Preface

This monograph sums up studies performed in developing the relativistic theory of gravity (RTG) and presented in refs. [3, 9, 38, 10, 5, 11, 6, 34, 12, 35, 36, 37, 31, 13]. Detailed references to earlier works, that to a certain extent served as scaffolding in the construction of RTG, are given in the monograph [10], written together with prof. M. A. Mestvirishvili and published in 1989. Therein, also, critical comments are presented concerning general relativity theory (GRT), which still remain in force. In order to facilitate reading in section 14 we provide elements of tensor analysis and Riemannian geometry. As a rule, we make use of the set of units in which $G = c = \hbar = 1$. However, in the final expressions we restore the dependence on the constants $G, c, \hbar$. Throughout the book, Greek letters assume values 0,1,2,3, while Latin letters assume —1,2,3.

The creation of this monograph advanced together with the completion of studies of individual issues, so it inevitably contains recurrences, especially concerning such issues that are important for understanding the essence of both RTG and GRT.

The hypothesis underlying RTG asserts that the gravitational field, like all other physical fields, develops in Minkowski space, while the source of this field is the conserved energy-momentum tensor of matter, including the gravitational field itself. This approach permits constructing, in a unique manner, the theory of the gravitational field as a gauge theory. Here, there arises an effective Riemannian space, which literally has a field nature. In GRT the space is considered to be Riemannian owing to the presence of matter, so gravity is considered a consequence of space–time exhibiting curvature. The RTG gravitational field has spins 2 and 0 and represents a physical field in the Faraday–Maxwell spirit. The complete set of RTG equations follows directly from the least action principle. Since all physical fields develop in Minkowski space, all fundamental principles of physics — the integral conservation laws of energy–momentum and of angular momentum — are strictly obeyed in RTG. In the theory the Mach principle is realized: an inertial system is determined by the distribution of matter. Unlike GRT, acceleration has an absolute sense. Inertial and gravitational forces are separated, and they differ in their nature. The theory, unlike GRT, provides a unique explanation for all gravitational effects in the Solar system. **GRT does not comply with the equivalence principle, does not explain the equality of the inert and active gravitational masses, and gives no unique prediction for gravitational effects. It does not contain the usual conservation laws of energy–momentum and of angular momentum of matter.**

It should be especially noted that the known post-Newtonian approximation do satisfy the equivalence principle, do provide a unique description of gravitational effects in the Solar system, and also establish the equality between the inertial and active gravitational masses. However, it does not follow uniquely from the GRT equations, since its derivation relies on additional assumptions, that do not follow from the theory, i.e. a departure occurs beyond the limits of GRT, which is based on the gravitational field being represented as a physical field, although this is not so in GRT. Therefore, this approximation cannot be considered a unique consequence of the GRT equations. It has rather been guessed, then derived from the theory, while, according to RTG, the post–Newtonian approximation follows uniquely from equations of the theory. Thus, the post–Newtonian approximation, previously applied for the description of gravitational effects follows directly from our theory. RTG introduces essential changes into the character of the development of the Universe and into the collapse of large masses.

Analysis of the development of a homogeneous and isotropic Universe within RTG leads to the conclusion that the Universe is infinite, and that it is “flat”. Its development proceeds cyclically from a certain maximum density down to a minimum and so on. Thus, no pointlike Big Bang occurred in the past. There existed a state of high density and high temperature at each point in space.

According to RTG, the so-called cosmological "expansion" of the Universe, observed by the red shift, is explained by changes in the gravitational field, but not by relative motion — galaxies escaping from each other, which actually does not take place. Matter in the Universe is in a state of rest relative to an inertial coordinate system. The peculiar velocities of galaxies relative to an inertial system arose owing to a certain structure of the inhomogeneity of the distribution of matter during the period, when the Universe became transparent. This means that in the past the distance between galaxies was never zero. The theory predicts the existence in the Universe of a large hidden mass of "dark matter". According to RGT, "black holes" cannot exist: a collapsing star cannot disappear beyond its gravitational radius. Objects with large masses can exist, and they are characterized not only by mass, but also by a distribution of matter density. Since, in accordance with GRT, objects with masses exceeding three solar masses transform, at the conclusive stage of their evolution, into "black holes", an object found to have a large mass is usually attributed to "black holes". Since RTG predictions concerning the behaviour of large masses differ essentially from GRT predictions, observational data of greater detail are required for testing the conclusions of theory. Thus, for example, in RTG spherically symmetric accretion of matter onto a body of large mass, that is at its conclusive stage of evolution (when the nuclear resources are exhausted), will be accompanied by a significant release of energy owing to the fall of matter onto the body's surface, while in GRT the energy release in the case of spherically symmetric accretion of matter onto a "black hole" is extremely small, since the falling matter takes the energy with it into the "black hole". Observational data on such objects could answer the question whether "black holes" exist in

Nature, or not. Field concepts of gravity necessarily require introduction of the graviton rest mass, which can be determined from observational data: the Hubble "constant" and the deceleration parameter q. According to the theory, the parameter q can only be positive, at present, i.e. deceleration of "expansion" of the Universe takes place, instead of acceleration. For this reason, the latest observational data on acceleration of the "expansion" must be checked carefully, since the conclusions of theory concerning "deceleration" follow from the general physical principles mentioned above.

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A.A. Logunov
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Introduction

Since construction of the relativistic theory of gravity (RTG) is based on special relativity theory (SRT), we shall deal with the latter in greater detail and in doing so we shall examine both the approach of Henri Poincaré and that of Albert Einstein. Such an analysis will permit a more profound comprehension of the difference between these approaches and will make it possible to formulate the essence of relativity theory.

In analyzing the Lorentz transformations, H. Poincaré showed that these transformations, together with all spatial rotations, form a **group** that does not alter the equations of electrodynamics. Richard Feynman wrote the following about this: *"Precisely Poincaré proposed investigating what could be done with the equations without altering their form. It was precisely his idea to pay attention to the symmetry properties of the laws of physics"*[1]. H.Poincaré did not restrict himself to studying electrodynamics; he discovered the equations of relativistic mechanics and extended the Lorentz transformations to all the forces of Nature. Discovery of the group, termed by H.Poincaré the Lorentz group, made it possible for him to introduce four-dimensional space-time with an **invariant** subsequently termed the interval

$$d\sigma^2 = (dX^0)^2 - (dX^1)^2 - (dX^2)^2 - (dX^3)^2 \,. \qquad (\alpha)$$

Precisely from the above it is absolutely clear that time and spatial length are **relative**.

Later, a further development in this direction was made by Herman Minkowski, who introduced the concepts of timelike and spacelike intervals. Following H.Poincaré and H.Minkowski exactly, the essence of relativity theory may be formulated thus: **all physical phenomena proceed in space–time, the geometry of which is pseudo-Euclidean and is determined by the interval** (α)**.** Here it is important to em-

[1] R.Feynman. The character of physical laws. M.:Mir, 1968, p.97.

phasize, that **the geometry of space-time reflects those general dynamic properties, that represent just what makes it universal.** In four-dimensional space (Minkowski space) one can adopt a quite arbitrary reference frame

$$X^\nu = f^\nu(x^\mu) \; ,$$

realizing a mutually unambiguous correspondence with a Jacobian differing from zero. Determining the differentials

$$dX^\nu = \frac{\partial f^\nu}{\partial x^\mu} dx^\mu \; ,$$

and substituting these expressions into (α) we find

$$d\sigma^2 = \gamma_{\mu\nu}(x) dx^\mu dx^\nu \; , \quad \text{where}$$

$$\gamma_{\mu\nu}(x) = \epsilon_\sigma \frac{\partial f^\sigma}{\partial x^\mu} \frac{\partial f^\sigma}{\partial x^\nu} \; , \; \epsilon_\sigma = (1, -1, -1, -1) \; . \qquad (\beta)$$

It is quite evident that the transition undergone to an arbitrary reference system did not lead us beyond the limits of pseudo-Euclidean geometry. But hence it follows that non-inertial reference systems can also be applied in SRT. The forces of inertia arising in transition to an accelerated reference system are expressed in terms of the Christoffel symbols of Minkowski space. The representation of SRT stemming from the work of H.Poincaré and H.Minkowski was more general and turned out to be extremely necessary for the construction of SRT, since it permitted introduction of the metric tensor $\gamma_{\mu\nu}(x)$ of Minkowski space in arbitrary coordinates and thus made it possible to introduce in a covariant manner the gravitational field, upon separation of the forces of inertia from gravity.

From the point of view of history it must be noted that in his earlier works [2], "The measurement of time" and "The

[2] H.Poincaré. The principle of relativity. M.:Atomizdat, 1973, pp.19, 33.

present and future of mathematical physics", H.Poincaré discussed in detail issues of the constancy of the velocity of light, of the simultaneity of events at different points of space determined by the synchronization of clocks with the aid of a light signal. Later, on the basis of the relativity principle, which he formulated in 1904 for all physical phenomena, as well as on the work published by H.Lorentz the same year, **H.Poincaré discovered a transformation group in 1905 and termed it the Lorentz group. This permitted him to give the following essentially accurate formulation of the relativity theory: the equations of physical processes must be invariant relative to the Lorentz group.** Precisely such a formulation was given by A.Einstein in 1948: *"With the aid of the Lorentz transformations the special principle of relativity can be formulated as follows: The laws of Nature are invariant relative to the Lorentz transformation (i.e. a law of Nature should not change if it is referred to a new inertial reference frame with the aid of the Lorentz transformation for* x, y, z, t)" [3].

The existence of a group of coordinate-time transformations signifies that there exists an infinite set of equivalent (inertial) reference frames related by the Lorentz transformations. From the invariance of equations it follows, in a trivial manner, that physical equations in the reference frames x and x', related by the Lorentz transformations, are identical. But this means that any phenomenon described both in x and x' reference systems under identical conditions will yield identical results, i.e. the relativity principle is satisfied in a trivial manner. Certain, even prominent, physicists understood this with difficulty not even long ago, while others have not even been able to. There is nothing strange in this fact, since any study requires certain professionalism. What is surprising is the following: they attempt to explain their incomprehen-

[3] Einstein A. Collection of scientific works, Moscow: Nauka, 1966, vol.2, art.133, p.660.

sion, or the difficulty they encountered in understanding, by H.Poincaré allegedly "not having taken the decisive step", "not having gone to the end". But these judgements, instead of the level of the outstanding results achieved by H.Poincaré in relativity theory, characterize their own level of comprehension of the problem.

Precisely for this reason W.Pauli wrote the following in 1955 in connection with the 50-th anniversary of relativity theory: "*Both Einstein and Poincaré relied on the preparatory works performed by H.A.Lorentz, who was very close to the final result, but was not able to take the last decisive step. In the results, obtained by Einstein and Poincaré independently of each other, being identical I see the profound meaning of the harmony in the mathematical method and analysis performed with the aid of thought experiments and based on the entire set of data of physical experiments"*[4].

Detailed investigation by H.Poincaré of the Lorentz group invariants resulted in his discovery of the pseudo-Euclidean geometry of space-time. Precisely on such a basis, he established the four-dimensionality of physical quantities: force, velocity, momentum, current. H.Poincaré's first short work appeared in the reports of the French Academy of sciences before A.Einstein's work was even submitted for publication. That work contained an accurate and rigorous solution of the problem of electrodynamics of moving bodies, and at the same time it extended the Lorentz transformations to all natural forces, of whatever origin they might be. Very often many historians, and, by the way, physicists, also, discuss priority issues. A very good judgement concerning this issue is due to academicians V.L.Ginzburg and Ya.B.Zel'dovich, who in 1967 wrote:"*Thus, no matter what a person has done himself, he cannot claim*

[4]W.Pauli. Essays in physics. M.:Nauka, 1975, p.189.

priority, if it later becomes known that the same result was obtained earlier by others" [5].

A.Einstein proceeded toward relativity theory from an analysis of the concepts of simultaneity and of synchronization for clocks at different points in space on the basis of the principle of constancy of the velocity of light. ¡¡*Each ray of light travels in a reference frame at "rest" with a certain velocity V, independently of whether this ray of light is emitted by a body at rest or by a moving body.*¿¿ But this point cannot be considered a principle, since it implies a certain choice of reference frame, while a physical principle should clearly not depend on the method of choosing the reference frame. In essence, A.Einstein accurately followed the early works of H.Poincaré. However, within such an approach it is impossible to arrive at non-inertial reference frames, since in such reference frames it is impossible to take advantage of clock synchronization, so the notion of simultaneity loses sense, and, moreover, the velocity of light cannot be considered constant.

In a reference frame undergoing acceleration the proper time $d\tau$, where

$$d\sigma^2 = d\tau^2 - s_{ik}dx^i dx^k,\ d\tau = \frac{\gamma_{0\alpha}dx^\alpha}{\sqrt{\gamma_{00}}},\ s_{ik} = -\gamma_{ik} + \frac{\gamma_{0i}\gamma_{0k}}{\gamma_{00}}$$

is not a complete differential, so the synchronization of clocks at different points in space depends on the synchronization path. This means that such a concept cannot be applied for reference frames undergoing acceleration. It must be stressed that the coordinates in expression (β) have no metric meaning, on their own. Physically measurable quantities must be constructed with the aid of coordinates and the metric coefficients $\gamma_{\mu\nu}$. But all this remained misunderstood for a long time in SRT, since it was usual to adopt A.Einstein's approach, instead of the one of H.Poincaré and H.Minkowski. Thus, the

[5]V.L.Ginzburg, Ya.B.Zel'dovich. Familiar and unfamiliar Zel'dovich. M.:Nauka, 1993, p.88.

starting points introduced by A.Einstein were of an exclusively limited and partial nature, even though they could create an illusion of simplicity. It was precisely for this reason that even in 1913 A.Einstein wrote: "*In usual relativity theory only linear orthogonal transformations are permitted*"[6]. Or somewhat later, in the same year, he writes: "*In the original relativity theory the independence of physical equations of the specific choice of reference system is based on postulating the fundamental invariant* $ds^2 = \sum dx_i^2$, *while now the issue consists in constructing a theory (general relativity theory is implied – A.L.), in which the role of the fundamental invariant is performed by a linear element of the general form*

$$ds^2 = \sum_{i,k} g_{ik} dx^i dx^k \text{ "}^{7}.$$

A.Einstein wrote something similar in 1930: "*In special relativity theory only such coordinate changes (transformations) are allowed that provide for the quantity* ds^2 *(a fundamental invariant) in the new coordinates having the form of the sum of square differentials of the new coordinates. Such transformations are called Lorentz transformations*"[8].

Hence it is seen that the approach adopted by A.Einstein did not lead him to the notion of space-time exhibiting a pseudo-Euclidean geometry. A comparison of the approaches of H.Poincaré and A.Einstein to the construction of SRT clearly reveals H.Poincaré's approach to be more profound and general, since precisely H.Poincaré had defined the pseudo-Euclidean structure of space-time. A.Einstein's approach essentially restricted the boundaries of SRT, but, since the exposition of SRT in the literature usually followed A.Einstein, SRT

[6]Einstein A. Collection of scientific works. Moscow: Nauka, 1965, vol.1, art.21, p.232.

[7]Einstein A. Collection of scientific works, Moscow: Nauka, 1965, vol.1, art.22,p.269.

[8]Einstein A. Collection of scientific works, Moscow: Nauka, 1966, vol.2, art.95, p.281.

was quite a long time considered valid only in inertial reference systems. Minkowski space was then treated like a useful geometric interpretation or like a mathematical formulation of the principles of SRT within the approach of Einstein. Let us now pass over to gravity. In 1905 H.Poincaré wrote: *"... that forces of whatever origin, for example, the forces of gravity, behave in the case of uniform motion (or, if you wish, under Lorentz transformations) precisely like electromagnetic forces"*[9]. This is precisely the path we shall follow.

A.Einstein, having noticed the equality of inertial and gravitational masses, was convinced that the forces of inertia and of gravity are related, since their action is independent of a body's mass. In 1913 he arrived at the conclusion that, if in expression (α) "*... we introduce new coordinates x_1, x_2, x_3, x_4, with the aid of some arbitrary substitution, then the motion of a point relative to the new reference frame will proceed in accordance with the equation*

$$\delta\{\int ds\} = 0 ,$$

and

$$ds^2 = \sum_{\mu,\nu} g_{\mu\nu} dx^\mu dx^\nu \text{ ."}$$

and he further pointed out: ¡¡*The motion of a material point in the new reference system is determined by the quantities $g_{\mu\nu}$, which in accordance with the preceding paragraphs should be understood as the components of the gravitational field, as soon as we decide to consider this new system to be "at rest"*¿¿[10]. **Identifying in such a manner the metric field, obtained from (α) with the aid of coordinate transformations, and the gravitational field is without physical grounds, since transformations of coordinates do not lead us beyond the framework of pseudo-Euclidean geometry.**

[9]H.Poincaré. Special relativity principle. M.:Atomizdat, 1973, p.152.

[10]Einstein A. Collection of scientific works, Moscow: Nauka, 1965, vol.1, art.23, p.286.

