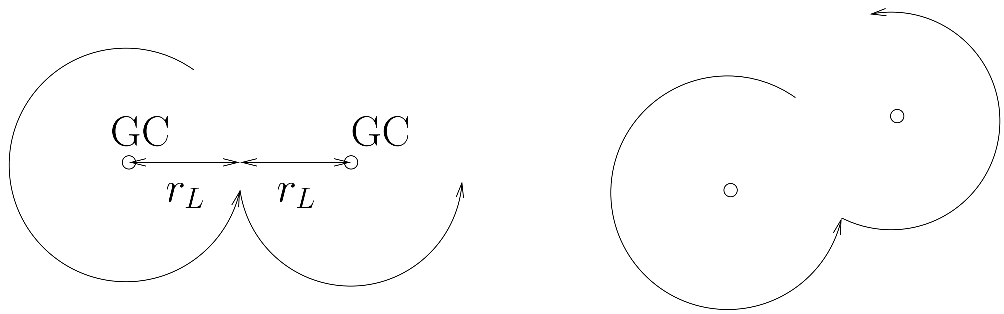

Figure 3.7: Cross-field diffusion by collisions causing a jump in the gyrocenter (GC) position.

Roughly speaking, if electron direction is changed by $\sim 90°$ the Guiding Centre moves by a distance $\sim r_L$. Hence we may think of this as a random walk with step size $\sim r_L$ and frequency $\overline{\nu}_{ei}$. Hence

$$D_{e\perp} \simeq r_{Le}^2 \overline{\nu}_{ei} \simeq \frac{v_{te}^2}{\Omega_e^2} \overline{\nu}_{ei} \tag{3.149}$$

Ion transport is similar but requires a discussion of the effects of *like* and *unlike* collisions.

Particle transport occurs only via *unlike* collisions. To show this we consider in more detail the change in guiding center position at a collision. Recall $m\dot{\mathbf{v}} = q\mathbf{v} \wedge \mathbf{B}$ which leads to

$$\mathbf{v}_\perp = \frac{q}{m} \mathbf{r}_L \wedge \mathbf{B} \qquad (\text{perp. velocity only}). \tag{3.150}$$

This gives

$$\mathbf{r}_L = \frac{\mathbf{B} \wedge m\mathbf{v}_\perp}{qB^2} \tag{3.151}$$

At a collision the particle position does not change (instantaneously) but the guiding center position ($\mathbf{r}_0$) does.

$$\mathbf{r}_0' + \mathbf{r}_L' = \mathbf{r}_0 + \mathbf{r}_L \Rightarrow \Delta\mathbf{r}_0 \equiv \mathbf{r}_0' - \mathbf{r}_0 = -(\mathbf{r}_L' - \mathbf{r}_L) \tag{3.152}$$

Change in $\mathbf{r}_L$ is due to the momentum change caused by the collision:

$$\mathbf{r}'_L - \mathbf{r}_L = \frac{\mathbf{B}}{qB^2} \wedge m(\mathbf{v}' - \mathbf{v}) \equiv \frac{\mathbf{B}}{qB^2} \wedge \mathbf{\Delta}(m\mathbf{v}) \tag{3.153}$$

So

$$\mathbf{\Delta r}_0 = -\frac{\mathbf{B}}{qB^2} \wedge \mathbf{\Delta}(m\mathbf{v}). \tag{3.154}$$

The total momentum conservation means that $\mathbf{\Delta}(m\mathbf{v})$ for the two particles colliding is equal and opposite. Hence, from our equation, for *like* particles, $\mathbf{\Delta r}_0$ is equal and opposite. The mean position of guiding centers of two colliding like particles $(\mathbf{r}_{01} + \mathbf{r}_{02})/2$ does not change.

No net cross field particle (guiding center) shift.

Unlike collisions (between particles of different charge q) *do* produce net transport of particles of either type. And indeed may move $\mathbf{r}_{01}$ and $\mathbf{r}_{02}$ in same direction if they have opposite charge.

$$D_{i\perp} \simeq r_{Li}^2 \, \overline{^p\nu_{ie}} \simeq \frac{v_{ti}^2}{\Omega_i^2} \, \overline{^p\nu_{ie}} \tag{3.155}$$

Notice that $r_{Li}^2/r_{Le}^2 \simeq m_i/m_e$; $\overline{^p\nu_{ie}}/\overline{\nu}_{ei} \simeq \frac{m_e}{m_i}$

So $D_{i\perp}/D_{e\perp} \simeq 1$ (for equal temperatures). Collisional diffusion rates of *particles* are same for ions and electrons.

However *energy* transport is different because it *can* occur by like-like collisins.

Thermal Diffusivity:

$$\chi_e \sim r_{Le}^2 \, (\overline{\nu}_{ei} + \overline{\nu}_{ee}) \sim r_{Le}^2 \, \overline{\nu}_{ei} \quad (\overline{\nu}_{ei} \sim \overline{\nu}_{ee}) \tag{3.156}$$

$$\chi_i \sim r_{Li}^2 \, (\overline{^p\nu_{ie}} + \overline{\nu}_{ii}) \simeq r_{Li}^2 \, \overline{\nu}_{ii} \quad (\overline{\nu}_{ii} >> \overline{\nu}_{ie}) \tag{3.157}$$

$$\chi_i/\chi_e \sim \frac{r_{Li}^2}{r_{Le}^2} \, \frac{\overline{\nu}_{ii}}{\overline{\nu}_{ei}} \simeq \frac{m_i}{m_e} \, \frac{m_e^{\frac{1}{2}}}{m_i^{\frac{1}{2}}} = \left(\frac{m_i}{m_e}\right)^{\frac{1}{2}} \quad \text{(equal T)} \tag{3.158}$$

Collisional *Thermal* transport by *Ions* is *greater* than by *electrons* [factor $\sim (m_i/m_e)^{\frac{1}{2}}$].

3.5.4 Energy Equilibration

If $T_e \neq T_i$ then there is an exchange of enegy between electrons and ions tending to make $T_e = T_i$. As we saw earlier

$$^K\nu_{ei} = \frac{2m_e}{m_i} \, {}^p\nu_{ei} = \frac{m_e}{m_i} \, {}^\perp\nu_{ei} \tag{3.159}$$

So applying this to averages.

$$\overline{^K\nu_{ei}} \simeq \frac{2m_e}{m_i} \, \overline{\nu}_{ei} \qquad (\simeq \overline{\nu}_{ie}) \tag{3.160}$$

Thermal energy exchange occurs $\sim m_e/m_i$ *slower* than momentum exchange. (Allows $T_e \neq T_i$). So

$$\frac{dT_e}{dt} = -\frac{dT_i}{dt} = -\overline{{}^K\nu_{ei}}\,(T_e - T_i) \tag{3.161}$$

From this one can obtain the heat exchange rate (per unit volume), H_{ei}, say:

$$\begin{aligned} H_{ei} &= -\frac{d}{dt}\left(\frac{3}{2}n_e T_e\right) = \frac{d}{dt}\left(\frac{3}{2}n_i T_i\right) && (3.162)\\ &= -\frac{3}{4}n\frac{d}{dt}(T_e - T_i) = \frac{3}{2}n\overline{{}^K\nu_{ei}}\,(T_e - T_i) && (3.163) \end{aligned}$$

Important point:

$$\overline{{}^K\nu_{ei}} \simeq \frac{m_e}{m_i}Z\nu_{ee} \simeq \frac{1}{Z^2}\left(\frac{M_e}{m_i}\right)^{\frac{1}{2}}\overline{\nu}_{ii}. \tag{3.164}$$

'Electrons and Ions equilibrate among themselves much faster than with each other'.

3.6 Some Orders of Magnitude

1. $\ln\Lambda$ is very slowly varying. Typically has value $\sim$ 12 to 16 for laboratory plasmas.

2. $\overline{\nu}_{ei} \approx 6 \times 10^{-11}(n_i/m^3)\,/\,(T_e/eV)^{\frac{3}{2}}$ $\quad(\ln\Lambda = 15, Z = 1)$.
   e.g. $= 2 \times 10^5 s^{-1}$ (when $n = 10^{20}m^{-3}$ and $T_e = 1keV$.) For phenomena which happen much faster than this, i.e. $\tau \ll 1/\nu_{ei} \sim 5\mu s$, collisions can be ignored.
   Examples: Electromagnetic Waves with high frequency.

3. *Resistivity.* Because most of the energy of a current carrying plasma is in the B field not the K.E. of electrons. Resistive decay of current can be much slower than $\overline{\nu}_{ei}$. E.g. Coaxial Plasma: (Unit length)
   Inductance $\quad L = \mu_o \ln\frac{b}{a}$
   Resistance $\quad R = \eta\ 1/\pi a^2$
   L/R decay time

$$\begin{aligned} \tau_R &\sim \frac{\mu_o \pi a^2}{\eta}\ \ln\frac{b}{a} \simeq \frac{n_e e^2}{m_e\overline{\nu}_{ei}}\mu_o\ \pi a^2 \ln\frac{b}{a}\\ &\sim \frac{n_e e^2}{m_e\epsilon_0}\ \frac{a^2}{c^2}\ \frac{1}{\overline{\nu}_{ei}} = \frac{\omega_p^2 a^2}{c^2}\cdot\frac{1}{\overline{\nu}_{ei}} >> \frac{1}{\overline{\nu}_{ei}}\ . && (3.165) \end{aligned}$$

Comparison 1 keV temperature plasma has $\sim$ same (conductivity/) resistivity as a slab of *copper* ($\sim 2 \times 10^{-8}\Omega m$).

Ohmic Heating Because $\eta \propto T_e^{-3/2}$, if we try to heat a plasma Ohmically, i.e. by simply passing a current through it, this works well at low temperatures but its effectiveness falls off rapidly at high temperature.

Result for most Fusion schemes it looks as if Ohmic heating does not quite yet get us to the required ignition temperature. We need auxilliary heating, e.g. Neutral Beams. (These slow down by collisions.)

Chapter 4

Fluid Description of Plasma

The single particle approach gets to be horribly complicated, as we have seen.

Basically we need a more statistical approach because we can't follow each particle separately. If the details of the distribution function in velocity space are important we have to stay with the Boltzmann equation. It is a kind of particle conservation equation.

4.1 Particle Conservation (In 3-d Space)

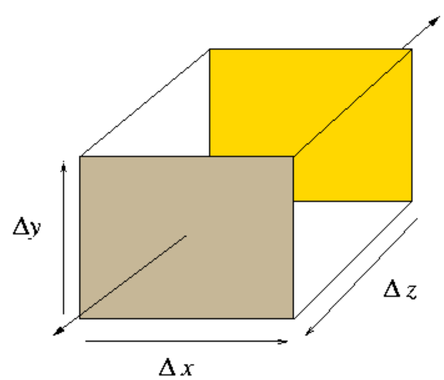

Figure 4.1: Elementary volume for particle conservation

Number of particles in box $\Delta x \Delta y \Delta z$ is the volume, $\Delta V = \Delta x \Delta y \Delta z$, times the density n. Rate of change of number is is equal to the number flowing across the boundary per unit time, the flux. (In absence of sources.)

$$-\frac{\partial}{\partial t}[\Delta x \Delta y \Delta z \; n] = \text{ Flow Out across boundary.} \tag{4.1}$$

Take particle velocity to be $\mathbf{v}(\mathbf{r})$ [no random velocity, only flow] and origin at the center of the box refer to flux density as $n\mathbf{v} = \mathbf{J}$.

$$\text{Flow Out } = [J_z(0,0,\Delta z/2) - J_z(0,0,-\Delta z/2)] \; \Delta x \Delta y \; + \; x \; + \; y \;\; . \tag{4.2}$$

Expand as Taylor series

$$J_z(0,0,\eta) = J_z(0) + \frac{\partial}{\partial z} J_z \; . \; \eta \tag{4.3}$$

So,

$$\begin{aligned} \text{flow out} \;\; &\simeq \;\; \frac{\partial}{\partial z}(nv_z)\Delta z \Delta x \Delta y \; + \; x \; + y \\ &= \;\; \Delta V \; \nabla \; . \; (n\mathbf{v}). \end{aligned} \tag{4.4}$$

Hence *Particle Conservation*

$$-\frac{\partial}{\partial t} n = \nabla . (n\mathbf{v}) \tag{4.5}$$

Notice we have essential proved an elementary form of Gauss's theorem

$$\int_v \nabla . \mathbf{A} d^3\mathbf{r} \; = \; \int_{\partial\gamma} \mathbf{A} . \mathbf{dS}. \tag{4.6}$$

The expression: *'Fluid Description'* refers to any simplified plasma treatment which does *not* keep track of v-dependence of f detail.

1. Fluid Descriptions are essentially 3-d ($\mathbf{r}$).
2. Deal with quantities averaged over velocity space (e.g. density, mean velocity, ...).
3. Omit some important physical processes (but describe others).
4. Provide tractable approaches to many problems.
5. Will occupy most of the rest of my lectures.

Fluid Equations can be derived mathematically by taking moments[1] of the Boltzmann Equation.

$$0^{th} \text{ moment} \quad \int \quad d^3\mathbf{v} \tag{4.7}$$

$$1st \text{ moment} \quad \int \quad \mathbf{v} d^3 v \tag{4.8}$$

$$2nd \text{ moment} \quad \int \quad \mathbf{vv} d^3 v \tag{4.9}$$

These lead, respectively, to (0) Particle (1) Momentum (2) Energy conservation equations. We shall adopt a more direct 'physical' approach.

[1] They are therefore sometimes called 'Moment Equations.'

4.2 Fluid Motion

The motion of a fluid is described by a vector velocity field $\mathbf{v}(\mathbf{r})$ (which is the mean velocity of all the individual particles which make up the fluid at $\mathbf{r}$). Also the particle density $n(r)$ is required. We are here discussing the motion of fluid of a *single type* of particle of mass/charge, m/q so the charge and mass density are qn and mn respectively.

The particle conservation equation we already know. It is also sometimes called the 'Continuity Equation'

$$\frac{\partial}{\partial t}\, n + \nabla.(n\mathbf{v}) = 0 \tag{4.10}$$

It is also possible to expand the $\nabla.$ to get:

$$\frac{\partial}{\partial t} n + (\mathbf{v}.\nabla)n + n\nabla.\mathbf{v} = 0 \tag{4.11}$$

The significance, here, is that the first two terms are the 'convective derivative" of n

$$\frac{D}{Dt} \equiv \frac{d}{dt} \equiv \frac{\partial}{\partial t} + \mathbf{v}.\nabla \tag{4.12}$$

so the continuity equation can be written

$$\frac{D}{Dt} n = -n\nabla.\mathbf{v} \tag{4.13}$$

4.2.1 Lagrangian & Eulerian Viewpoints

There are essentially 2 views.

1. Lagrangian. Sit on a fluid element and move with it as fluid moves.

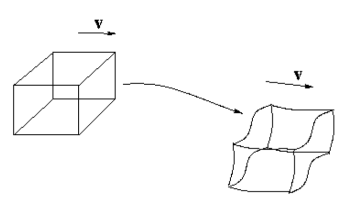

Figure 4.2: Lagrangean Viewpoint

2. Eulerian. Sit at a fixed point in space and watch fluid move through your volume element: "identity" of fluid in volume continually changing

   $\frac{\partial}{\partial t}$ means rate of change at *fixed* point (Euler).

   $\frac{D}{Dt} \equiv \frac{d}{dt} \equiv \frac{\partial}{\partial t} + \mathbf{v}.\nabla$ means rate of change at *moving* point (Lagrange).

   $\mathbf{v}.\nabla = \frac{dx}{\partial t}\frac{\partial}{\partial x} + \frac{dy}{\partial t}\frac{\partial}{\partial y} + \frac{dz}{\partial t}\frac{\partial}{\partial z}$: change due to motion.

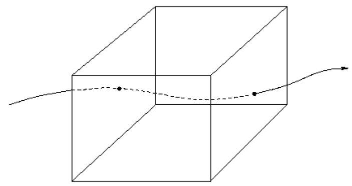

Figure 4.3: Eulerian Viewpoint

Our derivation of continuity was Eulerian. From the Lagrangian view

$$\frac{D}{Dt}\,n = \frac{d}{dt}\,\frac{\Delta N}{\Delta V} = -\frac{\Delta N}{\Delta V^2}\,\frac{d}{dt}\,\Delta V = -n\,\frac{1}{\Delta V}\,\frac{d\Delta V}{dt} \tag{4.14}$$

since total number of particles in volume element (ΔN) is constant (we are moving with them). ($\Delta V = \Delta x \Delta y \Delta z$.)

$$\text{Now } \frac{d}{dt}\Delta V = \frac{d\Delta x}{dt}\,\Delta y\Delta z + \frac{d\Delta y}{dt}\Delta z\Delta x + \frac{d\Delta z}{dt}\Delta y\Delta x \tag{4.15}$$

$$= \Delta V\left[\frac{1}{\Delta x}\,\frac{d\Delta x}{dt} + \frac{1}{\Delta y}\,\frac{d\Delta y}{dt} + \frac{1}{\Delta x}\,\frac{d\Delta z}{dt}\right] \tag{4.16}$$

$$\text{But } \frac{d\,(\Delta x)}{dt} = v_x\,(\Delta x/2) - v_x\,(-\Delta x/2) \tag{4.17}$$

$$\simeq \Delta x\,\frac{\partial v_x}{\partial x} \quad \text{etc.} \quad \ldots y \quad \ldots z \tag{4.18}$$

Hence

$$\frac{d}{dt}\Delta V = \Delta V\left[\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right] = \Delta V \nabla.\mathbf{v} \tag{4.19}$$

and so

$$\frac{D}{Dt}\,n = -n\nabla.\mathbf{v} \tag{4.20}$$

Lagrangian Continuity. Naturally, this is the same equation as Eulerian when one puts $\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{v}.\nabla$.

The quantity $-\nabla.\mathbf{v}$ is the rate of (Volume) compression of element.

4.2.2 Momentum (Conservation) Equation

Each of the particles is acted on by the Lorentz force $q[\mathbf{E} + \mathbf{u}_i \wedge \mathbf{B}]$ ($\mathbf{u}_i$ is individual particle's velocity).

Hence total force on the fluid element due to E-M fields is

$$\sum_i \left(q\left[\mathbf{E} + \mathbf{u}_i \wedge \mathbf{B}\right]\right) = \Delta N\; q\left(\mathbf{E} + \mathbf{v} \wedge \mathbf{B}\right) \tag{4.21}$$

(Using mean: $\mathbf{v} = \sum_i \mathbf{u}/\Delta N$.)

E-M Force density (per unit volume) is:

$$\mathbf{F}_{EM} = nq(\mathbf{E} + \mathbf{v} \wedge \mathbf{B}). \tag{4.22}$$

The total momentum of the element is

$$\sum_i m\mathbf{u}_i = m \ \Delta N \ \mathbf{v} = \Delta V \ mn\mathbf{v} \tag{4.23}$$

so *Momentum Density* is $mn\mathbf{v}$.

If no other forces are acting then clearly the equation of motion requires us to set the time derivative of $mn\mathbf{v}$ equal to $\mathbf{F}_{EM}$. Because we want to retain the identity of the particles under consideration we want D/Dt i.e. the convective derivative (Lagrangian picture).

In general there are additional forces acting.
(1) Pressure (2) Collisional Friction.

4.2.3 Pressure Force

In a gas $p(= nT)$ is the force per unit area arising from thermal motions. The surrounding fluid exerts this force on the element:

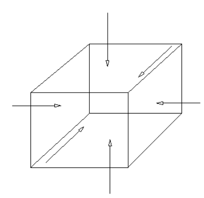

Figure 4.4: Pressure forces on opposite faces of element.

Net force in x direction is

$$\begin{aligned} & - \quad p\,(\Delta x/2)\,\Delta y \Delta z + p\,(-\Delta x/2)\,\Delta y \Delta z & (4.24) \\ & \simeq \quad -\Delta x \Delta y \Delta z \ \frac{\partial p}{\partial x} = -\Delta V \ \frac{\partial p}{\partial x} = -\Delta V \ (\nabla p)_x & (4.25) \end{aligned}$$

So (isotropic) pressure *force density* (/unit vol)

$$\mathbf{F}_p = -\nabla p \tag{4.26}$$

How does this arise in our picture above?
Answer: Exchange of momentum by particle thermal motion across the element boundary.

Although in Lagrangian picture we move with the element (as defined by mean velocity $\mathbf{v}$) individual particles also have thermal velocity so that the additional velocity they have is

$$\mathbf{w}_i = \mathbf{u}_i - \mathbf{v} \qquad \text{'peculiar' velocity} \tag{4.27}$$

Because of this, some cross the element boundary and exchange momentum with outside. (Even though there is no net change of number of particles in element.) Rate of exchange of momentum due to particles with peculiar velocity $\mathbf{w}, d^3\mathbf{w}$ across a surface element $\mathbf{ds}$ is

$$\underbrace{f(\mathbf{w})m\mathbf{w}\ d^3\mathbf{w}}_{\text{mom}^m \text{ density at } \mathbf{w}} \quad \times \quad \underbrace{\mathbf{w}\ .\ \mathbf{ds}}_{\text{flow rate across } \mathbf{ds}} \tag{4.28}$$

Integrate over distrib function to obtain the total momentum exchange rate:

$$\mathbf{ds}.\int m\mathbf{w}\mathbf{w}f(\mathbf{w})d^3\mathbf{w} \tag{4.29}$$

The thing in the integral is a tensor. Write

$$\mathsf{p} \ = \int m\mathbf{w}\mathbf{w}f(\mathbf{w})d^3\mathbf{w} \qquad \text{(Pressure Tensor)} \tag{4.30}$$

Then momentum exchange rate is

$$\mathsf{p}\ .\ \mathbf{ds} \tag{4.31}$$

Actually, if $f(\mathbf{w})$ is isotropic (e.g. Maxwellian) then

$$p_{xy} \ = \int m\ w_x\ w_y\ f(\mathbf{w})d^3\mathbf{w} = 0 \quad \text{etc.} \tag{4.32}$$

$$\text{and } p_{xx} \ = \int m w_x^2\ f(w)d^3\mathbf{w} \equiv nT (= p_{yy} = p_{zz} = \text{`}p\text{'}) \tag{4.33}$$

So then the exchange rate is $p\mathbf{ds}$. (Scalar Pressure).

Integrate $\mathbf{ds}$ over the whole ΔV then x component of mom^m exchange rate is

$$p\left(\frac{\Delta x}{2}\right)\Delta y \Delta z - -p\left(\frac{-\Delta x}{2}\right)\Delta y \Delta z = \Delta V\left(\nabla p\right)_x \tag{4.34}$$

and so

Total momentum loss rate due to exchange across the boundary per unit volume is

$$\nabla p \qquad (= -\mathbf{F}_p) \tag{4.35}$$

In terms of the momentum equation, either we put ∇p on the momentum derivative side or $\mathbf{F}_p$ on force side. The result is the same.

Ignoring Collisions, Momentum Equation is

$$\frac{D}{Dt}\left(mn\Delta V\mathbf{v}\right) = \left[\mathbf{F}_{EM} + \mathbf{F}_p\right]\Delta V \tag{4.36}$$

Recall that $n\Delta V = \Delta N$; $\frac{D}{Dt}(\Delta N) = 0$; so

$$L.H.S. = mn\Delta V\ \frac{D\mathbf{v}}{dt} \quad . \tag{4.37}$$

Thus, substituting for $\mathbf{F}'s$:

Momentum Equation.

$$mn\frac{D\mathbf{v}}{Dt} = mn\left(\frac{\partial\mathbf{v}}{\partial t} + \mathbf{v}.\nabla\mathbf{v}\right) = qn\left(\mathbf{E} + \mathbf{v}\wedge\mathbf{B}\right) - \nabla p \tag{4.38}$$

4.2.4 Momentum Equation: Eulerian Viewpoint

Fixed element in space. Plasma flows through it.

1. E.M. force on element (per unit vol.)

$$\mathbf{F}_{EM} = nq(\mathbf{E} + \mathbf{v} \wedge \mathbf{B}) \qquad \text{as before.} \tag{4.39}$$

2. Momentum flux across boundary (per unit vol)

$$= \nabla . \int m(\mathbf{v}+\mathbf{w})(\mathbf{v}+\mathbf{w})\ f(\mathbf{w})\ d^3\mathbf{w} \tag{4.40}$$

$$= \nabla . \{ \int m(\mathbf{vv} + \underbrace{\mathbf{vw}+\mathbf{wv}}_{\text{integrates to } 0} + \mathbf{ww})\ f(\mathbf{w})\ d^3\mathbf{w} \} \tag{4.41}$$

$$= \nabla . \{ mn\mathbf{vv} + \mathsf{p} \} \tag{4.42}$$

$$= mn(\mathbf{v}.\nabla)v + m\mathbf{v}\left[\nabla.(n\mathbf{v})\right] + \nabla p \tag{4.43}$$

(Take isotropic p.)

3. Rate of change of momentum within element (per unit vol)

$$= \frac{\partial}{\partial t}(mn\mathbf{v}) \tag{4.44}$$

Hence, total momentum balance:

$$\frac{\partial}{\partial t}(mn\mathbf{v}) + mn(\mathbf{v}.\nabla)\mathbf{v} + m\mathbf{v}\left[\nabla.(n\mathbf{v})\right] + \nabla p = \mathbf{F}_{EM} \tag{4.45}$$

Use the continuity equation:

$$\frac{\partial n}{\partial t} + \nabla.(n\mathbf{v}) = 0 \quad , \tag{4.46}$$

to cancel the third term and part of the 1st:

$$\frac{\partial}{\partial t}(mn\mathbf{v}) + m\mathbf{v}\left(\nabla.\left(n\mathbf{v}\right)\right) = m\mathbf{v}\{\frac{\partial n}{\partial t} + \nabla.\left(n\mathbf{v}\right)\} + mn\frac{\partial \mathbf{v}}{\partial t} = mn\frac{\partial \mathbf{v}}{\partial t} \tag{4.47}$$

Then take ∇p to RHS to get final form:
Momentum Equation:

$$mn\left[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}.\nabla)\,\mathbf{v}\right] = nq\left(\mathbf{E} + \mathbf{v}\wedge\mathbf{B}\right) - \nabla p\ . \tag{4.48}$$

As before, via Lagrangian formulation. (Collisions have been ignored.)

4.2.5 Effect of Collisions

First notice that *like* particle collisions *do not* change the total momentum (which is averaged over all particles of that species).

Collisions between *unlike* particles *do* exchange momentum between the species. Therefore once we realize that any quasi-neutral plasma consists of at least two different species (electrons and ions) and hence two different interpenetrating fluids we may need to account for another momentum loss (gain) term.

The rate of momentum density loss by species 1 colliding with species 2 is:

$$\nu_{12} n_1 m_1 (\mathbf{v}_1 - \mathbf{v}_2) \tag{4.49}$$

Hence we can immediately generalize the *momentum equation* to

$$m_1 n_1 \left[\frac{\partial \mathbf{v}_1}{\partial t} + (\mathbf{v}_1 . \nabla) \mathbf{v}_1 \right] = n_1 q_1 (\mathbf{E} + \mathbf{v}_1 \wedge \mathbf{B}) - \nabla p_1 - \nu_{12} n_1 m_1 (\mathbf{v}_1 - \mathbf{v}_2) \tag{4.50}$$

With similar equation for species 2.

4.3 The Key Question for Momentum Equation:

What do we take for p?

Basically $p = nT$ is determined by energy balance, which will tell how T varies. We could write an energy equation in the same way as momentum. However, this would then contain a term for heat flux, which would be unknown. In general, the k^{th} moment equation contains a term which is a $(k+1)^{th}$ moment.

Continuity, 0^{th} equation contains $\mathbf{v}$ determined by
Momentum, 1^{st} equation contains p determined by
Energy, 2^{nd} equation contains Q determined by ...

In order to get a sensible result we have to *truncate* this hierarchy. Do this by some sort of assumption about the heat flux. This will lead to an
Equation of State:

$$p n^{-\gamma} = \text{const.} \tag{4.51}$$

The value of γ to be taken depends on the heat flux assumption and on the isotropy (or otherwise) of the energy distribution.

Examples

1. Isothermal: $T = \text{const.}$: $\gamma = 1$.
2. Adiabatic/Isotropic: 3 degrees of freedom $\gamma = \frac{5}{3}$.
3. Adiabatic/1 degree of freedom $\gamma = 3$.

4. Adiabatic/2 degrees of freedom $\gamma = 2$.

In general, $n(\ell/2)\delta T = -p(\delta V/V)$ (Adiabatic ℓ degrees)

$$\frac{\ell}{2}\frac{\delta T}{T} = \frac{-\delta V}{V} = +\frac{\delta n}{n} \tag{4.52}$$

So

$$\frac{\delta p}{p} = \frac{\delta n}{n} + \frac{\delta T}{T} = \left(1 + \frac{2}{\ell}\right)\frac{\delta n}{n}, \tag{4.53}$$

i.e.

$$pn^{-\left(1+\frac{2}{\ell}\right)} = \text{const.} \tag{4.54}$$

In a normal gas, which 'holds together' by collisions, energy is rapidly shared between 3 space-degrees of freedom. Plasmas are often rather collisionless so compression in 1 dimension often stays confined to 1-degree of freedom. Sometimes heat transport is so rapid that the isothermal approach is valid. It depends on the exact situation; so let's leave γ undefined for now.

4.4 Summary of Two-Fluid Equations

Species j

Plasma Response

1. Continuity:

$$\frac{\partial n_j}{\partial t} + \nabla.(n_j \mathbf{v}_j) = 0 \tag{4.55}$$

2. Momentum:

$$m_j n_j \left[\frac{\partial \mathbf{v}_j}{\partial t} + (\mathbf{v}_j.\nabla)\,\mathbf{v}_j\right] = n_j q_j\,(\mathbf{E} + \mathbf{v}_j \wedge \mathbf{B}) - \nabla p_j - \bar{\nu}_{jk} n_j m_j\,(\mathbf{v}_j - \mathbf{v}_k) \tag{4.56}$$

3. Energy/Equation of State:

$$p_j n_j^{-\gamma} = \text{const.}. \tag{4.57}$$

(j = electrons, ions).

Maxwell's Equations

$$\nabla.\mathbf{B} = 0 \qquad \nabla.\mathbf{E} = \rho/\epsilon_o \tag{4.58}$$

$$\nabla \wedge \mathbf{B} = \mu_o \mathbf{j} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} \qquad \nabla \wedge \mathbf{E} = \frac{-\partial \mathbf{B}}{\partial t} \tag{4.59}$$

With

$$\rho = q_e n_e + q_i n_i = e\left(-n_e + Z n_i\right) \tag{4.60}$$

$$\mathbf{j} = q_e n_e \mathbf{v}_e + q_i n_i \mathbf{v}_i = e\left(-n_e \mathbf{v}_e + Z n_i \mathbf{v}_i\right) \tag{4.61}$$

$$= -e n_e \left(\mathbf{v}_e - \mathbf{v}_i r\right) \qquad \text{(Quasineutral)} \tag{4.62}$$

Accounting

Unknowns		Equations	
n_e, n_i	2	Continuity e, i	2
$\mathbf{v}_e, \mathbf{v}_i$	6	Momentum e, i	6
p_e, p_i	2	State e, i	2
$\mathbf{E}, \mathbf{B}$	6	Maxwell	8
	16		18

but 2 of Maxwell ($\nabla.$ equs) are redundant because can be deduced from others: e.g.

$$\nabla.\left(\nabla \wedge \mathbf{E}\right) = 0 = -\frac{\partial}{\partial t}\left(\nabla.\mathbf{B}\right) \tag{4.63}$$

$$\text{and } \nabla.\left(\nabla \wedge \mathbf{B}\right) = 0 = \mu_o \nabla.\mathbf{j} + \frac{1}{c^2}\frac{\partial}{\partial t}\left(\nabla.\mathbf{E}\right) = \frac{1}{c^2}\frac{\partial}{\partial t}\left(\frac{-\rho}{\epsilon_o} + \nabla.\mathbf{E}\right) \tag{4.64}$$

So 16 equs for 16 unknowns.

Equations still very difficult and complicated mostly because it is *Nonlinear*

In some cases can get a tractable problem by 'linearizing'. That means, take some known equilibrium solution and suppose the deviation (perturbation) from it is small so we can retain only the 1st linear terms and not the others.

4.5 Two-Fluid Equilibrium: Diamagnetic Current

Slab: $\frac{\partial}{\partial x} \neq 0 \quad \frac{\partial}{\partial y}, \frac{\partial}{\partial z} = 0.$

Straight B-field: $\mathbf{B} = \mathbf{B}\hat{\mathbf{z}}.$

Equilibrium: $\frac{\partial}{\partial t} = 0 \qquad (E = -\nabla\phi)$

Collisionless: $\nu \to 0.$

Momentum Equation(s):

$$m_j n_j (\mathbf{v}_j.\nabla)\mathbf{v}_j = n_j q_j (\mathbf{E} + \mathbf{v}_j \wedge \mathbf{B}) - \nabla p_j \tag{4.65}$$

Drop j suffix for now. Then take x, y components:

$$mn\, v_x \frac{d}{dx} v_x = nq(E_x + v_y B) - \frac{dp}{dx} \tag{4.66}$$

$$mn\, v_x \frac{d}{dx} v_y = nq(0 - v_x B) \tag{4.67}$$

Eq 4.67 is satisfied by taking $v_x = 0$. Then 4.66 →

$$nq(E_x + v_y B) - \frac{dp}{dx} = 0. \tag{4.68}$$

i.e.

$$v_y = \frac{-E_x}{B} + \frac{1}{nqB}\frac{dp}{dx} \tag{4.69}$$

or, in vector form:

$$\mathbf{v} = \underbrace{\frac{\mathbf{E}\wedge\mathbf{B}}{B^2}}_{\mathbf{E}\wedge\mathbf{B}\text{ drift}} - \underbrace{\frac{\nabla p}{nq}\wedge\frac{\mathbf{B}}{B^2}}_{\text{Diamagnetic Drift}} \tag{4.70}$$

Notice:

- In magnetic field (⊥) fluid velocity is determined by component of momentum equation *orthogonal* to it (and to **B**).
- Additional drift (diamagnetic) arises in standard $\mathbf{F}\wedge\mathbf{B}$ form from pressure force.
- Diagmagnetic drift is opposite for opposite signs of charge (electrons vs. ions).

Now restore species distinctions and consider electrons plus single ion species i. Quasineutrality says $n_i q_i = -n_e q_e$. Hence adding solutions

$$n_e q_e \mathbf{v}_e + n_i q_i \mathbf{v}_i = \frac{\mathbf{E}\wedge\mathbf{B}}{B^2}\underbrace{(n_i q_i + n_e q_e)}_{=0} - \nabla(p_e + p_i)\wedge\frac{\mathbf{B}}{B^2} \tag{4.71}$$

Hence current density:

$$\mathbf{j} = -\nabla(p_e + p_i)\wedge\frac{\mathbf{B}}{B^2} \tag{4.72}$$

This is the diamagnetic current. The electric field, **E**, disappears because of quasineutrality. (General case $\sum_j q_j n_j v_j = -\nabla(\sum p_j)\wedge\mathbf{B}/B^2$).

4.6 Reduction of Fluid Approach to the Single Fluid Equations

So far we have been using fluid equations which apply to electrons and ions *separately*. These are called *'Two Fluid'* equations because we always have to keep track of both fluids separately.

A further simplification is possible and useful sometimes by combining the electron and ion equations together to obtain equations governing the plasma viewed as a *'Single Fluid'*.

Recall 2-fluid equations:

$$\text{Continuity } (\mathrm{C_j}) \qquad \frac{\partial n_j}{\partial t} + \nabla.(n_j \mathbf{v}_j) = 0. \tag{4.73}$$

$$\text{Momentum } (\mathrm{M_j}) \qquad m_j n_j \left(\frac{\partial}{\partial t} + \mathbf{v}_j.\nabla \right) \mathbf{v}_j = n_j q_j \left(\mathbf{E} + \mathbf{v}_j \wedge \mathbf{B} \right) - \nabla p_j + \mathbf{F}_{jk} \tag{4.74}$$

(where we just write $\mathbf{F}_{jk} = -\overline{\mathbf{v}}_{jk} n_j m_j \left(\mathbf{v}_j - \mathbf{v}_k \right)$ for short.)

Now we rearrange these 4 equations (2×2 species) by adding and subtracting appropriately to get new equations governing the new variables:

Mass Density	ρ_m	$=$	$n_e m_e + n_i m_i$	(4.75)
C of M Velocity	$\mathbf{V}$	$=$	$\left(n_e m_e \mathbf{v}_e + n_i m_i \mathbf{v}_i \right) / \rho_m$	(4.76)
Charge density	ρ_q	$=$	$q_e n_e + q_i n_i$	(4.77)
Electric Current Density	$\mathbf{j}$	$=$	$q_e n_e \mathbf{v}_e + q_i n_i \mathbf{v}_i$	(4.78)
		$=$	$q_e n_e \left(\mathbf{v}_e - \mathbf{v}_i \right)$ by quasi neutrality	(4.79)
Total Pressure	p	$=$	$p_e + p_i$	(4.80)

1st equation: take $m_e \times C_e + m_i \times C_i \rightarrow$

$$(1) \qquad \frac{\partial \rho_m}{\partial t} + \nabla. \left(\rho_m \mathbf{V} \right) = 0 \qquad \text{Mass Conservation} \tag{4.81}$$

2nd take $q_e \times C_e + q_I \times C_i \rightarrow$

$$(2) \qquad \frac{\partial \rho_q}{\partial t} + \nabla.\mathbf{j} = 0 \qquad \text{Charge Conservation} \tag{4.82}$$

3rd take $M_e + M_i$. This is a bit more difficult. RHS becomes:

$$\sum n_j q_j \left(\mathbf{E} + \mathbf{v}_j \wedge \mathbf{B} \right) - \nabla_{pj} + F_{jk} = \rho_q \mathbf{E} + \mathbf{j} \wedge \mathbf{B} - \nabla \left(p_e + p_i \right) \tag{4.83}$$

(we use the fact that $F_{ei} - F_{ie}$ so no *net* friction). LHS is

$$\sum_j m_j n_j \left(\frac{\partial}{\partial t} + \mathbf{v}_j.\nabla \right) v_j \tag{4.84}$$

The difficulty here is that the convective term is non-linear and so does not easily lend itself to reexpression in terms of the new variables. But note that since $m_e << m_i$ the contribution from electron momentum is usually much less than that from ions. So we ignore it in this equation. To the same degree of approximation $\mathbf{V} \simeq \mathbf{v}_i$: the CM velocity is the ion velocity. Thus for the LHS of this momentum equation we take

$$\sum_j m_i n_i \left(\frac{\partial}{\partial t} + \mathbf{v}_j.\nabla \right) \mathbf{v}_j \simeq \rho_m \left(\frac{\partial}{\partial t} + \mathbf{V}.\nabla \right) \mathbf{V} \tag{4.85}$$

so:

$$(3) \qquad \rho_m \left(\frac{\partial}{\partial t} + \mathbf{V}.\nabla \right) \mathbf{V} = \rho_q \mathbf{E} + \mathbf{j} \wedge \mathbf{B} - \nabla p \tag{4.86}$$

Finally we take $\frac{q_e}{m_e} M_e + \frac{q_i}{m_i} M_i$ to get:

$$\sum_j n_j q_j \left[\frac{\partial}{\partial t} + (\mathbf{v}_j.\nabla) \right] \mathbf{v}_j = \sum_j \{ \frac{n_j q_j^2}{m_j} (\mathbf{E} + \mathbf{v}_j \wedge \mathbf{B}) - \frac{q_j}{m_j} \nabla p_j + \frac{q_j}{m_j} \mathbf{F}_{jk} \} \tag{4.87}$$

Again several difficulties arise which it is not very profitable to deal with rigorously. Observe that the LHS can be written (using quasineutrality $n_i q_i + n_e q_e = 0$) as $\rho_m \frac{\partial}{\partial t} \left(\frac{\mathbf{j}}{\rho_m} \right)$ provided we discard the term in $(\mathbf{v}.\nabla)\mathbf{v}$. (Think of this as a linearization of this question.) [The $(\mathbf{v}.\nabla)\mathbf{v}$ convective term is a term which is not satisfactorily dealt with in this approach to the single fluid equations.]

In the R.H.S. we use quasineutrality again to write

$$\sum_j \frac{n_j q_j^2}{m_j} \mathbf{E} = n_e^2 q_e^2 \left(\frac{1}{n_e m_e} + \frac{1}{n_i m_i} \right) \mathbf{E} = n_e^2 q_e^2 \frac{m_i n_i + m_e n_e}{n_e m_e n_i m_i} \mathbf{E} = -\frac{q_e q_i}{m_e m_i} \rho_m \mathbf{E}, \tag{4.88}$$

$$\begin{aligned} \sum \frac{n_j q_q^2}{m_j} \mathbf{v}_j &= \frac{n_e q_e^2}{m_e} \mathbf{v}_e + \frac{n_i q_i^2}{m_i} \mathbf{v}_i \\ &= \frac{q_e q_i}{m_e m_i} \{ \frac{n_e q_e m_i}{q_i} \mathbf{v}_e + \frac{n_i q_i m_e}{q_e} \mathbf{v}_i \} \\ &= -\frac{q_e q_i}{m_e m_i} \{ n_e m_e \mathbf{v}_e + n_i m_i \mathbf{v}_i - \left(\frac{m_i}{q_i} + \frac{m_e}{q_e} \right) (q_e n_e \mathbf{v}_e + q_i n_i \mathbf{v}_i) \} \\ &= -\frac{q_e q_i}{m_e m_i} \{ \rho_m \mathbf{V} - \left(\frac{m_i}{q_i} + \frac{m_e}{q_e} \right) \mathbf{j} \} \end{aligned} \tag{4.89}$$

Also, remembering $\mathbf{F}_{ei} = -\overline{\nu}_{ei} n_e m_i (\mathbf{v}_e - \mathbf{v}_i) = -\mathbf{F}_{ie}$,

$$\begin{aligned} \sum_j \frac{q_j}{m_j} \mathbf{F}_{jk} &= -\overline{\nu}_{ei} \left(n_e q_e - n_e q_i \frac{m_e}{m_i} \right) (\mathbf{v}_e - \mathbf{v}_i) \\ &= -\overline{\nu}_{ei} \left(1 - \frac{q_e}{q_i} \frac{m_e}{m_i} \right) \mathbf{j} \end{aligned} \tag{4.90}$$

So we get

$$\begin{aligned} \rho_m \frac{\partial}{\partial t} \left(\frac{j}{\rho_m} \right) &= -\frac{q_e q_i}{m_e m_i} \left[\rho_m \mathbf{E} + \left\{ \rho_m \mathbf{V} - \left(\frac{m_i}{q_i} + \frac{m_e}{q_e} \right) \mathbf{j} \right\} \wedge \mathbf{B} \right] \\ &- \frac{q_e}{m_e} \nabla p_e - \frac{q_i}{m_i} \nabla p_i - \left(1 - \frac{q_e}{q_i} \frac{m_e}{m_i} \right) \overline{\nu}_{ei} \mathbf{j} \end{aligned} \tag{4.91}$$

Regroup after multiplying by $\frac{m_e m_i}{q_e q_i \rho_m}$:

$$\begin{aligned} \mathbf{E} + \mathbf{V} \wedge \mathbf{B} &= -\frac{m_e m_i}{q_e q_i} \frac{\partial}{\partial t} \left(\frac{\mathbf{j}}{\rho_m} \right) + \frac{1}{\rho_m} \left(\frac{m_i}{q_i} + \frac{m_e}{q_e} \right) \mathbf{j} \wedge \mathbf{B} \\ &- \left(\frac{q_e}{m_e} \nabla p_e + \frac{q_i}{m_i} \nabla p_i \right) \frac{m_e m_i}{\rho_m q_e q_i} - \left(1 - \frac{q_e}{q_i} \frac{m_e}{m_i} \right) \frac{m_e m_i}{q_e q_i \rho_m} \overline{\nu}_{ei} \mathbf{j} \end{aligned} \tag{4.92}$$

Notice that this is an equation relating the Electric field in the frame moving with the fluid (L.H.S.) to things depending on current $\mathbf{j}$ i.e. this is a generalized form of *Ohm's Law*.

One essentially never deals with this full generalized Ohm's law. Make some approximations recognizing the physical significance of the various R.H.S. terms.

$$\frac{m_e m_i}{q_e q_i}\frac{\partial}{\partial t}\left(\frac{j}{\rho_m}\right) \text{ arises from } \textit{electron inertia.}$$

it will be negligible for low enough frequency.

$$\frac{1}{\rho_m}\left(\frac{m_i}{q_i}+\frac{m_e}{q_e}\right)\mathbf{j}\wedge\mathbf{B} \text{ is called the } \textit{Hall Term.}$$

and arises because current flow in a B-field tends to be diverted across the magnetic field. It is also often dropped but the justification for doing so is less obvious physically.

$$\frac{q_i}{m_i}\nabla p_i \text{ term } \ll \frac{q_e}{m_e}\nabla p_e \text{ for comparable pressures,}$$

and the latter is $\sim$ the Hall term; so ignore $q_i\nabla p_i/m_i$.

Last term in $\mathbf{j}$ has a coefficient, ignoring m_e/m_i c.f. 1 which is

$$\frac{m_e m_i \overline{\nu}_{ei}}{q_e q_i\left(n_i m_i\right)} = \frac{m_e\overline{\nu}_{ei}}{q_e^2 n_e} = \eta \quad \text{the resistivity.} \tag{4.93}$$

Hence dropping electron inertia, Hall term and pressure, the simplified Ohm's law becomes:

$$\mathbf{E}+\mathbf{V}\wedge\mathbf{B} = \eta\mathbf{j} \tag{4.94}$$

Final equation needed: state:

$$p_e n_e^{-\gamma_e} + p_i n_i^{-\gamma_i} = \text{constant.}$$

Take quasi-neutrality $\Rightarrow n_e \propto n_i \propto \rho_m$. Take $\gamma_e = \gamma_i$, then

$$p\rho_m^{-\gamma} = \text{const.} \tag{4.95}$$

4.6.1 Summary of Single Fluid Equations: M.H.D.

$$\text{Mass Conservation}: \quad \frac{\partial \rho_m}{\partial t} + \nabla\left(\rho_m \mathbf{V}\right) = 0 \tag{4.96}$$

$$\text{Charge Conservation}: \quad \frac{\partial \rho_q}{\partial t} + \nabla.\mathbf{j} = 0 \tag{4.97}$$

$$\text{Momentum}: \quad \rho_m\left(\frac{\partial}{\partial t}+\mathbf{V}.\nabla\right)\mathbf{V} = \rho_q\mathbf{E}+\mathbf{j}\wedge\mathbf{B}-\nabla p \tag{4.98}$$

$$\text{Ohm's Law}: \quad \mathbf{E}+\mathbf{V}\wedge\mathbf{B} = \eta\mathbf{j} \tag{4.99}$$

$$\text{Eq. of State}: \quad p\rho_m^{-\gamma} = \text{const.} \tag{4.100}$$

4.6.2 Heuristic Derivation/Explanation

Mass Charge: Obvious.

$$Mom^m \qquad \underbrace{\rho_m\left(\frac{\partial}{\partial t}+\mathbf{V}.\nabla\right)\mathbf{V}}_{\substack{\text{rate of change of}\\ \text{total momentum density}}} = \underbrace{\rho_q\mathbf{E}}_{\substack{\text{Electric}\\ \text{body force}}} + \underbrace{\mathbf{j}\wedge\mathbf{B}}_{\substack{\text{Magnetic Force}\\ \text{on current}}} - \underbrace{\nabla p}_{\text{Pressure}} \tag{4.101}$$

Ohm's Law

The electric field 'seen' by a moving (conducting) fluid is $\mathbf{E}+\mathbf{V}\wedge\mathbf{B}=\mathbf{E}_V$ electric field in frame in which fluid is at rest. This is equal to 'resistive' electric field $\eta\mathbf{j}$:

$$\mathbf{E}_V=\mathbf{E}+\mathbf{V}\wedge\mathbf{B}=\eta\mathbf{j} \tag{4.102}$$

The $\rho_q E$ term is generally dropped because it is much smaller than the $\mathbf{j}\wedge\mathbf{B}$ term. To see this, take orders of magnitude:

$$\nabla.\mathbf{E}=\rho_q/\epsilon_0 \quad \text{so} \quad \rho_q\sim E\epsilon_0/L \tag{4.103}$$

$$\nabla\wedge\mathbf{B}=\mu_0\mathbf{j}\left(+\frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t}\right) \quad \text{so} \quad \sigma E=j\sim B/\mu_0 L \tag{4.104}$$

Therefore

$$\frac{\rho_q E}{jB}\sim\frac{\epsilon_0}{L}\left(\frac{B}{\mu_0\sigma L}\right)^2\frac{L\mu_0}{B^2}\sim\frac{L^2/c^2}{(\mu_0\sigma L^2)^2}=\left(\frac{\text{light transit time}}{\text{resistive skin time}}\right)^2. \tag{4.105}$$

This is generally a very small number. For example, even for a small cold plasma, say $T_e=1$ eV ($\sigma\approx 2\times10^3$ mho/m), $L=1$ cm, this ratio is about 10^{-8}.

Conclusion: the $\rho_q E$ force is *much* smaller than the $\mathbf{j}\wedge\mathbf{B}$ force for essentially all practical cases. Ignore it.

Normally, also, one uses MHD only for low frequency phenomena, so the Maxwell displacement current, $\partial\mathbf{E}/c^2\partial t$ can be ignored.

Also we shall not need Poisson's equation because that is taken care of by quasi-neutrality.

4.6.3 Maxwell's Equations for MHD Use

$$\nabla.\mathbf{B}=0 \;\; ; \;\; \nabla\wedge\mathbf{E}=\frac{-\partial\mathbf{B}}{\partial t} \;\; ; \;\; \nabla\wedge\mathbf{B}=\mu_o\mathbf{j} \;\; . \tag{4.106}$$

The MHD equations find their major use in studying macroscopic magnetic confinement problems. In Fusion we want somehow to confine the plasma pressure away from the walls of the chamber, using the magnetic field. In studying such problems MHD is the major tool.

On the other hand if we focus on a small section of the plasma as we do when studying short-wavelength waves, other techniques: 2-fluid or kinetic are needed. Also, plasma is approx. uniform.

'Macroscopic' Phenomena MHD

'Microscopic' Phenomena 2-Fluid/Kinetic

4.7 MHD Equilibria

Study of how plasma can be 'held' by magnetic field. Equilibrium $\Rightarrow \mathbf{V} = \frac{\partial}{\partial t} = 0$. So equations reduce. Mass and Faraday's law are $\sim$ automatic. We are left with

$$(Mom^m) \rightarrow \text{'Force Balance'} \quad 0 = \mathbf{j} \wedge \mathbf{B} - \nabla p \tag{4.107}$$

$$\text{Ampere} \qquad \nabla \wedge \mathbf{B} = \mu_o \mathbf{j} \tag{4.108}$$

Plus $\nabla.\mathbf{B} = 0$, $\nabla.\mathbf{j} = 0$.

Notice that provided we don't ask questions about Ohm's law. **E** doesn't come into MHD equilibrium.

These deceptively simple looking equations are the subject of much of Fusion research. The hard part is taking into account complicated geometries.

We can do some useful calculations on simple geometries.

4.7.1 θ-pinch

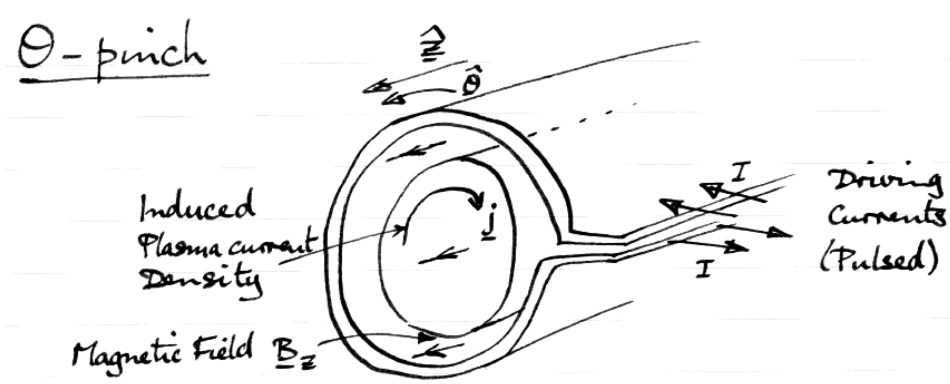

Figure 4.5: θ-pinch configuration.

So called because plasma currents flow in θ-direction.

Use MHD Equations

Take to be ∞ length, uniform in z-dir.

By symmetry $\mathbf{B}$ has only z component.

By symmetry $\mathbf{j}$ has only θ- comp.

By symmetry ∇p has only r comp.

So we only need

$$\begin{aligned}
\text{Force} \qquad (\mathbf{j}\wedge\mathbf{B})_r &- (\nabla p)_r = 0 && (4.109)\\
\text{Ampere} \qquad (\nabla\wedge\mathbf{B})_\theta &= (\mu_o\mathbf{j})_\theta && (4.110)\\
\text{i.e.} \qquad j_\theta B_z &- \frac{\partial}{\partial r}p = 0 && (4.111)\\
-\frac{\partial}{\partial r}B_z &= \mu_o j_\theta && (4.112)\\
\text{Eliminate } j: \quad -\frac{B_z}{\mu_o}\frac{\partial B_z}{\partial r} - \frac{\partial p}{\partial r} &= 0 && (4.113)
\end{aligned}$$

i.e.

$$\begin{aligned}
\frac{\partial}{\partial r}\left(\frac{B_z^2}{2\mu_o}+p\right) &= 0 && (4.114)\\
\text{Solution} \quad \frac{B_z^2}{2\mu_o}+p &= \text{const.} && (4.115)
\end{aligned}$$

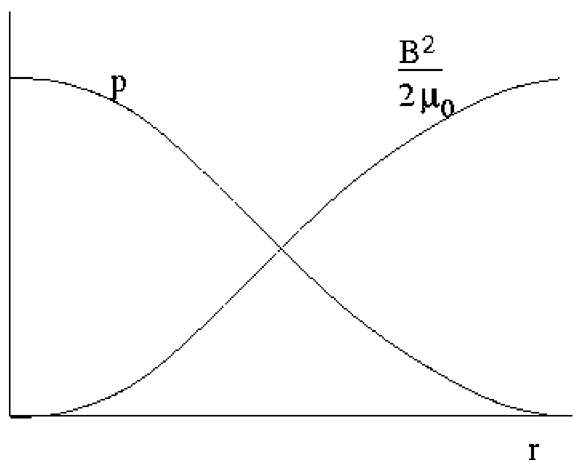

Figure 4.6: Balance of kinetic and magnetic pressure

$$\frac{B_z^2}{2\mu_o}+p = \frac{B_{z\ ext}^2}{2\mu_o} \qquad (4.116)$$

[Recall Single Particle Problem]

Think of these as a pressure equation. Equilibrium says *total* pressure = const.

$$\underbrace{\frac{B_z^2}{2\mu_o}}_{\text{magnetic pressure}} + \underbrace{p}_{\text{kinetic pressure}} = \text{const.} \qquad (4.117)$$

Ratio of kinetic to magnetic pressure is plasma 'β'.

$$\beta = \frac{2\mu_o p}{B_z^2} \tag{4.118}$$

measures 'efficiency' of plasma confinement by B. Want large β for fusion but limited by instabilities, etc.

4.7.2 Z-pinch

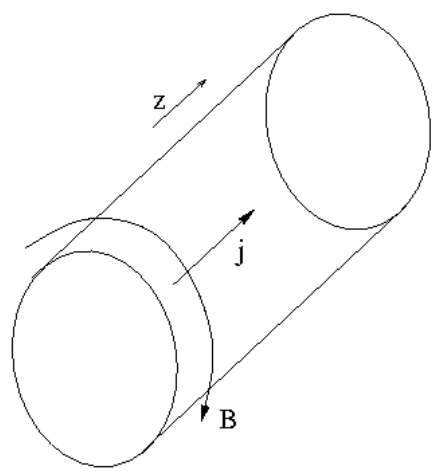

Figure 4.7: Z-pinch configuration.

so called because $\mathbf{j}$ flows in z-direction. Again take to be ∞ length and uniform.

$$\mathbf{j} = \mathbf{j}_z \hat{\mathbf{e}}_z \qquad \mathbf{B} = \mathbf{B}_\theta \hat{\mathbf{e}}_\theta \tag{4.119}$$

$$\text{Force} \quad (\mathbf{j} \wedge \mathbf{B})_r - (\nabla p)_r = -j_z B_\theta - \frac{\partial p}{\partial r} = 0 \tag{4.120}$$

$$\text{Ampere} \quad (\nabla \wedge \mathbf{B})_z - (\mu_o \mathbf{j})_z = \frac{1}{r}\frac{\partial}{\partial r}(rB_\theta) - \mu_o j_z = 0 \tag{4.121}$$

Eliminate j:

$$\frac{B_\theta}{\mu_o r}\frac{\partial}{\partial r}(rB_\theta) - \frac{\partial p}{\partial r} = 0 \tag{4.122}$$

or

$$\underbrace{\frac{B_\theta^2}{\mu_o r}}_{\text{Extra Term}} + \frac{\partial}{\partial r}\underbrace{\left(\frac{B_\theta^2}{2\mu_o} + p\right)}_{\text{Magnetic+Kinetic pressure}} = 0 \tag{4.123}$$

Extra term acts like a magnetic *tension* force. Arises because B-field lines are *curved*.

Can integrate equation

$$\int_a^b \frac{B_\theta^2}{\mu_o}\frac{dr}{r} + \left[\frac{B_\theta^2}{2\mu_o} + p(r)\right]_a^b = 0 \tag{4.124}$$

If we choose b to be edge ($p(b) = 0$) and set $a = r$ we get

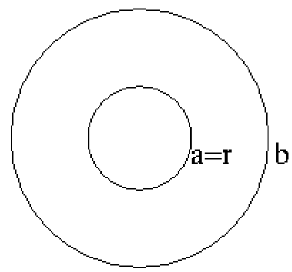

Figure 4.8: Radii of integration limits.

$$p(r) = \frac{B_\theta^2\,(b)}{2\mu_o} - \frac{B_\theta^2\,(r)}{2\mu_o} + \int_r^b \frac{B_\theta^2}{\mu_o}\frac{dr'}{r'} \tag{4.125}$$

Force balance in z-pinch is somewhat more complicated because of the tension force. We can't choose $p(r)$ and $j(r)$ independently; they have to be self consistent.

Example $\quad j = \text{const.}$

$$\frac{1}{r}\frac{\partial}{\partial r}(rB_\theta) = \mu_o j_z \Rightarrow B_\theta = \frac{\mu_o j_z}{2}\,r \tag{4.126}$$

Hence

$$p(r) = \frac{1}{2\mu_o}\left(\frac{\mu_o j_z}{2}\right)^2 \{b^2 - r^2 + \int_r^b 2r'dr'\} \tag{4.127}$$

$$= \frac{\mu_o j_z^2}{4}\{b^2 - r^2\} \tag{4.128}$$

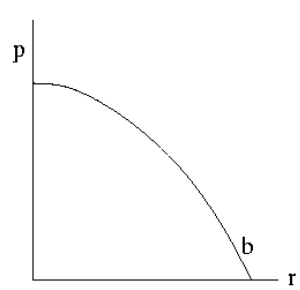

Figure 4.9: Parabolic Pressure Profile.