From our point of view, it is not permitted to consider such a metric field as the gravitational field, since this contradicts the very essence of the concept of a field as a physical reality. Therefore, it is impossible to agree with the following reasoning of A.Einstein: ¡¡*The gravitational field "exists" with respect to the system K' in the same sense as any other physical quantity that can be defined in a certain reference system, even though it does not exist in system K. There is nothing strange, here, and it may be readily demonstrated by the following example taken from classical mechanics. Nobody doubts the "reality" of kinetic energy, since otherwise it would be necessary to renounce energy in general. It is clear, however, that the kinetic energy of bodies depends on the state of motion of the reference system: by an appropriate choice of the latter it is evidently possible to provide for the kinetic energy of uniform motion of a certain body to assume, at a certain moment of time, a given positive or zero value set beforehand. In the special case, when all the masses have equal in value and equally oriented velocities, it is possible by an appropriate choice of the reference system to make the total kinetic energy equal to zero. In my opinion the analogy is complete*¿¿[11].

As we see, Einstein renounced the concept of a classical field, such as the Faraday–Maxwell field possessing density of energy-momentum, in relation to the gravitational field. Precisely this path led him up to the construction of GRT, to gravitational energy not being localizable, to introduction of the pseudotensor of the gravitational field. If the gravitational field is considered as a physical field, then it, like all other physical fields, is characterized by the energy-momentum tensor $t^{\mu\nu}$. If in some reference frame, for instance, K', there exists a gravitational field, this means that certain components (or all of them) of the tensor $t^{\mu\nu}$ differ from zero. The tensor $t^{\mu\nu}$ cannot be reduced to zero by a coordinate transformation,

[11] Einstein A. Collection of scientific works, Moscow: Nauka, 1965, vol.1, art.46, p.620.

i..e, if a gravitational field exists, then it represents a physical reality, and it cannot be annihilated by a choice of reference system. It is not correct to compare such a gravitational field with kinetic energy, since the latter is not characterized by a covariant quantity. It must be noted that such a comparison is not admissible, also, in GRT, since the gravitational field in this theory is characterized by the Riemann curvature tensor. If it differs from zero, then the gravitational field exists, and it cannot be annihilated by a choice of reference system, even locally.

Accelerated reference systems have played an important heuristic role in A.Einstein's creative work, although they have nothing to do with the essence of GRT. By identifying accelerated reference systems to the gravitational field, A.Einstein came to perceive the metric space-time tensor as the principal characteristic of the gravitational field. But the metric tensor reflects both the natural properties of geometry and the choice of reference system. In this way the possibility arises of explaining the force of gravity kinematically, by reducing it to the force of inertia. But in this case it is necessary to renounce the gravitational field as a physical field. "*Gravitational fields* (as A.Einstein wrote in 1918) *may be set without introducing tensions and energy density.*"[12]. But that is a serious loss, and one cannot consent to it. However, as we shall further see, this loss can be avoided in constructing RTG.

Surprisingly, even in 1933 A.Einstein wrote: ¡¡*In special Relativity theory — as shown by H.Minkowski — this metric was quasi-Euclidean, i.e. the square "length" ds of a linear element represented a certain quadratic function of the coordinate differentials. If, on the other hand, new coordinates are introduced with the aid of a linear transformation, then* ds^2 *remains a homogeneous function of the coordinate differentials, but the coefficients of this function* ($g_{\mu\nu}$) *will no longer be constant,*

[12]Einstein A. Collection of scientific works, Moscow: Nauka, 1965, vol.1, art.47, p.627.

but certain functions of the coordinates. From a mathematical point of view this means that the physical (four-dimensional) space possesses a Riemannian metric¿¿[13].

This is certainly wrong, since a pseudo-Euclidean metric cannot be transformed into a Riemannian metric by transformation of the coordinates. But the main point, here, consists in something else, namely, in that in this way, thanks to his profound intuition, A.Einstein arrived at the necessity of introducing precisely Riemannian space, since he considered the metric tensor $g_{\mu\nu}$ of this space to describe gravity. This was essentially how the tensor nature of gravity was revealed. The unity of the Riemannian metric and gravity is the main principle underlying general relativity theory. V.A.Fock wrote about this principle: "*... precisely this principle represents the essence of Einstein's theory of gravity*"[14]. From a general point of view, however, the answer to the following question still remains unclear: why is it necessary to relate gravity precisely to Riemannian space, and not to any other.

The introduction of Riemannian space permitted using the scalar curvature R as the Lagrangian function and, with the aid of the least action principle, to obtain the Hilbert–Einstein equation. Thus, the construction of Einstein's general relativity theory was completed. Here, as particularly stressed by J.L.Synge: "*In Einstein's theory the presence or absence of a gravitational field depends on whether the Riemann tensor differs from or equals to zero. This is an absolute property, which is in no way related to the world line of any observer*"[15].

In GRT, however, difficulties arose with the conservation laws of energy-momentum and angular momentum. D.Hilbert wrote in this connection: "*... I claim that within general rel-*

[13] Einstein A. Collection of scientific works, Moscow: Nauka, 1966, vol.2, art.110, p.405.

[14] V.A.Fock. Theory of space, time and gravity. M.:Gostekhizdat, 1961, p.308.

[15] J.L.Synge. Relativity: the general theory. M.:Foreign literature publishers, 1963, p.9.

ativity theory, i.e. in the case of general invariance of the Hamiltonian function, there definitely exist no energy equations ... corresponding to the energy equations in orthogonal-invariant theories, I could even point to this circumstance as a characteristic feature of general relativity theory"[16]. All the above is explained by the absence in Riemannian space of the ten-parameter group of motion of space-time, so it is essentially impossible to introduce energy-momentum and angular momentum conservation laws, similar to those that hold valid in any other physical theory.

Another feature peculiar to GRT, as compared to known theories, consists in the presence of second-order derivatives in the Lagrangian function R. About fifty years ago Nathan Rosen demonstrated that if, together with the Riemannian metric $g_{\mu\nu}$ one introduces the metric $\gamma_{\mu\nu}$ of Minkowski space, then it becomes possible to construct the scalar density of the Lagrangian of the gravitational field, which will not contain derivatives of orders higher than one. Thus, for example, he constructed such a density of the Lagrangian which led to the Hilbert-Einstein equations. Thus came into being the bimetric formalism. However, such an approach immediately complicated the problem of constructing a theory of gravity, since, when using the tensors $g_{\mu\nu}$ and $\gamma_{\mu\nu}$, one can write out a large number of scalar densities, and it is absolutely not clear which scalar density must be chosen as the Lagrangian density for constructing the theory of gravity. Although the GRT mathematical apparatus does permit introducing, instead of ordinary derivatives, covariant derivatives of Minkowski space, the metric $\gamma_{\mu\nu}$ not being present in the Hilbert-Einstein equations renders its utilization in GRT devoid of any physical meaning, because the solutions for the metric $g_{\mu\nu}$ are independent of the choice of $\gamma_{\mu\nu}$. It must be noted that substitution of covariant derivatives for ordinary derivatives in Minkowski space leaves the Hilbert-Einstein equations intact. This is explained by

[16]V.P.Vizgin. Relativistic theory of gravity. M.:Nauka, 1981, p.319.

the fact that, if in Minkowski space one substitutes covariant derivatives for ordinary ones in the Riemann curvature tensor, it will not change. Such a substitution in the Riemann tensor is nothing, but an identical transformation. Precisely for this reason such a freedom in writing out the Riemann tensor cannot be taken as an advantage within the framework of GRT, since the metric tensor of Minkowski space does not enter into the Hilbert-Einstein equations.

In constructing RTG, this freedom in writing the Riemann tensor turns out to be extremely necessary. But in this case the metric of Minkowski space enters into the equations of the gravitational field, and the field itself is considered as a physical field in Minkowski space. In GRT we only deal with the metric of Riemannian space as the main characteristic of gravity, in which both the features of geometry itself and the choice of reference frame are reflected. When the gravitational interaction is switched off, i.e. when the Riemann curvature tensor equals zero, we arrive at Minkowski space. It is precisely for this reason that in GRT the problem arises of satisfying the equivalence principle, since it is impossible to determine in which reference frame (inertial or accelerated) we happened to be when the gravitational field was switched off.

The relativistic theory of gravity, presented in this work is constructed as a field theory of the gravitational field within the framework of special relativity theory. The starting point is the hypothesis that the energy-momentum tensor — which is a universal characteristic of matter — serves as the source of gravity. The gravitational field is considered to be a universal physical field with spins 2 and 0, owing to the action of which the effective Riemannian space arises. This permits to find the gauge group and to construct unambiguously the Lagrangian density of the gravitational field. The set of equations of this theory is generally covariant and form-invariant with respect to the Lorentz group. Here, it is necessary in the theory to introduce the graviton mass. The graviton mass essentially influences the evolution of the Universe and alters the character of the gravitational collapse.

The goal of this work is a further development of the ideas by H.Poincaré, H.Minkowski, A.Einstein, D.Hilbert, N.Rosen, V.A.Fock, S.Gupta, V.Thirring, R.Feynman, S.Weinberg in the domain of theory of relativity and gravity.

1. The geometry of space-time

In Chapter II, "Space and time", of his book "Recent ideas", H.Poincaré wrote: "*The principle of physical relativity may serve for defining of space. It can be said to provide us with a novel instrument for measurement. I shall explain. How can a solid body serve for measuring or, to be more correct, for constructing space? The point is the following: in transferring a solid body from one place to another we, thus, note that it can be first applied to one figure, then to another, and we conventionally agree to consider these figures equal to each other. Geometry originated from this convention. Geometry is nothing but a science of mutual interrelationships between such transformations or, speaking in the mathematical language, a science of the structure of the group formed by these transformations, i.e. of the group of motions of solid bodies.*

Now, consider another group, the group of transformations not altering our differential equations. We arrive at a new way for defining the equality between two figures. We no longer say: two figures are equal, if one and the same solid body can be applied to both one and the other figures. We shall say: two figures are equal, when one and the same mechanical system, so distant from its neighbours that it may be considered isolated, being first thus situated so its material points reproduce the first figure, and then so they reproduce the second figure, behaves in the second case precisely like in the first. Do these two approaches differ in essence? No.

A solid body represents a mechanical system, just like any other. The only difference between the previous and the new definitions of space consists in that the latter is broader, since it allows substitution of any mechanical system for the solid body. Moreover, our new convention not only defines space, but time, also. It provides us with an explanation of what are *two equal time intervals or of what is represented by a time interval twice as long as another"* [17].

Precisely in this way, by discovering the group of transformations not altering the Maxwell–Lorentz equations, H.Poincaré introduced the notion of four-dimensional space-time exhibiting pseudo-Euclidean geometry. This concept of geometry was later developed by H.Minkowski.

We have chosen the pseudo-Euclidean geometry of space-time as the basis of the relativistic theory of gravity presently under development, since it is the fundamental Minkowski space for all physical fields, including the gravitational field. Minkowski space cannot be considered to exist a priori, since it reflects the properties of matter and, hence, cannot be separated from it. Although formally, precisely owing to the structure of space being independent of the form of matter, it is sometimes dealt with abstractly, separately from matter. In Galilean coordinates of an inertial reference system in Minkowski space, the interval that characterizes the structure of geometry and that is an invariant by construction, has the form

$$d\sigma^2 = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2 \; .$$

Here dx^ν represent differentials of the coordinates. In spite of the fact that the interval $d\sigma$, as a geometric characteristic of space-time, is independent of the choice of reference system, which is due to its very construction, one can still encounter in modern text-books on theoretical physics (see, for instance, Ref. [4]) "proofs" of the interval being the same in all inertial reference systems although it is an invariant and is independent of the choice of reference system.

Even such an outstanding physicist as L.I.Mandelstam wrote in his book [17]: "*... special relativity theory cannot answer the question, how a clock behaves when moving with acceleration and why it slows down, because it does not deal with reference systems moving with acceleration".* The incorrect as-

[17] H.Poincaré. On science. M.:Nauka, 1938, p.427.

sertions in [27, 19, 20, 30] can be explained by Minkowski space being considered by many people to be only some formal geometrical interpretation of SRT within A.Einstein's approach, instead of a revelation of the geometry of space-time. The issues of such limited concepts as the constancy of the speed of light, the synchronization of clocks, the speed of light being independent of the motion of its source became the most discussed topics. All this narrowed the scope of SRT and retarded the understanding of its essence. **And its essence actually consists only in that the geometry of space-time, in which all physical processes occur, is pseudo-Euclidean.**

In an arbitrary reference system the interval assumes the form

$$d\sigma^2 = \gamma_{\mu\nu}(x)dx^\mu dx^\nu \ ,$$

$\gamma_{\mu\nu}(x)$ is the metric tensor of Minkowski space. We note that one cannot, in principle, speak of the synchronization of clocks or of the constancy of the speed of light in an non-inertial reference system [7]. Most likely, precisely the lacking clarity on the essence of SRT led A.Einstein to concluding: "*that within the framework of special relativity theory there is no place for a satisfactory theory of gravity*"[18]. Free motion of a test body in an arbitrary reference system takes place along a geodesic line of Minkowski space:

$$\frac{DU^\nu}{d\sigma} = \frac{dU^\nu}{d\sigma} + \gamma^\nu_{\alpha\beta}U^\alpha U^\beta = 0 \ ,$$

where $U^\nu = \dfrac{dx^\nu}{d\sigma}$, $\gamma^\nu_{\alpha\beta}(x)$ are Christoffel symbols defined by the expression

$$\gamma^\nu_{\alpha\beta}(x) = \frac{1}{2}\gamma^{\nu\sigma}(\partial_\alpha\gamma_{\beta\sigma} + \partial_\beta\gamma_{\alpha\sigma} - \partial_\sigma\gamma_{\alpha\beta}) \ .$$

[18]Einstein A. Collection of scientific works. Moscow: Nauka, 1967, vol.4, art.76, p.282.

In 1921, in the article "Geometry and experiment", A. Einstein wrote: *"The issue of whether this continuum has an Euclidean, Riemannian or any other structure is a physical issue, which can only be settled by experiment, and not an issue of convention concerning a choice of simple expedience..."* [19]. This is, naturally, correct. But there immediately arises a question: what experiment? There may exist quite many experimental facts. Thus, for example, it is possible, in principle, by studying the motion of light and of test bodies, to establish unambiguously the geometry of space-time. Must a physical theory be based on it? At first sight, the answer to this question could be positive. And the issue would seem settled. Precisely such was the path that A.Einstein took in constructing GRT. Test bodies and light move along geodesic lines of Riemannian space-time. So he based the theory on Riemannian space. However, the situation is much more complex. All types of matter satisfy conservation laws of energy-momentum and of angular momentum. Precisely these laws, that originated from a generalization of numerous experimental data, characterize the general dynamic properties of all forms of matter by introducing universal characteristics permitting quantitative description of the transformation of some forms of matter into others. And all this also represents experimental facts, which have become fundamental physical principles. What should be done with them? If one follows A.Einstein and retains Riemannian geometry as the basis, then they must be discarded. That price would be too high. It is more natural to retain them for all physical fields, including the gravitational field. But, in this case, theory must, then, be based on Minkowski space, i.e. on the pseudo-Euclidean geometry of space-time. We have adopted precisely this approach, following H.Poincaré. The fundamental principles of physics, that reflect the numerous available experimental facts, indicate what geometry

[19] Einstein A. Collection of scientific works. Moscow: Nauka, 1965, vol.2, art.61, p.87.

of the space-time it is actually necessary to use as the basis of gravity theory. Thus, the issue of the structure of the space-time geometry is actually a physical issue, that should be resolved by experiment, and, from our point of view, the structure of the geometry of space-time is not determined by specific experimental data on the motion of test bodies and of light, but by fundamental physical principles based on the entire set of existing experimental facts. It is precisely here that our initial premises for constructing the theory of gravity differ completely from the ideas applied by A.Einstein as the basis of GRT. But they are fully consistent with the ideas of H.Poincaré.

We have chosen the pseudo-Euclidean geometry of space-time as the basis of the relativistic theory of gravity, but that certainly does not mean that the effective space will also be pseudo-Euclidean. The influence of the gravitational field may be expected to lead to a change in the effective space. We shall deal with this issue in detail in the next section. The metric of Minkowski space permits introducing the concepts of standard length and time intervals, when no gravitational field is present.