Also note $B_\theta(b) = \frac{\mu_o j_z b}{2}$ so

$$p = \frac{B_{\theta b}^2}{2\mu_o}\frac{2}{b^2}\{b^2 - r^2\} \tag{4.129}$$

4.7.3 'Stabilized Z-pinch'

Also called 'screw pinch', $\theta - z$ pinch or sometimes loosely just 'z-pinch'.
Z-pinch with some additional B_z as well as B_θ

$$(\text{Force})_r \quad j_\theta B_z - j_z B_\theta - \frac{\partial}{\partial r} = 0 \tag{4.130}$$

$$\text{Ampere :} \quad \frac{\partial}{\partial r} B_z = \mu_o j_\theta \tag{4.131}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(r b_\theta) = \mu_o j_z \tag{4.132}$$

Eliminate j:

$$-\frac{B_z}{\mu_o}\frac{\partial B_z}{\partial r} - \frac{B_\theta}{\mu_o r}\frac{\partial}{\partial r}(r B_\theta) - \frac{\partial p}{\partial r} = 0 \tag{4.133}$$

or

$$\underbrace{\frac{B_\theta^2}{\mu_o r}}_{\substack{\text{Mag Tension} \\ \theta \text{ only}}} + \frac{\partial}{\partial r}\underbrace{\left(\frac{B^2}{2\mu_o} + p\right)}_{\text{Mag } (\theta+z)+\text{Kinetic pressure}} = 0 \tag{4.134}$$

4.8 Some General Properties of MHD Equilibria

4.8.1 Pressure & Tension

$$\mathbf{j} \wedge \mathbf{B} - \nabla p = 0 \quad : \quad \nabla \wedge \mathbf{B} = \mu_o \mathbf{j} \tag{4.135}$$

We can eliminate $\mathbf{j}$ in the *general* case to get

$$\frac{1}{\mu_o}(\nabla \wedge \mathbf{B}) \wedge \mathbf{B} = \nabla p. \tag{4.136}$$

Expand the vector triple product:

$$\nabla p = \frac{1}{\mu_o}(\mathbf{B}.\nabla)\mathbf{B} - \frac{1}{2\mu_o}\nabla B^2 \tag{4.137}$$

put $\mathbf{b} = \frac{\mathbf{B}}{|B|}$ so that $\nabla \mathbf{B} = \nabla B \mathbf{b} = B \nabla \mathbf{b} + \mathbf{b} \nabla B$. Then

$$\nabla p = \frac{1}{\mu_o}\{B^2(\mathbf{b}.\nabla)\mathbf{b} + B\mathbf{b}(\mathbf{b}.\nabla)B\} - \frac{1}{2\mu_o}\nabla B^2 \tag{4.138}$$

$$= \frac{B^2}{\mu_o}(\mathbf{b}.\nabla)\mathbf{b} - \frac{1}{2\mu_o}(\nabla - \mathbf{b}(\mathbf{b}.\nabla))B^2 \tag{4.139}$$

$$= \frac{B^2}{\mu_o}(\mathbf{b}.\nabla)\mathbf{b} - \nabla_\perp\left(\frac{B^2}{2\mu_p}\right) \tag{4.140}$$

Now $\nabla_\perp\left(\frac{B^2}{2\mu_o}\right)$ is the perpendicular (to $\mathbf{B}$) derivative of *magnetic pressure* and $(\mathbf{b}.\nabla)\mathbf{b}$ is the *curvature* of the magnetic field line giving *tension*.

$\mid(\mathbf{b}.\nabla)\mathbf{b}\mid$ has value $\frac{1}{R}$. R: radius of curvature.

4.8.2 Magnetic Surfaces

$$0=\mathbf{B}.\left[\mathbf{j}\wedge\mathbf{B}-\nabla p\right]=-\mathbf{B}.\nabla p \tag{4.141}$$

*Pressure is constant on a field line (in MHD situation).

(Similarly, $0=\mathbf{j}.[\mathbf{j}\wedge\mathbf{B}-\nabla p]=\mathbf{j}.\nabla p$.)

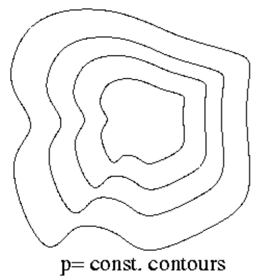

Figure 4.10: Contours of pressure.

Consider some arbitrary volume in which $\nabla p\neq 0$. That is, some plasma of whatever shape. Draw contours (surfaces in 3-d) on which $p=$ const. At any point on such an isoberic surface ∇p is perp to the surface. But $\mathbf{B}.\nabla p=0$ implies that $\mathbf{B}$ is also perp to ∇p.

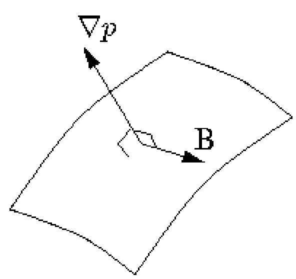

Figure 4.11: $\mathbf{B}$ is perpendicular to ∇p and so lies in the isobaric surface.

Hence

$\mathbf{B}$ lies in the surface $p=$ const.

In equilibrium *isobaric surfaces* are *'magnetic surfaces'*.

[This argument does not work if $p=$ const. i.e. $\nabla p=0$. Then there need be no magnetic surfaces.]

4.8.3 'Current Surfaces'

Since $\mathbf{j}.\nabla p = 0$ in equilibrium the same argument applies to current density. That is

$\mathbf{j}$ lies in the surface $p =$ const.

Isobaric Surfaces are 'Current Surfaces'.

Moreover it is clear that

'Magnetic Surfaces' are 'Current Surfaces'.

(since both coincide with isobaric surfaces.)

[It is important to note that the existence of magnetic surfaces is guaranteed only in the MHD approximation when $\nabla p \neq 0 >$ Taking account of corrections to MHD we may not have magnetic surfaces even if $\nabla p \neq 0$.]

4.8.4 Low β equilibria: Force-Free Plasmas

In many cases the ratio of kinetic to magnetic pressure is small, $\beta << 1$ and we can approximately *ignore* ∇p. Such an equilibrium is called 'force free'.

$$\mathbf{j} \wedge \mathbf{B} = 0 \tag{4.142}$$

implies $\mathbf{j}$ and $\mathbf{B}$ are parallel.

i.e.

$$\mathbf{j} = \mu(\mathbf{r})\mathbf{B} \tag{4.143}$$

Current flows *along* field lines *not across*. Take divergence:

$$0 = \nabla.\mathbf{j} = \nabla.\left(\mu\left(\mathbf{r}\right)\mathbf{B}\right) = \mu\left(\mathbf{r}\right)\nabla.\mathbf{B} + \left(\mathbf{B}.\nabla\right)\mu \tag{4.144}$$

$$= \left(\mathbf{B}.\nabla\right)\mu. \tag{4.145}$$

The ratio $j/B = \mu$ is *constant along field lines.*

μ is constant on a magnetic surface. If there are no surfaces, μ is constant *everywhere.*

Example: Force-Free Cylindrical Equil.

$$j \wedge \mathbf{B} = \quad \Leftrightarrow \quad \mathbf{j} = \mu(r)\mathbf{B} \tag{4.146}$$

$$\nabla \wedge \mathbf{B} = \mu_o\mathbf{j} = \mu_o\mu(r)\mathbf{B} \tag{4.147}$$

This is a somewhat more convenient form because it is linear in $\mathbf{B}$ (for specified $\mu(r)$).

$$\text{Constant-}\mu: \qquad \nabla \wedge \mathbf{B} = \mu_o\mu\mathbf{B} \tag{4.148}$$

leads to a Bessel function solution

$$B_z = B_o J_o(\mu_o \mu r) \quad (4.149)$$
$$B_\theta = B_o J_1(\mu_o \mu r) \quad (4.150)$$

for $\mu_o \mu r >$ 1st zero of J_o the toroidal field reverses. There are plasma confinement schemes with $\mu \simeq$ const. 'Reversed Field Pinch'.

4.9 Toroidal Equilibrium

Bend a z-pinch into a torus

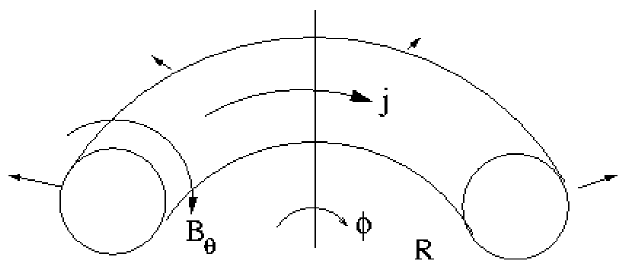

Figure 4.12: Toroidal z-pinch

B_θ fields due to current are stronger at small R side $\Rightarrow$ Pressure (Magnetic) Force *outwards*. Have to balance this by applying a *vertical field* $\mathbf{B}_v$ to push plasma back by $\mathbf{j}_\phi \wedge \mathbf{B}_v$.

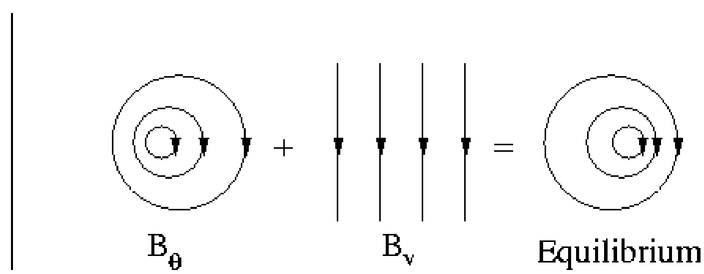

Figure 4.13: The field of a toroidal loop is not an MHD equilibrium. Need to add a vertical field.

Bend a θ-pinch into a torus: $B\phi$ is stronger at small R side $\Rightarrow$ outward force.

Cannot be balanced by $\mathbf{B}_v$ because no j_ϕ. No equilibrium for a toroidally symmetric θ-pinch.

Underlying Single Particle reason:

Toroidal θ-pinch has B_ϕ only. As we have seen before, curvature drifts are uncompensated in such a configuration and lead to rapid *outward* motion.

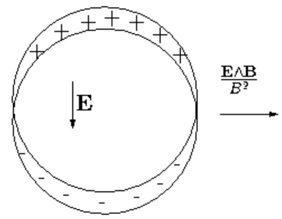

Figure 4.14: Charge-separation giving outward drift is equivalent to the lack of MHD toroidal force balance.

We know how to solve this: Rotational Transform: get some B_θ. Easiest way: add $\mathbf{j}_\phi$. From MHD viewpoint this allows you to push the plasma back by $\mathbf{j}_\phi \wedge \mathbf{B}_v$ force. Essentially, this is Tokamak.

4.10 Plasma Dynamics (MHD)

When we want to analyze *non*-equilibrium situations we must retain the momentum terms. This will give a dynamic problem. Before doing this, though, let us analyse some purely Kinematic Effects.

'Ideal MHD' $\Leftrightarrow$ Set $eta = 0$ in Ohm's Law.
A good approximation for high frequencies, i.e. times shorter than resistive decay time.

$$\mathbf{E} + \mathbf{V} \wedge \mathbf{B} = 0. \qquad \text{Ideal Ohm's Law.} \tag{4.151}$$

Also

$$\nabla \wedge \mathbf{E} = \frac{-\partial \mathbf{B}}{\partial t} \qquad \text{Faraday's Law.} \tag{4.152}$$

Together these two equations imply constraints on how the magnetic field can change with time: Eliminate $\mathbf{E}$:

$$+\nabla \wedge (\mathbf{V} \wedge \mathbf{B}) = +\frac{\partial \mathbf{B}}{\partial t} \tag{4.153}$$

This shows that the changes in $\mathbf{B}$ are completely determined by the flow, $\mathbf{V}$.

4.11 Flux Conservation

Consider an arbitrary closed contour C and spawning surface S in the fluid.

Flux linked by C is

$$\Phi = \int_S \mathbf{B}.\mathbf{ds} \tag{4.154}$$

Let C and S move with fluid:

Total rate of change of Φ is given by two terms:

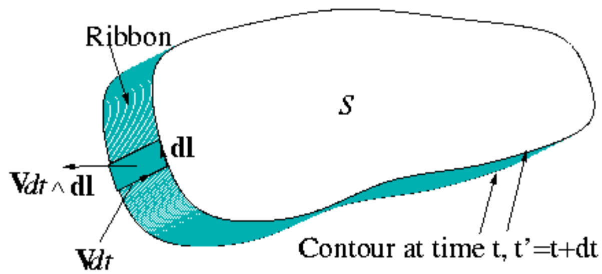

Figure 4.15: Motion of contour with fluid gives convective flux derivative term.

$$\dot{\Phi} = \int_S \underbrace{\frac{\partial \mathbf{B}}{\partial t}.\mathbf{ds}}_{\text{Due to changes in }\mathbf{B}} + \underbrace{\oint_C \mathbf{B}.(\mathbf{V}\wedge\mathbf{dl})}_{\text{Due to motion of C}} \quad (4.155)$$

$$= -\int_S \nabla\wedge\mathbf{E}.\mathbf{ds} - \oint_C (\mathbf{V}\wedge\mathbf{B}).\mathbf{dl} \quad (4.156)$$

$$= -\oint_C (\mathbf{E}+\mathbf{V}\wedge\mathbf{B}).\mathbf{dl} = 0 \quad \text{by Ideal Ohm's Law.} \quad (4.157)$$

Flux through any surface moving with fluid is conserved.

4.12 Field Line Motion

Think of a field line as the intersection of two surfaces both tangential to the field everywhere:

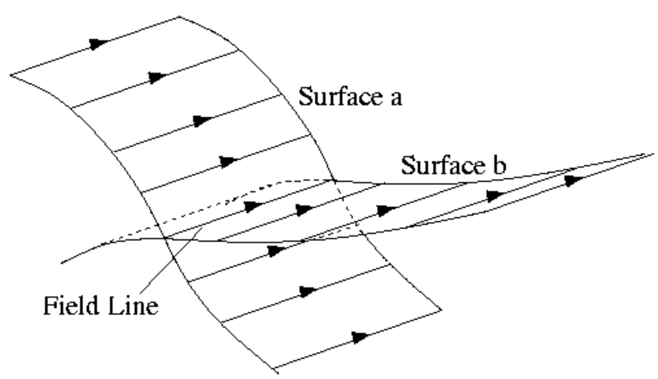

Figure 4.16: Field line defined by intersection of two flux surfaces tangential to field.

Let surfaces move with fluid.

Since all parts of surfaces had zero flux crossing at start, they also have zero after, (by flux conserv.).

Surfaces are tangent after motion

⇒ Their intersection defines a field line after.

We think of the new field line as the same line as the old one (only moved).

Thus:

1. Number of field lines (≡ flux) through any surface is constant. (Flux Cons.)

2. A line of fluid that starts as a field line remains one.

4.13 MHD Stability

The fact that one can find an MHD *equilibrium* (e.g. z-pinch) does not guarantee a useful confinement scheme because the equil. might be unstable. Ball on hill analogies:

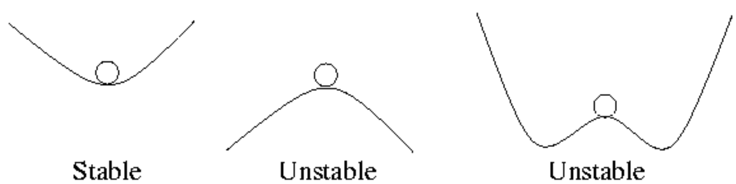

Figure 4.17: Potential energy curves

An equilibrium is *unstable* if the curvature of the 'Potential energy surface' is downward away from equil. That is if $\frac{d^2}{dx^2}\{W_{\rm pot}\} < 0$.

In MHD the potential energy is Magnetic + Kinetic Pressure (usually mostly magnetic).

If we can find *any* type of perturbation which *lowers* the potential energy then the equil is *unstable*. It will not remain but will rapidly be lost.

Example Z-pinch

We know that there is an equilibrium: Is it stable?

Consider a perturbation thus:

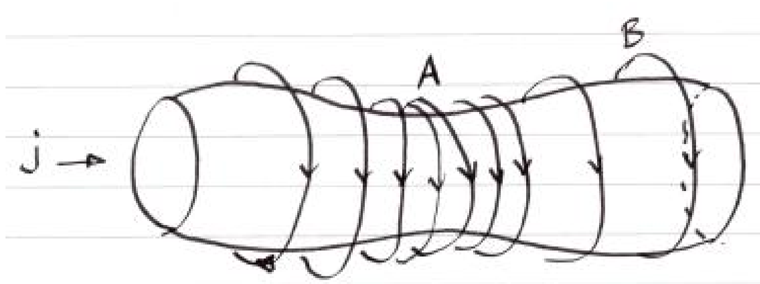

Figure 4.18: 'Sausage' instability

Simplify the picture by taking the current all to flow in a skin. We know that the pressure is supported by the combination of $B^2/2\mu_o$ pressure and $\frac{B^2}{\mu_o r}$ tension forces.

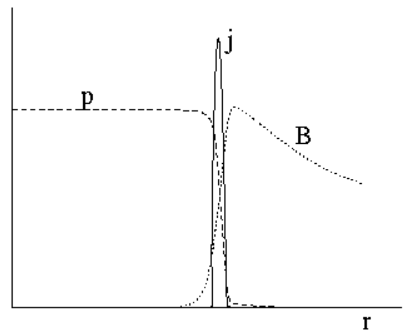

Figure 4.19: Skin-current, sharp boundary pinch.

At the place where it pinches *in* (A)
B_θ and $\frac{1}{r}$ *increase* $\rightarrow$ Mag. pressure & tension increase $\Rightarrow$ inward force no longer balance by $p \Rightarrow$ perturbation grows.

At place where it bulges *out* (B)
B_θ & $\frac{1}{r}$ *decrease* $\rightarrow$ Pressure & tension $\Rightarrow$ perturbation grows.

Conclusion a small perturbation induces a force tending to increase itself. *Unstable* ($\equiv \delta W < 0$).

4.14 General Perturbations of Cylindrical Equil.

Look for things which go like $\exp[i(kz + m\theta)]$. [Fourier (Normal Mode) Analysis].

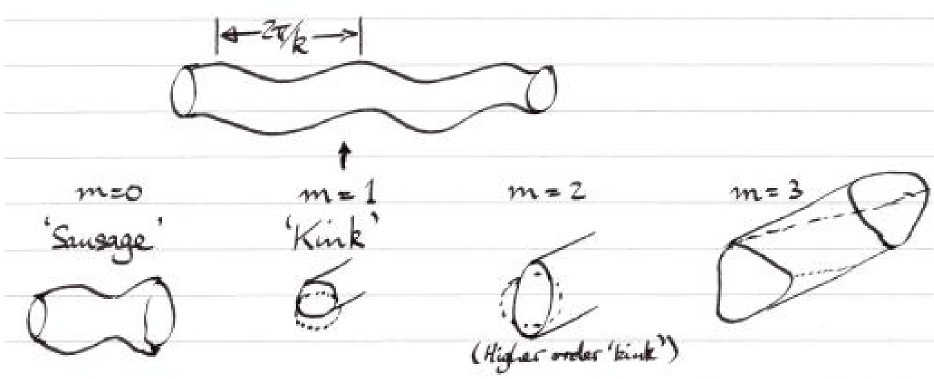

Figure 4.20: Types of kink perturbation.

Generally Helical in form (like a screw thread). Example: $m = 1 \quad k \neq 0$ z-pinch

4.15 General Principles Governing Instabilities

(1) They try *not* to bend field lines. (Because bending takes energy). Perturbation

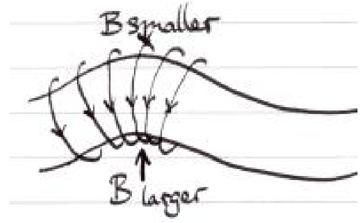

Figure 4.21: Driving force of a kink. Net force tends to increase perturbation. Unstable.

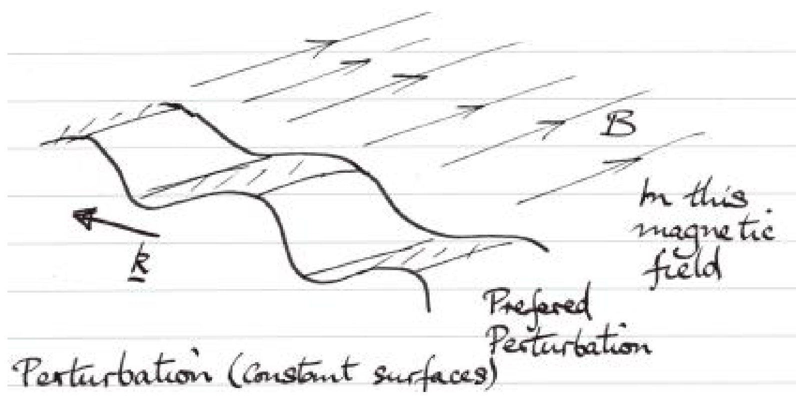

Figure 4.22: Alignment of perturbation and field line minimizes bending energy.

(Constant surfaces) lies *along* magnetic field.

Example: θ-pinch type plasma column:

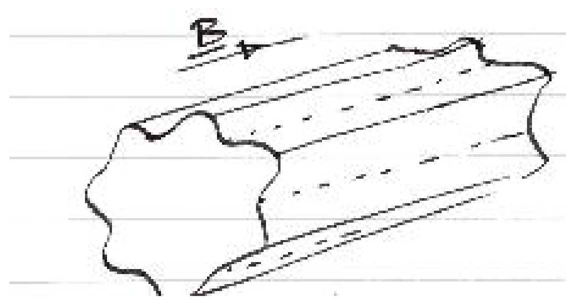

Figure 4.23: 'Flute' or 'Interchange' modes.

Preferred Perturbations are 'Flutes' as per Greek columns → 'Flute Instability.' [Better name: 'Interchange Instability', arises from idea that plasma and vacuum change places.]

(2) Occur when a 'heavier' fluid is supported by a 'lighter' (Gravitational analogy).

Why does water fall out of an inverted glass? Air pressure could sustain it but does not because of Rayleigh-Taylor instability.

Similar for supporting a plasma by mag field.

(3) Occur when $\mid B \mid$ decreases *away* from the plasma region.

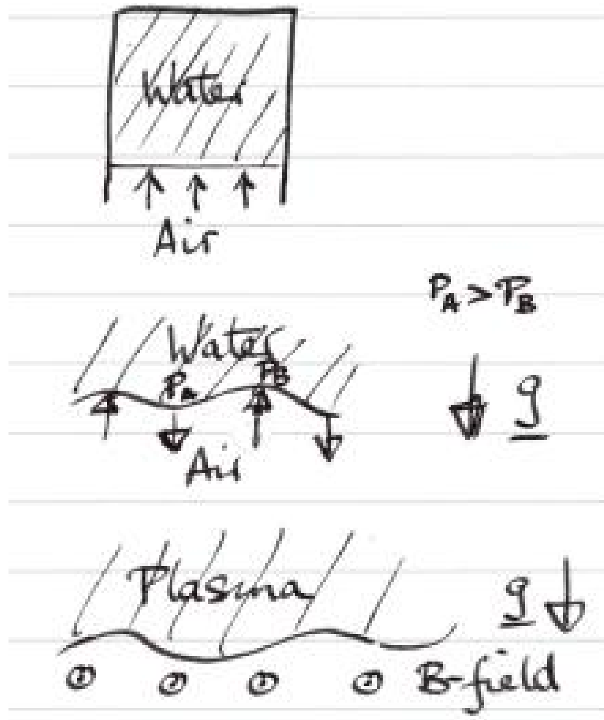

Figure 4.24: Inverted water glass analogy. Rayleigh Taylor instability.

$$\frac{B_A^2}{2\mu_o} < \frac{B_B^2}{2\mu_o} \tag{4.158}$$

⇒ Perturbation Grows.

(4) Occur when field line curvature is *towards* the plasma (Equivalent to (3) because of $\nabla \wedge \mathbf{B} = 0$ in a vacuum).

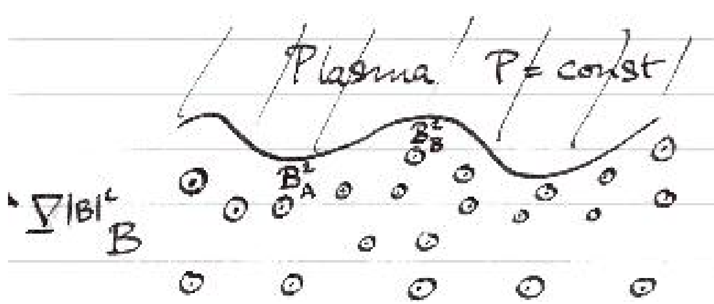

Figure 4.25: Vertical upward field gradient is unstable.

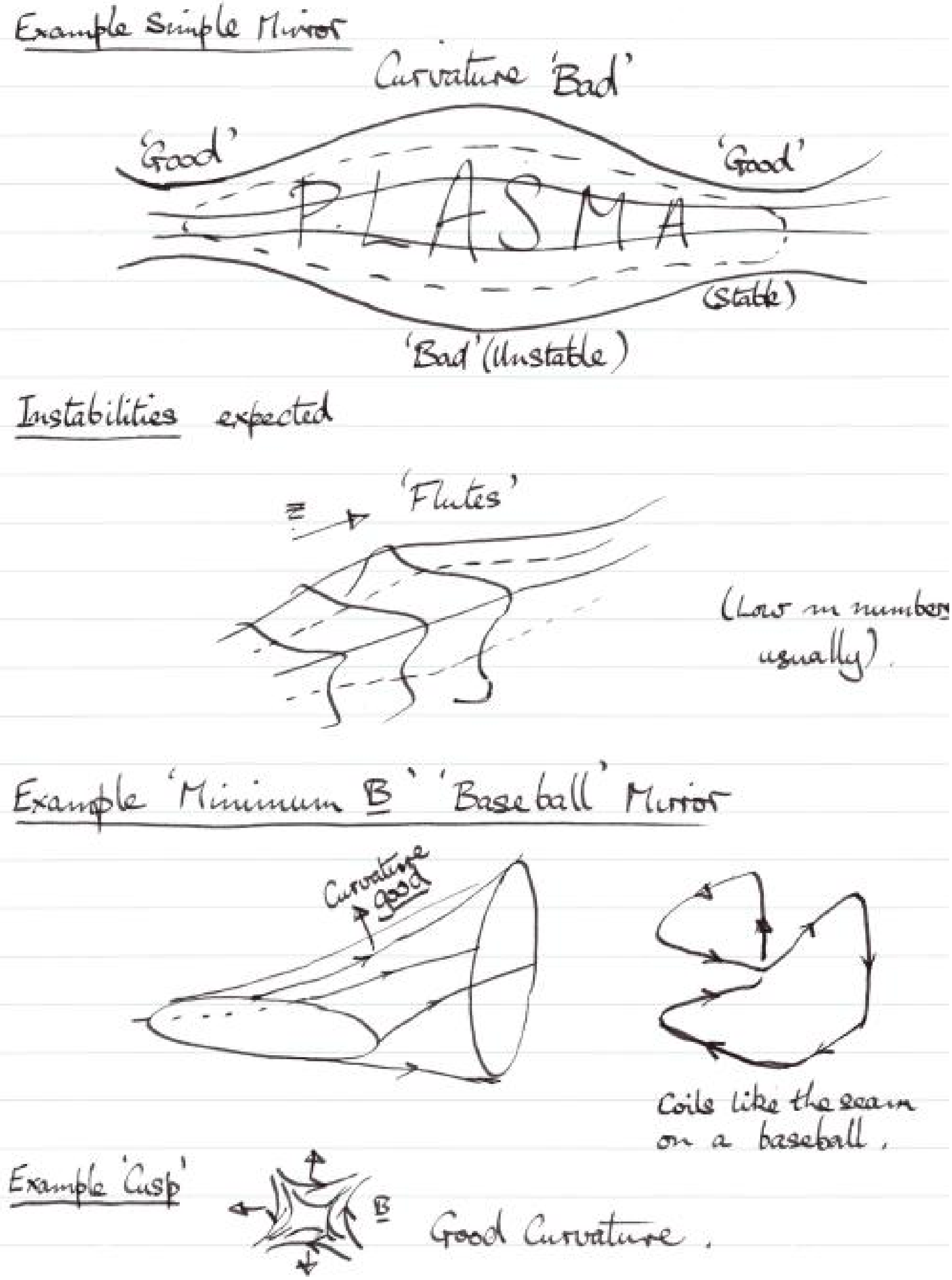

Figure 4.26: Examples of magnetic configurations with good and bad curvature.

4.16 Quick and Simple Analysis of Pinches

θ-pinch $\quad | B | =$ const. outside pinch
$\equiv$ No field line curvature. *Neutral stability*

z-pinch $\quad \nabla | B |$ away from plasma outside
$\equiv$ Bad Curvature (Towards plasma) $\Rightarrow$ *Instability.*

Generally it is difficult to get the curvature to be good everywhere. Often it is sufficient to make it good *on average* on a field line. This is referred to as 'Average Minimum B'. Tokamak has this.

General idea is that if field line is only in bad curvature over part of its length then to perturb in that region and not in the good region requires field line bending:

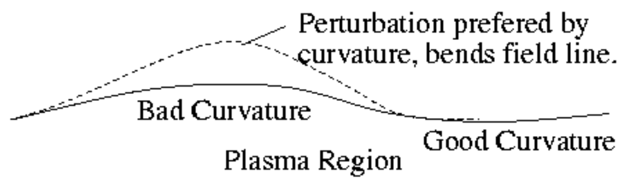

Figure 4.27: Parallel localization of perturbation requires bending.

But bending is *not* preferred. So this may stabilize.

Possible way to stabilize configuration with bad curvature: *Shear*
<u>Shear of Field Lines</u>

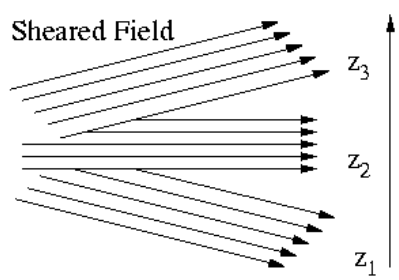

Figure 4.28: Depiction of field shear.

Direction of B changes. A perturbation along B at z_3 is *not* along $\mathbf{B}$ at z_2 or z_1 so it would have to *bend* field there $\rightarrow$ Stabilizing effect.

General Principle: Field line bending is stabilizing.

<u>Example: Stabilized z-pinch</u>

Perturbations (e.g. sausage or kink) *bend* $\mathbf{B}_z$ so the tension in B_z acts as a restoring force to prevent instability. If wave length very *long* bending is less. $\Rightarrow$ Least stable tends to be longest wave length.

<u>Example: 'Cylindrical Tokamak'</u>

Tokamak is in some ways like a periodic cylindrical stabilized pinch. Longest allowable wave length = 1 turn round torus the long way, i.e.

$$kR = 1 : \qquad \lambda = 2\pi R. \tag{4.159}$$

Express this in terms of a toroidal mode number, n (s.t. perturbation $\propto \exp i(n\phi + m\theta)$: $\phi = \frac{z}{R} \quad n = kR$.

Most unstable mode *tends* to be $n = 1$.

[Careful! Tokamak has important toroidal effects and some modes can be localized in the bad curvature region ($n \neq 1$).

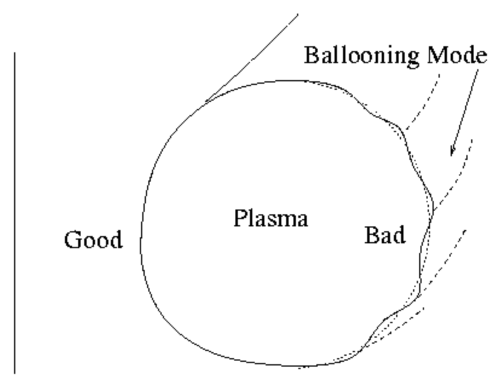

Figure 4.29: Ballooning modes are localized in the outboard, bad curvature region.

Chapter 5

Electromagnetic Waves in Plasmas

5.1 General Treatment of Linear Waves in Anisotropic Medium

Start with general approach to waves in a linear Medium: Maxwell:

$$\nabla \wedge \mathbf{B} = \mu_o j + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} \quad ; \quad \nabla \wedge \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{5.1}$$

we keep all the medium's response *explicit* in $\mathbf{j}$. Plasma is (infinite and) uniform so we Fourier analyze in space and time. That is we seek a solution in which all variables go like

$$\exp i(\mathbf{k}.\mathbf{x} - \omega t) \quad \text{[real part of]} \tag{5.2}$$

It is really the linearised equations which we treat this way; if there is some equilibrium field OK but the equations above mean implicitly the perturbations $\mathbf{B}$, $\mathbf{E}$, $\mathbf{j}$, etc.

Fourier analyzed:

$$i\mathbf{k} \wedge \mathbf{B} = \mu_o \mathbf{j} + \frac{-i\omega}{c^2}\mathbf{E} \quad ; \quad i\mathbf{k} \wedge \mathbf{E} = i\omega \mathbf{B} \tag{5.3}$$

Eliminate $\mathbf{B}$ by taking $\mathbf{k}\wedge$ second eq. and $\omega\times$ 1st

$$i\mathbf{k} \wedge (\mathbf{k} \wedge \mathbf{E}) = \omega \mu_o \mathbf{j} - \frac{i\omega^2}{c^2}\mathbf{E} \tag{5.4}$$

So

$$\mathbf{k} \wedge (\mathbf{k} \wedge \mathbf{E}) + \frac{\omega^2}{c^2}\mathbf{E} + i\omega\mu_o \mathbf{j} = 0 \tag{5.5}$$

Now, in order to get further we must have some relationship between $\mathbf{j}$ and $\mathbf{E}(\mathbf{k}, \omega)$. This will have to come from solving the plasma equations but for now we can just write the most general *linear* relationship $\mathbf{j}$ and $\mathbf{E}$ as

$$\mathbf{j} = \boldsymbol{\sigma}.\mathbf{E} \tag{5.6}$$

$\boldsymbol{\sigma}$ is the 'conductivity tensor'. Think of this equation as a matrix e.g.:

$$\begin{pmatrix} j_x \\ j_y \\ j_z \end{pmatrix} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \dots \\ \dots & \dots & \dots \\ \dots & \dots & \sigma_{zz} \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} \tag{5.7}$$

This is a general form of Ohm's Law. Of course if the plasma (medium) is isotropic (same in all directions) all off-diagonal $\sigma's$ are zero and one gets $\mathbf{j} = \sigma\mathbf{E}$.

Thus

$$\mathbf{k}(\mathbf{k}.\mathbf{E}) - k^2\mathbf{E} + \frac{\omega^2}{c^2}\mathbf{E} + i\omega\mu_o\boldsymbol{\sigma}.\mathbf{E} = 0 \tag{5.8}$$

Recall that in elementary E&M, dielectric media are discussed in terms of a dielectric constant ϵ and a "polarization" of the medium, $\mathbf{P}$, caused by modification of atoms. Then

$$\epsilon_o\mathbf{E} = \underbrace{\mathbf{D}}_{\text{Displacement}} - \underbrace{\mathbf{P}}_{\text{Polarization}} \quad \text{and} \quad \nabla.\mathbf{D} = \underbrace{\rho_{\text{ext}}}_{\text{externalcharge}} \tag{5.9}$$

and one writes

$$\mathbf{P} = \underbrace{\chi}_{\text{susceptibility}} \epsilon_o\mathbf{E} \tag{5.10}$$

Our case is completely analogous, except we have chosen to express the response of the medium in terms of current density, $\mathbf{j}$, rather than "polarization" $\mathbf{P}$ For such a dielectric medium, Ampere's law would be written:

$$\frac{1}{\mu_o}\nabla \wedge \mathbf{B} = \mathbf{j}_{\text{ext}} + \frac{\partial\mathbf{D}}{\partial t} = \frac{\partial}{\partial t}\epsilon\epsilon_o\mathbf{E}, \quad \text{if } \mathbf{j}_{\text{ext}} = 0 \quad , \tag{5.11}$$

where the dielectric constant would be $\epsilon = 1 + \chi$.

Thus, the explicit polarization current can be expressed in the form of an equivalent dielectric expression if

$$\mathbf{j} + \epsilon_o\frac{\partial\mathbf{E}}{\partial t} = \boldsymbol{\sigma}.\mathbf{E} + \epsilon\frac{\partial\mathbf{E}}{\partial t} = \frac{\partial}{\partial t}\epsilon_o\boldsymbol{\epsilon}.\mathbf{E} \tag{5.12}$$

or

$$\boldsymbol{\epsilon} = \mathbf{1} + \frac{\boldsymbol{\sigma}}{-i\omega\epsilon_o} \tag{5.13}$$

Notice the dielectric constant is a tensor because of anisotropy. The last two terms come from the RHS of Ampere's law:

$$\mathbf{j} + \frac{\partial}{\partial t}(\epsilon_o\mathbf{E}) \quad . \tag{5.14}$$

If we were thinking in terms of a dielectric medium with no explicit currents, only implicit (in ϵ) we would write this $\frac{\partial}{\partial t}(\epsilon\epsilon_o\mathbf{E})$; ϵ the dielectric constant. Our medium is possibly anisotropic so we need $\frac{\partial}{\partial t}(\epsilon_o\boldsymbol{\epsilon}.\mathbf{E})$ dielectric *tensor*. The obvious thing is therefore to define

$$\boldsymbol{\epsilon} = \mathbf{1} + \frac{1}{-i\omega\epsilon_o}\boldsymbol{\sigma} = \mathbf{1} + \frac{i\mu_o c^2}{\omega}\boldsymbol{\sigma} \tag{5.15}$$

Then

$$\mathbf{k}(\mathbf{k}.\mathbf{E}) - k^2\mathbf{E} + \frac{\omega^2}{c^2}\boldsymbol{\epsilon}.\mathbf{E} = 0 \tag{5.16}$$

and we may regard $\boldsymbol{\epsilon}(\mathbf{k},\omega)$ as the *dielectric tensor*.

Write the equation as a tensor multiplying $\mathbf{E}$:

$$\mathbf{D}.\mathbf{E} = 0 \tag{5.17}$$

with

$$\mathbf{D} = \{\mathbf{kk} - k^2\mathbf{1} + \frac{\omega^2}{c^2}\boldsymbol{\epsilon}\} \tag{5.18}$$

Again this is a matrix equation i.e. 3 simultaneous homogeneous eqs. for E.

$$\begin{pmatrix} D_{xx} & D_{xy} & \ldots \\ D_{yx} & \ldots & \ldots \\ \ldots & \ldots & D_{zz} \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = 0 \tag{5.19}$$

In order to have a non-zero $\mathbf{E}$ solution we must have

$$\det \mid \mathbf{D} \mid = 0. \tag{5.20}$$

This will give us an equation relating $\mathbf{k}$ and ω, which tells us about the possible wavelengths and frequencies of waves in our plasma.

5.1.1 Simple Case. Isotropic Medium

$$\boldsymbol{\sigma} = \sigma \; \mathbf{1} \tag{5.21}$$

$$\boldsymbol{\epsilon} = \epsilon \; \mathbf{1} \tag{5.22}$$

Take $\mathbf{k}$ in z direction then write out the Dispersion tensor $\mathbf{D}$.

$$\mathbf{D} = \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & kk \end{pmatrix}}_{\mathbf{kk}} - \underbrace{\begin{pmatrix} k^2 & 0 & 0 \\ 0 & k^2 & 0 \\ 0 & 0 & k^2 \end{pmatrix}}_{k^2\mathbf{1}} + \underbrace{\begin{pmatrix} \frac{\omega^2}{c^2}\epsilon & 0 & 0 \\ 0 & \frac{\omega^2}{c^2}\epsilon & 0 \\ 0 & 0 & \frac{\omega^2}{c^2}\epsilon \end{pmatrix}}_{\frac{\omega^2}{c^2}\boldsymbol{\epsilon}}$$

$$= \begin{bmatrix} -k^2 + \frac{\omega^2}{c^2}\epsilon & 0 & 0 \\ 0 & -k^2 + \frac{\omega^2}{c^2}\epsilon & 0 \\ 0 & 0 & \frac{\omega^2}{c^2}\epsilon \end{bmatrix} \tag{5.23}$$

Take determinant:

$$\det \; \mid D \mid = \left(-k^2 + \frac{\omega^2}{c^2}\epsilon\right)^2 \frac{\omega^2}{c^2}\epsilon = 0. \tag{5.24}$$

Two possible types of solution to this dispersion relation:

(A)

$$-k^2 + \frac{\omega^2}{c^2}\epsilon = 0. \tag{5.25}$$

$$\Rightarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{\omega^2}{c^2}\epsilon \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = 0 \qquad \Rightarrow E_z = 0. \tag{5.26}$$

Electric field is *transverse* ($\mathbf{E}.\mathbf{k} = 0$)

Phase velocity of the wave is

$$\frac{\omega}{k} = \frac{c}{\sqrt{\epsilon}} \tag{5.27}$$

This is just like a regular EM wave traveling in a medium with *refractive index*

$$N \equiv \frac{kc}{\omega} = \sqrt{\epsilon} \;\; . \tag{5.28}$$

(B)

$$\frac{\omega^2}{c^2}\epsilon = 0 \qquad \text{i.e. } \epsilon = 0 \tag{5.29}$$

$$\Rightarrow \begin{pmatrix} D_{xx} & 0 & 0 \\ 0 & D_{yy} & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = 0 \Rightarrow E_x = E_y = 0. \tag{5.30}$$

Electric Field is *Longitudinal* ($\mathbf{E} \wedge \mathbf{k} = 0$) $\quad \mathbf{E} \parallel \mathbf{k}$.

This has no obvious counterpart in optics etc. because ϵ is not usually zero. In plasmas $\epsilon = 0$ *is* a relevant solution. Plasmas can support longitudinal waves.

5.1.2 General Case (k in z-direction)

$$\mathbf{D} = \frac{\omega^2}{c^2} \begin{bmatrix} -N^2 + \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{yx} & -N^2 + \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{zx} & \epsilon_{zy} & \epsilon_{zz} \end{bmatrix} \quad , \quad \left(N^2 = \frac{k^2 c^2}{\omega^2} \right) \tag{5.31}$$

When we take determinant we shall get a quadratic in N^2 (for given ω) provided $\boldsymbol{\epsilon}$ is not explicitly dependent on k. So for any ω there are two values of N^2. Two 'modes'. The polarization $\mathbf{E}$ of these modes will be in general partly longitudinal and partly transverse. *The point:* separation into distinct longitudinal and transverse modes is not possible in anisotropic media (e.g. plasma with B_o).

All we have said applies to general linear medium (crystal, glass, dielectric, plasma). Now we have to get the correct expression for $\boldsymbol{\sigma}$ and hence $\boldsymbol{\epsilon}$ by analysis of the plasma (fluid) equations.

5.2 High Frequency Plasma Conductivity

We want, now, to calculate the current for given (Fourier) electric field $\mathbf{E}(\mathbf{k}, \omega)$, to get the conductivity, $\boldsymbol{\sigma}$. It won't be the same as the DC conductivity which we calculated before (for collisions) because the *inertia* of the species will be important. In fact, provided

$$\omega \gg \bar{\nu}_{ei} \tag{5.32}$$

we can ignore collisions altogether. Do this for simplicity, although this approach can be generalized.

Also, under many circumstances we can ignore the pressure force $-\nabla p$. In general will be true if $\frac{\omega}{k} \gg v_{te,i}$ We take the plasma equilibrium to be at rest: $\mathbf{v}_o = 0$. This gives a manageable problem with wide applicability.

Approximations:

$$\begin{array}{ll} \text{Collisionless} & \bar{\nu}_{ei} = 0 \\ \text{`Cold Plasma'} & \nabla p = 0 \quad (e.g.\ T \simeq 0) \\ \text{Stationary Equil} & \mathbf{v}_o = 0 \end{array} \tag{5.33}$$

5.2.1 Zero B-field case

To start with take $\mathbf{B}_o = 0$: *Plasma isotropic* Momentum equation (for electrons first)

$$mn\left[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}.\nabla)\mathbf{v}\right] = nq\mathbf{E} \tag{5.34}$$

Notice the characteristic of the cold plasma approx. that we can cancel n from this equation and on linearizing get essentially the single particle equation.

$$m\frac{\partial \mathbf{v}_1}{\partial t} = q\mathbf{E} \qquad \text{(Drop the 1 suffix now).} \tag{5.35}$$

This can be solved for given ω as

$$\mathbf{v} = \frac{q}{-i\omega m}\mathbf{E} \tag{5.36}$$

and the current (due to this species, electrons) is

$$\mathbf{j} = nq\mathbf{v} = \frac{nq^2}{-i\omega m}\mathbf{E} \tag{5.37}$$

So the conductivity is

$$\sigma = i\frac{nq^2}{\omega m} \tag{5.38}$$

Hence dielectric constant is

$$\epsilon = 1 + \frac{i}{\omega\epsilon_o}\sigma = 1 - \left(\frac{nq^2}{m\epsilon_o}\right)\frac{1}{\omega^2} = 1 + \chi \tag{5.39}$$

Longitudinal Waves ($\mathbf{B}_o = 0$)

Dispersion relation we know is

$$\epsilon = 0 = 1 - \left(\frac{nq^2}{m\epsilon_o}\right)\frac{1}{\omega^2} \tag{5.40}$$

[Strictly, the ϵ we want here is the total ϵ including both electron and ion contributions to the conductivity. But

$$\frac{\sigma_e}{\sigma_i} \simeq \frac{m_i}{m_e} \quad \text{(for z = 1)} \tag{5.41}$$

so to a first approximation, ignore ion motions.]
Solution

$$\omega^2 = \left(\frac{n_e q_e^2}{m_e \epsilon_o}\right). \tag{5.42}$$

In this approx. longitudinal oscillations of the electron fluid have a single unique frequency:

$$\omega_p = \left(\frac{n_e e^2}{m_e \epsilon_o}\right)^{\frac{1}{2}}. \tag{5.43}$$

This is called the *'Plasma Frequency'* (more properly ω_{pe} the *'electron'* plasma frequency). If we allow for *ion motions* we get an ion conductivity

$$\sigma_i = \frac{i n_i q_i^2}{\omega m_i} \tag{5.44}$$

and hence

$$\begin{aligned} \epsilon_{\text{tot}} &= 1 + \frac{i}{\omega\epsilon_o}(\sigma_e + \sigma_i) = 1 - \left(\frac{n_e q_e^2}{\epsilon_o m_e} + \frac{n_i q_i^2}{\epsilon_o m_i}\right)\frac{1}{\omega^2} \\ &= 1 - \left(\omega_{pe}^2 + \omega_{pi}^2\right)/\omega^2 \end{aligned} \tag{5.45}$$

where

$$\omega_{pi} \equiv \left(\frac{n_i q_i^2}{\epsilon_o m_i}\right)^{\frac{1}{2}} \tag{5.46}$$

is the 'Ion Plasma Frequency'.

Simple Derivation of Plasma Oscillations

Take ions stationary; perturb a slab of plasma by shifting electrons a distance x. Charge built up is $n_e q x$ per unit area. Hence electric field generated

$$E = -\frac{n_e q_e x}{\epsilon_o} \tag{5.47}$$

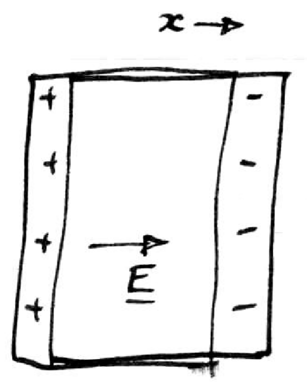

Figure 5.1: Slab derivation of plasma oscillations

Equation of motion of electrons

$$m_e \frac{dv}{dt} = -\frac{n_e q_e^2 x}{\epsilon_o}; \tag{5.48}$$

i.e.

$$\frac{d^2x}{dt^2} + \left(\frac{n_e q_e^2}{\epsilon_o m_e}\right) x = 0 \tag{5.49}$$

Simple harmonic oscillator with frequency

$$\omega_{pe} = \left(\frac{n_e q_e^2}{\epsilon_o m_e}\right)^{\frac{1}{2}} \qquad \text{Plasma Frequency.} \tag{5.50}$$

The Characteristic Frequency of Longitudinal Oscillations in a plasma. Notice

1. $\omega = \omega_p$ for all k in this approx.
2. Phase velocity $\frac{\omega}{k}$ can have any value.
3. Group velocity of wave, which is the velocity at which information/energy travel is

$$v_g = \frac{d\omega}{dk} = 0 \text{ !!} \tag{5.51}$$

In a way, these oscillations can hardly be thought of as a 'proper' wave because they do not transport energy or information. (In Cold Plasma Limit). [Nevertheless they do emerge from the wave analysis and with less restrictive approxs do have finite v_g.]

Transverse Waves ($B_o = 0$)

Dispersion relation:

$$-k^2 + \frac{\omega^2}{c^2}\epsilon = 0 \tag{5.52}$$

or

$$
\begin{aligned}
N^2 &\equiv \frac{k^2c^2}{\omega^2} = \epsilon = 1 - \left(\omega_{pe}^2 + \omega_{pi}^2\right)/\omega^2 \\
&\simeq 1 - \omega_{pe}^2/\omega^2
\end{aligned} \tag{5.53}
$$

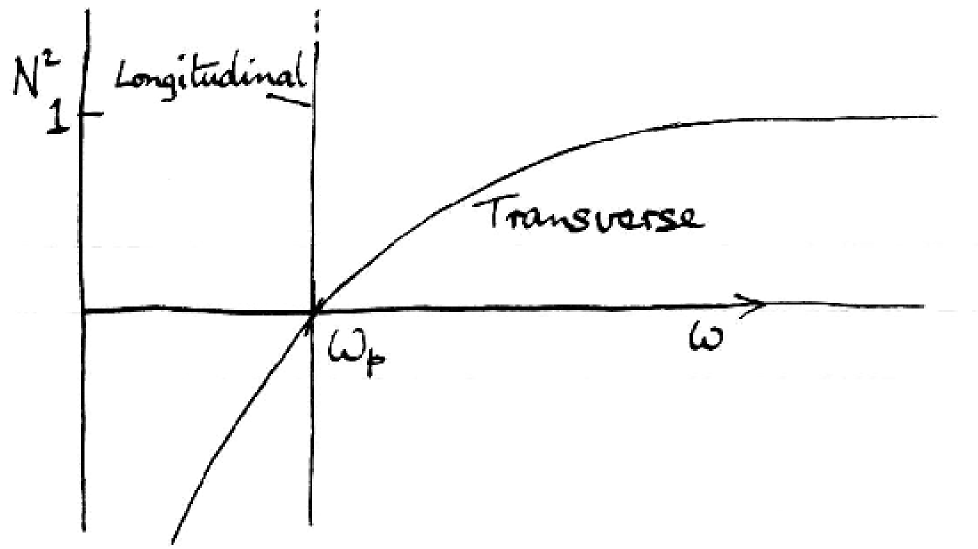

Figure 5.2: Unmagnetized plasma transverse wave.

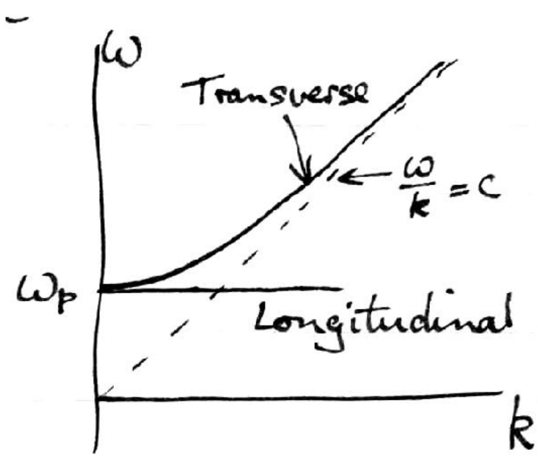

Figure 5.3: Alternative dispersion plot.

Alternative expression:

$$
-k^2 + \frac{\omega^2}{c^2} - \frac{\omega_p^2}{c^2} = 0 \tag{5.54}
$$

which implies

$$\omega^2 = \omega_p^2 + k^2c^2 \quad (5.55)$$

$$\omega = \left(\omega_p^2 + k^2c^2\right)^{\frac{1}{2}}. \quad (5.56)$$

5.2.2 Meaning of Negative N^2: Cut Off

When $N^2 < 0$ (for $\omega < \omega_p$) this means N is pure imaginary and hence so is k for real ω. Thus the wave we have found goes like

$$\exp\{\pm \mid k \mid x - i\omega t\} \quad (5.57)$$

i.e. its space dependence is exponential *not oscillatory*. Such a wave is said to be *'Evanescent'* or 'Cut Off'. It does not truly propagate through the medium but just damps exponentially.

Example:

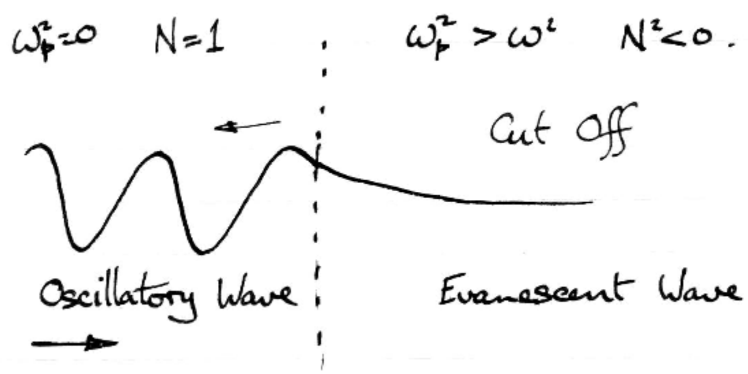

Figure 5.4: Wave behaviour at cut-off.

A wave incident on a plasma with $\omega_p^2 > \omega^2$ is simply reflected, no energy is transmitted through the plasma.

5.3 Cold Plasma Waves (Magnetized Plasma)

Objective: calculate $\boldsymbol{\epsilon}$,$\mathbf{D}, \mathbf{k}(\omega)$, using known plasma equations.

Approximation: Ignore thermal motion of particles.

Applicability: Most situations where (1) plasma pressure and (2) absorption are negligible. Generally requires wave phase velocity $\gg v_{\text{thermal}}$.

5.3.1 Derivation of Dispersion Relation

Can "derive" the cold plasma approx from fluid plasma equations. Simpler just to say that all particles (of a specific species) just move together obeying Newton's 2nd law:

$$m\frac{\partial \mathbf{v}}{\partial t} = q(\mathbf{E} + \mathbf{v} \wedge \mathbf{B}) \tag{5.58}$$

Take the background plasma to have $\mathbf{E}_0 = 0$, $\mathbf{B} = \mathbf{B}_0$ and zero velocity. Then all motion is due to the wave and also the wave's magnetic field can be ignored provided the particle speed stays small. (This is a linearization).

$$m\frac{\partial \mathbf{v}}{\partial t} = q(\mathbf{E} + \mathbf{v} \wedge \mathbf{B}_0), \tag{5.59}$$

where $\mathbf{v}$, $\mathbf{E} \propto \exp i(\mathbf{k}.\mathbf{x} - \omega t)$ are wave quantities.

Substitute $\frac{\partial}{\partial t} \to -i\omega$ and write out equations. Choose axes such that $\mathbf{B}_0 = B_0(0,0,1)$.

$$\begin{aligned} -i\omega m v_x &= q(E_x + v_y B_0) \\ -i\omega m v_y &= q(E_y - v_x B_0) \\ -i\omega m v_z &= qE_z \end{aligned} \tag{5.60}$$

Solve for $\mathbf{v}$ in terms of $\mathbf{E}$.

$$\begin{aligned} v_x &= \frac{q}{m}\left(\frac{i\omega E_x - \Omega E_y}{\omega^2 - \Omega^2}\right) \\ v_y &= \frac{q}{m}\left(\frac{\Omega E_x + i\omega E_y}{\omega^2 - \Omega^2}\right) \\ v_z &= \frac{q}{m}\frac{i}{\omega}E_z \end{aligned} \tag{5.61}$$

where $\Omega = \frac{qB_0}{m}$ is the gyrofrequency but its sign is that of the charge on the particle species under consideration.

Since the current is $\mathbf{j} = q\mathbf{v}n = \boldsymbol{\sigma}.\mathbf{E}$ we can identify the conductivity tensor for the species (j) as:

$$\boldsymbol{\sigma}_j = \begin{bmatrix} \frac{q_j^2 n_j}{m_j}\frac{i\omega}{\omega^2-\Omega_j^2} & -\frac{q_j^2 n_j}{m_j}\frac{\Omega_j}{\omega^2-\Omega_j^2} & 0 \\ \frac{q_j^2 n_j}{m_j}\frac{\Omega_j}{\omega^2-\Omega_j^2} & \frac{q_j^2 n_j}{m_j}\frac{i\omega}{\omega^2-\Omega_j^2} & 0 \\ 0 & 0 & \frac{iq_j^2}{m}\frac{n_j}{\omega} \end{bmatrix} \tag{5.62}$$

The total conductivity, due to all species, is the sum of the conductivities for each

$$\boldsymbol{\sigma} = \sum_j \boldsymbol{\sigma}_j \tag{5.63}$$

So

$$\sigma_{xx} = \sigma_{yy} = \sum_j \frac{q_1^2 n_j}{m_j}\frac{i\omega}{\omega^2 - \Omega_j^2} \tag{5.64}$$

$$\sigma_{xy} = -\sigma_{yx} = -\sum_j \frac{q_j^2 n_j}{m_j} \frac{\Omega_j}{\omega^2 - \Omega_j^2} \tag{5.65}$$

$$\sigma_{zz} = \sum_j \frac{q_j^2 n_j}{m_j} \frac{i}{\omega} \tag{5.66}$$

Susceptibility $\chi = \frac{1}{-i\omega\epsilon_o}\boldsymbol{\sigma}$.

$$\boldsymbol{\epsilon} = \begin{bmatrix} \epsilon_{xx} & \epsilon_{xy} & 0 \\ \epsilon_{yx} & \epsilon_{yy} & 0 \\ 0 & 0 & \epsilon_{zz} \end{bmatrix} = \begin{bmatrix} S & -iD & 0 \\ iD & S & 0 \\ 0 & 0 & P \end{bmatrix} \tag{5.67}$$

where

$$\epsilon_{xx} = \epsilon_{yy} = S = 1 - \sum_j \frac{\omega_{pj}^2}{\omega^2 - \Omega_j^2} \tag{5.68}$$

$$i\epsilon_{xy} = -i\epsilon_{yx} = D = \sum_j \frac{\Omega_j}{\omega} \frac{\omega_{pj}^2}{\omega^2 - \Omega_j^2} \tag{5.69}$$

$$\epsilon_{zz} = P = 1 - \sum_j \frac{\omega_{pj}^2}{\omega^2} \tag{5.70}$$

and

$$\omega_{pj}^2 \equiv \frac{q_j^2 n_j}{\epsilon_o m_j} \tag{5.71}$$

is the "plasma frequency" for that species.