2. The energy-momentum tensor of matter as the source of the gravitational field

Owing to the existence in Minkowski space of the Poincaré ten-parameter group of motion, there exist for any closed physical system ten integrals of motion, i.e. the conservation laws of energy-momentum and angular momentum hold valid. Any physical field in Minkowski space is characterized by the density of the energy-momentum tensor $t^{\mu\nu}$, which is a general universal characteristic of all forms of matter that satisfies both local and integral conservation laws. In an arbitrary reference system the local conservation law is written in the form

$$D_\mu t^{\mu\nu} = \partial_\mu t^{\mu\nu} + \gamma^\nu_{\alpha\beta} t^{\alpha\beta} = 0 \ .$$

Here $t^{\mu\nu}$ is the total conserved density of the energy-momentum tensor for all the fields of matter; D_μ represents the covariant derivative in Minkowski space. Here and further we shall always deal with the densities of scalar and tensor quantities defined in accordance with the rule

$$\tilde{\phi} = \sqrt{-\gamma}\phi \ , \ \tilde{\phi}^{\mu\nu} = \sqrt{-\gamma}\phi^{\mu\nu}, \ \gamma = \det(\gamma_{\mu\nu}) \ .$$

The introduction of densities is due to an invariant volume element in Minkowski space being determined by the expression

$$\sqrt{-\gamma} d^4x \ ,$$

while an invariant volume element in Riemannian space is given by the expression

$$\sqrt{-g} d^4x \ , g = \det(g_{\mu\nu}) \ .$$

Therefore, the principle of least action assumes the form

$$\delta S = \delta \int L d^4x = 0 \ ,$$

where L is the scalar density of the Lagrangian of matter. In deriving Euler's equations with the aid of the principle of least action we shall automatically have to deal precisely with the variation of the Lagrangian density. According to D.Hilbert, the density of the energy-momentum tensor $t^{\mu\nu}$ is expressed via the scalar density of the Lagrangian L as follows:

$$t^{\mu\nu} = -2\frac{\delta L}{\delta \gamma_{\mu\nu}} \ , \tag{2.1}$$

where

$$\frac{\delta L}{\delta \gamma_{\mu\nu}} = \frac{\partial L}{\partial \gamma_{\mu\nu}} - \partial_\sigma \left(\frac{\partial L}{\partial \gamma_{\mu\nu,\sigma}} \right) \ , \ \gamma_{\mu\nu,\sigma} = \frac{\partial \gamma_{\mu\nu}}{\partial x^\sigma} \ .$$

Owing to gravity being universal, it would be natural to assume the conserved density of the energy-momentum tensor of all fields of matter, $t^{\mu\nu}$, to be the source of the gravitational field. Further, we shall take advantage of the analogy with electrodynamics, in which the conserved density of the charged vector current serves as the source of the electromagnetic field, while the field itself is described by the density of the vector potential $\tilde{A}^\nu$:

$$\tilde{A}^\nu = (\tilde{\phi}, \tilde{A}) \ .$$

In the absence of gravity, Maxwell's equations of electrodynamics will have the following form in arbitrary coordinates:

$$\gamma^{\alpha\beta} D_\alpha D_\beta \tilde{A}^\nu + \mu^2 \tilde{A}^\nu = 4\pi j^\nu,$$
$$D_\nu \tilde{A}^\nu = 0 \ ,$$

Here, for generalization we have introduced the parameter μ, which, in the system of units $\hbar = c = 1$ is the photon rest mass.

Since we have decided to consider the conserved density of the energy-momentum $t^{\mu\nu}$ to be the source of the gravitational field, it is natural to consider the gravitational field a tensor field and to describe it by the density of the symmetric tensor $\tilde{\phi}^{\mu\nu}$:

$$\tilde{\phi}^{\mu\nu} = \sqrt{-\gamma}\phi^{\mu\nu},$$

and in complete analogy with Maxwell's electrodynamics the equations for the gravitational field can be written in the form

$$\gamma^{\alpha\beta} D_\alpha D_\beta \tilde{\phi}^{\mu\nu} + m^2 \tilde{\phi}^{\mu\nu} = \lambda t^{\mu\nu} \ , \quad (2.2)$$

$$D_\mu \tilde{\phi}^{\mu\nu} = 0 \ . \quad (2.3)$$

Here λ is a certain constant which, in accordance with the principle of correspondence to Newton's law of gravity, should be equal to 16π. Equation (2.3) excludes spins 1 and $0'$, only retaining those polarizational properties of the field, that correspond to spins 2 and 0.

The density of the energy-momentum tensor of matter $t^{\mu\nu}$ consists of the density of the energy-momentum tensor of the gravitational field, $t_g^{\mu\nu}$, and of the energy-momentum tensor of matter, $t_M^{\mu\nu}$. We understand matter to comprise all the fields of matter, with the exception of the gravitational field,

$$t^{\mu\nu} = t_g^{\mu\nu} + t_M^{\mu\nu} \ .$$

The interaction between the gravitational field and matter is taken into account in the density of the energy-momentum tensor of matter, $t_M^{\mu\nu}$.

Back in 1913 A.Einstein wrote [28]: "*... the tensor of the gravitational field $\vartheta_{\mu\nu}$ is the source of a field together with the tensor of material systems $\Theta_{\mu\nu}$. The energy of the gravitational occupying a special position as compared with all other forms of energy would result in inadmissible consequences*" [28]. We have adopted precisely this idea of A.Einstein as the basis for constructing the relativistic theory of gravity (RTG). In constructing general relativity theory (GRT) A.Einstein was not successful, since instead of the energy-momentum tensor of the gravitational field there arose in GRT the pseudotensor of the gravitational field. All this happened because

A.Einstein did not consider the gravitational field a physical field (such as the Faraday–Maxwell field) in Minkowski space. Precisely for this reason the equations of GRT do not contain the metric of Minkowski space. From equations (2.2) it follows that they will also be non-linear for the gravitational field proper, since the density of the tensor $t_g^{\mu\nu}$ is the source of the gravitational field.

Equations (2.2) and (2.3), which we formally declared the equations of gravity by analogy with electrodynamics, must be derived from the principle of least action, since only in this case we will have an explicit expression for the density of the energy-momentum tensor of the gravitational field and of the fields of matter. But, to this end it is necessary to construct the density of the Lagrangian of matter and of the gravitational field. Here it is extremely important to realize this construction on the basis of general principles. Only in this case one can speak of the theory of gravity. The initial scalar density of the Lagrangian of matter may be written in the form

$$L = L_g(\gamma_{\mu\nu}, \tilde{\phi}^{\mu\nu}) + L_M(\gamma_{\mu\nu}, \tilde{\phi}^{\mu\nu}, \phi_A) \ ,$$

here L_g is the density of the Lagrangian of the gravitational field; L_M is the density of the Lagrangian of the fields of matter; ϕ_A represents the fields of matter.

The equations for the gravitational field and the fields of matter have, in accordance with the principle of least action, the form

$$\frac{\delta L}{\delta \tilde{\phi}^{\mu\nu}} = 0 \ , \tag{2.4}$$

$$\frac{\delta L_M}{\delta \phi_A} = 0 \ . \tag{2.5}$$

Equations (2.4) differ from equations (2.2), first of all, in that the variational derivative of the density of the Lagrangian there is the derivative with respect to the field $\tilde{\phi}^{\mu\nu}$, while the variational derivative in equations (2.2) is, in agreement with definition (2.1), taken from the density of the Lagrangian over the metric $\gamma_{\mu\nu}$. For equations (2.4) to reduce to equations (2.2) for any form of matter it is necessary to assume the tensor density $\tilde{\phi}^{\mu\nu}$ to be always present in the density of the Lagrangian together with the tensor density $\tilde{\gamma}^{\mu\nu}$ via some common density $\tilde{g}^{\mu\nu}$ in the form

$$\tilde{g}^{\mu\nu} = \tilde{\gamma}^{\mu\nu} + \tilde{\phi}^{\mu\nu}, \; \tilde{g}^{\mu\nu} = \sqrt{-g} g^{\mu\nu}. \tag{2.6}$$

Thus arises the effective Riemannian space with the metric $g^{\mu\nu}(x)$. Since the gravitational field $\tilde{\phi}^{\mu\nu}(x)$, like all other physical fields in Minkowski space, can be described within a sole coordinate map, it is evident from expression (2.6) that the quantity $\tilde{g}^{\mu\nu}(x)$ can also be fully defined in a sole coordinate map. For description of the effective Riemannian space due to the influence of the gravitational field, no atlas of maps is required, which is usually necessary for describing Riemannian space of the general form. This means that our effective Riemannian space has a simple topology. In GRT topology is not simple. Precisely for this reason, GRT cannot, in principle, be constructed on the basis of ideas considering gravity a physical gravitational field in Minkowski space.

If condition (2.6) is taken into account, the density of the Lagrangian L assumes the form

$$L = L_g(\gamma_{\mu\nu}, \tilde{g}^{\mu\nu}) + L_M(\gamma_{\mu\nu}, \tilde{g}^{\mu\nu}, \phi_A) \; .$$

It must be stressed that condition (2.6) permits substituting the variational derivative with respect to $\tilde{g}^{\mu\nu}$ for the variational derivative with respect to $\tilde{\phi}^{\mu\nu}$, and to express the variational derivative with respect to $\gamma_{\mu\nu}$ through the variational derivative with respect to $\tilde{g}^{\mu\nu}$ and the variational derivative with respect to $\gamma_{\mu\nu}$ entering explicitly into the Lagrangian L.

Indeed,

$$\frac{\delta L}{\delta\tilde{\phi}^{\mu\nu}} = \frac{\delta L}{\delta\tilde{g}^{\mu\nu}} = 0\ , \tag{2.7}$$

$$\frac{\delta L}{\delta\gamma_{\mu\nu}} = \frac{\delta^\star L}{\delta\gamma_{\mu\nu}} + \frac{\delta L}{\delta\tilde{g}^{\alpha\beta}} \cdot \frac{\partial\tilde{g}^{\alpha\beta}}{\partial\gamma_{\mu\nu}}\ . \tag{2.8}$$

The derivation of the latter formula is presented in detail in Appendix (A.17). The asterisk in formula (2.8) indicates the variational derivative of the density of the Lagrangian with respect to the metric $\gamma_{\mu\nu}$ which is explicitly present in L. In agreement with (2.1), formula (2.8) can be written in the form

$$t^{\mu\nu} = -2\frac{\delta L}{\delta\tilde{g}^{\alpha\beta}} \cdot \frac{\partial\tilde{g}^{\alpha\beta}}{\partial\gamma_{\mu\nu}} - 2\frac{\delta^\star L}{\delta\gamma_{\mu\nu}}\ .$$

Taking equation (2.7) into account in the above expression we obtain

$$t^{\mu\nu} = -2\frac{\delta^\star L}{\delta\gamma_{\mu\nu}}\ . \tag{2.9}$$

Comparing equations (2.9) and (2.2) we obtain the condition

$$-2\frac{\delta^\star L}{\delta\gamma_{\mu\nu}} = \frac{1}{16\pi}[\gamma^{\alpha\beta}D_\alpha D_\beta\tilde{\phi}^{\mu\nu} + m^2\tilde{\phi}^{\mu\nu}]\ , \tag{2.10}$$

which, in case it is fulfilled, makes it possible to derive the equations of the gravitational field, (2.2) and (2.3), directly from the principle of least action. Since the fields of matter are not present in the right-hand side of (2.10), this means that the variation in density of the Lagrangian of matter, L_M, with respect to the explicitly present metric $\gamma_{\mu\nu}$ must be zero. For no additional restrictions on the motion of matter determined by equations (2.5) to arise, it hence follows directly that the tensor $\gamma_{\mu\nu}$ does not explicitly enter into the expression for the density of the Lagrangian of matter L_M. Expression (2.10) then assumes the form

$$-2\frac{\delta^\star L_g}{\delta\gamma_{\mu\nu}} = \frac{1}{16\pi}[\gamma^{\alpha\beta}D_\alpha D_\beta\tilde{\phi}^{\mu\nu} + m^2\tilde{\phi}^{\mu\nu}]\ . \tag{2.11}$$

Thus, everything reduces to finding the density of the Lagrangian of the gravitational field proper, L_g, which would satisfy condition (2.11).

At the same time, from the previous arguments we arrive at the important conclusion that the density of the Lagrangian of matter, L, has the form

$$L = L_g(\gamma_{\mu\nu}, \tilde{g}^{\mu\nu}) + L_M(\tilde{g}^{\mu\nu}, \phi_A) \ . \qquad (2.12)$$

Thus, from the requirement that the density of the energy-momentum tensor of matter be the source of the gravitational field it follows in a natural way that the motion of matter should take place in effective Riemannian space. This assertion has the character of a theorem. Hence it becomes clear, why Riemannian space arose, instead of some other. Precisely this circumstance provides us with the possibility of formulating, in section 3, the gauge group, and then to construct the density of the Lagrangian (4.24) satisfying condition (2.11), in accordance with (B.20).

An interesting picture arises consisting in that the motion of matter in Minkowski space with the metric $\gamma_{\mu\nu}$ under the influence of the gravitational field $\phi^{\mu\nu}$ is identical to the motion of matter in effective Riemannian space with the metric $g_{\mu\nu}$, determined by expression (2.6). We term such interaction of the gravitational field with matter the g e o m e t i z a t i o n p r i n c i p l e. The geometrization principle is a consequence of the initial assumption that a universal characteristic of matter — the density of the energy-momentum tensor — serves as the source of the gravitational field. Such a density structure of the Lagrangian of matter indicates that a unique possibility is realized for the gravitational field to be attached inside the Lagrangian density of matter directly to the density of the tensor $\tilde{\gamma}^{\mu\nu}$.

The effective Riemannian space is literally of a field origin, owing to the presence of the gravitational field. Thus, the reason that the effective space is Riemannian, and not any other, lies in the hypothesis that a universal conserved quantity — the density of the energy-momentum tensor — is the source of gravity. We shall explain this fundamental property of gravitational forces by comparing them with the electromagnetic forces.

In the case of a homogeneous magnetic field, a charged particle in Minkowski space is known to undergo, due to the Lorentz force, motion along a circle in the plane perpendicular to the magnetic field. However, this motion is far from identical even for charged particles, if their charge-to-mass ratio differ. Moreover, there exist neutral particles, and their trajectories in a magnetic field are just straight lines. Therefore, owing to the non-universal character of electromagnetic forces their action cannot be reduced to the geometry of space-time. Gravity is another issue. It is universal, any test bodies move along identical trajectories given identical initial conditions. In this case, owing to the hypothesis claiming the energy-momentum tensor of matter to be the source of the gravitational field, it turns out to be possible to describe these trajectories by geodesic lines in the effective Riemannian space-time due to the presence of the gravitational field in Minkowski space. In those regions of space, where a whatever small gravitational field is present, we have metric properties of space which up to a high precision approach the actually observed properties of pseudo-Euclidean space. On the other hand, when the gravitational fields are strong, the metric properties of the effective space become Riemannian. But in this case, also, the pseudo-Euclidean geometry does not vanish without trace — it is observable and manifests itself in that the motion of bodies in effective Riemannian space is not free by inertia, but proceeds with acceleration with respect to pseudo-Euclidean space in Galilean coordinates. Precisely for this reason, acceleration in RTG, unlike GRT, has an absolute sense. Consequently, "Einstein's lift" cannot serve as an inertial reference frame. This is manifested in that a charge at rest in "Einstein's lift" will emit electromagnetic waves. This physical phenomenon should also testify in favour of the existence of Minkowski space. As we shall further see, the metric of Minkowski space can be defined from studies of the distribution of matter and of the motion of test bodies and light in effective Riemannian space. We shall raise this issue again in section 7.

The equation of motion of matter does not include the metric tensor $\gamma_{\mu\nu}$ of Minkowski space. Minkowski space will only influence the motion of matter by means of the metric tensor $g_{\mu\nu}$ of Riemannian space, derived, as we shall further see, from the equations of gravity, which contain the metric tensor $\gamma_{\mu\nu}$ of Minkowski space. Since the effective Riemannian metric arises on the basis of the physical field given in Minkowski space, hence it follows that effective Riemannian space has a simple topology and is presented in a single map. If, for instance, matter is concentrated in a region of the island-type, then in Galilean coordinates of an inertial reference system the gravitational field $\tilde{\phi}^{\mu\nu}$ cannot decrease slower than $1/r$, but this circumstance imposes a strong restriction on the asymptotic behaviour of the metric $g_{\mu\nu}$ of effective Riemannian geometry

$$g_{\mu\nu} = \eta_{\mu\nu} + 0\left(\frac{1}{r}\right), \text{ here } \eta_{\mu\nu} = (1, -1, -1, -1) \ . \qquad (2.13)$$

If, on the other hand, one simply takes as the starting point the Riemannian metric, without assuming it to have originated from the action of a physical field, then such restrictions do not arise, since the asymptotics of the metric $g_{\mu\nu}$ even depends on the choice of three-dimensional space coordinates. Physical quantity, however, in principle, cannot depend on the choice of the three-dimensional space coordinates. RTG imposes no restrictions on the choice of reference system. The reference system may be arbitrary, if only it realizes a one-to-one correspondence between all the points of the inertial reference system in Minkowski space and provides for the following inequalities, necessary for introducing the concepts of time and spatial length, to be satisfied:

$$\gamma_{00} > 0, \ dl^2 = s_{ik} dx^i \, dx^k > 0; \ i, k = 1, 2, 3,$$

where

$$s_{ik} = -\gamma_{ik} + \frac{\gamma_{0i}\gamma_{0k}}{\gamma_{00}}.$$

In our theory of gravity the geometrical characteristics of Riemannian space arise as field quantities in Minkowski space, and for this reason their transformational properties become tensor properties, even if this was previously not so, from the conventional point of view. Thus, for instance, the Christoffel symbols, given as field quantities in Galilean coordinates of Minkowski space become tensors of the third rank. In a similar manner, ordinary derivatives of tensor quantities in Cartesian coordinates of Minkowski space are also tensors.