S & D stand for "Sum" and "Difference":

$$S = \frac{1}{2}(R + L) \qquad D = \frac{1}{2}(R - L) \tag{5.72}$$

where R & L stand for "Right-hand" and "Left-hand" and are:

$$R = 1 - \sum_j \frac{\omega_{pj}^2}{\omega(\omega + \Omega_j)} \quad , \quad L = 1 - \sum_j \frac{\omega_{pj}^2}{\omega(\omega - \Omega_j)} \tag{5.73}$$

The R & L terms arise in a derivation based on expressing the field in terms of rotating polarizations (right & left) rather than the direct Cartesian approach.

We now have the dielectric tensor from which to obtain the dispersion relation and solve it to get $\mathbf{k}(\omega)$ and the polarization. Notice, first, that $\boldsymbol{\epsilon}$ is indeed independent of $\mathbf{k}$ so the dispersion relation (for given ω) is a quadratic in N^2 (or k^2).

Choose convenient axes such that $k_y = N_y = 0$. Let θ be angle between $\mathbf{k}$ and $\mathbf{B}_0$ so that

$$N_z = N\cos\theta \ , \qquad N_x = N\sin\theta \ . \tag{5.74}$$

Then

$$\mathbf{D} = \begin{bmatrix} -N^2\cos^2\theta + S & -iD & N^2\sin\theta\cos\theta \\ +iD & -N^2 + S & 0 \\ N^2\sin\theta\cos\theta & 0 & -N^2\sin^2\theta + P \end{bmatrix} \tag{5.75}$$

and

$$\| \mathbf{D} \| = AN^4 - BN^2 + C \tag{5.76}$$

where

$$A \equiv S\sin^2\theta + P\cos^2\theta \tag{5.77}$$
$$B \equiv RL\sin^2\theta + PS(1+\cos^2\theta) \tag{5.78}$$
$$C \equiv PRL \tag{5.79}$$

Solutions are

$$N^2 = \frac{B \pm F}{2A}, \tag{5.80}$$

where the discriminant, F, is given by

$$F^2 = (RL - PS)^2 \sin^4\theta + 4P^2D^2\cos^2\theta \tag{5.81}$$

after some algebra. This is often, for historical reasons, written in the equivalent form (called the Appleton-Hartree dispersion relation)

$$N^2 = 1 - \frac{2(A - B + C)}{2A - B \pm F} \tag{5.82}$$

The quantity F^2 is generally $+ve$, so N^2 is real $\Rightarrow$ "propagating" or "evanescent" *no* wave absorption for cold plasma.

Solution can also be written

$$\tan^2\theta = -\frac{P\,(N^2 - R)\,(N^2 - L)}{(SN^2 - RL)\,(N^2 - P)} \tag{5.83}$$

This compact form makes it easy to identify the dispersion relation at $\theta = 0$ & $\frac{\pi}{2}$ i.e. parallel and perpendicular propagation $\tan\theta = 0, \infty$.

Parallel: $P = 0$, $N^2 = R$ $N^2 = L$

Perp: $N^2 = \frac{RL}{S}$ $N^2 = P$.

Example: Right-hand wave

$N^2 = R$. (Single Ion Species).

$$N^2 = 1 - \frac{\omega_{pe}^2}{\omega\,(\omega - |\Omega_e|)} - \frac{\omega_{pi}^2}{\omega\,(\omega + |\Omega_i|)} \tag{5.84}$$

This has a wave resonance $N^2 \to \infty$ at $\omega = |\Omega_e|$, only. Right-hand wave also has a cutoff at $R = 0$, whose solution proves to be

$$\omega = \omega_R = \frac{|\Omega_e| - |\Omega_i|}{2} + \left[\left(\frac{|\Omega_e| + |\Omega_i|}{2}\right)^2 + \omega_{pe}^2 + \omega_{pi}^2\right]^{1/2} \tag{5.85}$$

Since $m_i \gg m_e$ this can be approximated as:

$$\omega_R \simeq \frac{|\Omega_e|}{2}\left\{1+\left(1+4\frac{\omega_{pe}^2}{|\Omega_e|^2}\right)^{\frac{1}{2}}\right\} \tag{5.86}$$

This is always above $|\Omega_e|$.

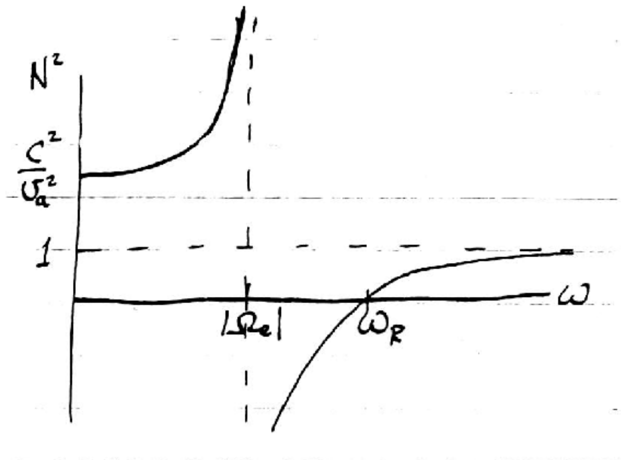

Figure 5.5: The form of the dispersion relation for RH wave.

One can similarly investigate LH wave and perp propagating waves. The resulting wave resonances and cut-offs depend only upon 2 properties (for specified ion mass) (1) Density $\leftrightarrow \omega_{pe}^2$ (2) Magnetic Field $\leftrightarrow |\Omega_e|$. [Ion values ω_{pi}, $|\Omega_i|$ are got by $\frac{m_i}{m_e}$ factors.]

These resonances and cutoffs are often plotted on a 2-D plane $\frac{|\Omega_e|}{\omega}$, $\frac{\omega_p^2}{\omega^2}$ ($\propto B, n$) called the C M A Diagram.

We don't have time for it here.

5.3.2 Hybrid Resonances Perpendicular Propagation

"Extraordinary" wave $\quad N^2 = \frac{RL}{S}$

$$N^2 = \frac{\left[(\omega+\Omega_e)(\omega+\Omega_i) - \frac{\omega_{pe}^2}{\omega}(\omega+\Omega_i) - \frac{\omega_{pi}^2}{\omega}(\omega+\Omega_e)\right]\left[(\omega-\Omega_e)(\omega-\Omega_i) - \frac{\omega_{pe}^2}{\omega}(\omega-\Omega_i)\ldots\right]}{(\omega^2-\Omega_e^2)(\omega^2-\Omega_i^2) - \omega_{pe}^2(\omega^2-\Omega_i^2) - \omega_{pi}^2(\omega^2-\Omega_e^2)} \tag{5.87}$$

Resonance is where denominator = 0. Solve the quadratic in ω^2 and one gets

$$\omega^2 = \frac{\omega_{pe}^2+\Omega_e^2+\omega_{pi}^2+\Omega_i^2}{2} \pm \sqrt{\left(\frac{\omega_{pe}^2+\Omega_e^2-\omega_{pi}^2-\Omega_i^2}{2}\right)^2 + \omega_{pe}^2\omega_{pi}^2} \tag{5.88}$$

Neglecting terms of order $\frac{m_e}{m_i}$ (e.g. $\frac{\omega_{pi}^2}{\omega_{pe}^2}$) one gets solutions

$$\omega_{UH}^2 = \omega_{pe}^2 + \Omega_e^2 \qquad \text{Upper Hybrid Resonance.} \tag{5.89}$$

$$\omega_{LH}^2 = \frac{\Omega_e^2 \omega_{pi}^2}{\Omega_e^2 + \omega_{pe}^2} \qquad \text{Lower Hybrid Resonance..} \tag{5.90}$$

At very high density, $\omega_{pe}^2 \gg \Omega_e^2$

$$\omega_{LH}^2 \simeq |\Omega_e||\Omega_i| \tag{5.91}$$

geometric mean of cyclotron frequencies.
At very low density, $\omega_{pe}^2 \ll \Omega_e^2$

$$\omega_{LH}^2 \simeq \omega_{pi}^2 \tag{5.92}$$

ion plasma frequency
Usually in tokamaks $\omega_{pe}^2 \sim \Omega_e^2$. Intermediate.

Summary Graph ($\Omega > \omega_p$)

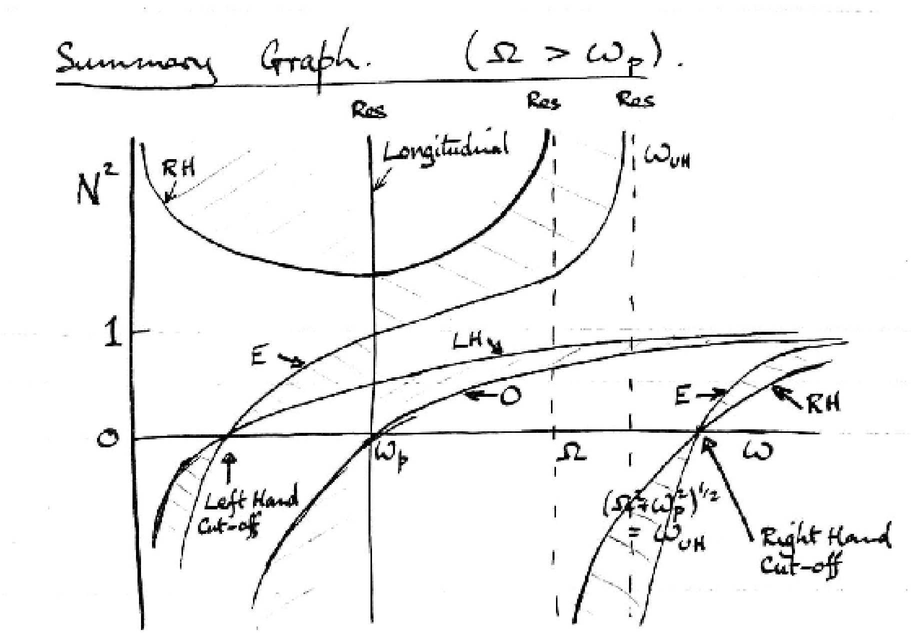

Figure 5.6: Summary of magnetized dispersion relation

Cut-offs are where $N^2 = 0$.
Resonances are where $N^2 \to \infty$.
Intermediate angles of propagation have refractive indices between the $\theta = 0$, $\frac{\pi}{2}$ lines, in the shaded areas.

5.3.3 Whistlers

(Ref. R.A. Helliwell, "Whistlers & Related Ionospheric Phenomena," Stanford UP 1965.)
For $N^2 \gg 1$ the right hand wave can be written

$$N^2 \simeq \frac{-\omega_{pe}^2}{\omega\left(\omega - |\Omega_e|\right)} \quad , \qquad (N = kc/\omega) \tag{5.93}$$

Group velocity is

$$v_g = \frac{d\omega}{dk} = \left(\frac{dk}{d\omega}\right)^{-1} = \left[\frac{d}{d\omega}\left(\frac{N\omega}{c}\right)\right]^{-1}. \tag{5.94}$$

Then since

$$N = \frac{\omega_p}{\omega^{\frac{1}{2}}\left(|\Omega_e| - \omega\right)^{\frac{1}{2}}} \quad , \tag{5.95}$$

we have

$$\begin{aligned} \frac{d}{d\omega}(N\omega) &= \frac{d}{d\omega}\frac{\omega_p \omega^{\frac{1}{2}}}{\left(|\Omega_e| - \omega\right)^{\frac{1}{2}}} = \omega_p\left\{\frac{\frac{1}{2}}{\omega^{\frac{1}{2}}\left(|\Omega_e| - \omega\right)^{\frac{1}{2}}} + \frac{\frac{1}{2}\omega^{\frac{1}{2}}}{\left(|\Omega_e| - \omega\right)^{\frac{3}{2}}}\right\} \\ &= \frac{\omega_p/2}{\left(|\Omega_e| - \omega\right)^{\frac{3}{2}}\omega^{\frac{1}{2}}}\left\{\left(|\Omega_e| - \omega\right) + \omega\right\} \\ &= \frac{\omega_p|\Omega_e|/2}{\left(|\Omega_e| - \omega\right)^{\frac{3}{2}}\omega^{\frac{1}{2}}} \end{aligned} \tag{5.96}$$

Thus

$$v_g = \frac{c\,2\left(|\Omega_e| - \omega\right)^{\frac{3}{2}}\omega^{\frac{1}{2}}}{\omega_p|\Omega_e|} \tag{5.97}$$

Group Delay is

$$\frac{L}{v_g} \propto \frac{1}{\omega^{\frac{1}{2}}\left(|\Omega_e| - \omega\right)^{\frac{3}{2}}} \propto \frac{1}{\left(\frac{\omega}{|\Omega_e|}\right)^{\frac{1}{2}}\left(1 - \frac{\omega}{|\Omega_e|}\right)^{\frac{3}{2}}} \tag{5.98}$$

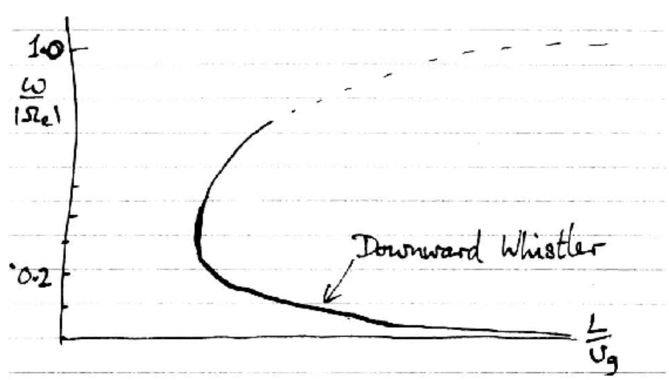

Figure 5.7: Whistler delay plot

Plot with $\frac{L}{v_g}$ as x-axis.

Resulting form explains downward whistle.

Lightning strike $\sim$ δ-function excites all frequencies.

Lower ones arrive later.

Examples of actual whistler sounds can be obtained from `http://www-istp.gsfc.nasa.gov/istp/polar/polar_pwi_sounds.html`.

5.4 Thermal Effects on Plasma Waves

The cold plasma approx is only good for high frequency, $N^2 \sim 1$ waves. If ω is low or $N^2 \gg 1$ one may have to consider thermal effects. From the fluid viewpoint, this means *pressure*. Write down the momentum equation. (We shall go back to $B_0 = 0$) linearized

$$mn\frac{\partial \mathbf{v}_1}{\partial t} = nq\mathbf{E}_1 - \nabla p_1 \quad ; \tag{5.99}$$

remember these are the perturbations:

$$p = p_0 + p_1 \quad . \tag{5.100}$$

Fourier Analyse (drop 1's)

$$mn(-i\omega)\mathbf{v} = nq\mathbf{E} - i\mathbf{k}p \tag{5.101}$$

The *key question*: how to relate p to $\mathbf{v}$

Answer: *Equation of state + Continuity*

State

$$pn^{-\gamma} = \text{const.} \;\Rightarrow (p_0 + p_1)\,(n_0 + n_1)^{-\gamma} = p_0 n_0^{-\gamma} \tag{5.102}$$

Use Taylor Expansion

$$(p_0 + p_1)\,(n_0 + n_1)^{-\gamma} \simeq p_0 n_0^{-\gamma}\left[1 + \frac{p_1}{p_0} - \gamma\frac{n_1}{n_0}\right] \tag{5.103}$$

Hence

$$\frac{p_1}{p_0} = \gamma\frac{n_1}{n_0} \tag{5.104}$$

Continuity

$$\frac{\partial n}{\partial t} + \nabla.\,(n\mathbf{v}) = 0 \tag{5.105}$$

Linearise:

$$\frac{\partial n_1}{\partial t} + \nabla.\,(n_0\mathbf{v}_1) = 0 \quad \Rightarrow \quad \frac{\partial n}{\partial t} + n_0\nabla.\mathbf{v} = 0 \tag{5.106}$$

Fourier Transform

$$-\,i\omega n_1 + n_0 i\mathbf{k}.\mathbf{v}_1 = 0 \tag{5.107}$$

i.e.

$$n_1 = n_0 \frac{\mathbf{k}.\mathbf{v}}{\omega} \tag{5.108}$$

Combine State & Continuity

$$p_1 = p_0 \gamma \frac{n_1}{n_0} = p_0 \gamma \frac{n_0 \frac{\mathbf{k}.\mathbf{v}}{\omega}}{n_o} = p_0 \gamma \frac{\mathbf{k}.\mathbf{v}}{\omega} \tag{5.109}$$

Hence Momentum becomes

$$mn\left(-i\omega\right)\mathbf{v} = nq\mathbf{E} - \frac{ikp_0\gamma}{\omega}\mathbf{k}.\mathbf{v} \tag{5.110}$$

Notice *Transverse waves* have $\mathbf{k}.\mathbf{v} = 0$; so they are *unaffected by pressure*.

Therefore we need only consider the longitudinal wave. However, for consistency let us proceed as before to get the dielectric tensor etc.

Choose axes such that $\mathbf{k} = k\hat{\mathbf{e}}_z$ then obviously:

$$v_x = \frac{iq}{\omega m} E_x \qquad v_y = \frac{iq}{\omega m} E_y \tag{5.111}$$

$$v_z = \frac{q}{m} \frac{E_z}{-i\omega + (ik^2\gamma p_0/mn\omega)} \tag{5.112}$$

Hence

$$\boldsymbol{\sigma} = \frac{inq^2}{\omega m} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{1-\frac{k^2 p_0 \gamma}{mn\omega^2}} \end{bmatrix} \tag{5.113}$$

$$\boldsymbol{\epsilon} = 1 + \frac{i\boldsymbol{\sigma}}{\epsilon_0 \omega} = \begin{bmatrix} 1-\frac{\omega_p^2}{\omega^2} & 0 & 0 \\ 0 & 1-\frac{\omega_p^2}{\omega^2} & 0 \\ 0 & 0 & 1-\frac{w_p^2}{\omega^2 - k^2 \frac{p_0\gamma}{mn}} \end{bmatrix} \tag{5.114}$$

(Taking account only of 1 species, electrons, for now.)

We have confirmed the previous comment that the transverse waves (E_x, E_y) are unaffected. The longitudinal wave *is*. Notice that $\boldsymbol{\epsilon}$ now depends on $\mathbf{k}$ as well as ω. This is called *'spatial dispersion'*.

For completeness, note that the dielectric tensor can be expressed in general tensor notation as

$$\begin{aligned} \boldsymbol{\epsilon} &= \mathbf{1} - \frac{\omega_p^2}{\omega^2}\left(\mathbf{1} + \mathbf{kk}\left[\frac{1}{1-\frac{k^2}{\omega^2}\frac{p_0\gamma}{mn}} - 1\right]\right) \\ &= \mathbf{1} - \frac{\omega_p^2}{\omega^2}\left(\mathbf{1} + \mathbf{kk}\frac{1}{\frac{w^2 mn}{k^2 p_0 \gamma} - 1}\right) \end{aligned} \tag{5.115}$$

This form shows isotropy with respect to the medium: there is no preferred direction in space for the wave vector $\mathbf{k}$.

But once $\mathbf{k}$ is chosen, $\boldsymbol{\epsilon}$ is not isotropic. The direction of $\mathbf{k}$ becomes a special direction.

Longitudinal Waves: dispersion relation is

$$\epsilon_{zz} = 0 \qquad \text{(as before)} \tag{5.116}$$

which is

$$1 - \frac{\omega_p^2}{\omega^2 - \frac{k^2 p_0 \gamma}{mn}} = 0 \quad . \tag{5.117}$$

or

$$\omega^2 = \omega_p^2 + k^2 \frac{p_0 \gamma}{mn} \tag{5.118}$$

Recall $p_0 = n_0 T = nT$; so this is usually written:

$$\omega^2 = \omega_p^2 + k^2 \frac{\gamma T}{m} = \omega_p^2 + k^2 \gamma v_t^2 \tag{5.119}$$

[The appropriate value of γ to take is 1 dimensional adiabatic i.e. $\gamma = 3$. This seems plausible since the electron motion is 1-d (along k) and may be demonstrated more rigorously by kinetic theory.]

The above dispersion relation is called the *Bohm-Gross* formula for *electron plasma waves*. Notice the group velocity:

$$v_g = \frac{d\omega}{dk} = \frac{1}{2\omega}\frac{d\omega^2}{dk} = \frac{\gamma k v_t^2}{\left(\omega_p^2 + \gamma k^2 v_t^2\right)^{\frac{1}{2}}} \neq 0. \tag{5.120}$$

and for $kv_t > \omega_p$ this tends to $\gamma^{\frac{1}{2}} v_t$. In this limit energy travels at the electron thermal speed.

5.4.1 Refractive Index Plot

Bohm Gross electron plasma waves:

$$N^2 = \frac{c^2}{\gamma_e \mathbf{v}_{te}^2}\left(1 - \frac{\omega_p^2}{\omega^2}\right) \tag{5.121}$$

Transverse electromagnetic waves:

$$N^2 = \left(1 - \frac{\omega_p^2}{\omega^2}\right) \tag{5.122}$$

These have just the same shape except the electron plasma waves have much larger vertical scale:

On the E-M wave scale, the plasma wave curve is nearly vertical. In the cold plasma it was *exactly* vertical.

We have relaxed the Cold Plasma approximation.

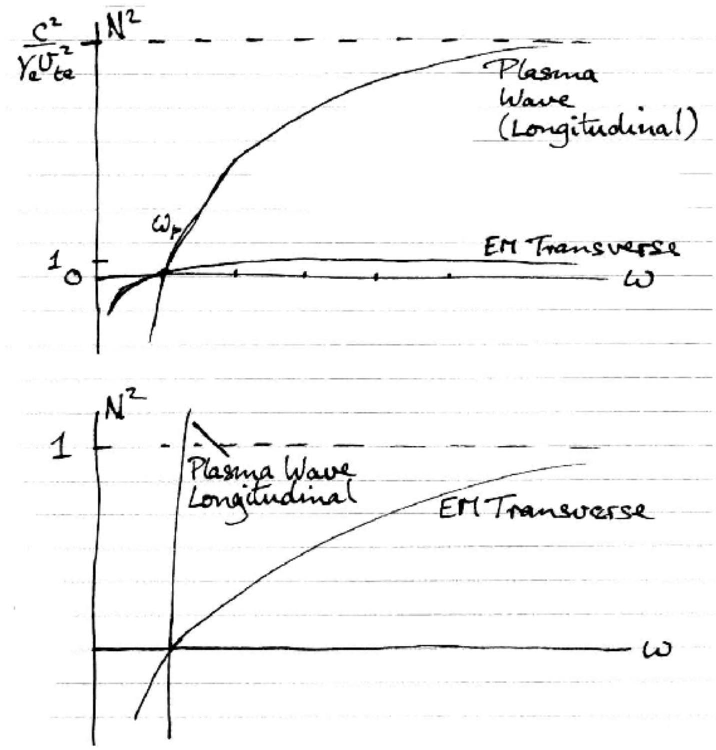

Figure 5.8: Refractive Index Plot. Top plot on the scale of the Bohm-Gross Plasma waves. Bottom plot, on the scale of the E-M transverse waves

5.4.2 Including the ion response

As an example of the different things which can occur when ions are allowed to move include longitudinal ion response:

$$0 = \epsilon_{zz} = 1 - \frac{\omega_{pe}^2}{\omega^2 - \frac{k^2 p_e \gamma_e}{m_e n_e}} - \frac{\omega_{pi}^2}{\omega^2 - \frac{k^2 p_i \gamma_i}{m_i n_i}} \tag{5.123}$$

This is now a quadratic equation for ω^2 so there are *two* solutions possible for a given ω. One will be in the vicinity of the electron plasma wave solution and the inclusion of ω_{pi}^2 which is $\ll \omega_{pe}^2$ will give a small correction.

Second solution will be where the third term is same magnitude as second (both will be $\gg 1$). This will be at low frequency. So we may write the dispersion relation approximately as:

$$-\frac{\omega_{pi}^2}{-\frac{k^2 p_e \gamma_e}{m_e n_e}} - \frac{\omega_{pi}^2}{\omega^2 - \frac{k^2 p_i \gamma_i}{m_i n_i}} = 0 \tag{5.124}$$

i.e.

$$\omega^2 = \frac{k^2 p_i \gamma_i}{m_i n_i} + \frac{\omega_{pi}^2}{\omega_{pe}^2} \frac{k^2 p_e \gamma_e}{m_e n_e}$$

$$= \quad k^2 \left[\left(\frac{\gamma_i p_i}{n_i} + \frac{\gamma_e p_e}{n_e} \right) \frac{1}{m_i} \right]$$

$$= \quad k^2 \left[\frac{\gamma_i T_i + \gamma_e T_e}{m_i} \right] \tag{5.125}$$

[In this case the electrons have time to stream through the wave in 1 oscillation so they tend to be isothermal: i.e. $\gamma_e = 1$. What to take for γ_i is less clear, and less important because kinetic theory shows that these waves we have just found are strongly damped unless $T_i \ll T_e$.]

These are *'ion-acoustic'* or 'ion-sound' waves

$$\frac{\omega^2}{k^2} = c_s^2 \tag{5.126}$$

c_s is the sound speed

$$c_s^2 = \frac{\gamma_i T_i + T_e}{m_i} \simeq \frac{T_e}{m_i} \tag{5.127}$$

Approximately non-dispersive waves with phase velocity c_s.

5.5 Electrostatic Approximation for (Plasma) Waves

The dispersion relation is written generally as

$$\mathbf{N} \wedge (\mathbf{N} \wedge \mathbf{E}) + \boldsymbol{\epsilon}.\mathbf{E} = \mathbf{N}(\mathbf{N}.\mathbf{E}) - N^2\mathbf{E} + \boldsymbol{\epsilon}.\mathbf{E} = 0 \tag{5.128}$$

Consider $\mathbf{E}$ to be expressible as longitudinal and transverse components $\mathbf{E}_\ell$, $\mathbf{E}_t$ such that $\mathbf{N} \wedge \mathbf{E}_\ell = 0$, $\mathbf{N}.\mathbf{E}_t = 0$. Then the dispersion relation can be written

$$\mathbf{N}\left(\mathbf{N}.\mathbf{E}_\ell\right) - N^2\left(\mathbf{E}_\ell + \mathbf{E}_t\right) + \boldsymbol{\epsilon}.\left(\mathbf{E}_\ell + \mathbf{E}_t\right) = -N^2\mathbf{E}_t + \boldsymbol{\epsilon}.\mathbf{E}_t + \boldsymbol{\epsilon}.E_\ell = 0 \tag{5.129}$$

or

$$\left(N^2 - \boldsymbol{\epsilon}\right).\mathbf{E}_t = \boldsymbol{\epsilon}.\mathbf{E}_\ell \tag{5.130}$$

Now the electric field can always be written as the sum of a curl-free component plus a divergenceless component, e.g. conventionally

$$\mathbf{E} = \underbrace{-\nabla\phi}_{\substack{Curl-free \\ Electrostatic}} + \underbrace{\dot{\mathbf{A}}}_{\substack{Divergence-free \\ Electromagnetic}} \tag{5.131}$$

and these may be termed electrostatic and electromagnetic parts of the field.

For a plane wave, these two parts are clearly the same as the longitudinal and transverse parts because

$$-\nabla\phi = -i\mathbf{k}\phi \quad \text{is longitudinal} \tag{5.132}$$

and if $\nabla.\dot{\mathbf{A}} = 0$ (because $\nabla.\mathbf{A} = 0$ (w.l.o.g.)) then $\mathbf{k}.\dot{\mathbf{A}} = 0$ so $\dot{\mathbf{A}}$ is transverse.

'Electrostatic' waves are those that are describable by the electrostatic part of the electric field, which is the longitudinal part: $|E_\ell| \gg |E_t|$.

If we simply say $\mathbf{E}_t = 0$ then the dispersion relation becomes $\boldsymbol{\epsilon}.\mathbf{E}_\ell = 0$. This is *not* the most general dispersion relation for electrostatic waves. It is too restrictive. In general, there is a more significant way in which to get solutions where $|E_\ell| \gg |E_t|$. It is for N^2 to be very large compared to all the components of $\boldsymbol{\epsilon}$: $N^2 \gg\| \epsilon \|$.

If this is the case, then the dispersion relation is approximately

$$N^2\mathbf{E}_t = \boldsymbol{\epsilon}.\mathbf{E}_\ell \quad ; \tag{5.133}$$

$\mathbf{E}_t$ is small but not zero.

We can then annihilate the $\mathbf{E}_t$ term by taking the $\mathbf{N}$ component of this equation; leaving

$$\mathbf{N}.\boldsymbol{\epsilon}.\mathbf{E}_\ell = (\mathbf{N}.\boldsymbol{\epsilon}.\mathbf{N})\, E_\ell = 0 \quad : \quad \mathbf{k}.\boldsymbol{\epsilon}.\mathbf{k} = 0\ . \tag{5.134}$$

When the medium is isotropic there is no relevant difference between the electrostatic dispersion relation:

$$\mathbf{N}.\boldsymbol{\epsilon}.\mathbf{N} = 0 \tag{5.135}$$

and the purely longitudinal case $\boldsymbol{\epsilon}.\mathbf{N} = \mathbf{0}$. If we choose axes such that $\mathbf{N}$ is along $\hat{\mathbf{z}}$, then the medium's isotropy ensures the off-diagonal components of $\boldsymbol{\epsilon}$ are zero so $\mathbf{N}.\boldsymbol{\epsilon}.\mathbf{N} = 0$ requires $\epsilon_{zz} = 0 \Rightarrow \boldsymbol{\epsilon}.\mathbf{N} = \mathbf{0}$. However if the medium is *not* isotropic, then even if

$$\mathbf{N}.\boldsymbol{\epsilon}.\mathbf{N}\left(= N^2\epsilon_{zz}\right) = 0 \tag{5.136}$$

there may be off-diagonal terms of $\boldsymbol{\epsilon}$ that make

$$\boldsymbol{\epsilon}.\mathbf{N} \neq 0 \tag{5.137}$$

In other words, in an anisotropic medium (for example a magnetized plasma) the electrostatic approximation can give waves that have non-zero transverse electric field (of order $||\boldsymbol{\epsilon}||/N^2$ times E_ℓ) even though the waves are describable in terms of a scalar potential.