The question may arise: why is no division of the metric, like (2.6), performed in GRT by introduction of the concept of gravitational field in Minkowski space? The Hilbert-Einstein equations only contain the quantity $g_{\mu\nu}$, so, consequently, it is impossible to say unambiguously with the help of which metric $\gamma_{\mu\nu}$ of Minkowski space we should define, in accordance with (2.6), the gravitational field. But the difficulty consists not only in the above, but, also, in that the solutions of Hilbert-Einstein equations are generally found not in one map, but in a whole atlas of maps. Such solutions for $g_{\mu\nu}$ describe Riemannian space with a complex topology, while the Riemannian spaces, obtained by representation of the gravitational field in Minkowski space, are described in a sole map and have a simple topology. It is precisely for these reasons that field representations are not compatible with GRT, since they are extremely rigorous. But this means that no field formulation of GRT in Minkowski space can exist, in principle, no matter how much someone and who might want this to happen. The apparatus of Riemannian geometry is inclined towards the possibility of introducing covariant derivatives in Minkowski space, which we took advantage of in constructing RTG. But to implement this, it was necessary to introduce the metric of Minkowski space into the gravitational equations, and it, thus, turned out to be possible to realize the functional relationship of the metric of Riemannian space, $g_{\mu\nu}$, with the metric of Minkowski space, $\gamma_{\mu\nu}$. But this will be dealt with in detail in subsequent sections.

3. The gauge group of transformations

Since the density of the Lagrangian of matter has the form

$$L_M(\tilde{g}^{\mu\nu},\, \phi_A), \tag{3.1}$$

it is easy to find the group of transformations, under which the density of the Lagrangian of matter is only changed by the divergence. To this end we shall take advantage of the action

$$S_M = \int L_M(\tilde{g}^{\mu\nu},\, \phi_A)\, d^4x \tag{3.2}$$

being invariant under infinitesimal transformations of coordinates,

$$x'^{\alpha} = x^{\alpha} + \xi^{\alpha}(x), \tag{3.3}$$

where ξ^{α} is the four-vector of an infinitesimal displacement. The field functions $\tilde{g}^{\mu\nu}$, ϕ_A vary as follows under these transformations of coordinates:

$$\begin{aligned} \tilde{g}'^{\mu\nu}(x') &= \tilde{g}^{\mu\nu}(x) + \delta_\xi \tilde{g}^{\mu\nu}(x) + \xi^{\alpha}(x) D_\alpha \tilde{g}^{\mu\nu}(x), \\ \phi'_A(x') &= \phi_A(x) + \delta_\xi \phi_A(x) + \xi^{\alpha}(x) D_\alpha \phi_A(x), \end{aligned} \tag{3.4}$$

where the expressions

$$\begin{aligned} \delta_\xi \tilde{g}^{\mu\nu}(x) &= \tilde{g}^{\mu\alpha} D_\alpha \xi^{\nu}(x) + \tilde{g}^{\nu\alpha} D_\alpha \xi^{\mu}(x) - D_\alpha(\xi^{\alpha} \tilde{g}^{\mu\nu}), \\ \delta_\xi \phi_A(x) &= -\xi^{\alpha}(x) D_\alpha \phi_A(x) + F^{B;\alpha}_{A;\beta}\, \phi_B(x) D_\alpha \xi^{\beta}(x) \end{aligned} \tag{3.5}$$

are Lie variations.

The operators δ_ξ satisfy the conditions of Lie algebras, i.e. the commutation relation

$$[\delta_{\xi_1},\, \delta_{\xi_2}](\cdot) = \delta_{\xi_3}\,(\cdot) \tag{3.6}$$

and the Jacobi identity

$$[\delta_{\xi_1},\, [\delta_{\xi_2},\, \delta_{\xi_3}]] + [\delta_{\xi_3},\, [\delta_{\xi_1},\, \delta_{\xi_2}]] + [\delta_{\xi_2},\, [\delta_{\xi_3},\, \delta_{\xi_1}]] = 0, \tag{3.7}$$

where

$$\xi_3^\nu = \xi_1^\mu D_\mu \xi_2^\nu - \xi_2^\mu D_\mu \xi_1^\nu = \xi_1^\mu \partial_\mu \xi_2^\nu - \xi_2^\mu \partial_\mu \xi_1^\nu.$$

For (3.6) to hold valid the following conditions must be satisfied:

$$F_{A;\,\nu}^{B;\,\mu}\, F_{B;\,\beta}^{C;\,\alpha} - F_{A;\,\beta}^{B;\,\alpha}\, F_{B;\,\nu}^{C;\,\mu} = f_{\nu\beta;\,\sigma}^{\mu\alpha;\,\tau}\, F_{A;\,\tau}^{C;\,\sigma}\,, \tag{3.8}$$

where the structure constants f are

$$f_{\nu\beta;\,\sigma}^{\mu\alpha;\,\tau} = \delta_\beta^\mu \delta_\sigma^\alpha \delta_\nu^\tau - \delta_\nu^\alpha \delta_\sigma^\mu \delta_\beta^\tau. \tag{3.9}$$

It is readily verified that they satisfy the Jacobi equality

$$f_{\beta\mu;\,\tau}^{\alpha\nu;\,\sigma}\, f_{\sigma\varepsilon;\,\delta}^{\tau\rho;\,\omega} + f_{\mu\varepsilon;\,\tau}^{\nu\rho;\,\sigma}\, f_{\sigma\beta;\,\delta}^{\tau\alpha;\,\omega} + f_{\varepsilon\beta;\,\tau}^{\rho\alpha;\,\sigma}\, f_{\sigma\mu;\,\delta}^{\tau\nu;\,\omega} = 0 \tag{3.10}$$

and have the property of antisymmetry,

$$f_{\beta\mu;\,\sigma}^{\alpha\nu;\,\rho} = -f_{\mu\beta;\,\sigma}^{\nu\alpha;\,\rho}.$$

The variation of action under the coordinate transformation (3.3) equals zero:

$$\delta_c S = \int\limits_{\Omega'} L'(x')\, d^4x' - \int\limits_{\Omega} L_M(x)\, d^4x = 0. \tag{3.11}$$

The first integral in (3.11) can be written in the form

$$\int\limits_{\Omega'} L'_M(x')\, d^4x' = \int\limits_{\Omega} J\, L'_M(x')\, d^4x,$$

where

$$J = \det\left(\frac{\partial x'^\alpha}{\partial\, x^\beta}\right).$$

In the first order of ξ^α the determinant J equals

$$J = 1 + \partial_\alpha \xi^\alpha(x). \tag{3.12}$$

Taking into account the expansion

$$L'_M(x') = L'_M(x) + \xi^\alpha(x) \frac{\partial L_M}{\partial x^\alpha}\,,$$

as well as (3.12), one can represent the expression for the variation in the form

$$\delta_c S_M = \int\limits_{\Omega} \left[\delta L_M(x) + \partial_\alpha(\xi^\alpha L_M(x))\right] d^4x = 0.$$

Owing to the integration volume Ω being arbitrary, we have the identity

$$\delta\, L_M(x) = -\partial_\alpha(\xi^\alpha(x) L_M(x)), \tag{3.13}$$

where the Lie variation δL_M is

$$\delta L_M(x) = \frac{\partial L_M}{\partial \tilde{g}^{\mu\nu}}\,\delta\tilde{g}^{\mu\nu} + \frac{\partial L_M}{\partial(\partial_\alpha \tilde{g}^{\mu\nu})}\,\delta(\partial_\alpha \tilde{g}^{\mu\nu}) + \\ + \frac{\partial L_M}{\partial \phi_A}\,\delta\phi_A + \frac{\partial L_M}{\partial(\partial_\alpha \phi_A)}\,\delta(\partial_\alpha \phi_A). \tag{3.14}$$

Hence, for instance, it follows that if the scalar density depends only on $\tilde{g}^{\mu\nu}$ and its derivatives, it will vary under transformation (3.5) only by the divergence

$$\delta L(\tilde{g}^{\mu\nu}(x)) = -\partial_\alpha(\xi^\alpha(x)\, L(\tilde{g}^{\mu\nu}(x))), \tag{3.13a}$$

where the Lie variation δL is

$$\delta L(\tilde{g}^{\mu\nu}(x)) = \frac{\partial L}{\partial \tilde{g}^{\mu\nu}}\,\delta\tilde{g}^{\mu\nu} + \frac{\partial L}{\partial(\partial_\alpha \tilde{g}^{\mu\nu})}\,\delta(\partial_\alpha \tilde{g}^{\mu\nu}) + \\ + \frac{\partial L}{\partial(\partial_\alpha \partial_\beta \tilde{g}^{\mu\nu})}\delta(\partial_\alpha \partial_\beta \tilde{g}^{\mu\nu}). \tag{3.14a}$$

The Lie variations (3.5) were established within the context of the coordinate transformations (3.3). But one may also adopt another standpoint, in accordance with which transformations (3.5) can be considered gauge transformations. In this case an arbitrary infinitesimal four-vector $\xi^\alpha(x)$ will already be a gauge vector, but no longer the displacement vector of the coordinates. To stress the difference between the gauge group and the group of coordinate transformations, we shall further use the notation $\varepsilon^{\alpha}(x)$ for the group parameter and call the transformation of field functions

$$\begin{aligned} &\tilde{g}^{\mu\nu}(x) \to \tilde{g}^{\mu\nu}(x) + \delta\tilde{g}^{\mu\nu}(x), \\ &\phi_A(x) \to \phi_A(x) + \delta\phi_A(x) \end{aligned} \tag{3.15}$$

with the variations

$$\begin{aligned} &\delta_\varepsilon \tilde{g}^{\mu\nu}(x) = \tilde{g}^{\mu\alpha} D_\alpha \varepsilon^\nu(x) + \tilde{g}^{\nu\alpha} D_\alpha \varepsilon^\mu(x) - D_\alpha(\varepsilon^\alpha \tilde{g}^{\mu\nu}), \\ &\delta_\varepsilon \phi_A(x) = -\varepsilon^\alpha(x) D_\alpha \phi_A(x) + F^{B;\,\alpha}_{A;\,\beta}\, \phi_B(x)\, D_\alpha\, \varepsilon^\beta(x) \end{aligned} \tag{3.16}$$

gauge transformations.

In full compliance with formulae (3.6) and (3.7), the operators satisfy the same Lie algebra, i.e. the commutation relation

$$[\delta_{\varepsilon_1},\ \delta_{\varepsilon_2}](\cdot) = \delta_{\varepsilon_3}(\cdot) \tag{3.17}$$

and the Jacobi identity

$$[\delta_{\varepsilon_1}, [\delta_{\varepsilon_2}, \delta_{\varepsilon_3}]] + [\delta_{\varepsilon_3}, [\delta_{\varepsilon_1}, \delta_{\varepsilon_2}]] + [\delta_{\varepsilon_2}, [\delta_{\varepsilon_3}, \delta_{\varepsilon_1}]] = 0. \tag{3.18}$$

Like in the preceding case, we have

$$\varepsilon_3^\nu = \varepsilon_1^\mu D_\mu \varepsilon_2^\nu - \varepsilon_2^\mu D_\mu \varepsilon_1^\nu = \varepsilon_1^\mu \partial_\mu \varepsilon_2^\nu - \varepsilon_2^\mu \partial_\mu \varepsilon_1^\nu.$$

The gauge group arose from the geometrized structure of the scalar density of the Lagrangian of matter, $L_M(\tilde{g}^{\mu\nu},\ \phi_A)$, which owing to identity (3.13) only changes by the divergence under gauge transformations (3.16). Thus, the geometrization principle, which determined the universal character of the interaction of matter and of the gravitational field, has provided us with the possibility of formulating the non-commutative infinite-dimensional gauge group (3.16).

The essential difference between the gauge and coordinate transformations will manifest itself at the decisive point of the theory in the course of construction of the scalar density of the Lagrangian of the gravitational field proper. The difference arises owing to the metric tensor $\gamma_{\mu\nu}$ not changing under gauge transformation, and, consequently, owing to (2.6) we have

$$\delta_\varepsilon \tilde{g}^{\mu\nu}(x) = \delta_\varepsilon \tilde{\phi}^{\mu\nu}(x).$$

From (3.16) the transformation for the field follows

$$\delta_\varepsilon \tilde{\phi}^{\mu\nu}(x) = \tilde{g}^{\mu\alpha}\, D_\alpha\, \varepsilon^\nu(x) + \tilde{g}^{\nu\alpha}\, D_\alpha\, \varepsilon^\mu(x) - D_\alpha(\varepsilon^\alpha\, \tilde{g}^{\mu\nu}),$$

but this transformation for the field differs essentially from its transformation in the case of displacement of the coordinates:

$$\delta_\xi\, \tilde{\phi}^{\mu\nu}(x) = \tilde{\phi}^{\mu\alpha}\, D_\alpha \xi^\nu(x) + \tilde{\phi}^{\nu\alpha}\, D_\alpha \xi^\mu(x) - D_\alpha(\xi^\alpha\, \tilde{\phi}^{\mu\nu}).$$

Under the gauge transformations (3.16) the equations of motion for matter do not change, because under any such transformations the density of the Lagrangian of matter is altered only by the divergence.

4. Density of the Lagrangian and the equations of motion for the gravitational field proper

It is known to be impossible, only using the sole tensor $g_{\mu\nu}$, to construct the scalar density of the Lagrangian of the gravitational field proper with respect to arbitrary coordinate transformations in the form of a quadratic form of derivatives of order not exceeding the first. Therefore, such a density of the Lagrangian will certainly contain the metric $\gamma_{\mu\nu}$ together with the metric $g_{\mu\nu}$. But, since the metric $\gamma_{\mu\nu}$ is not altered under the gauge transformation (3.16), there, consequently, must be imposed strong restrictions on the structure of the density of the Lagrangian of the gravitational field proper for it to change only by the divergence under this transformation. It is precisely here that there arises an essential difference between gauge and coordinate transformations.

While coordinate transformations impose nearly no restrictions on the structure of the scalar density of the Lagrangian of the gravitational field proper, gauge transformations will permit us to find the density of the Lagrangian. A straightforward general method for constructing the Lagrangian is presented in the monograph [10].

Here we shall choose a more simple method for constructing the Lagrangian. On the basis of (3.13a) we conclude that the most simple scalar densities $\sqrt{-g}$ and $\tilde{R} = \sqrt{-g}R$, where R is the scalar curvature of effective Riemannian space, vary as follows under the gauge transformation (3.16):

$$\sqrt{-g} \to \sqrt{-g} - D_\nu(\varepsilon^\nu \sqrt{-g}), \tag{4.1}$$

$$\tilde{R} \to \tilde{R} - D_\nu(\varepsilon^\nu \tilde{R}). \tag{4.2}$$

The scalar density $\tilde{R}$ is expressed via the Christoffel symbols

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma}(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}) \tag{4.3}$$

as follows:

$$\tilde{R} = -\tilde{g}^{\mu\nu}(\Gamma^{\lambda}_{\mu\nu}\,\Gamma^{\sigma}_{\lambda\sigma} - \Gamma^{\lambda}_{\mu\sigma}\,\Gamma^{\sigma}_{\nu\lambda}) - \partial_\nu(\tilde{g}^{\mu\nu}\,\Gamma^{\sigma}_{\mu\sigma} - \tilde{g}^{\mu\sigma}\,\Gamma^{\nu}_{\mu\sigma}). \quad (4.4)$$

Since the Christoffel symbols are not tensor quantities, no summand in (4.4) is a scalar density. However, if one introduces the tensor quantities $G^{\lambda}_{\mu\nu}$

$$G^{\lambda}_{\mu\nu} = \frac{1}{2}\,g^{\lambda\sigma}(D_\mu g_{\sigma\nu} + D_\nu g_{\sigma\mu} - D_\sigma g_{\mu\nu}), \quad (4.5)$$

then the scalar density can be identically written in the form

$$\tilde{R} = -\tilde{g}^{\mu\nu}(G^{\lambda}_{\mu\nu}\,G^{\sigma}_{\lambda\sigma} - G^{\lambda}_{\mu\sigma}\,G^{\sigma}_{\nu\lambda}) - D_\nu(\tilde{g}^{\mu\nu}\,G^{\sigma}_{\mu\sigma} - \tilde{g}^{\mu\sigma}\,G^{\nu}_{\mu\sigma}). \quad (4.6)$$

Note that under arbitrary coordinate transformations each group of terms in (4.6) individually exhibits the same behaviour as scalar density. We see that the apparatus of Riemannian geometry is inclined toward the introduction of covariant, instead of ordinary, derivatives in Minkowski space, but the metric tensor $\gamma_{\mu\nu}$, used for determining the covariant derivatives, is in no way fixed here.