To approach this more directly, from Maxwell's equations, applied to a dielectric medium of dielectric tensor $\boldsymbol{\epsilon}$, the electrostatic part of the electric field is derived from the electric displacement

$$\nabla.\mathbf{D} = \nabla.\left(\epsilon_0\boldsymbol{\epsilon}.\mathbf{E}\right) = \rho = 0 \quad \text{(no free charges)} \tag{5.138}$$

So for plane waves $0 = \mathbf{k}.\mathbf{D} = \mathbf{k}.\boldsymbol{\epsilon}.\mathbf{E} = i\mathbf{k}.\boldsymbol{\epsilon}.\mathbf{k}\phi$.

The electric displacement, $\mathbf{D}$, is purely transverse (not zero) but the electric field, $\mathbf{E}$ then gives rise to an electromagnetic field via $\nabla \wedge \mathbf{H} = \partial\mathbf{D}/\partial t$. If $N^2 \gg\| \epsilon \|$ then this magnetic (inductive) component can be considered as a benign passive coupling to the electrostatic wave.

In summary, the electrostatic dispersion relation is $\mathbf{k}.\boldsymbol{\epsilon}.\mathbf{k} = 0$, or in coordinates where $\mathbf{k}$ is in the z-direction, $\epsilon_{zz} = 0$.

5.6 Simple Example of MHD Dynamics: Alfven Waves

Ignore Pressure & Resistance.

$$\rho \frac{D\mathbf{V}}{Dt} = \mathbf{j} \wedge \mathbf{B} \tag{5.139}$$

$$\mathbf{E} + \mathbf{V} \wedge \mathbf{B} = 0 \tag{5.140}$$

Linearize:

$$\mathbf{V} = \mathbf{V}_1, \quad \mathbf{B} = \mathbf{B}_0 + \mathbf{B}_1 \text{ } (\mathbf{B}_0 \text{ uniform}), \quad \mathbf{j} = \mathbf{j}_1. \tag{5.141}$$

$$\rho \frac{\partial \mathbf{V}}{\partial t} = \mathbf{j} \wedge \mathbf{B}_0 \tag{5.142}$$

$$\mathbf{E} + \mathbf{V} \wedge \mathbf{B}_0 = 0 \tag{5.143}$$

Fourier Transform:

$$\rho(-i\omega)\mathbf{V} = \mathbf{j} \wedge \mathbf{B}_0 \tag{5.144}$$

$$\mathbf{E} + \mathbf{V} \wedge \mathbf{B}_0 = 0 \tag{5.145}$$

Eliminate V by taking 5.144 $\wedge \mathbf{B}_0$ and substituting from 5.145.

$$\mathbf{E} + \frac{1}{-i\omega\rho} (\mathbf{j} \wedge \mathbf{B}_0) \wedge \mathbf{B}_0 = 0 \tag{5.146}$$

or

$$\mathbf{E} = -\frac{1}{-i\omega\rho} \{(\mathbf{j}.\mathbf{B}_0)\, \mathbf{B}_0 - B_0^2 \mathbf{j}\} = \frac{B_0^2}{-i\omega\rho} \mathbf{j}_\perp \tag{5.147}$$

So conductivity tensor can be written (z in B direction).

$$\boldsymbol{\sigma} = \frac{-i\omega\rho}{B_0^2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \infty \end{bmatrix} \tag{5.148}$$

where ∞ implies that $E_\parallel = 0$ (because of Ohm's law). Hence Dielectric Tensor

$$\boldsymbol{\epsilon} = 1 + \frac{\boldsymbol{\sigma}}{-i\omega\epsilon_0} = \left(1 + \frac{\rho}{\epsilon_0 B^2}\right) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \infty \end{bmatrix}. \tag{5.149}$$

Dispersion tensor in general is:

$$\mathbf{D} = \frac{\omega^2}{c^2} \left[\mathbf{NN} - N^2 + \boldsymbol{\epsilon}\right] \tag{5.150}$$

Dispersion Relation taking $N_\perp = N_x$, $N_y = 0$

$$|\mathbf{D}| = \left| \begin{bmatrix} -N_\parallel^2 + 1 + \frac{\rho}{\epsilon_0 B^2} & 0 & N_\perp N_\parallel \\ 0 & -N_\parallel^2 - N_\perp^2 + 1 + \frac{\rho}{\epsilon_0 B^2} & 0 \\ N_\perp N_\parallel & 0 & \infty \end{bmatrix} \right| = 0 \tag{5.151}$$

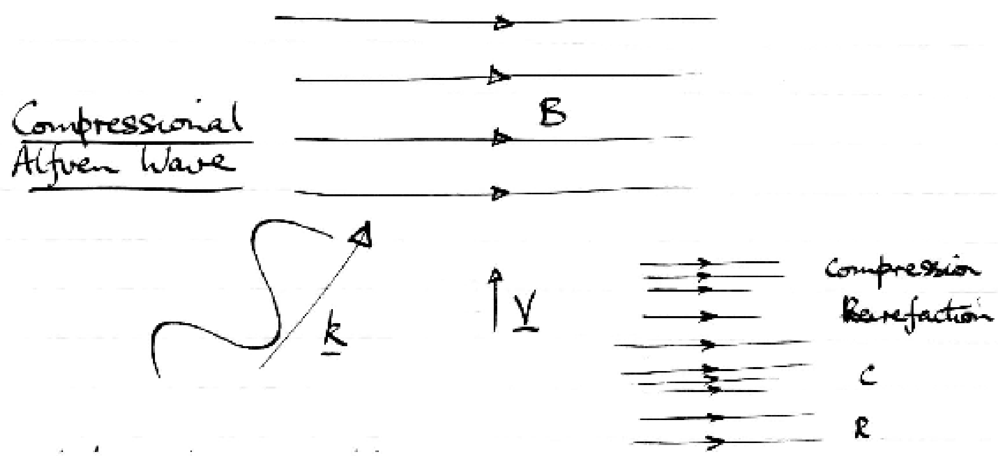

Figure 5.9: Compressional Alfven Wave. Works by magnetic pressure (primarily).

Meaning of ∞ is that the cofactor must be zero i.e.

$$\left(-N_\parallel^2 + 1 + \frac{\rho}{\epsilon_0 B^2}\right)\left(-N^2 + 1 + \frac{\rho}{\epsilon_0 B^2}\right) = 0 \tag{5.152}$$

The 1's here come from Maxwell displacement current and are usually negligible ($N_\perp^2 \gg 1$). So final waves are

1. $N^2 = \frac{\rho}{\epsilon_0 B^2} \Rightarrow$ Non-dispersive wave with phase and group velocities

$$v_p = v_g = \frac{c}{N} = \left(\frac{c^2\epsilon_0 B^2}{\rho}\right)^{\frac{1}{2}} = \left[\frac{B^2}{\mu_0\rho}\right]^{\frac{1}{2}} \tag{5.153}$$

where we call

$$\left[\frac{B^2}{\mu_0\rho}\right]^{\frac{1}{2}} \equiv v_A \qquad \text{the 'Alfven Speed'} \tag{5.154}$$

Polarization:

$$E_\parallel = E_z = 0, \quad E_x = 0. \quad E_y \neq 0 \quad \Rightarrow V_y = 0 \quad V_x \neq 0 \quad (V_z = 0) \tag{5.155}$$

Party longitudinal (velocity) wave $\rightarrow$ Compression *"Compressional Alfven Wave"*.

2. $N_\parallel^2 = \frac{\rho}{\epsilon_0 B^2} = \frac{k_\parallel^2 c^2}{\omega^2}$
Any ω has unique $k_\parallel$. Wave has unique velocity in $\parallel$ direction: v_A.
Polarization

$$E_z = E_y = 0 \quad E_x \neq 0 \quad \Rightarrow V_x = 0 \quad V_y \neq 0 \quad (V_z = 0) \tag{5.156}$$

Transverse velocity: *"Shear Alfven Wave"*.

Works by field line bending (Tension Force) (no compression).

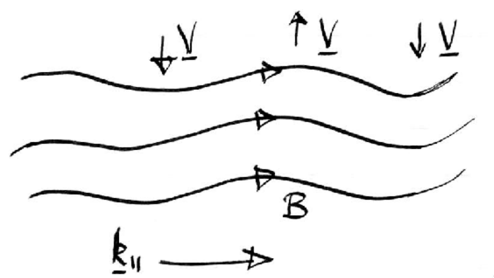

Figure 5.10: Shear Alfven Wave

5.7 Non-Uniform Plasmas and wave propagation

Practical plasmas are not infinite & homogeneous. So how does all this plane wave analysis apply practically?

If the spatial variation of the plasma is slow c.f. the wave length of the wave, then coupling to other waves will be small (negligible).

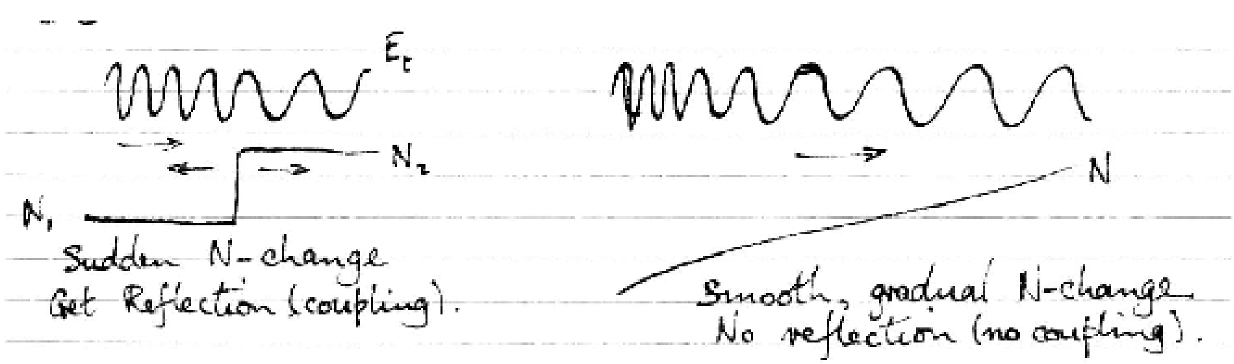

Figure 5.11: Comparison of sudden and gradualy refractive index change.

For a given ω, slowly varying plasma means $N/\frac{dN}{dx} \gg \lambda$ or $kN/\frac{dN}{dx} \gg 1$. Locally, the plasma appears uniform.

Even if the coupling is small, so that locally the wave propagates as if in an infinite uniform plasma, we still need a way of calculating how the solution propagates from one place to the other. This is handled by the 'WKB(J)' or 'eikonal' or 'ray optic' or 'geometric optics' approximation.

WKBJ solution

Consider the model 1-d wave equation (for field ω)

$$\frac{d^2E}{dx^2} + k^2 E = 0 \tag{5.157}$$

with k now a slowly varying function of x. Seek a solution in the form

$$E = \exp\left(i\phi\left(x\right)\right) \qquad (-i\omega t \text{ implied}) \tag{5.158}$$

ϕ is the wave phase ($= kx$ in uniform plasma).

Differentiate twice

$$\frac{d^2E}{dx^2} = \{i\frac{d^2\phi}{dx^2} - \left(\frac{d\phi}{dx}\right)^2\}e^{i\phi} \tag{5.159}$$

Substitute into differential equation to obtain

$$\left(\frac{d\phi}{dx}\right)^2 = k^2 + i\frac{d^2\phi}{dx^2} \tag{5.160}$$

Recognize that in uniform plasma $\frac{d^2\phi}{dx^2} = 0$. So in slightly non-uniform, 1st approx is to ignore this term.

$$\frac{d\phi}{dx} \simeq \pm k(x) \tag{5.161}$$

Then obtain a second approximation by substituting

$$\frac{d^2\phi}{dx^2} \simeq \pm\frac{dk}{dx} \tag{5.162}$$

so

$$\left(\frac{d\phi}{dx}\right)^2 \simeq k^2 \pm i\frac{dk}{dx} \tag{5.163}$$

$$\frac{d\phi}{dx} \simeq \pm\left(k \pm \frac{i}{2k}\frac{dk}{dx}\right) \quad \text{using Taylor expansion.} \tag{5.164}$$

Integrate:

$$\phi \simeq \pm\int^x kdx + i\ln\left(k^{\frac{1}{2}}\right) \tag{5.165}$$

Hence E is

$$E = e^{i\phi} = \frac{1}{k^{\frac{1}{2}}}\exp\left(\pm i\int^x kdx\right) \tag{5.166}$$

This is classic WKBJ solution. Originally studied by Green & Liouville (1837), the Green of Green's functions, the Liouville of Sturm Liouville theory.

Basic idea of this approach: (1) solve the local dispersion relation as if in infinite homogeneous plasma, to get $k(x)$, (2) form approximate solution for all space as above.

Phase of wave varies as integral of kdx.

In addition, amplitude varies as $\frac{1}{k^{\frac{1}{2}}}$. This is required to make the total energy flow uniform.

5.8 Two Stream Instability

An example of waves becoming unstable in a non-equilibrium plasma. Analysis is possible using Cold Plasma techniques.

Consider a plasma with two participating cold species but having *different* average velocities.

These are two "streams".

$$\begin{array}{cc} Species1 & Species2 \\ . \rightarrow & . \\ Moving. & Stationary. \\ Speed\ v & \end{array} \tag{5.167}$$

We can look at them in different inertial frames, e.g. species (stream) 2 stationary or 1 stationary (or neither).

We analyse by obtaining the susceptibility for each species and adding together to get total dielectric constant (scalar 1-d if *unmagnetized*).

In a frame of reference in which it is stationary, a stream j has the (Cold Plasma) susceptibility

$$\chi_j = \frac{-\omega_{pj}^2}{\omega^2} \quad . \tag{5.168}$$

If the stream is moving with velocity v_j (*zero order*) then its susceptibility is

$$\chi_j = \frac{-\omega_{pj}^2}{(\omega - kv_j)^2} \quad . \quad (\mathbf{k}\ \&\ \mathbf{v}_j \text{ in same direction}) \tag{5.169}$$

Proof from equation of motion:

$$\frac{q_j}{m_j}\mathbf{E} = \frac{\partial \tilde{\mathbf{v}}}{\partial t} + \mathbf{v}.\nabla\tilde{\mathbf{v}} = (-i\omega + i\mathbf{k}.\mathbf{v}_j)\,\tilde{\mathbf{v}} = -i\,(\omega - kv_j)\,\tilde{\mathbf{v}}\ . \tag{5.170}$$

Current density

$$\mathbf{j} = \rho_j\mathbf{v}_j + \rho_j.\tilde{\mathbf{v}} + \tilde{\rho}\mathbf{v}_j. \tag{5.171}$$

Substitute in

$$\nabla.\mathbf{j} + \frac{\partial\rho}{\partial t} = i\mathbf{k}.\tilde{\mathbf{v}}\rho_j + i\mathbf{k}.\tilde{\mathbf{v}}\rho - i\omega\tilde{\rho} = 0 \tag{5.172}$$

$$\tilde{\rho}_j = \rho_j\frac{\mathbf{k}.\tilde{\mathbf{v}}}{\omega - \mathbf{k}.\mathbf{v}_j} \tag{5.173}$$

Hence substituting for $\tilde{\mathbf{v}}$ in terms of $\mathbf{E}$:

$$-\chi_j\epsilon_0\nabla.\mathbf{E} = \tilde{\rho}_j = \frac{\rho_j q_j}{m_j}\frac{\mathbf{k}.\mathbf{E}}{-i\,(\omega - \mathbf{k}.\mathbf{v}_j)^2}, \tag{5.174}$$

which shows the longitudinal susceptibility is

$$\chi_j = -\frac{\rho_j q_j}{m_j\epsilon_0}\frac{1}{(\omega^2 - kv_j)^2} = \frac{-\omega_{pj}^2}{(\omega - kv_j)^2} \tag{5.175}$$

Proof by transforming frame of reference:

Consider Galileean transformation to a frame moving with the stream at velocity $\mathbf{v}_j$.

$$\mathbf{x} = \mathbf{x}' + \mathbf{v}_j t \quad ; \quad t' = t \tag{5.176}$$

$$\exp i\left(\mathbf{k}.\mathbf{x}-\omega t\right)=\exp i\left(\mathbf{k}.\mathbf{x}'-\left(\omega-\mathbf{k}.\mathbf{v}_j\right)t'\right) \tag{5.177}$$

So in frame of the stream, $\omega' = \omega - \mathbf{k}.\mathbf{v}_j$.
Substitute in stationary cold plasma expression:

$$\chi_j = -\frac{\omega_{pj}^2}{\omega'^2} = -\frac{\omega_{pj}^2}{\left(\omega - kv_j\right)^2}. \tag{5.178}$$

Thus for n streams we have

$$\epsilon = 1 + \sum_j \chi_j = 1 - \sum_j \frac{\omega_{pj}^2}{\left(\omega - kv_j\right)^2} \quad . \tag{5.179}$$

Longitudinal wave dispersion relation is

$$\epsilon = 0. \tag{5.180}$$

Two streams

$$0 = \epsilon = 1 - \frac{\omega_{p1}^2}{\left(\omega - kv_1\right)^2} - \frac{\omega_{P2}^2}{\left(\omega - kv_2\right)^2} \tag{5.181}$$

For given real k this is a quartic in ω. It has the form:

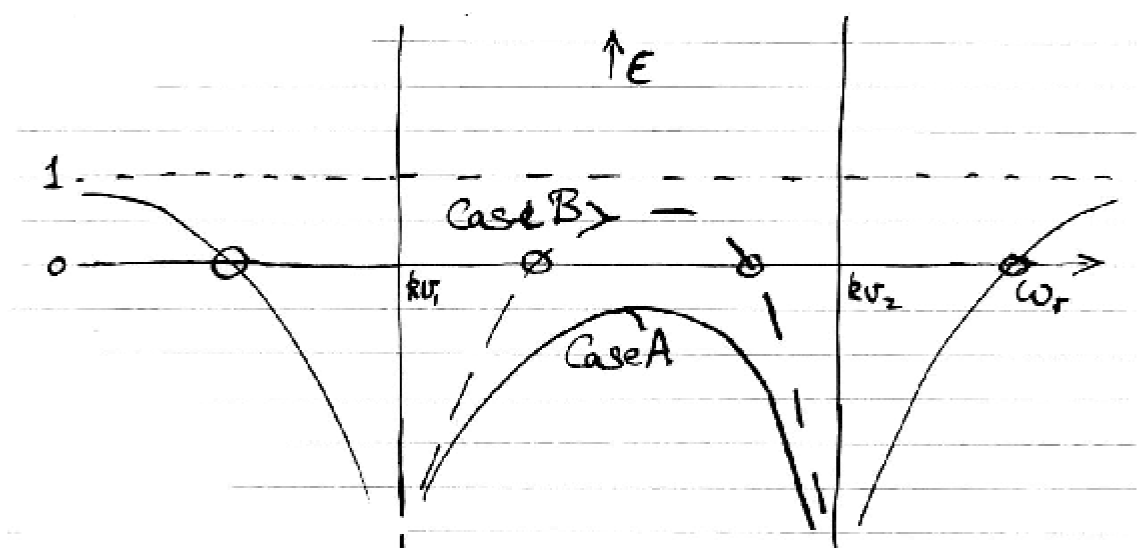

Figure 5.12: Two-stream stability analysis.

If ϵ crosses zero between the wells, then $\exists$ 4 real solutions for ω. (Case B).
If not, then 2 of the solutions are complex: $\omega = \omega_r \pm i\omega_i$ (Case A).
The time dependence of these complex roots is

$$\exp\left(-i\omega t\right) = \exp\left(-i\omega_r t \pm \omega_i t\right). \tag{5.182}$$

The $+ve$ sign is growing in time: *instability.*

It is straightforward to show that Case A occurs if

$$|k(v_2 - v_1)| < \left[\omega_{p1}^{\frac{2}{3}} + \omega_{p2}^{\frac{2}{3}}\right]^{\frac{3}{2}} \quad . \tag{5.183}$$

Small enough k (long enough wavelength) is always unstable.

Simple interpretation ($\omega_{p2}^2 \ll \omega_{p1}^2$, $v_1 = 0$) a tenuous beam in a plasma sees a negative ϵ if $|kv_2| \lesssim \omega_{p1}$.

Negative ϵ implies charge perturbation causes E that enhances itself: charge (spontaneous) bunching.

5.9 Kinetic Theory of Plasma Waves

Wave damping is due to wave-particle resonance. To treat this we need to keep track of the particle distribution in velocity space → kinetic theory.

5.9.1 Vlasov Equation

Treat particles as moving in 6-D phase space $\mathbf{x}$ position, $\mathbf{v}$ velocity. At any instant a particle occupies a unique position in phase space $(\mathbf{x}, \mathbf{v})$.

Consider an elemental volume $d^3\mathbf{x}d^3\mathbf{v}$ of phase space $[dxdydzdv_xdv_ydv_z]$, at $(\mathbf{x}, \mathbf{v})$. Write down an equation that is conservation of particles for this volume

$$\begin{aligned} -\frac{\partial}{\partial t}\left(fd^3\mathbf{x}d^3\mathbf{v}\right) &= \left[v_x f\left(\mathbf{x}+dx\hat{\mathbf{x}},\mathbf{v}\right) - v_x f\left(\mathbf{x},\mathbf{v}\right)\right] dydzd^3\mathbf{v} \\ &+ \text{ same for } dy, dz \\ &+ \left[a_x f\left(\mathbf{x},\mathbf{v}+dv_x\hat{\mathbf{x}}\right) - a_x f\left(\mathbf{x},\mathbf{v}\right)\right] d^3\mathbf{x}dv_ydv_z \\ &+ \text{ same for } dv_y, dv_z \end{aligned} \tag{5.184}$$

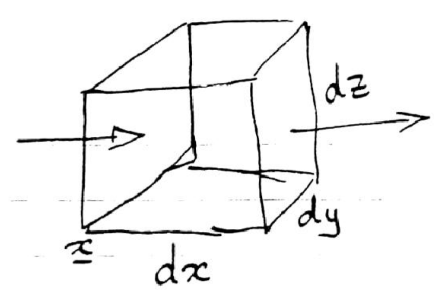

Figure 5.13: Difference in flow across x-surfaces $(+y+z)$.

$\mathbf{a}$ is "velocity space motion", i.e. acceleration.

Divide through by $d^3\mathbf{x}d^3\mathbf{v}$ and take limit

$$\begin{aligned} -\frac{\partial f}{\partial t} &= \frac{\partial}{\partial x}(v_x f) + \frac{\partial}{\partial y}(v_y f) + \frac{\partial}{\partial z}(v_z f) + \frac{\partial}{\partial v_x}(a_x f) + \frac{\partial}{\partial v_y}(a_y f) + \frac{\partial}{\partial v_z}(a_z f) \\ &= \nabla.(\mathbf{v}f) + \nabla_v.(\mathbf{a}f) \end{aligned} \tag{5.185}$$

[Notation: Use $\frac{\partial}{\partial \mathbf{x}} \leftrightarrow \nabla; \frac{\partial}{\partial \mathbf{v}} \leftrightarrow \nabla_v$].
Take this simple continuity equation in phase space and expand:

$$\frac{\partial f}{\partial t} + (\nabla.\mathbf{v}) f + (\mathbf{v}.\nabla) f + (\nabla_v.\mathbf{a}) f + (\mathbf{a}.\nabla_v) f = 0. \tag{5.186}$$

Recognize that ∇ means here $\frac{\partial}{\partial x}$ etc. *keeping* $\mathbf{v}$ *constant* so that $\nabla.\mathbf{v} = 0$ by definition. So

$$\frac{\partial f}{\partial t} + \mathbf{v}.\frac{\partial f}{\partial \mathbf{x}} + \mathbf{a}.\frac{\partial f}{\partial \mathbf{v}} = -f(\nabla_v.\mathbf{a}) \tag{5.187}$$

Now we want to couple this equation with Maxwell's equations for the fields, and the Lorentz force

$$\mathbf{a} = \frac{q}{m}(\mathbf{E} + \mathbf{v} \wedge \mathbf{B}) \tag{5.188}$$

Actually we don't want to use the $\mathbf{E}$ retaining all the local effects of individual particles. We want a smoothed out field. Ensemble averaged $\mathbf{E}$.
Evaluate

$$\nabla_v.\mathbf{a} = \nabla_v.\frac{q}{m}(\mathbf{E} + \mathbf{v} \wedge \mathbf{B}) = \frac{q}{m}\nabla_v.(\mathbf{v} \wedge \mathbf{B}) \tag{5.189}$$

$$= \frac{q}{m}\mathbf{B}.(\nabla_v \wedge \mathbf{v}) = 0. \tag{5.190}$$

So RHS is zero. However in the use of smoothed out E we have ignored local effect of one particle on another due to the graininess. That is *collisions*.

Boltzmann Equation:

$$\frac{\partial f}{\partial t} + \mathbf{v}.\frac{\partial f}{\partial \mathbf{x}} + \mathbf{a}.\frac{\partial f}{\partial \mathbf{v}} = \left(\frac{\partial f}{\partial t}\right)_{\text{collisions}} \tag{5.191}$$

Vlasov Equation $\equiv$ Boltzman Eq *without* collisions. For electromagnetic forces:

$$\frac{\partial f}{\partial t} + \mathbf{v}.\frac{\partial f}{\partial \mathbf{x}} + \frac{q}{m}(\mathbf{E} + \mathbf{v} \wedge \mathbf{B})\frac{\partial f}{\partial \mathbf{v}} = 0. \tag{5.192}$$

Interpretation:
Distribution function is constant along particle orbit in phase space: $\frac{d}{dt}f = 0$.

$$\frac{d}{dt}f = \frac{\partial f}{\partial t} + \frac{d\mathbf{x}}{dt}.\frac{\partial f}{\partial \mathbf{x}} + \frac{\partial \mathbf{v}}{dt}.\frac{\partial f}{\partial \mathbf{v}} \tag{5.193}$$

Coupled to Vlasov equation for each particle species we have Maxwell's equations.

Vlasov-Maxwell Equations

$$\frac{\partial f_j}{\partial t} + \mathbf{v}.\frac{\partial f_j}{\partial \mathbf{x}} + \frac{q_j}{m_j}\left(\mathbf{E} + \mathbf{v}\wedge\mathbf{B}\right).\frac{\partial f_j}{\partial \mathbf{v}_j} = 0 \tag{5.194}$$

$$\nabla\wedge\mathbf{E} = \frac{-\partial\mathbf{B}}{\partial t} \quad , \quad \nabla\wedge\mathbf{B} = \mu_0\mathbf{j} + \frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t} \tag{5.195}$$

$$\nabla.\mathbf{E} = \frac{\rho}{\epsilon_0} \quad , \quad \nabla.\mathbf{B} = 0 \tag{5.196}$$

Coupling is completed via charge & current densities.

$$\rho = \sum_j q_j n_j = \sum_J q_j \int f_j d^3\mathbf{v} \tag{5.197}$$

$$\mathbf{j} = \sum_j q_j n_j \mathbf{V}_j = \sum_j q_j \int f_j \mathbf{v} d^3\mathbf{v}. \tag{5.198}$$

Describe phenomena in which collisions are not important, keeping track of the (statistically averaged) particle distribution function.

Plasma waves are the most important phenomena covered by the Vlasov-Maxwell equations.

6-dimensional, nonlinear, time-dependent, integral-differential equations!

5.9.2 Linearized Wave Solution of Vlasov Equation

Unmagnetized Plasma

Linearize the Vlasov Eq by supposing

$$f = f_0(\mathbf{v}) + f_1(v)\exp i\left(\mathbf{k}.\mathbf{x} - \omega t\right) \quad , \quad f_1 \text{ small.} \tag{5.199}$$

$$\text{also } \mathbf{E} = \mathbf{E}_1 \exp i\left(\mathbf{k}.\mathbf{x} - \omega t\right) \quad \mathbf{B} = \mathbf{B}_1 \exp i\left(\mathbf{k}.\mathbf{x} - \omega t\right) \tag{5.200}$$

Zeroth order f_0 equation satisfied by $\frac{\partial}{\partial t}, \frac{\partial}{\partial x} = 0$. First order:

$$-i\omega f_1 + \mathbf{v}.i\mathbf{k}f_1 + \frac{q}{m}\left(\mathbf{E}_1 + \mathbf{v}\wedge\mathbf{B}_1\right).\frac{\partial f_0}{\partial \mathbf{v}} = 0. \tag{5.201}$$

[Note $\mathbf{v}$ is not per se of any order, it is an independent variable.]