With account of (4.1) and (4.2), the expression

$$\lambda_1(\tilde{R} + D_\nu\,Q^\nu) + \lambda_2\sqrt{-g} \quad (4.7)$$

varies only by the divergence under arbitrary gauge transformations. Choosing the vector density Q^ν to be

$$Q^\nu = \tilde{g}^{\mu\nu}\,G^{\sigma}_{\mu\sigma} - \tilde{g}^{\mu\sigma}\,G^{\nu}_{\mu\sigma},$$

we exclude from the preceding expression terms containing derivatives of orders higher, than the first, and obtain the following density of the Lagrangian:

$$-\lambda_1\tilde{g}^{\mu\nu}(G^{\lambda}_{\mu\nu}\,G^{\sigma}_{\lambda\sigma} - G^{\lambda}_{\mu\sigma}\,G^{\sigma}_{\nu\lambda}) + \lambda_2\sqrt{-g}. \quad (4.8)$$

Thus, we see that the requirement for the density of the Lagrangian of the gravitational field proper to vary under the gauge transformation (3.16) only by the divergence, unambiguously determines the structure of the Lagrangian's density (4.8). But, if one restricts oneself only to considering this density, then the equations of the gravitational field will be gauge invariant, while the metric of Minkowski space, $\gamma_{\mu\nu}$, will not be present in the set of equations determined by the density of the Lagrangian (4.8). Since within such an approach the metric of Minkowski space disappears, the possibility of representing the gravitational field as a physical field of the Faraday–Maxwell type in Minkowski space disappears also.

In the case of the density of the Lagrangian (4.8), introduction of the metric $\gamma_{\mu\nu}$ with the aid of equations (2.3) will not save the situation, since physical quantities — the interval and the curvature tensor of Riemannian space, as well as the tensor $t_g^{\mu\nu}$ of the gravitational field — will depend on the choice of gauge, which is inadmissible from a physical point of view. Thus, for example,

$$\delta_\epsilon R_{\mu\nu} = -R_{\mu\sigma} D_\nu \epsilon^\sigma - R_{\nu\sigma} D_\mu \epsilon^\sigma - \epsilon^\sigma D_\sigma R_{\mu\nu} \ ,$$
$$\delta_\epsilon R_{\mu\nu\alpha\beta} = R_{\sigma\nu\alpha\beta} D_\mu \epsilon^\sigma - R_{\mu\sigma\alpha\beta} D_\nu \epsilon^\sigma -$$
$$-R_{\mu\nu\sigma\beta} D_\alpha \epsilon^\sigma - R_{\mu\nu\alpha\sigma} D_\beta \epsilon^\sigma - \epsilon^\sigma D_\sigma R_{\mu\nu\alpha\beta}.$$

To retain the concept of a field in Minkowski space and to exclude the above ambiguity it is necessary to add, in the density of the Lagrangian of the gravitational field, a term violating the gauge group. It is precisely here that there arises an essentially new way, which for a long time evaded being revealed. At first sight, it may seem that a significant arbitrariness should arise here, since the group can be violated in extremely diverse ways. However, it turns out not to be so, because our physical requirement, concerning the polarization properties of the gravitational field which is a field of spins 2 and 0, imposed by equations (2.3), results in the term violating the group (3.16) being necessarily chosen so as to make equations (2.3) a consequence of the set of equations of the gravitational field and of fields of matter, since only in this case we have no over-determined set of differential equations arising. To this end we introduce into the scalar density of the Lagrangian of the gravitational field a term of the form

$$\gamma_{\mu\nu}\tilde{g}^{\mu\nu}, \tag{4.9}$$

which, given conditions (2.3), also varies under transformations (3.16) by the divergence, but only on the class of vectors satisfying the condition

$$g^{\mu\nu}D_\mu D_\nu \varepsilon^\sigma(x) = 0. \tag{4.10}$$

In electrodynamics a nearly analogous situation occurs with the photon rest mass differing from zero. With account of (4.8)-(4.9) the general scalar density of the Lagrangian has the form:

$$\begin{aligned} L_g = -\lambda_1 \tilde{g}^{\mu\nu}(G^\lambda_{\mu\nu}\,G^\sigma_{\lambda\sigma} - G^\lambda_{\mu\sigma}\,G^\sigma_{\nu\lambda}) + \\ +\lambda_2\sqrt{-g} + \lambda_3\,\gamma_{\mu\nu}\,\tilde{g}^{\mu\nu} + \lambda_4\sqrt{-\gamma}. \end{aligned} \tag{4.11}$$

We have introduced the last constant term in (4.11) in order to use it for reducing to zero the density of the Lagrangian in absence of the gravitational field. The narrowing of the class of gauge vectors due to introduction of the term (4.9) automatically results in equations (2.3) being a consequence of the equations of the gravitational field. We shall further verify this directly.

In accordance with the principle of least action, the equations for the gravitational field proper are of the form

$$\frac{\delta L_g}{\delta \tilde{g}^{\mu\nu}} = \lambda_1 R_{\mu\nu} + \frac{1}{2}\,\lambda_2\,g_{\mu\nu} + \lambda_3\,\gamma_{\mu\nu} = 0, \tag{4.12}$$

here

$$\frac{\delta L_g}{\delta \tilde{g}^{\mu\nu}} = \frac{\partial L_g}{\partial \tilde{g}^{\mu\nu}} - \partial_\sigma\left(\frac{\partial L}{\partial(\partial_\sigma \tilde{g}^{\mu\nu})}\right),$$

where we write the Ricci tensor $R_{\mu\nu}$ in the form

$$R_{\mu\nu} = D_\lambda\,G^\lambda_{\mu\nu} - D_\mu\,G^\lambda_{\nu\lambda} + G^\sigma_{\mu\nu}\,G^\lambda_{\sigma\lambda} - G^\sigma_{\mu\lambda}\,G^\lambda_{\nu\sigma}. \tag{4.13}$$

Since in absence of the gravitational field equations (4.12) must be satisfied identically, hence follows

$$\lambda_2 = -\,2\,\lambda_3. \qquad (4.14)$$

Let us now find the density of the energy-momentum for the gravitational field in Minkowski space

$$t\,^{\mu\nu}_{g} = -2\frac{\delta L_g}{\delta\,\gamma_{\mu\nu}} 2\sqrt{-\gamma}(\gamma^{\mu\alpha}\gamma^{\nu\beta} - \frac{1}{2}\gamma^{\mu\nu}\gamma^{\alpha\beta})\,\frac{\delta L_g}{\delta \tilde{g}^{\alpha\beta}} + $$
$$+\,\lambda_1\,J^{\mu\nu} - 2\lambda_3\,\tilde{g}^{\mu\nu} - \lambda_4\,\tilde{\gamma}^{\mu\nu}, \qquad (4.15)$$

where

$$J^{\mu\nu} = D_\alpha D_\beta(\gamma^{\alpha\mu}\tilde{g}^{\beta\nu} + \gamma^{\alpha\nu}\tilde{g}^{\beta\mu} - \gamma^{\alpha\beta}\,\tilde{g}^{\mu\nu} - \gamma^{\mu\nu}\,\tilde{g}^{\alpha\beta}). \quad (4.16)$$

(see Appendix (B.19)). If the dynamic equations (4.12) are taken into account in expression (4.15), then we obtain equations for the gravitational field proper in the form

$$\lambda_1\,J^{\mu\nu} - 2\,\lambda_3\tilde{g}^{\mu\nu} - \lambda_4\,\tilde{\gamma}^{\mu\nu} = t^{\mu\nu}_{g}. \qquad (4.17)$$

For this equation to be satisfied identically in the absence of the gravitational field, it is necessary to set

$$\lambda_4 = -2\,\lambda_3. \qquad (4.18)$$

Since the equality

$$D_\mu\, t^{\mu\nu}_{g} = 0, \qquad (4.19)$$

always holds valid for the gravitational field proper, from equation (4.17) it follows that

$$D_\mu\,\tilde{g}^{\mu\nu} = 0. \qquad (4.20)$$

Thus, equations (2.3) determining the polarization states of the field follow directly from equations (4.17). With account of equations (4.20), one can write the field equations (4.17) in the form

$$\gamma^{\alpha\beta}\,D_\alpha\,D_\beta\,\tilde{\phi}^{\mu\nu} - \frac{\lambda_4}{\lambda_1}\,\tilde{\phi}^{\mu\nu} = -\frac{1}{\lambda_1}\,t^{\mu\nu}_{g}. \qquad (4.21)$$

In Galilean coordinates this equation has the simple form

$$\square\tilde{\phi}^{\mu\nu} - \frac{\lambda_4}{\lambda_1}\,\tilde{\phi}^{\mu\nu} = -\frac{1}{\lambda_1}\,t_g^{\mu\nu}. \tag{4.22}$$

It is natural to consider the numerical factor $-\frac{\lambda_4}{\lambda_1} = m^2$ to represent the square graviton mass and to set the value of $-1/\lambda_1$ equal to 16π, in accordance with the equivalence principle. Thus, all the unknown constants present in the density of the Lagrangian have been defined:

$$\lambda_1 = -\frac{1}{16\pi}, \quad \lambda_2 = \lambda_4 = -2\,\lambda_3 = \frac{m^2}{16\,\pi}. \tag{4.23}$$

The constructed scalar density of the Lagrangian of the gravitational field proper will have the form

$$\begin{aligned} L_g = & \frac{1}{16\,\pi}\,\tilde{g}^{\mu\nu}(G^\lambda_{\mu\nu}\,G^\sigma_{\lambda\sigma} - G^\lambda_{\mu\sigma}\,G^\sigma_{\nu\lambda}) - \\ & - \frac{m^2}{16\,\pi}\left(\frac{1}{2}\,\gamma_{\mu\nu}\tilde{g}^{\mu\nu} - \sqrt{-g} - \sqrt{-\gamma}\right). \end{aligned} \tag{4.24}$$

The corresponding to it dynamic equations for the gravitational field proper can be written down in the form

$$J^{\mu\nu} - m^2\,\tilde{\phi}^{\mu\nu} = -16\pi\,t_g^{\mu\nu}, \tag{4.25}$$

or

$$R^{\mu\nu} - \frac{m^2}{2}(g^{\mu\nu} - g^{\mu\alpha}g^{\nu\beta}\gamma_{\alpha\beta}) = 0. \tag{4.26}$$

These equations impose significant limits on the class of gauge transformations, retaining only the trivial ones satisfying the Killing conditions in Minkowski space. Such transformations are a consequence of Lorentz invariance and are present in any theory.

The density of the Lagrangian constructed above leads to equations (4.26) from which it follows that equations (4.20) are their consequence, and, therefore, outside matter we shall have ten equations for ten unknown field functions. The unknown field functions $\phi^{0\alpha}$ are readily expressed with the aid of equations (4.20) via the field functions ϕ^{ik}, where the indices i and k run through the values 1, 2, 3.

Thus, the structure of the mass term violating the gauge group in the density of the Lagrangian of the gravitational field proper is unambiguously determined by the polarization properties of the gravitational field. **The field approach to gravity, that declares the energy-momentum tensor of all matter to be the source of the field, necessarily requires introduction of the graviton rest mass in the theory.**

5. Equations of motion for the gravitational field and for matter

The total density of the Lagrangian of matter and of the gravitational field is

$$L = L_g + L_M(\tilde{g}^{\mu\nu},\, \phi_A), \tag{5.1}$$

where L_g is determined by expression (4.24).

On the basis of (5.1) we shall obtain, with the aid of the least action principle, the complete set of equations for matter and for the gravitational field:

$$\frac{\delta L}{\delta \tilde{g}^{\mu\nu}} = 0, \tag{5.2}$$

$$\frac{\delta L_M}{\delta \phi_A} = 0. \tag{5.3}$$

Since in the case of an arbitrary infinitesimal variation of the coordinates the variation of the action, $\delta_c S_M$, is zero,

$$\delta_c S_M = \delta_c \int L_M(\tilde{g}^{\mu\nu},\, \phi_A)\, d^4x = 0,$$

it is hence possible to obtain an identity (see Appendix (C.16)) in the form

$$g_{\mu\nu}\nabla_\lambda T^{\lambda\nu} = -D_\nu\Big(\frac{\delta L_M}{\delta \phi_A}\, F^{B;\,\nu}_{A;\,\mu}\, \phi_B(x)\Big) - \frac{\delta L_M}{\delta \phi_A}\, D_\mu\, \phi_A(x). \tag{5.4}$$

Here $T^{\lambda\nu} = -2\frac{\delta L_M}{\delta g_{\lambda\nu}}$ is the density of the tensor of matter in Riemannian space; ∇_λ is the covariant derivative in this space with the metric $g_{\lambda\nu}$. From identity (5.4) it follows that, if the equations of motion of matter (5.3) are satisfied, then the following equation occurs:

$$\nabla_\lambda\, T^{\lambda\nu} = 0. \tag{5.5}$$

When the number of equations (5.3) for matter equals four, the equivalent equations (5.5) may be used, instead. Since we shall further only deal with such equations for matter, we shall always make use of the equations for matter in the form (5.5). Thus, the complete set of equations for matter and for the gravitational field will have the form

$$\frac{\delta L}{\delta \tilde{g}^{\mu\nu}} = 0 \ , \tag{5.6}$$

$$\nabla_\lambda T^{\lambda\nu} = 0. \tag{5.7}$$

Matter will be described by velocity $\vec{v}$, the density of matter ρ, and pressure p. The gravitational field will be determined by ten components of the tensor $\phi^{\mu\nu}$.

Thus, we have 15 unknowns. For determining them it is necessary to add to the 14 equations (5.6),(5.7) the equation of state for matter. If the relations (see Appendices B*.18,B*.19)

$$\frac{\delta L_g}{\delta \tilde{g}^{\mu\nu}} = -\frac{1}{16\,\pi}\, R_{\mu\nu} + \frac{m^2}{32\,\pi}\,(g_{\mu\nu} - \gamma_{\mu\nu}), \tag{5.8}$$

$$\frac{\delta L_M}{\delta \tilde{g}^{\mu\nu}} = \frac{1}{2\sqrt{-g}}\Big(T_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu}\, T\Big), \tag{5.9}$$

are taken into account, then the set of equations (5.6), (5.7) may be represented as

$$\Big(R^{\mu\nu} - \frac{1}{2}\, g^{\mu\nu} R\Big) + \frac{m^2}{2}\Big[g^{\mu\nu} + (g^{\mu\alpha} g^{\nu\beta} - \frac{1}{2}\, g^{\mu\nu} g^{\alpha\beta})\,\gamma_{\alpha\beta}\Big] = \frac{8\,\pi}{\sqrt{-g}}\, T^{\mu\nu}, \tag{5.10}$$

$$\nabla_\lambda T^{\lambda\nu} = 0. \tag{5.11}$$

Owing to the Bianchi identity

$$\nabla_\mu (R^{\mu\nu} - \frac{1}{2}\, g^{\mu\nu}\, R) = 0$$

from equations (5.10) we have

$$m^2\sqrt{-g}(g^{\mu\alpha}g^{\nu\beta} - \frac{1}{2}\,g^{\mu\nu}g^{\alpha\beta})\nabla_\mu\,\gamma_{\alpha\beta} = 16\,\pi\,\nabla_\mu\,T^{\mu\nu}. \quad (5.12)$$

Taking into account expression

$$\nabla_\mu\,\gamma_{\alpha\beta} = -G^\sigma_{\mu\alpha}\,\gamma_{\sigma\beta} - G^\sigma_{\mu\beta}\,\gamma_{\sigma\alpha}, \quad (5.13)$$

where $G^\sigma_{\mu\alpha}$ is defined by formula (4.5), we find

$$(g^{\mu\alpha}g^{\nu\beta} - \frac{1}{2}\,g^{\mu\nu}g^{\alpha\beta})\,\nabla_\mu\,\gamma_{\alpha\beta} =$$
$$= \gamma_{\mu\lambda}g^{\mu\nu}(D_\sigma\,g^{\sigma\lambda} + G^\beta_{\alpha\beta}\,g^{\alpha\lambda}), \quad (5.14)$$

but since (see formulae (B*.20))

$$\sqrt{-g}(D_\sigma\,g^{\sigma\lambda} + G^\beta_{\alpha\beta}\,g^{\alpha\lambda}) = D_\sigma\,\tilde{g}^{\lambda\sigma}, \quad (5.15)$$

expression (5.14) assumes the form

$$\sqrt{-g}(g^{\mu\alpha}g^{\nu\beta} - \frac{1}{2}\,g^{\mu\nu}g^{\alpha\beta})\,\nabla_\mu\,\gamma_{\alpha\beta} = \gamma_{\mu\lambda}\,g^{\mu\nu}\,D_\sigma\,\tilde{g}^{\lambda\sigma}. \quad (5.16)$$

With the aid of (5.16) expression (5.12) can be represented in the form

$$m^2\gamma_{\mu\lambda}\,g^{\mu\nu}D_\sigma\tilde{g}^{\lambda\sigma} = 16\pi\,\nabla_\mu\,T^{\mu\nu}.$$

This expression can be rewritten in the form

$$m^2D_\sigma\tilde{g}^{\lambda\sigma} = 16\,\pi\,\gamma^{\lambda\nu}\nabla_\mu\,T^\mu_\nu. \quad (5.17)$$

With the aid of this relation, equation (5.11) can be replaced by the equation

$$D_\sigma\tilde{g}^{\nu\sigma} = 0. \quad (5.18)$$

Therefore, the set of equations (5.10), (5.11) is reduced to the set of gravitational equations in the form

$$\left(R^{\mu\nu} - \frac{1}{2}\,g^{\mu\nu}R\right) + \frac{m^2}{2}\left[g^{\mu\nu} + (g^{\mu\alpha}g^{\nu\beta} - \right.$$
$$\left. -\frac{1}{2}\,g^{\mu\nu}\,g^{\alpha\beta})\gamma_{\alpha\beta}\right] = \frac{8\,\pi}{\sqrt{-g}}\,T^{\mu\nu}, \quad (5.19)$$

$$D_\mu\,\tilde{g}^{\mu\nu} = 0. \quad (5.20)$$

These equations are universally covariant with respect to arbitrary transformations of coordinates and form-invariant only with respect to such transformations of coordinates that leave the Minkowski metric $\gamma_{\mu\nu}(x)$ form-invariant. Hence, for instance, it follows that in any inertial (Galilean) reference system phenomena are described by identical equations. Equations involving the graviton mass had arisen previously; however, owing to misunderstanding of the fundamental fact that special relativity theory is also valid in non-inertial reference systems, they were not considered seriously, since they were not universally covariant. Usually, following A.Einstein, the metric $\eta_{\alpha\beta} = (1, -1, -1, -1)$ was considered to be a tensor only with respect to the Lorentz transformations. But, actually, the metric of Minkowski space, $\gamma_{\mu\nu}(x)$, is a tensor with respect to arbitrary transformations of coordinates. The set of equations (5.19) and (5.20) is hyperbolic. In the case of static problems, it is elliptic. By adding the equation of state to the set of equations (5.19) and (5.20) we obtain a complete set of equations for determining the unknown physical quantities $g_{\mu\nu},\ \ \vec{v},\ \ \rho, p$ for one or another formulations of the problem.

A concrete inertial Galilean reference system is singled out by formulation of the actual physical problem (by the initial and boundary conditions). The descriptions of a given formulated physical problem in different inertial (Galilean) reference systems are different, naturally, but this does not contradict the relativity principle. If one introduces the tensor

$$N^{\mu\nu} = R^{\mu\nu} - \frac{m^2}{2}[g^{\mu\nu} - g^{\mu\alpha}g^{\nu\beta}\gamma_{\alpha\beta}],\ \ N = N^{\mu\nu}g_{\mu\nu},$$

then the set of equations (5.19) and (5.20) can be written in the form

$$N^{\mu\nu} - \frac{1}{2}\,g^{\mu\nu}N = \frac{8\,\pi}{\sqrt{-g}}\,T^{\mu\nu}, \qquad (5.19a)$$

$$D_\mu\,\tilde{g}^{\mu\nu} = 0. \qquad (5.20a)$$

It may also be represented in the form

$$N^{\mu\nu} = \frac{8\,\pi}{\sqrt{-g}}(T^{\mu\nu} - \frac{1}{2}\,g^{\mu\nu}\,T), \tag{5.21}$$

$$D_\mu \tilde{g}^{\mu\nu} = 0, \tag{5.22}$$

or

$$N_{\mu\nu} = \frac{8\,\pi}{\sqrt{-g}}(T_{\mu\nu} - \frac{1}{2}\,g_{\mu\nu}\,T), \tag{5.21a}$$

$$D_\mu \tilde{g}^{\mu\nu} = 0. \tag{5.22a}$$

It must be especially stressed that both sets of equations (5.21) and (5.22) contain the metric tensor of Minkowski space.

Transformations of coordinates, which leave the metric of Minkowski space form-invariant, relate physically equivalent reference systems. The most simple of these are inertial reference systems. For this reason, possible gauge transformations satisfying the Killing conditions

$$D_\mu \varepsilon_\nu + D_\nu \varepsilon_\mu = 0,$$

do not remove us from the class of physically equivalent reference systems.

Let us deal with this issue in more detail. To this end we write the equation of RTG, (5.21) and (5.22), in the expanded form:

$$R^{\mu\nu}(x) - \frac{m^2}{2}[g^{\mu\nu}(x) - g^{\mu\alpha}g^{\nu\beta}\gamma_{\alpha\beta}(x)] =$$
$$= 8\pi\left[T^{\mu\nu}(x) - \frac{1}{2}g^{\mu\nu}T(x)\right], \tag{5.23}$$

$$D_\mu \tilde{g}^{\mu\nu} = 0. \tag{5.24}$$

Consider that, given appropriate conditions of the problem, these equations have the solution $g^{\mu\nu}(x)$ in Galilean coordinates x in an inertial reference system, when the distribution of matter is $T^{\mu\nu}(x)$. In another inertial reference system, in Galilean coordinates x' satisfying the condition

$$\begin{aligned} &x'^{\nu} = x^{\nu} + \epsilon^{\nu}(x), \\ &D^{\mu}\epsilon^{\nu} + D^{\nu}\epsilon^{\mu} = 0. \end{aligned} \tag{5.25}$$

We obtain with the aid of tensor transformations the following:

$$\begin{aligned} &R'^{\mu\nu}(x') - \frac{m^2}{2}[g'^{\mu\nu}(x') - g'^{\mu\alpha}g'^{\nu\beta}\gamma_{\alpha\beta}(x')] = \\ &= 8\pi\left[T'^{\mu\nu}(x') - \frac{1}{2}g'^{\mu\nu}T'(x')\right]. \end{aligned} \tag{5.26}$$

Since equations (5.23) are form-invariant with respect to the Lorentz transformations, we can return to the initial variables x:

$$\begin{aligned} &R'^{\mu\nu}(x) - \frac{m^2}{2}[g'^{\mu\nu}(x) - g'^{\mu\alpha}g'^{\nu\beta}\gamma_{\alpha\beta}(x)] = \\ &= 8\pi\left[T'^{\mu\nu}(x) - \frac{1}{2}g'^{\mu\nu}T'(x)\right] \end{aligned} \tag{5.27}$$

Hence it is clear that the solution $g'^{\mu\nu}(x)$ does not correspond to the distribution of matter $T^{\mu\nu}(x)$, but to another distribution $T'^{\mu\nu}(x)$. The quantity $g'^{\mu\nu}(x)$ in equations (5.27) is

$$g'^{\mu\nu}(x) = g^{\mu\nu}(x) + \delta_\epsilon g^{\mu\nu}, \tag{5.28}$$

where

$$\delta_\epsilon g^{\mu\nu} = g^{\mu\lambda}D_\lambda\epsilon^{\nu} + g^{\nu\lambda}D_\lambda\epsilon^{\mu} - \epsilon^{\lambda}D_\lambda g^{\mu\nu}. \tag{5.29}$$

In the case of transformations (5.25) we have

$$\begin{aligned} &R'^{\mu\nu}(x) = R^{\mu\nu}(x) + \delta_\epsilon R^{\mu\nu}, \\ &T'^{\mu\nu}(x) = T^{\mu\nu}(x) + \delta_\epsilon T^{\mu\nu}, \\ &T'(x) = T(x) + \delta_\epsilon T. \end{aligned} \tag{5.30}$$

Here

$$\begin{aligned}
&\delta_\epsilon R^{\mu\nu} = R^{\mu\lambda} D_\lambda \epsilon^\nu + R^{\nu\lambda} D_\lambda \epsilon^\mu - \epsilon^\lambda D_\lambda R^{\mu\nu}, \\
&\delta_\epsilon T^{\mu\nu} = T^{\mu\lambda} D_\lambda \epsilon^\nu + T^{\nu\lambda} D_\lambda \epsilon^\mu - \epsilon^\lambda D_\lambda T^{\mu\nu}, \\
&\delta_\epsilon T = -\epsilon^\lambda D_\lambda T = -\epsilon^\lambda \partial_\lambda T.
\end{aligned} \tag{5.31}$$

We obtained expression (5.27) with the aid of the coordinate transformations (5.25), but identical equations are also obtained in the case of the gauge transformation (5.29) with the vectors ϵ^λ satisfying condition (5.25). Thus, gauge transformations result in the metric field $g'^{\mu\nu}(x)$ in the case of matter exhibiting the distribution $T'^{\mu\nu}(x)$. Although we considered the transition from one inertial reference system in Galilean coordinates to another, our formulae (5.25), (5.31) are of a general nature, they are valid, also, for a noninertial reference system in Minkowski space. The same situation occurs in electrodynamics.

In GRT the situation is completely different, since owing to the Hilbert–Einstein equations being form-invariant with respect to arbitrary transformations of coordinates, there exist, for one and the same distribution of matter, $T^{\mu\nu}(x)$, any amount of metrics $g_{\mu\nu}(x), g'_{\mu\nu}(x)$... satisfying the equations. It is precisely for this reason that in GRT there arises an ambiguity in the description of gravitational phenomena.

If one imagines that it is possible to perform experimental measurements of the characteristics of Riemannian space and of the motion of matter with whatever high precision, then it would be possible, on the basis of equations (5.21a) and (5.22a), to determine the metric of Minkowski space and to find Galilean (inertial) reference systems. Thus, Minkowski space is observable, in principle.

The existence of Minkowski space is reflected in the conservation laws, and, therefore, testing their validity in physical phenomena serves at the same time for testing the structure of space-time.

It must be especially noted that both sets of equations

(5.19) and (5.20) contain the metric tensor of Minkowski space. The presence of the cosmological term in the equations of GRT is known not to be obligatory, and this issue is still being discussed. The presence of the cosmological term in the equations of gravity is obligatory in RTG. However, the cosmological term in equations (5.19) arises in combination with the term related to the metric $\gamma_{\mu\nu}$ of Minkowski space, and with the same constant factor equal to a half of the square graviton mass. The presence in equations (5.19) of the term connected with the metric $\gamma_{\mu\nu}$ significantly alters the character of the collapse and development of the Universe. In accordance with equations (5.19), in absence of matter and of the gravitational field, the metric of the space becomes the Minkowski metric, and it coincides exactly with the one previously chosen in formulating the physical problem. If the metric of Minkowski space were absent in the equations of the gravitational field, then it would be absolutely unclear, in which reference system of Minkowski space we would happen to be in the absence of matter and of the gravitational field.

The graviton mass is essential for this theory, since only its introduction permits construction of the theory of gravity in Minkowski space. The graviton mass violates the gauge group or, in other words, it removes the degeneracy. Therefore, one cannot exclude the possibility of the graviton mass tending toward zero in the final results, when gravitational effects are studied. However, the theory with a graviton mass and the theory involving violation of the gauge group [8] (with the graviton mass subsequently tending toward zero) are essentially different theories. Thus, for example, while in the first theory the Universe is homogeneous and isotropic, no such Universe can exist in the second one.

Let us now touch upon the equivalence principle. Any physical theory must comply with the equivalence principle. Gravitational interactions alter the equations of motion of matter. The requirement imposed by the equivalence principle reduces to the requirement that, when the gravitational interaction is switched off, i.e. when the Riemann curvature tensor turns to zero, these equations of motion become ordinary equations of motion of SRT in the chosen reference system.

In formulating the physical problem in RTG we choose a certain reference system with the metric tensor of Minkowski space, $\gamma_{\mu\nu}(x)$. In RTG, the equation of motion of matter in effective Riemannian space with the metric tensor $g_{\mu\nu}(x)$, determined by the equations of the gravitational field (5.19) and (5.20), has the form

$$\nabla_\mu T^{\mu\nu}(x) = 0. \qquad (\sigma)$$

For example, we shall take dustlike matter with the energy-momentum tensor $T^{\mu\nu}$ equal to

$$T^{\mu\nu}(x) = \rho U^\mu U^\nu, \quad U^\nu = \frac{dx^\nu}{ds},$$

ds is the interval in Riemannian space.

On the basis of equations (σ), using the expression for $T^{\mu\nu}$, we find the equation for the geodesic line in Riemannian space,

$$\frac{dU^\nu}{ds} + \Gamma^\nu_{\alpha\beta}(x) U^\alpha U^\beta = 0.$$

When the gravitational interaction is switched off, i.e. when the Riemann curvature tensor turns to zero, from the equations of the gravitational field (5.19) and (5.20) it follows that the Riemannian metric $g_{\mu\nu}(x)$ transforms into the previously chosen metric of Minkowski space, $\gamma_{\mu\nu}(x)$. In this case the equation of motion of matter (σ) assumes the form

$$D_\mu t^{\mu\nu}(x) = 0. \qquad (\lambda)$$

Here the energy-momentum tensor $t^{\mu\nu}(x)$ is

$$t^{\mu\nu}(x) = \rho u^\mu u^\nu, \; u^\nu = \frac{dx^\nu}{d\sigma},$$

$d\sigma$ is the interval in Minkowski space.

On the basis of (λ), using the expression for $t^{\mu\nu}$, we find the equations for the geodesic line in Minkowski space,

$$\frac{du^\nu}{d\sigma} + \gamma^\nu_{\alpha\beta} u^\alpha u^\beta = 0,$$

i.e., we have arrived at the ordinary equations for the free motion of particles in SRT in the previously chosen reference system with the metric tensor $\gamma_{\mu\nu}(x)$. Thus, the equation of motion of matter in a gravitational field in the chosen reference system automatically transforms, when the gravitational interaction is switched off, i.e., when the Riemann curvature tensor turns to zero, into the equation of motion of matter in Minkowski space in the same reference system with the metric tensor $\gamma_{\mu\nu}(x)$, i.e., the equivalence principle is obeyed. This assertion in RTG is of a general character, since when the Riemann tensor turns to zero the density of the Lagrangian of matter in the gravitational field, $L_M(\tilde{g}^{\mu\nu}, \Phi_A)$, transforms into the ordinary SRT density of the Lagrangian, $L_M(\tilde{\gamma}^{\mu\nu}, \Phi_A)$, in the chosen reference system.

In GRT the equation of motion of matter also has the form (σ). But, since the Hilbert–Einstein equations do not contain the metric tensor of Minkowski space, then when the gravitational interaction is switched off, i.e., the Riemann curvature tensor turns to zero, it is impossible to say in which reference system (inertial or accelerated) of Minkowski space we happen to be, and therefore it is impossible to determine which equation of motion of matter in Minkowski space we will obtain when the gravitational interaction is switched off. Precisely for this reason, the equivalence principle cannot be complied with in GRT within the framework of this theory. Usually such correspondence in GRT is achieved precisely within the field approach, when a weak gravitational field is considered to be a physical field in Minkowski space in Galilean coordinates. Thus, GRT is made to include what it essentially does not contain, since, as A. Einstein wrote, "*gravitational fields can be* *given without introducing tensions and energy density*"[20], so no existence of any physical field in GRT can even be spoken of.

In conclusion, we note that RTG revives all those concepts (inertial reference system, the law of inertia, acceleration relative to space, the conservation laws of energy-momentum and of angular momentum), which occurred in classical Newtonian mechanics and in special relativity theory, and which had to be renounced by A. Einstein in constructing GRT.

In 1955 A.Einstein wrote: ¡¡*A significant achievement of general relativity theory consists in that it rids physics of the necessity of introducing the "inertial system" (or "inertial systems")¿¿*[21]. From our point of view, the fields of inertia and of gravity cannot be identified with each other even locally, since they are of totally different natures. While the former can be removed by a choice of the reference system, no choice whatever of the reference system can remove the fields of gravity. Regrettably, this circumstance is not understood by many persons, since they do not apprehend that "*in Einstein's theory*", as especially stressed by J.L.Synge, "*the existence or absence of a gravitational field depends on whether the Riemann curvature tensor differs from or equals zero*"[22].

[20]Einstein A. Collection of scientific works, Moscow: Nauka, 1965, vol.1, art.47, p.627.

[21]Einstein A. Collection of scientific works, Moscow: Nauka, 1965, vol.2, art.146, p.854.

[22]J.L.Synge. Relativity: the general theory. M.:Foreign literature publishers, 1963, p.9.

6. The causality principle in RTG

RTG was constructed within the framework of SRT, like the theories of other physical fields. According to SRT, any motion of a pointlike test body (including the graviton) always takes place within the causality light cone of Minkowski space. Consequently, non-inertial reference systems, realized by test bodies, must also be inside the causality cone of pseudo-Euclidean space-time. This fact determines the entire class of possible non-inertial reference systems. Local equality between the three-dimensional force of inertia and gravity in the case of action on a material pointlike body will occur, if the light cone of the effective Riemannian space does not go beyond the limits of the causality light cone of Minkowski space. Only in this case can the three-dimensional force of the gravitational field acting on the test body be locally compensated by transition to the admissible non-inertial reference system, connected with this body.

If the light cone of the effective Riemannian space were to reach beyond the causality light cone of Minkowski space, this would mean that for such a "gravitational field" no admissible non-inertial reference system exists, within which this "force field" could be compensated in the case of action on a material pointlike body. In other words, local compensation of the 3-force of gravity by the force of inertia is possible only when the gravitational field, acting as a physical field on particles, does not lead their world lines outside the causality cone of pseudo-Euclidean space-time. This condition should be considered the causality principle permitting selection of solutions of the set of equations (5.19) and (5.20) having physical sense and corresponding to the gravitational fields.