Solution:

$$f_1 = \frac{1}{i\left(\omega - \mathbf{k}.\mathbf{v}\right)}\frac{q}{m}\left(\mathbf{E}_1 + \mathbf{v}\wedge\mathbf{B}_1\right).\frac{\partial f_0}{\partial \mathbf{v}} \tag{5.202}$$

For convenience, assume f_0 is *isotropic*. Then $\frac{\partial f_0}{\partial \mathbf{v}}$ is in direction $\mathbf{v}$ so $\mathbf{v}\wedge\mathbf{B}_1.\frac{\partial f_0}{\partial \mathbf{v}} = 0$

$$f_1 = \frac{\frac{q}{m}\mathbf{E}_1.\frac{\partial f_0}{\partial \mathbf{v}}}{i\left(\omega - \mathbf{k}.\mathbf{v}\right)} \tag{5.203}$$

We want to calculate the conductivity $\boldsymbol{\sigma}$. Do this by simply integrating:

$$\mathbf{j} = \int q f_1 \mathbf{v} d^3 v = \frac{q^2}{im}\int \frac{\mathbf{v}\frac{\partial f_0}{\partial \mathbf{v}}}{\omega - \mathbf{k}.\mathbf{v}} d^3 v \ .\mathbf{E}_1. \tag{5.204}$$

Here the electric field has been taken outside the v-integral but its dot product is with $\partial f_0/\partial \mathbf{v}$. Hence we have the tensor conductivity,

$$\boldsymbol{\sigma} = \frac{q^2}{im}\int \frac{\mathbf{v}\frac{\partial f_0}{\partial \mathbf{v}}}{\omega - \mathbf{k}.\mathbf{v}} d^3v \tag{5.205}$$

Focus on zz component:

$$1 + \chi_{zz} = \epsilon_{zz} = 1 + \frac{\sigma_{zz}}{-i\omega\epsilon_0} = 1 + \frac{q^2}{\omega m \epsilon_0}\int \frac{v_z \frac{\partial f_0}{\partial v_z}}{\omega - \mathbf{k}.\mathbf{v}} d^3\mathbf{v} \tag{5.206}$$

Such an expression applies for the conductivity (susceptibility) of each species, if more than one needs to be considered.

It looks as if we are there! Just do the integral!

Now the problem becomes evident. The integrand has a zero in the denominator. At least we can do 2 of 3 integrals by defining the 1-dimensional distribution function

$$f_z(v_z) \equiv \int f(\mathbf{v}) dv_x dv_y \qquad (\mathbf{k} = k\hat{\mathbf{z}}) \tag{5.207}$$

Then

$$\chi = \frac{q^2}{\omega m \epsilon_0}\int \frac{v_z \frac{\partial f_z}{\partial v_z}}{\omega - k v_z} dv_z \tag{5.208}$$

(drop the z suffix from now on. 1-d problem).

How do we integrate through the pole at $v = \frac{\omega}{k}$? Contribution of resonant particles. Crucial to get right.

Path of velocity integration

First, realize that the solution we have found is not complete. In fact a more general solution can be constructed by adding any solution of

$$\frac{\partial f_1}{\partial t} + v\frac{\partial f_1}{\partial z} = 0 \tag{5.209}$$

[We are dealing with 1-d Vlasov equation: $\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial z} + \frac{qE}{m}\frac{\partial f}{\partial v} = 0$.] Solution of this is

$$f_1 = g(vt - z, v) \tag{5.210}$$

where g is an arbitrary function of its arguments. Hence general solution is

$$f_1 = \frac{\frac{q}{m}E\frac{\partial f_0}{\partial v}}{i(\omega - kv)} \exp i(kz - \omega t) + g(vt - z, v) \tag{5.211}$$

and g must be determined by initial conditions. In general, if we start up the wave suddenly there will be a transient that makes g non-zero.

So instead we consider a case of *complex* ω (real k for simplicity) where $\omega = \omega_r + i\omega_i$ and $\omega_i > 0$.

This case corresponds to a growing wave:

$$\exp(-i\omega t) = \exp(-i\omega_r t + \omega_i t) \tag{5.212}$$

Then we can take our initial condition to be $f_1 = 0$ at $t \to -\infty$. This is satisfied by taking $g = 0$.

For $\omega_i > 0$ the complementary function, g, is zero.

Physically this can be thought of as treating a case where there is a very gradual, smooth start up, so that no transients are generated.

Thus if $\omega_i > 0$, the solution is simply the velocity integral, taken along the real axis, with no additional terms. For

$$\omega_i > 0, \qquad \chi = \frac{q^2}{\omega m \epsilon_o} \int_C \frac{v \frac{\partial f}{\partial v}}{\omega - kv} dv \tag{5.213}$$

where there is now no difficulty about the integration because ω is complex.

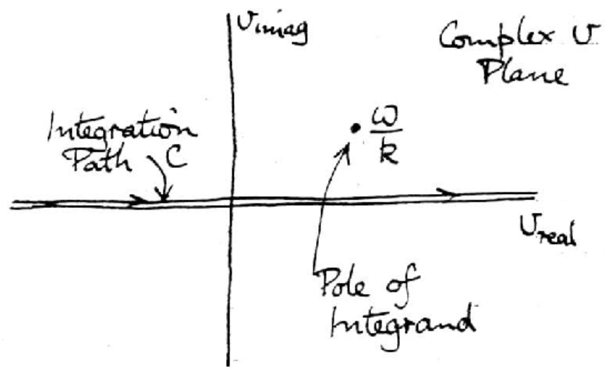

Figure 5.14: Contour of integration in complex v-plane.

The pole of the integrand is at $v = \frac{\omega}{k}$ which is above the real axis.

The question then arises as to how to do the calculation if $\omega_i \leq 0$. The answer is by "analytic continuation", regarding all quantities as complex.

"Analytic Continuation" of χ is accomplished by allowing ω/k to move (e.g. changing the ω_i) but never allowing any poles to cross the integration contour, as things change continuously.

Remember (Fig 5.15)

$$\oint_c F dz = \sum \text{residues} \ \times 2\pi i \tag{5.214}$$

(Cauchy's theorem)

Where residues $= \lim_{z \to z_k} [F(z)/(z - z_k)]$ at the poles, z_k, of $F(z)$. We can deform the contour how we like, provided no poles cross it. Hence contour (Fig 5.16)

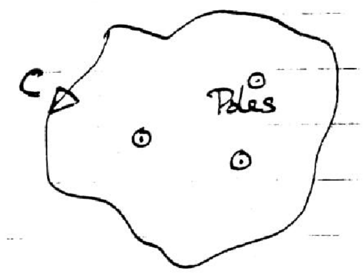

Figure 5.15: Cauchy's theorem.

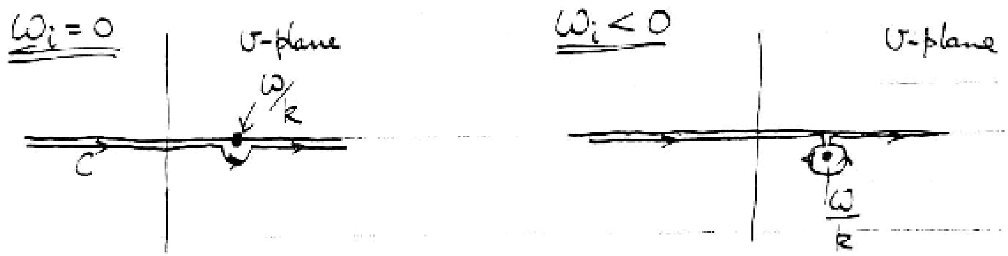

Figure 5.16: Landau Contour

We conclude that the integration contour for $\omega_i < 0$ is *not* just along the real v axis. It includes the pole also.

To express our answer in a universal way we use the notation of "Principal Value" of a singular integral defined as the average of paths above and below

$$\wp\int \frac{F}{v-v_0}dv = \frac{1}{2}\left[\int_{C_1} + \int_{C_2}\right]\frac{F}{v-v_0}dv \tag{5.215}$$

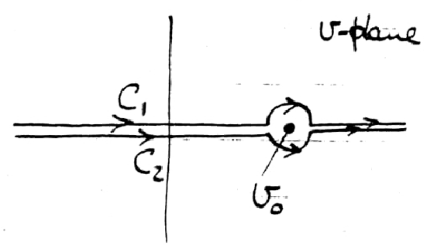

Figure 5.17: Two halves of principal value contour.

Then

$$\chi = \frac{1^2}{\omega m\epsilon_o}\{\wp\int \frac{v\frac{\partial f_0}{\partial v}}{\omega - kv}dv - \frac{1}{2}2\pi i\frac{\omega}{k^2}\left.\frac{\partial f_0}{\partial v}\right|_{v=\frac{\omega}{k}}\} \tag{5.216}$$

Second term is half the normal residue term; so it is half of the integral round the pole.

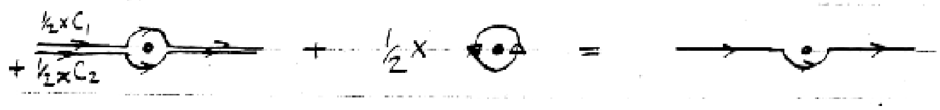

Figure 5.18: Contour equivalence.

Our expression is only short-hand for the (Landau) prescription:
"Integrate below the pole". (Nautilus).

Contribution from the pole can be considered to arise from the complementary function $g(vt-z, v)$. If g is to be proportional to $\exp(ikz)$, then it must be of the form $g = \exp[ik(z-vt)]h(v)$ where $h(v)$ is an arbitrary function. To get the result previously calculated, the value of $h(v)$ must be (for real ω)

$$h(v) = \pi \frac{q}{m}\frac{1}{k}\left.\frac{\partial f_0}{\partial v}\right|_{\frac{\omega}{k}} \delta\left(v - \frac{\omega}{k}\right) \tag{5.217}$$

$$\text{(so that } \int \frac{q}{-i\omega\epsilon_o} vg dv = \left(\pi i \frac{\omega}{k^2}\left.\frac{\partial f_0}{\partial v}\right|_{\frac{\omega}{k}}\right)\frac{q^2}{\omega m \epsilon_o} \quad .) \tag{5.218}$$

This Dirac delta function says that the complementary function is limited to particles with "exactly" the wave phase speed $\frac{\omega}{k}$. It is the resonant behaviour of these particles and the imaginary term they contribute to χ that is responsible for wave damping.

We shall see in a moment, that the standard case will be $\omega_i < 0$, so the opposite of the prescription $\omega_i > 0$ that makes $g = 0$. Therefore there will generally be a complementary function, non-zero, describing resonant effects. We don't have to calculate it explicitly because the Landau prescription takes care of it.

5.9.3 Landau's original approach. (1946)

Corrected Vlasov's assumption that the correct result was just the principal value of the integral. Landau recognized the importance of initial conditions and so used Laplace Transform approach to the problem

$$\tilde{A}(p) = \int_0^\infty e^{-pt} A(t) dt \tag{5.219}$$

The Laplace Transform inversion formula is

$$A(t) = \frac{1}{2\pi i}\int_{s-i\infty}^{s+i\infty} e^{pt}\tilde{A}(p)dp \tag{5.220}$$

where the path of integration must be chosen to the right of any poles of $\tilde{A}(p)$ (i.e. s large enough). Such a prescription seems reasonable. If we make $\Re(p)$ large enough then the $\tilde{A}(p)$ integral will presumably exist. The inversion formula can also be proved rigorously so that gives confidence that this is the right approach.

If we identify $p \to -i\omega$, then the transform is $\tilde{A} = \int e^{i\omega t} A(t) dt$, which can be identified as the Fourier transform that would give component $\tilde{A} \propto e^{-i\omega t}$, the wave we are discussing. Making $\Re(p)$ positive enough to be to the right of all poles is then equivalent to making $\Im(\omega)$ positive enough so that the path in ω-space is above all poles, in particular $\omega_i > \Im(kv)$. For real velocity, v, this is precisely the condition $\omega_i > 0$, we adopted before to justify putting the complementary function zero.

Either approach gives the same prescription. It is all bound up with satisfying causality.

5.9.4 Solution of Dispersion Relation

We have the dielectric tensor

$$\epsilon = 1 + \chi = 1 + \frac{q^2}{\omega m \epsilon_0} \left\{ \wp \int \frac{v \frac{\partial f_0}{\partial v}}{\omega - kv} dv - \pi i \frac{\omega}{k^2} \left. \frac{\partial f_0}{\partial v} \right|_{\frac{\omega}{k}} \right\} \quad , \tag{5.221}$$

for a general isotropic distribution. We also know that the dispersion relation is

$$\begin{bmatrix} -N^2 + \epsilon_t & 0 & 0 \\ 0 & -N^2 + \epsilon_t & 0 \\ 0 & 0 & \epsilon \end{bmatrix} = \left(-N^2 + \epsilon_t\right)^2 \epsilon = 0 \tag{5.222}$$

Giving transverse waves $N^2 = \epsilon_t$ and
longitudinal waves $\epsilon = 0$.
Need to do the integral and hence get ϵ.

Presumably, if we have done this right, we ought to be able to get back the cold-plasma result as an approximation in the appropriate limits, plus some corrections. We previously argued that cold-plasma is valid if $\frac{\omega}{k} \gg v_t$. So regard $\frac{kv}{\omega}$ as a small quantity and expand:

$$\begin{aligned} \wp \int \frac{v \frac{\partial f_0}{dv}}{\omega \left(1 - \frac{kv}{\omega}\right)} dv &= \frac{1}{\omega} \int v \frac{\partial f_0}{\partial v} \left[1 + \frac{kv}{\omega} + \left(\frac{kv}{\omega}\right)^2 + ... \right] dv \\ &= \frac{-1}{\omega} \int f_o \left[1 + \frac{2kv}{\omega} + 3 \left(\frac{kv}{\omega}\right)^2 + ... \right] dv \quad \text{(by parts)} \\ &\simeq \frac{-1}{\omega} \left[n + \frac{3nT}{m} \frac{k^2}{\omega^2} \right] + ... \end{aligned} \tag{5.223}$$

Here we have assumed we are in the particles' average rest frame (no bulk velocity) so that $\int f_0 v dv = 0$ and also we have used the temperature definition

$$nT = \int m v^2 f_0 dv \quad , \tag{5.224}$$

appropriate to one degree of freedom (1-d problem). Ignoring the higher order terms we get:

$$\epsilon = 1 - \frac{\omega_p^2}{\omega^2} \left\{ 1 + 3 \frac{T}{m} \frac{k^2}{\omega^2} + \pi i \frac{\omega^2}{k^2} \frac{1}{n} \left. \frac{\partial f_0}{\partial v} \right|_{\frac{\omega}{k}} \right\} \tag{5.225}$$

This is just what we expected. Cold plasma value was $\epsilon = 1 - \frac{\omega_p^2}{\omega^2}$. We have two corrections

1. To real part of ϵ, correction $3\frac{T}{m}\frac{k^2}{\omega^2} = 3\left(\frac{v_t}{v_p}\right)^2$ due to finite temperature. We could have got this from a fluid treatment *with* pressure.

2. Imaginary part $\rightarrow$ antihermitian part of $\epsilon \rightarrow$ dissipation.

Solve the dispersion relation for longitudinal waves $\epsilon = 0$ (again assuming k real ω complex). Assume $\omega_i \ll \omega_r$ then

$$
\begin{aligned}
(\omega_r + i\omega_i)^2 &\simeq \omega_r^2 + 2\omega_r\omega_i i = \omega_p^2\{1 + 3\frac{T}{m}\frac{k^2}{\omega^2} + \pi i \frac{\omega^2}{k^2}\frac{1}{n}\frac{\partial f_0}{\partial v}|_{\frac{\omega}{k}}\} \\
&\simeq \omega_p^2\{1 + 3\frac{T}{m}\frac{k^2}{\omega_r^2} + \pi i \frac{\omega_r^2}{k^2}\frac{1}{n}\frac{\partial f_o}{\partial v}|_{\frac{\omega_r}{k}}\} \qquad (5.226)
\end{aligned}
$$

$$
\text{Hence } \omega_1 \simeq \frac{1}{2\omega_r i}\omega_p^2 \pi i \frac{\omega_r^2}{k^2}\frac{1}{n}\frac{\partial f_0}{\partial v}|_{\frac{\omega_r}{k}} = \omega_p^2 \frac{\pi}{2}\frac{\omega_r}{k^2}\frac{1}{n}\frac{\partial f_0}{\partial v}|_{\frac{\omega_r}{k}} \qquad (5.227)
$$

For a Maxwellian distribution

$$
f_0 = \left(\frac{m}{2\pi T}\right)^{\frac{1}{2}} \exp\left(-\frac{mv^2}{2T}\right) n \qquad (5.228)
$$

$$
\frac{\partial f_0}{\partial v} = \left(\frac{m}{2\pi T}\right)^{\frac{1}{2}} \left(-\frac{mv}{T}\right) \exp\left(-\frac{mv^2}{2T}\right) n \qquad (5.229)
$$

$$
\omega_i \simeq -\omega_p^2 \frac{\pi}{2}\frac{\omega_r^2}{k^3}\left(\frac{m}{2\pi T}\right)^{\frac{1}{2}} \frac{m}{T} \exp\left(-\frac{m\omega_r^2}{2Tk^2}\right) \qquad (5.230)
$$

The difference between ω_r and ω_p may not be important in the outside but ought to be retained inside the exponential since

$$
\frac{m}{2T}\frac{\omega_p^2}{k^2}\left[1 + 3\frac{T}{m}\frac{k^2}{\omega_p^2}\right] = \frac{m\omega_p^2}{2Tk^2} + \frac{3}{2} \qquad (5.231)
$$

$$
\text{So} \quad \omega_i \simeq -\omega_p \left(\frac{\pi}{8}\right)^{\frac{1}{2}} \frac{\omega_p^3}{k^3}\frac{1}{v_t^3} \exp\left(-\frac{m\omega_p^2}{2Tk^2} - \frac{3}{2}\right) \qquad (5.232)
$$

Imaginary part of ω is *negative* $\Rightarrow$ damping. This is Landau Damping.

Note that we have been treating a single species (electrons by implication) but if we need more than one we simply add to χ. Solution is then more complex.

5.9.5 Direct Calculation of Collisionless Particle Heating

(Landau Damping without complex variables!)

We show by a direct calculation that net energy is transferred to electrons.

Suppose there exists a longitudinal wave

$$\mathbf{E} = E\cos(kz - \omega t)\hat{\mathbf{z}} \tag{5.233}$$

Equations of motion of a particle

$$\frac{dv}{dt} = \frac{q}{m}E\cos(kz - \omega t) \tag{5.234}$$

$$\frac{dz}{dt} = v \tag{5.235}$$

Solve these assuming E is small by a perturbation expansion $v = v_0 + v_1 + ...$, $z = z_0(t) + z_1(t) + ...$.

Zeroth order:

$$\frac{dv_o}{dt} = 0 \Rightarrow v_0 = \text{const} \quad , \quad z_0 = z_i + v_0 t \tag{5.236}$$

where $z_i = $ const is the initial position.

First Order

$$\frac{dv_1}{dt} = \frac{q}{m}E\cos\left(kz_0 - \omega t\right) = \frac{q}{m}E\cos\left(k\left(z_i + v_0 t\right) - \omega t\right) \tag{5.237}$$

$$\frac{dz_1}{dt} = v_1 \tag{5.238}$$

Integrate:

$$v_1 = \frac{qE}{m}\frac{\sin\left(kz_i + kv_0 - \omega t\right)}{kv_0 - \omega} + \text{const.} \tag{5.239}$$

take initial conditions to be $v_1, v_2 = 0$. Then

$$v_1 = \frac{qE}{m}\frac{\sin\left(kz_i + \Delta\omega t\right) - \sin\left(kz_i\right)}{\Delta\omega} \tag{5.240}$$

where $\Delta\omega \equiv kv_0 - \omega$, is (-) the frequency at which the particle feels the wave field.

$$z_1 = \frac{qE}{m}\left[\frac{\cos kz_i - \cos\left(kz_i + \Delta\omega t\right)}{\Delta\omega^2} - t\frac{\sin kz_i}{\Delta\omega}\right] \tag{5.241}$$

(using $z_1(0) = 0$).

2nd Order (Needed to get energy right)

$$\begin{aligned}\frac{dv_2}{dt} &= \frac{qE}{M}\{\cos\left(kz_i + kv_0 t - \omega t + kz_1\right) - \cos\left(kz_i + kv_0 t - \omega t\right)\} \\ &= \frac{qE}{m}kz_i\{-\sin\left(kz_i + \Delta\omega t\right)\} \qquad (kz_1 \ll 1)\end{aligned} \tag{5.242}$$

Now the gain in kinetic energy of the particle is

$$\begin{aligned}\frac{1}{2}mv^2 - \frac{1}{2}mv_0^2 &= \frac{1}{2}m\{(v_0 + v_1 + v_2 + ...)^2 - v_0^2\} \\ &= \frac{1}{2}\{2v_0 v_1 + v_1^2 + 2v_0 v_2 + \text{higher order}\}\end{aligned} \tag{5.243}$$

and the rate of increase of K.E. is

$$\frac{d}{dt}\left(\frac{1}{2}mv^2\right) = m\left(v_0\frac{dv_1}{dt} + v_1\frac{dv_1}{dt} + v_0\frac{dv_2}{dt}\right) \tag{5.244}$$

We need to average this over space, i.e. over z_i. This will cancel any component that simply oscillates with z_i.

$$\left\langle\frac{d}{dt}\left(\frac{1}{2}mv^2\right)\right\rangle = \left\langle v_0\frac{dv_1}{dt} + v_1\frac{dv_1}{dt} + v_0\frac{dv_2}{dt}\right\rangle m \tag{5.245}$$

$$\begin{aligned}
\left\langle v_0\frac{dv_1}{dt}\right\rangle &= 0 && (5.246)\\
\left\langle v_1\frac{dv_1}{dt}\right\rangle &= \left\langle\frac{q^2E^2}{m^2}\left[\frac{\sin(kz_i+\Delta\omega t) - \sin kz_i}{\Delta\omega}\cos(kz_i+\Delta\omega t)\right]\right\rangle \\
&= \frac{q^2E^2}{m^2}\left\langle\frac{\sin(kz_i+\Delta\omega t) - \sin(kz_i+\Delta\omega t)\cos\Delta\omega t + \cos(kz_i+\Delta\omega t)\sin\Delta\omega t}{\Delta\omega}\right. \\
&\qquad \left.\cos(kz_i+\Delta\omega t)\right\rangle \\
&= \frac{q^2E^2}{m^2}\left\langle\frac{\sin\Delta\omega t}{\Delta\omega}\cos^2(kz_i+\Delta\omega t)\right\rangle \\
&= \frac{q^2E^2}{m^2}\frac{1}{2}\frac{\sin\Delta\omega t}{\Delta\omega} && (5.247)\\
\left\langle v_0\frac{dv_2}{dt}\right\rangle &= \frac{-q^2E^2}{m^2}kv_0\left\langle\left(\frac{\cos kz_i - \cos(kz_i+\Delta\omega t)}{\Delta\omega^2} - t\frac{\sin kz_i}{\Delta\omega}\right)\sin(kz_i+\Delta\omega t)\right\rangle \\
&= \frac{-q^2E^2}{m^2}kv_0\left\langle\left(\frac{\sin\Delta\omega t}{\Delta\omega^2} - t\frac{\cos\Delta\omega t}{\Delta\omega}\right)\sin^2(kz_i+\Delta\omega t)\right\rangle \\
&= \frac{q^2E^2}{m^2}\frac{kv_0}{2}\left[-\frac{\sin\Delta\omega t}{\Delta\omega^2} + t\frac{\cos\Delta\omega t}{\Delta\omega}\right] && (5.248)
\end{aligned}$$

Hence

$$\begin{aligned}
\left\langle\frac{d}{dt}\frac{1}{2}mv^2\right\rangle &= \frac{q^2E^2}{2m}\left[\frac{\sin\Delta\omega t}{\Delta\omega} - kv_0\frac{\sin\Delta\omega t}{\Delta\omega^2} + kv_0t\frac{\cos\Delta\omega t}{\Delta\omega}\right] && (5.249)\\
&= \frac{q^2E^2}{2m}\left[\frac{-\omega\sin\Delta\omega t}{\Delta\omega^2} + \frac{\omega t}{\Delta\omega}\cos\Delta\omega t + t\cos\Delta\omega t\right] && (5.250)
\end{aligned}$$

This is the space-averaged power into particles of a specific velocity v_0. We need to integrate over the distribution function. A trick identify helps:

$$\begin{aligned}
\frac{-\omega}{\Delta\omega^2}\sin\Delta\omega t &+ \frac{\omega t}{\Delta\omega}\cos\Delta\omega t + t\cos\Delta\omega t = \frac{\partial}{\partial\Delta\omega}\left(\frac{\omega\sin\Delta\omega t}{\Delta\omega} + \sin\Delta\omega t\right) && (5.251)\\
&= \frac{1}{k}\frac{\partial}{\partial v_0}\left(\frac{\omega\sin\Delta\omega t}{\Delta\omega} + \sin\Delta\omega t\right) && (5.252)
\end{aligned}$$

Hence power per unit volume is

$$\begin{aligned} P &= \int \left\langle \frac{d}{dt}\frac{1}{2}mv^2 \right\rangle f(v_0)\, dv_0 \\ &= \frac{q^2E^2}{2mk}\int f(v_0)\frac{\partial}{\partial v_0}\left(\frac{\omega \sin\Delta\omega t}{\Delta\omega} + \sin\Delta\omega t\right) dv_0 \\ &= -\frac{q^2E^2}{2mk}\int \left(\frac{\omega \sin\Delta\omega t}{\Delta\omega} + \sin\Delta\omega t\right)\frac{\partial f}{\partial v_0} dv_0 \end{aligned} \tag{5.253}$$

As t becomes large, $\sin\Delta\omega t = \sin(kv_0 - \omega)t$ becomes a rapidly oscillating function of v_0. Hence second term of integrand contributes negligibly and the first term,

$$\propto \frac{\omega \sin\Delta\omega t}{\Delta\omega} = \frac{\sin\Delta\omega t}{\Delta\omega t} \quad \omega t \tag{5.254}$$

becomes a highly localized, delta-function-like quantity. That enables the rest of the integrand to be evaluated just where $\Delta\omega = 0$ (i.e. $kv_0 - \omega = 0$).

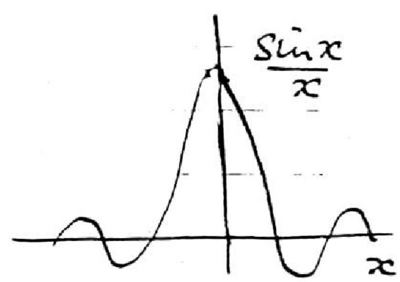

Figure 5.19: Localized integrand function.

So:

$$P = -\frac{q^2E^2}{2mk}\frac{\omega}{k}\frac{\partial f}{\partial v}\Big|_{\frac{\omega}{k}}\int \frac{\sin x}{x}dx \tag{5.255}$$

$x = \Delta\omega t = (kv_0 - \omega)t$.
and $\int \frac{\sin x}{x}dz = \pi$ so

$$P = -E\frac{\pi q^2\omega}{2mk^2}\frac{\partial f_0}{\partial v}\Big|_{\frac{\omega}{k}} \tag{5.256}$$

We have shown that there is a net transfer of energy to particles at the resonant velocity $\frac{\omega}{k}$ from the wave. (Positive if $\frac{\partial f}{\partial v}|$ is negative.)

5.9.6 Physical Picture

$\Delta\omega$ is the frequency in the particles' (unperturbed) frame of reference, or equivalently it is kv_0' where v_0' is particle speed in wave frame of reference. The latter is easier to deal with. $\Delta\omega t = kv_0't$ is the phase the particle travels in time t. We found that the energy gain was of the form

$$\int \frac{\sin\Delta\omega t}{\Delta\omega t} d(\Delta\omega t)\,. \tag{5.257}$$

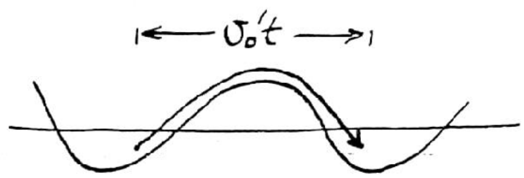

Figure 5.20: Phase distance traveled in time t.

This integrand becomes small (and oscillatory) for $\Delta\omega t \gg 1$. Physically, this means that if particle moves through many wavelengths its energy gain is small. Dominant contribution is from $\Delta\omega t < \pi$. These are particles that move through less than $\frac{1}{2}$ wavelength during the period under consideration. These are the resonant particles.

Figure 5.21: Dominant contribution

Particles moving slightly *faster* than wave are *slowed* down. This is a second-order effect.

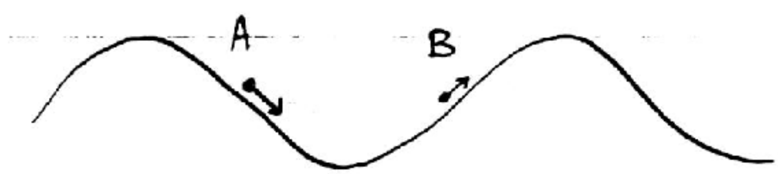

Figure 5.22: Particles moving slightly faster than the wave.

Some particles of this v_0 group are being accelerated (A) some slowed (B). Because A's are then going faster, they spend less time in the 'down' region. B's are slowed; they spend more time in up region. Net effect: tendency for particle to move its speed toward that of wave.

Particles moving slightly *slower* than wave are *speeded* up. (Same argument). But this is only true for particles that have "caught the wave".

Summary: Resonant particles' velocity is drawn toward the wave phase velocity.

Is there net energy when we average both slower and faster particles? Depends which type has most.