The causality principle is not satisfied automatically. There is nothing unusual in this fact, since both in electrodynamics, and in other physical theories, as well, the causality condition for matter in the form $d\sigma^2 = \gamma_{\mu\nu} dx^\mu dx^\nu \geq 0$ is always added (but not always noted) to the main equations, which actually provides for it being impossible for any form of matter to undergo motion with velocities exceeding the speed of light. In our case it is necessary to take into account that the gravitational interaction enters into the coefficients of the second-order derivatives in the field equations, i.e. there arises an effective geometry of space-time. This feature is only peculiar to the gravitational field. The interaction of all other known physical fields usually does not involve the second-order derivatives of the field equations, and therefore does not alter the initial pseudo-Euclidean geometry of space-time.

We shall now present an analytical formulation of the causality principle in RTG. Since in RTG the motion of matter under the action of the gravitational field in pseudo-Euclidean space-time is equivalent to the motion of matter in the corresponding effective Riemannian space-time, we must for events (world lines of particles and of light) related by causality, on the one hand, have the condition

$$d\,s^2 = g_{\mu\nu}\,dx^\mu\,dx^\nu \;\geq\; 0, \tag{6.1}$$

and, on the other hand, the following inequality must hold valid for such events:

$$d\sigma^2 = \gamma_{\mu\nu}\,dx^\mu\,dx^\nu \;\geq\; 0. \tag{6.2}$$

The following condition is valid for the chosen reference system realized by physical bodies:

$$\gamma_{00} \;>\; 0. \tag{6.3}$$

We single out in expression (6.2) the time- and spacelike parts:

$$d\,\sigma^2 = \left(\sqrt{\gamma_{00}}\,dt + \frac{\gamma_{0\,i}\,dx^i}{\sqrt{\gamma_{00}}}\right)^2 - s_{i\,k}\,dx^i\,dx^k, \tag{6.4}$$

here the Latin indices i, k run through the values 1, 2, 3;

$$s_{i\,k} = -\,\gamma_{i\,k} + \frac{\gamma_{0\,i}\gamma_{0\,k}}{\gamma_{00}}, \tag{6.5}$$

s_{ik} is the metric tensor of three-dimensional space in four-dimensional pseudo-Euclidean space-time. The square spatial distance is determined by the expression

$$d\,l^2 = s_{i\,k}\,dx^i\,dx^k. \tag{6.6}$$

Now we represent the velocity $v^i = \frac{dx^i}{dt}$ as $v^i = ve^i$, where v is the absolute value of the velocity and e^i is an arbitrary unit vector in three-dimensional space,

$$s_{i\,k}e^i\,e^k = 1. \tag{6.7}$$

In absence of the gravitational field the velocity of light in the chosen reference system is readily determined from expression (6.4) by setting it equal to zero:

$$\left(\sqrt{\gamma_{00}}\,dt + \frac{\gamma_{0i}\,dx^i}{\sqrt{\gamma_{00}}}\right)^2 = s_{i\,k}\,dx^i\,dx^k.$$

Hence, we find

$$v = \sqrt{\gamma_{00}}\Big/\left(1 - \frac{\gamma_{0i}\,e^i}{\sqrt{\gamma_{00}}}\right). \tag{6.8}$$

Thus, an arbitrary four-dimensional isotropic vector in Minkowski space, u^ν, is

$$u^\nu = (1,\,v\,e^i). \tag{6.9}$$

For both conditions (6.1), (6.2) to be satisfied simultaneously, it is necessary and sufficient that for any isotropic vector

$$\gamma_{\mu\nu}\,u^\mu\,u^\nu = 0 \tag{6.10}$$

the causality condition

$$g_{\mu\nu}\,u^\mu\,u^\nu \;\leq\; 0, \tag{6.11}$$

hold valid, which precisely indicates that the light cone of the effective Riemannian space does not go beyond the causality light cone of pseudo-Euclidean space-time. The causality condition may be written in the following form:

$$g_{\mu\nu}\, v^{\mu}\, v^{\nu} = 0, \tag{6.10a}$$

$$\gamma_{\mu\nu} v^{\mu}\, v^{\nu} \;\geq\; 0. \tag{6.11a}$$

In GRT, physical meaning is also attributed to such solutions of the Hilbert-Einstein equations, which satisfy the inequality

$$g < 0,$$

as well as the requirement known as the energodominance condition, which is formulated as follows. For any timelike vector K_{ν} the inequality

$$T^{\mu\nu}\, K_{\mu}\, K_{\nu} \;\geq\; 0,$$

must be valid, and the quantity $T^{\mu\nu}\, K_{\nu}$ must form, for the given vector K_{ν}, a non-spacelike vector.

In our theory, such solutions of equations (5.21a) and (5.22a) have physical meaning, which, besides these requirements, must also satisfy the causality condition (6.10a) and (6.11a). The latter can be written, on the basis of equation (5.21a), in the following form:

$$R_{\mu\nu}\, K^{\mu} K^{\nu} \;\leq\; \frac{8\,\pi}{\sqrt{-g}}(T_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu}\, T)\, K^{\mu} K^{\nu} +$$
$$+\frac{m^2}{2}\, g_{\mu\nu}\, K^{\mu} K^{\nu}. \tag{6.12}$$

If the density of the energy-momentum tensor is taken in the form:

$$T_{\mu\nu} = \sqrt{-g}[(\rho + p) U_{\mu} U_{\nu} - p g_{\mu\nu}],$$

then on the basis of (5.21a) it is possible to establish between the interval of Minkowski space, $d\sigma$, and the interval of the effective Riemannian space, ds, the following relationship:

$$\frac{m^2}{2}d\sigma^2 = ds^2[4\pi(\rho + 3p) + \frac{m^2}{2} - R_{\mu\nu}U^\mu U^\nu],$$

here $U^\mu = \frac{dx^\mu}{ds}$.

Owing to the causality principle the inequality

$$R_{\mu\nu}U^\mu U^\nu < 4\pi(\rho + 3p) + \frac{m^2}{2},$$

which is a special case of inequality (6.12), or

$$\sqrt{-g}R_{\mu\nu}\, v^\mu v^\nu \ \le\ 8\,\pi T_{\mu\nu}v^\mu v^\nu \qquad (6.12a)$$

must hold valid.

Let us now consider the motion of a test body under the influence of gravity in GRT and RTG. In 1918 A. Einstein gave the following formulation of the equivalence principle: "*Inertia and gravity are identical; hence and from the results of special relativity theory it inevitably follows that the symmetric ≪fundamental tensor≫ $g_{\mu\nu}$ determines the metric properties of space, of the motion of bodies due to inertia in it, and, also, the influence of gravity*" [23]. Identifying in GRT the gravitational field and the metric tensor $g_{\mu\nu}$ of Riemannian space permits, by an appropriate choice of the reference system, to equate to zero all the components of the Christoffel symbol at all points of an arbitrary non-selfintersecting line. Precisely for this reason, motion along a geodesic line in GRT is considered free. But, in this case, the choice of reference system cannot remove the gravitational field in GRT, also, because the motion of two close material pointlike bodies will not be free due to the existence of the curvature tensor, which can never be equated to zero by a choice of reference system owing to its tensor properties.

[23] Einstein A. Collection of scientific works, Moscow: Nauka, 1965, vol.1, art.45, p.613.

The gravitational field in RTG is a physical field in the spirit of Faraday–Maxwell, so the gravitational force is described by a four-vector, and, consequently, the forces of inertia can be made to compensate the three-dimensional part of the force of gravity by a choice of reference system only if conditions (6.10) and (6.11) are satisfied. Now, the motion of a material pointlike body in the gravitational field can never be free. This is especially evident, if the equation of the geodesic line is written in the form [41]

$$\frac{DU^\nu}{d\sigma} = -G^\rho_{\alpha\beta} U^\alpha U^\beta (\delta^\nu_\rho - U^\nu U_\rho).$$

Here

$$d\sigma^2 = \gamma_{\mu\nu} dx^\mu dx^\nu,\ U^\nu = \frac{dx^\nu}{d\sigma}.$$

Free motion in Minkowski space is described by the equation:

$$\frac{DU^\nu}{d\sigma} = \frac{dU^\nu}{d\sigma} + \gamma^\nu_{\mu\lambda} U^\mu U^\lambda = 0,$$

$\gamma^\nu_{\mu\lambda}$ are the Christoffel symbols of Minkowski space. We see that motion along a geodesic line of Riemannian space is the motion of a test body under the action of the force F^ν:

$$F^\nu = -G^\rho_{\alpha\beta} U^\alpha U^\beta (\delta^\nu_\rho - U^\nu U_\rho),$$

and this force is a four-vector. If the test body were charged, it would emit electromagnetic waves, since it moves with acceleration.

In SRT there exists an essential difference between the forces of inertia and physical forces. The forces of inertia can always be made equal to zero by a simple choice of reference system, while essentially no choice of reference system can turn physical forces into zero, since they are of a vector nature in Minkowski space. Since in RTG all forces, including gravitational forces, are of a vector nature, this means that they cannot be equated to zero by a choice of reference system. A choice of reference system can only make the force of inertia compensate a three-dimensional force, acting on a material pointlike body, the force being of any nature, including gravitational. In GRT, as noted by J.L.Synge [23], "*... the concept of the force of gravity does not exist, since gravitational properties are organically present in the structure of space-time and are manifested in the curvature of space-time, i.e. in that the Riemann tensor R_{ijkm} differs from zero.*" Precisely in this connection, J.L.Synge wrote: "*According to the famous legend, Newton was inspired to create his theory of gravity, when he once observed an apple falling from the branch of a tree, and those who study Newtonian physics even now are ready to claim that the acceleration (980 cm/s^2) of a falling apple is due to the gravitational field. In accordance with relativity theory* (GRT is intended — A.L.), *this point of view is completely erroneous. We shall undertake a thorough investigation of this problem and verify that the gravitational field (i.e., the Riemann tensor) actually plays an extremely insignificant role in the phenomenon of a falling apple, while the acceleration 980 cm/s^2 is really due to the curvature of the world line of the tree's branch.*"

According to RTG, the gravitational field is a physical field, and therefore, unlike the case of GRT, it fully retains the concept of the force of gravity. Precisely owing to the force of gravity does the free fall of bodies occur, i.e., everything proceeds like in Newtonian physics. Moreover, all gravitational effects in the Solar system (the displacement of the perihelion of Mercury, the deflection of light by the Sun, the time delay of a radiosignal, the precession of a gyroscope) are caused precisely by the action of the force of gravity, but not by the curvature tensor of space-time, which in the Solar system is quite small.

The local identity between inertia and gravity was seen by Einstein as the main reason for the inertial and gravitational masses to be equal to each other. However, in our opinion, as it can be seen from equations (2.2), the reason for this equality lies in that the source of the gravitational field is the conversed total density of the tensor of matter and of the gravitational field. Precisely for this reason, the inertial and gravitational masses being equal to each other does not require the forces of gravity and of inertia to be locally identical.

7. Mach's principle

In formulating the laws of mechanics Newton introduced the notion of absolute space, which always remains the same and is motionless. He defined the acceleration of a body precisely with respect to this space. This acceleration had an absolute character. The introduction of such an abstraction as absolute space turned out to be extremely fruitful. Hence, for instance, arose the concepts of inertial reference systems in the entire space, the relativity principle for mechanical processes, and the idea came into being of states of motion, that are physically singled out. In this connection Einstein wrote the following in 1923: "*Reference systems that are in such states of motion are characterized by the laws of Nature formulated in these coordinates assuming the most simple form."* And further: *¡¡...according to classical mechanics there exists "relativity of velocity", but not "relativity of acceleration"¿¿*[24].

Thus in theory was the notion established of inertial reference systems, in which material pointlike bodies, not subject to the action of forces, do not experience acceleration and remain at rest or in their state of uniform motion along a straight line. However, Newton's absolute space or inertial reference system were actually introduced a priori, without any relation to the character of the distribution of matter in the Universe.

Mach displayed much courage in seriously criticizing the main points of Newton's mechanics. He later wrote that he succeeded in publishing his ideas with difficulty. Although Mach did not construct any physical theory free of the disadvantages he himself pointed out, he greatly influenced the development of physical theory. He drew the attention of scientists to the analysis of the main physical concepts.

We shall quote some statements made by Mach [18], which in the literature have been termed the "Mach principle". *"No*

[24]Einstein A. Collection of scientific works, Moscow: Nauka, 1965, vol.2, art.70, p.122.

one can say anything about absolute space and absolute motion, this is only something that can be imagined and is not observable in experiments". And further: *"Instead of referring a moving body to space (to some reference system), we shall directly consider its relation to b o d i e s of the world, only by which it is possible to d e f i n e a reference system. ...even in the most simple case, when we apparently consider the interaction between only t w o masses, it is i m p o s s i b l e to become distracted from the rest of the world. ... If a body revolves with respect to the sky of motionless stars, then there arise centrifugal forces, while if it revolves round a n o t h e r body, instead of the sky of motionless stares, no centrifugal forces will arise. I have nothing against calling the first revolution a b s o l u t e, if only one does not forget that this signifies nothing but revolution r e l a t i v e to the sky of motionless stars."*

Therefore Mach wrote: *"...there is no necessity for relating the Law of inertia to some special absolute space. The most natural approach of a true naturalist is the following: first to consider the law of inertia as quite an approximate law, then to establish its relationship in space to the motionless sky of stars, ...and then one should expect corrections or some development of our knowledge on the basis of further experiments. Not long ago Lange published a critical article, in which he exposes how it would be possible, in accordance with his principles, to introduce a n e w reference system, if the ordinary rough reference to the motionless starry sky were to become no longer suitable owing to more precise astronomical observations. There exists no difference between the opinion of Lange and my own relative to the t h e o r e t i c a l formal value of Lange's conclusions, namely, that at present the motionless starry sky is the only p r a c t i c a l l y suitable reference system, and, also, relative to the method of defining a new reference system by gradually introducing corrections."* [18]. Further, Mach quotes S. Neumann: *"Since all motions must be referred to the reference system alpha (the reference sys-*

tem of inertia), it evidently represents an indirect relationship between all the processes taking place in the Universe, and, consequently, it contains, so to say, a universal law which is just as mysterious as it is complex". In this connection Mach notes: *"I think anyone will agree with this"* [18].

From Mach's statements it is obvious that, since the issue concerns the law of inertia, in accordance with which, following Newton, *"...each individual body, being left to itself, retains its state of rest or uniform motion along a straight line..."*, there naturally arises the question of inertial reference systems and of their relations to the distribution of matter. Mach and his contemporaries quite clearly understood that such a relation should exist in Nature. Precisely this meaning will further be attributed to the concept of "Mach's principle".

Mach wrote: *"Although I think that at the beginning astronomical observations will necessitate only very insignificant corrections, I anyhow do think it possible that the law of inertia in the simple form given it by Newton plays for us, human beings, only a limited and transient role."* [18]. As we shall further see, Mach did not turn out to be right, here. Mach did not give a mathematical formulation of his idea, and therefore very often diverse authors attribute to Mach's principle diverse meanings. We shall try, here, to retain the meaning, attributed to it by Mach himself.

Poincaré, and later Einstein, generalized the relativity principle to all physical phenomena. Poincaré's formulation [40] goes as follows: *"...the relativity principle, according to which the laws governing physical phenomena should be identical for an observer at rest and for an observer undergoing uniform motion along a straight line, so we have and can have no method for determining whether we are undergoing similar motion or not."* Application of this principle to electromagnetic phenomena led Poincaré, and then Minkowski, to the discovery of the pseudo-Euclidean geometry of space-time and thus even more reinforced the hypothesis of inertial reference systems existing throughout the entire space. Such reference systems are physically singled out, and therefore acceleration relative to them has an absolute sense.

In general relativity theory (GRT) no inertial reference systems exist in all space. Einstein wrote about this in 1929: *"The starting point of theory is the assertion that there exists no singled out state of motion, i.e. not only velocity, but acceleration has no absolute sense"*[25].

Mach's principle, in his own formulation, turned out not to have any use. It must, however, be noted that the ideas of inertial reference systems throughout the space have quite a weighty basis, since, for instance, in passing from a reference system bound to the Earth to a reference system bound to the Sun and, then, further to the Metagalaxy we approach, with an increasing precision, the inertial reference system. Therefore, there are no reasons for renouncing such an important concept as the concept of an inertial reference system. On the other hand, the existence of the fundamental conservation laws of energy-momentum and of angular momentum also leads with necessity to the existence of inertial reference systems in the entire space. The pseudo-Euclidean geometry of space reflects the general dynamic properties of matter and at the same time introduces inertial reference systems. Although the pseudo-Euclidean geometry of space-time resulted from studies of matter, and therefore cannot be separated from it, nevertheless, it is possible to speak of Minkowski space in the absence of matter. However, like earlier in Newtonian mechanics, in special relativity theory no answer exists to the question of how inertial reference systems are related to the distribution of matter in the Universe.