Our Complex variables wave treatment and our direct particle energy calculation give consistent answers. To show this we need to show energy conservation. Energy density of

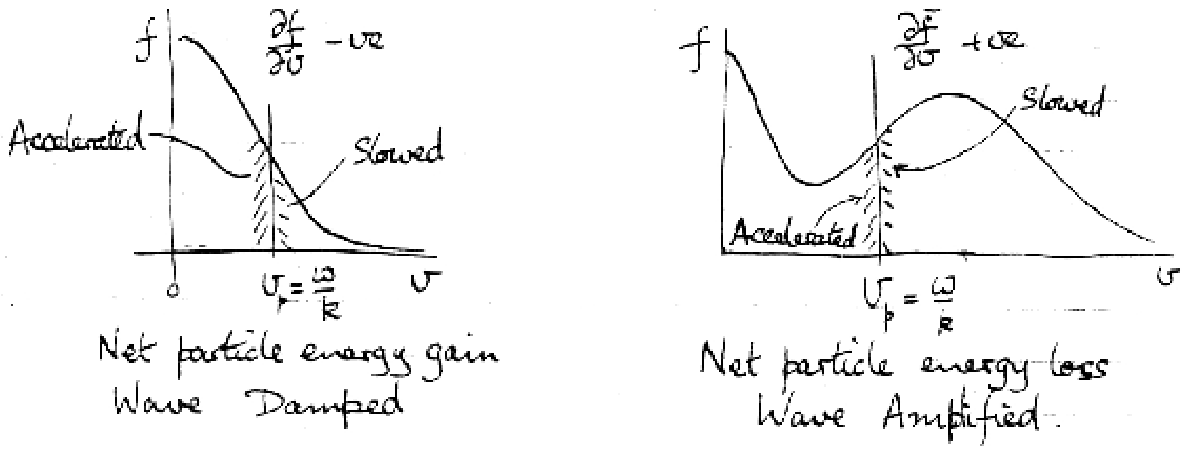

Figure 5.23: Damping or growth depends on distribution slope

wave:

$$\overline{W} = \underbrace{\frac{1}{2}}_{<\sin^2>} [\underbrace{\frac{1}{2}\epsilon_0|E^2|}_{Electrostatic} + \underbrace{n\frac{1}{2}m|\tilde{v}^2|}_{Particle\ Kinetic}] \tag{5.258}$$

Magnetic wave energy zero (negligible) for a longitudinal wave. We showed in Cold Plasma treatment that the velocity due to the wave is $\tilde{v} = \frac{qE}{-i\omega m}$ Hence

$$\overline{W} \simeq \frac{1}{2}\frac{\epsilon_0 E^2}{2}\left[1 + \frac{\omega_p^2}{\omega^2}\right] \qquad \text{(again electrons only)} \tag{5.259}$$

When the wave is damped, it has imaginary part of ω, ω_i and

$$\frac{d\overline{W}}{dt} = \overline{W}\frac{1}{E^2}\frac{dE^2}{dt} = 2\omega_i\overline{W} \tag{5.260}$$

Conservation of energy requires that this equal minus the particle energy gain rate, P. Hence

$$\omega_i = \frac{-P}{2W} = \frac{+E^2\frac{\pi q^2\omega}{2mk^2}\frac{\partial f_0}{\partial v}|_{\frac{\omega}{k}}}{\frac{\epsilon_0 E^2}{1}\left[1+\frac{\omega_p^2}{\omega^2}\right]} = \omega_p^2\frac{\pi}{2}\frac{\omega}{k^2}\frac{1}{n}\frac{\partial f_0}{\partial v}|_{\frac{\omega}{k}} \times \frac{2}{1+\frac{\omega_p^2}{\omega^2}} \tag{5.261}$$

So for waves such that $\omega \sim \omega_p$, which is the dispersion relation to lowest order, we get

$$\omega_i = \omega_p^2\frac{\pi}{2}\frac{\omega_r}{k^2}\frac{1}{n}\left.\frac{\partial f_0}{\partial v}\right|_{\frac{\omega_r}{k}} \quad . \tag{5.262}$$

This exactly agrees with the damping calculated from the complex dispersion relation using the Vlasov equation.

This is the Landau damping calculation for longitudinal waves in a (magnetic) field-free plasma. Strictly, just for electron plasma waves.

How does this apply to the general magnetized plasma case with multiple species?

Doing a complete evaluation of the dielectric tensor using kinetic theory is feasible but very heavy algebra. Our direct intuitive calculation gives the correct answer more directly.

5.9.7 Damping Mechanisms

Cold plasma dielectric tensor is Hermitian. [Complex conjugate*, transpose^T = original matrix.] This means *no damping* (dissipation).

The proof of this fact is simple but instructive. Rate of doing work on plasma per unit volume is $P = \mathbf{E}.\mathbf{j}$. However we need to observe notation.

Notation is that $\mathbf{E}(\mathbf{k},\omega)$ is amplitude of wave which is really $\Re(\mathbf{E}(\mathbf{k},\omega)\exp i(\mathbf{k}.\mathbf{x}-\omega t))$ and similarly for $\mathbf{j}$. Whenever products are taken: must *take real part first.* So

$$\begin{aligned} P &= \Re\left(\mathbf{E}\exp i\left(\mathbf{k}.\mathbf{x}-\omega t\right)\right).\Re\left(\mathbf{j}\exp i\left(\mathbf{k}.\mathbf{x}-\omega t\right)\right) \\ &= \frac{1}{2}\left[\mathbf{E}e^{i\phi}+\mathbf{E}^*e^{-i\phi}\right].\frac{1}{2}\left[\mathbf{j}e^{i\phi}+\mathbf{j}^*e^{-i\phi}\right] \quad (\phi = \mathbf{k}.\mathbf{x}-\omega t.) \\ &= \frac{1}{4}\left[\mathbf{E}.\mathbf{j}e^{2i\phi}+\mathbf{E}.\mathbf{j}^*+\mathbf{E}^*.\mathbf{j}+\mathbf{E}^*.\mathbf{j}^*e^{-2i\phi}\right] \end{aligned} \tag{5.263}$$

The terms $e^{2i\phi}$ & $e^{-2i\phi}$ are rapidly varying. We usually average over at least a period. These average to zero. Hence

$$\langle P\rangle = \frac{1}{4}\left[\mathbf{E}.\mathbf{j}^*+\mathbf{E}^*.\mathbf{j}\right] = \frac{1}{2}\Re\left(\mathbf{E}.\mathbf{j}^*\right) \tag{5.264}$$

Now recognize that $\mathbf{j} = \boldsymbol{\sigma}.\mathbf{E}$ and substitute

$$\langle P\rangle = \frac{1}{4}\left[\mathbf{E}.\boldsymbol{\sigma}^*.\mathbf{E}^*+\mathbf{E}^*.\boldsymbol{\sigma}.\mathbf{E}\right] \tag{5.265}$$

But for arbitrary matrices and vectors:

$$\mathbf{A}.\mathbf{M}.\mathbf{B} = \mathbf{B}.\mathbf{M}^T.\mathbf{A}; \tag{5.266}$$

(in our dyadic notation we don't explicitly indicate transposes of vectors). So

$$\mathbf{E}.\boldsymbol{\sigma}^*.\mathbf{E}^* = \mathbf{E}^*.\boldsymbol{\sigma}^{*T}.\mathbf{E} \tag{5.267}$$

hence

$$\langle P\rangle = \frac{1}{4}\mathbf{E}^*.\left[\boldsymbol{\sigma}^{*T}+\boldsymbol{\sigma}\right].\mathbf{E} \tag{5.268}$$

If $\boldsymbol{\epsilon} = \mathbf{1} + \frac{1}{-i\omega\epsilon_0}\boldsymbol{\sigma}$ is hermitian $\boldsymbol{\epsilon}^{*T} = \boldsymbol{\epsilon}$, then the conductivity tensor is antihermitian $\boldsymbol{\sigma}^{*T} = -\boldsymbol{\sigma}$ (if ω is real). In that case, equation 5.268 shows that $< P >= 0$. No dissipation. Any dissipation of wave energy is associated with an antihermitian part of $\boldsymbol{\sigma}$ and hence $\boldsymbol{\epsilon}$. Cold Plasma has none.

Collisions introduce damping. Can be included in equation of motion

$$m\frac{d\mathbf{v}}{dt} = q\left(\mathbf{E}+\mathbf{v}\wedge\mathbf{B}\right) - m\mathbf{v}\ \nu \tag{5.269}$$

where ν is the collision frequency.

Whole calculation can be followed through replacing $m(-i\omega)$ with $m(\nu - i\omega)$ everywhere. This introduces complex quantity in S, D, P.

We shall not bother with this because in fusion plasmas collisional damping is usually *negligible*. See this physically by saying that transit time of a wave is

$$\frac{\text{Size}}{\text{Speed}} \sim \frac{1 \text{ meter}}{3 \times 10^{+8} m/s} \simeq 3 \times 10^{-9} \text{ seconds.} \tag{5.270}$$

(Collision frequency)$^{-1} \sim 10\mu s \rightarrow 1ms$, depending on T_e, n_e.

When is the conductivity tensor Antihermitian?

Cold Plasma:

$$\boldsymbol{\epsilon} = \begin{bmatrix} S & -iD & 0 \\ iD & S & 0 \\ 0 & 0 & P \end{bmatrix} \quad \text{where} \quad \begin{array}{c} S = 1 - \sum_j \frac{\omega_{pj}^2}{\omega^2 - \Omega_j^2} \\ D = \sum_j \frac{\Omega_j}{\omega} \frac{\omega_{pj}}{\omega^2 - \Omega_j^2} \\ P = 1 - \sum_j \frac{\omega_{pj}^2}{\omega^2} \end{array} \tag{5.271}$$

This is manifestly Hermitian *if ω is real*, and then σ is anti-Hermitian.

This observation is sufficient to show that if the plasma is driven with a steady wave, there is no damping, and k does not acquire a complex part.

Two stream Instability

$$\epsilon_{zz} = 1 - \sum_j \frac{\omega_{pj}^2}{(\omega - kv_j)^2} \tag{5.272}$$

In this case, the relevant component is Hermitian (i.e. real) if *both* ω and k are real.

But that just begs the question: If ω and k are real, then there's no damping by definition.

So we can't necessarily detect damping or growth just by inspecting the dieletric tensor form when it depends on *both* ω and k.

Electrostatic Waves in general have $\epsilon = 0$ which is Hermitian. So really it is not enough to deal with ϵ or χ. We need to deal with $\sigma = -i\omega\epsilon_o\chi$, which indeed has a Hermitian component for the two-stream instability (even though χ is Hermitian) because ω is complex.

5.9.8 Ion Acoustic Waves and Landau Damping

We previously derived ion acoustic waves based on fluid treatment giving

$$\epsilon_{zz} = 1 - \frac{\omega_{pe}^2}{\omega^2 - \frac{k^2 p_e \gamma_e}{m_e n_e}} - \frac{\omega_{pi}^2}{\omega^2 - \frac{k^2 p_i \gamma_i}{m_i n_i}} \tag{5.273}$$

Leading to $\omega^2 \simeq k^2 \left[\frac{\gamma_i T_i + \gamma_e T_e}{m_i}\right]$.

Kinetic treatment adds the extra ingredient of Landau Damping. Vlasov plasma, unmagnetized:

$$\epsilon_{zz} = 1 - \frac{\omega_{pe}^2}{k^2}\int_C \frac{1}{v-\frac{\omega}{k}}\frac{\partial f_{oe}}{\partial v}\frac{dv}{n} - \frac{\omega_{pi}^2}{k^2}\int_C \frac{1}{v-\frac{\omega}{k}}\frac{\partial f_{oi}}{\partial v}\frac{dv}{n} \tag{5.274}$$

Both electron and ion damping need to be considered as possibly important.

Based on our fluid treatment we know these waves will have *small* phase velocity relative to *electron* thermal speed. Also c_s is somewhat larger than the *ion* thermal speed.

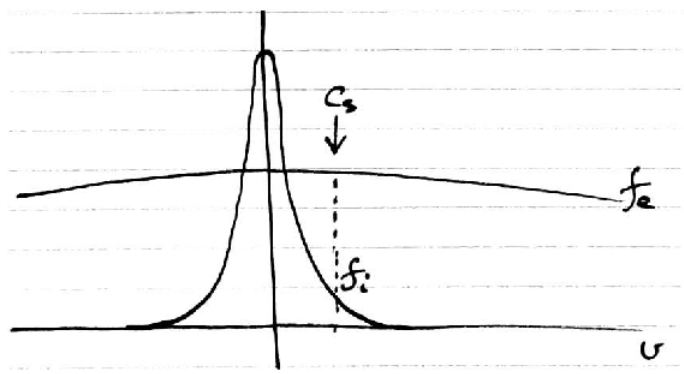

Figure 5.24: Distribution functions of ions and electrons near the sound wave speed.

So we adopt approximations

$$v_{te} \gg \frac{\omega}{k}, \quad v_{ti} < (<)\frac{\omega}{k} \tag{5.275}$$

and expand in opposite ways.

Ions are in the standard limit, so

$$\chi_i \simeq -\frac{\omega_{pi}^2}{\omega^2}\left[1 + \frac{3T_i}{m}\frac{k^2}{\omega^2} + \pi i\frac{\omega^2}{k^2}\frac{1}{n_i}\frac{\partial f_{oi}}{\partial_v}|_{w/k}\right] \tag{5.276}$$

Electrons: we regard $\frac{\omega}{k}$ as small and write

$$\begin{aligned}\wp\int \frac{1}{v-\frac{\omega}{k}}\frac{\partial f_{oe}}{\partial v}\frac{dv}{n} &\simeq \wp\int \frac{1}{v}\frac{\partial f_{oe}}{\partial v}\frac{dv}{n} \\ &= \frac{2}{n}\int \frac{\partial f_{oe}}{\partial v^2}dv \\ &= \frac{2}{n}\int -\frac{m_e}{2T_e}f_{oe}dv \qquad \text{for Maxwellian.} \\ &= -\frac{m_e}{T_e}\end{aligned} \tag{5.277}$$

Write $F_0 = f_o/n$.

Contribution from the pole is as usual so

$$\chi_e = -\frac{\omega_{pe}^2}{k^2}\left[-\frac{m_e}{T_e} + \pi i\left.\frac{\partial F_{oe}}{\partial v}\right|_{\omega/k}\right] \tag{5.278}$$

Collecting real and imaginary parts (at real ω)

$$\varepsilon_r(\omega_r) = 1 + \frac{\omega_{pe}^2}{k^2}\frac{m_e}{T_e} - \frac{\omega_{pi}^2}{\omega_r^2}\left[1 + \frac{3T_i}{m}\frac{k^2}{\omega_r^2}\right] \quad (5.279)$$

$$\varepsilon_i(\omega_r) = -\pi\frac{1}{k^2}\left[\omega_{pe}^2\frac{\partial F_{oe}}{\partial v}|_{\omega/k} + \omega_{pi}^2\frac{\partial F_{oi}}{\partial v}|_{\omega/k}\right] \quad (5.280)$$

The real part is essentially the same as before. The extra Bohm Gross term in ions appeared previously in the denominator as

$$\frac{\omega_{pi}^2}{\omega^2 - \frac{k^2 p_i \gamma_i}{m_i}} \leftrightarrow \frac{\omega_{pi}^2}{\omega^2}\left[1 + \frac{3T_i}{m_i}\frac{k^2}{\omega^2}\right] \quad (5.281)$$

Since our kinetic form is based on a rather inaccurate Taylor expansion, it is not clear that it is a better approx. We are probably better off using

$$\frac{\omega_{pi}^2}{\omega^2}\frac{1}{1 - \frac{3T_i k^2}{m_i \omega^2}}. \quad (5.282)$$

Then the solution of $\varepsilon_r(\omega_r) = 0$ is

$$\frac{\omega_r^2}{k^2} = \left[\frac{T_e + 3T_i}{m_i}\right]\frac{1}{1 + k^2\lambda_{De}^2} \quad (5.283)$$

as before, but we've proved that $\gamma_e = 1$ is the correct choice, and kept the $k^2\lambda_{De}^2$ term (1st term of ε_r).

The imaginary part of ε gives damping.

General way to solve for damping when small

We want to solve $\varepsilon(\mathbf{k}, \omega) = 0$ with $\omega = \omega_r + i\omega_i$, ω_i small.

Taylor expand ε about real ω_r:

$$\varepsilon(\omega) \simeq \varepsilon(\omega_r) + i\omega_i\frac{d\varepsilon}{d\omega}|_{\omega_r} \quad (5.284)$$

$$= \varepsilon(\omega_r) + i\omega_i\frac{\partial}{\partial\omega_r}\varepsilon(\omega_r) \quad (5.285)$$

Let ω_r be the solution of $\varepsilon_r(\omega_r) = 0$; then

$$\varepsilon(\omega) = i\varepsilon_i(\omega_r) + i\omega_i\frac{\partial}{\partial\omega_r}\varepsilon(\omega_r). \quad (5.286)$$

This is equal to zero when

$$\omega_i = -\frac{\varepsilon_i(\omega_r)}{\frac{\partial\varepsilon(\omega_r)}{\partial\omega_r}} \quad . \quad (5.287)$$

If, by presumption, $\varepsilon_i \ll \varepsilon_r$, or more precisely (in the vicinity of $\varepsilon = 0$), $\partial\varepsilon_i/\partial\omega_r \ll \partial\varepsilon_r/\partial\omega_r$ then this can be written to lowest order:

$$\omega_i = -\frac{\varepsilon_i(\omega_r)}{\frac{\partial\varepsilon_r(\omega_r)}{\partial\omega_r}} \tag{5.288}$$

Apply to ion acoustic waves:

$$\frac{\partial\varepsilon_r(\omega_r)}{\partial\omega_r} = \frac{\omega_{pi}^2}{\omega_r^3}\left[2 + 4\frac{4T_i}{m_i}\frac{k^2}{\omega_r^2}\right] \tag{5.289}$$

so

$$\omega_i = \frac{\pi}{k^2}\frac{\omega_r^3}{\omega_{pi}^2}\left[\frac{1}{2 + 4\frac{4T_i}{m_i}\frac{k^2}{\omega_r^2}}\right]\left[\omega_{pe}^2\frac{\partial F_{oe}}{\partial v}|_{\omega/k} + \omega_{pi}^2\frac{\partial F_{oi}}{\partial v}|_{\omega/k}\right] \tag{5.290}$$

For Maxwellian distributions, using our previous value for ω_r,

$$\begin{aligned}
\frac{\partial F_{oe}}{\partial v}|_{\frac{\omega_r}{k}} &= \left[-\left(\frac{m_e}{2\pi T_e}\right)^{\frac{1}{2}}\frac{m_e v}{T_e}e^{-\frac{m_e v^2}{2T_e}}\right]_{v=\frac{\omega_r}{k}} \\
&= -\frac{1}{\sqrt{2\pi}}\left(\frac{m_e}{T_e}\right)^{\frac{3}{2}}\left[\frac{T_e + 3T_i}{m_i}\right]^{\frac{1}{2}}\frac{1}{\sqrt{1+k^2\lambda_D^2}}\exp\left(-\frac{m_e}{2m_i}\frac{1+\frac{3T_i}{T_e}}{1+k^2\lambda_D^2}\right) \\
&= -\frac{1}{\sqrt{2\pi}}\left(\frac{m_e}{m_i}\right)^{\frac{1}{2}}\frac{m_e}{T_e}\frac{\left[1+\frac{3T_i}{T_e}\right]^{\frac{1}{2}}}{\sqrt{1+k^2\lambda_{De}^2}}\,,
\end{aligned} \tag{5.291}$$

where the exponent is of order m_e/m_i here, and so the exponential is 1. And

$$\frac{\partial F_{oi}}{\partial v}|_{\frac{\omega_r}{k}} = -\frac{1}{\sqrt{2\pi}}\frac{m_i}{T_i}\frac{\left[1+\frac{3T_i}{T_e}\right]^{\frac{1}{2}}}{\sqrt{1+k^2\lambda_D^2}}\left(\frac{T_e}{T_i}\right)^{\frac{1}{2}}\exp\left(-\frac{T_e}{2T_i}\frac{1+\frac{3T_i}{T_e}}{1+k^2\lambda_D^2}\right) \tag{5.292}$$

Hence

$$\begin{aligned}
\frac{\omega_i}{\omega_r} &= -\frac{\pi}{\sqrt{2\pi}}\frac{\omega_r^2}{k^2}\left[\frac{1}{2+4\frac{3T_i}{m_i}\frac{k^2}{\omega_r^2}}\right]\frac{\left[1+\frac{3T_i}{T_e}\right]^{\frac{1}{2}}}{\sqrt{1+k^2\lambda_D^2}} \times \\
&\quad \left[\frac{m_i}{m_e}\left(\frac{m_e}{m_i}\right)^{\frac{1}{2}}\frac{m_e}{T_e} + \frac{m_i}{T_i}\left(\frac{T_e}{T_i}\right)^{\frac{1}{2}}\exp\left(-\frac{T_e}{2T_i}\frac{1+\frac{3T_i}{T_e}}{1+k^2\lambda_D^2}\right)\right]
\end{aligned} \tag{5.293}$$

$$\begin{aligned}
\frac{\omega_i}{\omega_r} &= -\sqrt{\frac{\pi}{2}}\frac{1}{[1+k^2\lambda_{De}^2]^{\frac{3}{2}}}\frac{\left[1+\frac{3T_i}{T_e}\right]^{\frac{3}{2}}}{2+4\frac{3T_i}{T_e+3T_i}} \times \\
&\quad \left[\underbrace{\left(\frac{m_e}{m_i}\right)^{\frac{1}{2}}}_{electron} + \underbrace{\left(\frac{T_e}{T_i}\right)^{\frac{3}{2}}\exp\left(-\frac{T_e}{2T_i}\frac{1+\frac{3T_i}{T_e}}{1+k^2\lambda_{De}^2}\right)}_{ion\ damping}\right].
\end{aligned} \tag{5.294}$$

[Note: the coefficient on the first line of equation 5.294 for ω_i/ω_r reduces to $\simeq -\sqrt{\pi/8}$ for $T_i/T_e \ll 1$ and $k\lambda_D e \ll 1$.]
Electron Landau damping of ion acoustic waves is rather small: $\frac{\omega_i}{\omega_r} \sim \sqrt{\frac{m_e}{m_i}} \sim \frac{1}{70}$.
Ion Landau damping is *large*, ~ 1 unless the term in the exponent is large. That is

$$\text{unless} \quad \frac{T_e}{T_i} \gg 1 \quad . \tag{5.295}$$

Physics is that large $\frac{T_e}{T_i}$ pulls the phase velocity of the wave: $\sqrt{\frac{T_e+3T_i}{m_i}} = c_s$ above the ion thermal velocity $v_{ti} = \sqrt{\frac{T_i}{m_i}}$. If $c_s \gg v_{ti}$ there are few resonant ions to damp the wave.
[Note. Many texts drop terms of order $\frac{T_i}{T_e}$ early in the treatment, but that is not really accurate. We have kept the first order, giving extra coefficient

$$\left[1 + \frac{3T_i}{T_e}\right]^{\frac{3}{2}} \left[\frac{T_e + 3T_i}{T_e + 6T_i}\right] \simeq 1 + \frac{3}{2}\frac{T_i}{T_e} \tag{5.296}$$

and an extra factor $1 + \frac{3T_i}{T_e}$ in the exponent. When $T_i \sim T_e$ we ought really to use full solutions based on the Plasma Dispersion Function.]

5.9.9 Alternative expressions of Dielectric Tensor Elements

This subsection gives some useful algebraic relationships that enable one to transform to different expressions sometimes encountered.

$$\chi_{zz} = \frac{q^2}{\omega m \epsilon_o} \int_C \frac{v \frac{\partial f_o}{\partial v}}{\omega - kv} dv = \frac{q^2}{\omega^2 m \epsilon_o} \frac{\omega}{k} \int_C \left(\frac{\omega}{\omega - kv} - 1\right) \frac{\partial f_o}{\partial v} dv \tag{5.297}$$

$$= \frac{q^2}{m\epsilon_o} \frac{1}{k^2} \int_C \frac{1}{\frac{\omega}{k} - v} \frac{\partial f_o}{\partial v} dv \tag{5.298}$$

$$= \frac{\omega_p^2}{k^2} \int_C \frac{1}{\frac{\omega}{k} - v} \frac{1}{n} \frac{\partial f_o}{\partial v} dv \tag{5.299}$$

$$= \frac{\omega_p^2}{k^2} \left[\wp \int \frac{1}{\frac{\omega}{k} - v} \frac{\partial F_o}{2v} dv - \pi i \frac{\partial F_o}{\partial v}|_{\frac{\omega}{k}}\right] \tag{5.300}$$

where $F_o = \frac{f_o}{n}$ is the *normalized* distribution function. Other elements of χ involve integrals of the form

$$\chi_{jl} \frac{\omega m \epsilon_o}{q^2} = \int \frac{v_j \frac{\partial f_o}{\partial v_l}}{\omega - \mathbf{k}.\mathbf{v}} d^3v \ . \tag{5.301}$$

When $\mathbf{k}$ is in z-direction, $\mathbf{k}.\mathbf{v} = k_z v_z$. (Multi dimensional distribution f_0).
If (e.g., χ_{xy}) $l \neq z$ and $j \neq l$ then the integral over v_l yields $\int \frac{\partial f_o}{\partial v_l} dv_l = 0$. If $j = l \neq z$ then

$$\int v_j \frac{\partial f_o}{\partial v_j} dv_j = -\int f_o dv_j \ , \tag{5.302}$$

by parts. So, recalling the definition $f_z \equiv \int f dv_x dv_y$,

$$\begin{aligned}\chi_{xx} = \chi_{yy} &= -\frac{q^2}{\omega m \epsilon_o} \int \frac{f_{oz}}{\omega - \mathbf{k}.\mathbf{v}} dv_z \\ &= -\frac{\omega_p^2}{\omega} \int \frac{F_{oz}}{\omega - \mathbf{k}.\mathbf{v}} dv_z. \end{aligned} \tag{5.303}$$

The fourth type of element is

$$\chi_{xz} = \frac{q^2}{\omega m \epsilon_o} \int \frac{v_x \frac{\partial f_o}{\partial v_z}}{\omega - k_z v_z} d^3 v \ . \tag{5.304}$$

This is not zero unless f_o is isotropic ($= f_o(v)$).
If f is isotropic

$$\frac{\partial f_o}{\partial v_z} = \frac{df_o}{dv} \frac{\partial v}{\partial v_z} = \frac{v_z}{v} \frac{df_o}{dv} \tag{5.305}$$

Then

$$\begin{aligned}\int \frac{v_x \frac{\partial f_o}{\partial v_z}}{\omega - k_z v_z} d^3 v &= \int \frac{v_x v_z}{\omega - k_z v_z} \frac{1}{v} \frac{df_o}{dv} d^3 v \\ &= \int \frac{v_z}{\omega - k_z v_z} \frac{\partial f_o}{\partial v_x} d^3 v = 0 \end{aligned} \tag{5.306}$$

(since the v_x-integral of $\partial f_o / \partial v_x$ is zero). Hence for isotropic $F_o = f_0/n$, with $\mathbf{k}$ in the z-direction,

$$\chi = \begin{bmatrix} -\frac{\omega_p^2}{\omega} \int_C \frac{F_{oz}}{\omega - k v_z} dv_z & 0 & 0^+ \\ 0 & -\frac{\omega_p^2}{\omega} \int_C \frac{F_{oz}}{\omega - k v_z} dv_z & 0^+ \\ 0 & 0 & \frac{\omega_p^2}{k} \int_C \frac{1}{\omega - k v_z} \frac{\partial F_{oz}}{\partial v_z} dv_z \end{bmatrix} \tag{5.307}$$

(and the terms 0^+ are the ones that need isotropy to make them zero).

$$\boldsymbol{\epsilon} = \begin{bmatrix} \epsilon_t & 0 & 0 \\ 0 & \epsilon_t & 0 \\ 0 & 0 & \epsilon_l \end{bmatrix} \tag{5.308}$$

where

$$\epsilon_t = 1 - \frac{\omega_p^2}{\omega} \int_C \frac{F_{oz}}{\omega - k v_z} dv_z \tag{5.309}$$

$$\epsilon_l = 1 - \frac{\omega_p^2}{k^2} \int_C \frac{1}{v - \frac{\omega}{k}} \frac{\partial F_{oz}}{\partial v_z} dv_z \tag{5.310}$$

All integrals are along the Landau contour, passing *below* the pole.

5.9.10 Electromagnetic Waves in unmagnetized Vlasov Plasma

For transverse waves the dispersion relation is

$$\frac{k^2c^2}{\omega^2} = N^2 = \epsilon_t = 1 - \frac{\omega_p^2}{\omega}\frac{1}{n}\int_C \frac{f_{oz}dv_z}{(\omega - k_z v_z)} \tag{5.311}$$

This has, in principle, a contribution from the pole at $\omega - kv_z = 0$. However, for a non-relativistic plasma, thermal velocity is $\ll c$ and the EM wave has phase velocity $\sim c$. Consequently, for all velocities v_z for which f_{oz} is non-zero $kv_z \ll \omega$. We have seen with the cold plasma treatment that the wave phase velocity is actually greater than c. Therefore a proper relativistic distribution function will have no particles at all in resonance with the wave.

Therefore:

1. The imaginary part of ϵ_t from the pole is negligible. And relativisitically zero.

2.

$$\begin{aligned}
\epsilon_t &\simeq 1 - \frac{\omega_p^2}{\omega^2}\frac{1}{n}\int_{-\infty}^{\infty} f_{oz}\left(1 + \frac{kv_z}{\omega} + \frac{k^2v_z^2}{\omega^2} + ...\right) dv_z \\
&= 1 - \frac{\omega_p^2}{\omega^2}\left[1 + \frac{k^2}{\omega^2}\frac{T}{m} + ...\right] \\
&\simeq 1 - \frac{\omega_p^2}{\omega^2}\left[1 + \frac{k^2v_t^2}{\omega^2}\right] \\
&\simeq 1 - \frac{\omega_p^2}{\omega^2}
\end{aligned} \tag{5.312}$$

Thermal correction to the refractive index N is small because $\frac{k^2v_t^2}{\omega^2} \ll 1$.

Electromagnetic waves are hardly affected by Kinetic Theory treatment in unmagnetized plasma. Cold Plasma treatment is generally good enough.