The discovery of the pseudo-Euclidean geometry of space and time permitted considering not only inertial, but accelerated reference systems, also, from a unique standpoint. A large difference was revealed between the forces of inertia and forces

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[25] Einstein A. Collection of scientific works, Moscow: Nauka, 1966, vol.2, art.92, p.264.

caused by physical fields. It consists in that the forces of inertia can always be equated to zero by choosing an appropriate reference system, while forces caused by physical fields cannot, in principle, be made equal to zero by a choice of reference system, since they are of a vector nature in four-dimensional space-time. Since the gravitational field in RTG is a physical field in the spirit of Faraday-Maxwell, forces caused by such a field cannot be equated to zero by a choice of reference system.

Owing to the gravitational field having a rest mass, the main equations of RTG, (5.19) and (5.20), contain, together with the Riemannian metric, the metric tensor of Minkowski space, also, but this means that, in principle, the metric of this space can be expressed via the geometric characteristics of the effective Riemannian space and, also, via quantities characterizing the distribution of matter in the Universe. This is readily done by passing in equations (5.19) from contravariant to covariant quantities. In this way we obtain

$$\frac{m^2}{2}\,\gamma_{\mu\nu}(x) = \frac{8\,\pi}{\sqrt{-g}}(T_{\mu\nu} - \frac{1}{2}\,g_{\mu\nu}\,T) - R_{\mu\nu} + \frac{m^2}{2}\,g_{\mu\nu}\,. \quad (7.1)$$

Hence, we see that in the right-hand side of the equations there occur only geometric characteristics of the effective Riemannian space and quantities determining the distribution of matter in this space.

Experimental investigation of the motion of particles, and of light, in Riemannian space, in principle, allows to find the metric tensor of Minkowski space and, consequently, to construct an inertial reference system, also. Thus, RTG constructed within the framework of special relativity theory permits to establish the relation between an inertial reference system and the distribution matter. For this reason, motion relative to space is motion relative to matter in the Universe. The existence of an inertial reference system, determined by the the distribution of matter, makes acceleration absolute. We see that the special relativity principle is of general significance, independent of the form of matter.

The requirements of this principle in the case of the gravitational field are expressed by the condition that equations (5.19) and (5.20) be form-invariant relative to the Lorentz group. Lorentz form-invariance of physical equations remains a most important physical principle in constructing a theory, since precisely this principle provides the possibility of introducing universal characteristics for all forms of matter.

A. Einstein wrote in his work of 1950: *"...should one not finally try to retain the concept of an inertial system, renouncing all attempts at explaining the fundamental feature of gravitational phenomena, which manifests itself in Newton's system as the equivalence of inert and gravitating masses?"*[26]. The concept of an inertial system is retained in RTG, and at the same time it is shown that the equivalence of inert and gravitating masses is a direct consequence of the hypothesis that the conserved density of the energy-momentum tensor of matter is the source of the gravitational field. Thus, the equality between inert and gravitating masses in no way contradicts the existence of an inertial reference system. Moreover, these conditions organically complement each other and underlie RTG.

Contrary to our conclusion, A.Einstein gave the following answer to his own question: "*Who believes in the comprehensibility of Nature should answer — no.*" The existence of inertial reference systems permits resolving Mach's paradox, since only in this case can one speak of acceleration relative to space. V.A.Fock wrote in this connection: ¡¡*As to Mach's paradox, it is known to be based on the consideration of a rotating liquid, having the shape of an ellipsoid, and of a spherical body that does not rotate. The paradox arises, here, only if the concept "rotation relative to space" is considered to be senseless; then, indeed, both bodies (the rotating one and the one not rotating) are apparently equivalent, and it becomes incomprehensible why one of them is spherical and the other one is not.*

[26]Einstein A. Collection of scientific works, Moscow: Nauka, 1966, vol.2, art.137, p.724.

But the paradox vanishes as soon as we acknowledge the legitimacy of the concept of "acceleration relative to space"¿¿ [27].

Mach's ideas profoundly influenced Einstein's views on gravity during the construction of general relativity theory. Einstein wrote in one of his works: *"Mach's principle: the G-field is fully determined by the masses of bodies."* But this statement turns out to be not valid in GRT, since there exist solutions in the absence of matter, also. Attempts at eliminating this circumstance by introduction of the λ-term did not lead to the desired result. It turned out to be that equations with the λ-term also have solutions differing from zero in the absence of matter. We see that Einstein attached a totally different meaning to the concept of "Mach's principle". But within such an interpretation, also, no place was found in GRT for Mach's principle.

Is there any place in RTG for Mach's principle as formulated by Einstein? Unlike GRT, in this theory spacelike surfaces are present throughout the entire space (global Cauchy surfaces), owing to the causality principle. And if no matter is present on one of such surfaces, then the requirement of energodominance imposed on the tensor of matter will result in matter always being absent [26]. It will be shown in section 10 that a gravitational field cannot arise without matter.

Only solutions of the set of inhomogeneous gravitational equations have a physical sense, when matter exists in some part of space or throughout the entire space. This means that the gravitational field and the effective Riemannian space in the actual Universe could not arise without the matter that produced them. Solution of the equations for the metric of effective Riemannian space in the absence of matter can, for example, be considered a limit case of the solution obtained for a homogeneous and isotropic distribution of matter in space, as the density of matter tends subsequently toward zero. We

[27] V.A.Fock. Theory of space, time and gravity. M.:Gostekhizdat, 1961, p.499.

see that Mach's principle, even as formulated by Einstein, is realized in relativistic theory of gravity.

There exists, however, an essential difference between the understanding of the G-field in our theory and in GRT. Einstein understood the G-field to be the Riemannian metric, while in our opinion the gravitational field is a physical field. Such a field is present in the Riemannian metric together with the plane metric, and therefore the metric does not vanish in the absence of matter and of the gravitational field, but remains the metric of Minkowski space.

In the literature there also exist other formulations of Mach's principle, differing in meaning from the ideas of both Mach and Einstein, but since, in our opinion, these formulations are not sufficiently clear, we have not dealt with them. Since gravitational forces in RTG are due to a physical field of the Faraday–Maxwell type, any common unique essence of the forces of inertia and of gravity is, in principle, out of the question.

Sometimes the essence of Mach's principle is seen to consist in that the forces of inertia are determined, allegedly in compliance with this principle, by interaction with matter in the Universe. From a field standpoint such a principle cannot exist in Nature. The point is that, although inertial reference systems, as we have seen above, are related to the distribution of matter in the Universe, forces of inertia do not result from the interaction with matter in the Universe, because any influence of matter can only be exerted via physical fields, but this means that the forces produced by these fields, owing to their vector nature, cannot be made equal to zero by a choice of reference system. Thus, forces of inertia are directly determined not by physical fields, but by a rigorously defined structure of geometry and by the choice of reference system.

The pseudo-Euclidean geometry of space–time, which reflects dynamic properties common to all forms of matter, on the one hand confirmed the hypothetical existence of inertial reference systems, and on the other hand revealed that forces of inertia, arising under an appropriate choice of reference system, are expressed via the Christoffel symbols of Minkowski space. Therefore, they are independent of the nature of the body. All this became clear when it was shown that special relativity theory is applicable not only in inertial reference systems, but also in non-inertial (accelerated) systems.

This made it possible to provide in Ref. [7] a more general formulation of the relativity principle: *"Whatever physical reference system (inertial or non-inertial we choose, it is always possible to indicate an infinite set of such other reference systems, in which all physical phenomena proceed like in the initial reference system, so we have and can have no experimental for determining precisely in which reference system of this infinite set we happen to be."* Mathematically this is expressed as follows: consider the interval in a certain reference system of Minkowski space to be

$$d\sigma^2 = \gamma_{\mu\nu}(x)dx^\mu dx^\nu,$$

then there exists another reference system x':

$$x'^\nu = f^\nu(x),$$

in which the interval assumes the form

$$d\sigma^2 = \gamma_{\mu\nu}(x')dx'^\mu dx'^\nu,$$

where the metric coefficients $\gamma_{\mu\nu}$ have the same functional form as in the initial reference system. In this case it is said that the **metric is form-invariant** relative to such transformations, and all **physical equations are also form-invariant**, i.e. they have the same form both in the primed and in the not primed reference systems. The transformations of coordinates that leave the metric form-invariant form a group. In the case of Galilean coordinates in an inertial reference system these are the usual Lorentz transformations.

In RTG there exists an essential difference between the forces of inertia and the forces of gravity consisting in that as the distance from bodies increases, the gravitational field becomes weaker, while the forces of inertia may become indefinitely large, depending on the choice of reference system. And only in an inertial reference system are they equal to zero. Therefore, it is a mistake to consider forces of inertia inseparable from forces of gravity. In everyday life the difference between them is nearly obvious.

The construction of RTG has permitted to establish the relationship between an inertial reference system and the distribution of matter in the Universe and, thus, to understand more profoundly the nature of forces of inertial and their difference from material forces. In our theory forces of inertia are assigned the same role as the one they assume in any other field theories.

8. Post-Newtonian approximation

The post-Newtonian approximation is quite sufficient for studying gravitational effects in the Solar system. In this section we shall construct this approximation. Technically, our construction takes advantage of many results previously obtained by V.A.Fock [24], and it turns out to be possible to further simplify the method of deriving the post-Newtonian approximation.

We shall write the main equations of theory in the form (see Appendix D)

$$\tilde{\gamma}^{\alpha\beta} D_\alpha D_\beta \tilde{\Phi}^{\epsilon\lambda} + m^2 \sqrt{-\gamma}\tilde{\Phi}^{\epsilon\lambda} = -16\pi g (T_M^{\epsilon\lambda} + \tau_g^{\epsilon\lambda}), \qquad (8.1)$$

$$D_\lambda \tilde{\Phi}^{\epsilon\lambda} = 0\ . \qquad (8.2)$$

where $T_M^{\epsilon\lambda}$ is the energy-momentum tensor of matter; $\tau_g^{\epsilon\lambda}$ is the energy-momentum tensor of the gravitational field.

The expression for the energy-momentum tensor of the gravitational field can be represented in the form

$$\begin{aligned}
-16\pi g \tau_g^{\epsilon\lambda} &= \frac{1}{2}(\tilde{g}^{\epsilon\alpha}\tilde{g}^{\lambda\beta} - \frac{1}{2}\tilde{g}^{\epsilon\lambda}\tilde{g}^{\alpha\beta})(\tilde{g}_{\nu\sigma}\tilde{g}_{\tau\mu} - \frac{1}{2}\tilde{g}_{\tau\sigma}\tilde{g}_{\nu\mu}) D_\alpha \tilde{\Phi}^{\tau\sigma} \times \\
&\times D_\beta \tilde{\Phi}^{\mu\nu} + \tilde{g}^{\alpha\beta}\tilde{g}_{\tau\sigma} D_\alpha \tilde{\Phi}^{\epsilon\tau} D_\beta \tilde{\Phi}^{\lambda\sigma} - \tilde{g}^{\epsilon\beta}\tilde{g}_{\tau\sigma} D_\alpha \tilde{\Phi}^{\lambda\sigma} D_\beta \tilde{\Phi}^{\alpha\tau} - \\
&- \tilde{g}^{\lambda\alpha}\tilde{g}_{\tau\sigma} D_\alpha \tilde{\Phi}^{\beta\sigma} D_\beta \tilde{\Phi}^{\epsilon\tau} + \frac{1}{2}\tilde{g}^{\epsilon\lambda}\tilde{g}_{\tau\sigma} D_\alpha \tilde{\Phi}^{\sigma\beta} D_\beta \tilde{\Phi}^{\alpha\tau} + \\
&+ D_\alpha \tilde{\Phi}^{\epsilon\beta} D_\beta \tilde{\Phi}^{\lambda\alpha} - \tilde{\Phi}^{\alpha\beta} D_\alpha D_\beta \tilde{\Phi}^{\epsilon\lambda} - \\
&- m^2 (\sqrt{-g}\tilde{g}^{\epsilon\lambda} - \sqrt{-\gamma}\tilde{\Phi}^{\epsilon\lambda} + \tilde{g}^{\epsilon\alpha}\tilde{g}^{\lambda\beta}\gamma_{\alpha\beta} - \frac{1}{2}\tilde{g}^{\epsilon\lambda}\tilde{g}^{\alpha\beta}\gamma_{\alpha\beta})\ . \qquad (8.3)
\end{aligned}$$

This expression is written in an arbitrary reference system in Minkowski space. We shall further perform all computations in the Galilean coordinates of the inertial reference system,

$$\gamma_{\mu\nu} = (1, -1, -1, -1)\ . \qquad (8.4)$$

In constructing the series of perturbation theory it is natural to apply as a small parameter such a quantity ϵ that

$$v \sim \epsilon,\ U \sim \epsilon^2,\ \Pi \sim \epsilon^2,\ p \sim \epsilon^2\ . \qquad (8.5)$$

Here U is the Newtonian potential of the gravitational field; Π is the specific internal energy of the body considered; p is the specific pressure.
For the Solar system the parameter ϵ^2 is of the order of

$$\epsilon^2 \sim 10^{-6} \ . \tag{8.6}$$

We shall use the expansions of the components of the density of the tensor:

$$\tilde{g}^{00} = 1 + \overset{(2)}{\tilde{\Phi}}{}^{00} + \overset{(4)}{\tilde{\Phi}}{}^{00} + ..., \tag{8.7}$$

$$\tilde{g}^{0i} = \overset{(3)}{\tilde{\Phi}}{}^{0i} + \overset{(5)}{\tilde{\Phi}}{}^{0i} + ..., \tag{8.8}$$

$$\tilde{g}^{ik} = \tilde{\gamma}^{ik} + \overset{(2)}{\tilde{\Phi}}{}^{ik} + \overset{(4)}{\tilde{\Phi}}{}^{ik} + ... \tag{8.9}$$

We shall adopt the ideal fluid model of matter, the energy-momentum tensor of which has the form

$$T^{\epsilon\lambda} = [p + \rho(1+\Pi)]u^\epsilon u^\lambda - pg^{\epsilon\lambda}, \tag{8.10}$$

where ρ is the invariant density of an ideal fluid;
p is the specific isotropic pressure;
u^λ is the velocity four-vector.

We shall now write the expansion in the small parameter ϵ for the energy-momentum tensor of matter:

$$T_M^{00} = \overset{(0)}{T}{}^{00} + \overset{(2)}{T}{}^{00} + ..., \tag{8.11}$$

$$T_M^{0i} = \overset{(1)}{T}{}^{0i} + \overset{(3)}{T}{}^{0i} + ..., \tag{8.12}$$

$$T_M^{ik} = \overset{(2)}{T}{}^{ik} + \overset{(4)}{T}{}^{ik} + ... \tag{8.13}$$

In the Newtonian approximation, i.e., when we neglect the forces of gravity, we have for the four-vector the following:

$$u^0 = 1 + 0(\epsilon^2), \ u^i = v^i(1 + 0(\epsilon^2)) \ . \tag{8.14}$$

Substituting these expressions into (8.10) we find

$$\overset{(0)}{T}{}^{00} = \rho,\ \overset{(1)}{T}{}^{0i} = \rho v^i,\ \overset{(0)}{T}{}^{ik} = 0\ . \qquad (8.15)$$

In this approximation, on the basis of (5.7), we have

$$\partial_0 \rho + \partial_i(\rho v^i) = 0\ . \qquad (8.16)$$

Hence it can be seen that in the Newtonian approximation the total inert mass of a body is a conserved quantity:

$$M = \int \rho d^3x\ . \qquad (8.17)$$

On the basis of equations (8.1) we have in the Newtonian approximation:

$$\Delta\ \overset{(2)}{\tilde{\Phi}}{}^{00} = -16\pi\rho\ , \qquad (8.18)$$

$$\Delta\ \overset{(3)}{\tilde{\Phi}}{}^{0i} = -16\pi\rho v^i\ , \qquad (8.19)$$

$$\Delta\ \overset{(2)}{\tilde{\Phi}}{}^{ik} = 0\ . \qquad (8.20)$$

In an inertial reference system the mass of the graviton, owing to its smallness, plays an insignificant role for effects in the Solar system, and therefore in deriving equations (8.18) – (8.20) we did not take it into account. But even in this case its influence is manifested in that equations (8.2) have to exist together with the set of equations (8.1). Such equations in Galilean coordinates were also applied in V.A.Fock's theory of gravity, but unlike the case of RTG, they did not follow from the least action principle, so it was not clear why precisely they had to be applied, instead of some other equations. V.A. Fock chose them as coordinate conditions and applied them in studying island systems. In RTG these equations arise from the least action principle, and for this reason they are universal. It is precisely owing to equations (8.2) that we obtain a complete set of equations for determining physical quantities. It must be noted that in the general case of a non-inertial reference system or in the case of strong gravitational fields the term with the graviton mass m can no longer be dropped. Thus, for example, even for a static body in the region close to the Schwarzschild sphere the influence of the graviton mass is very significant, so it can no longer be neglected.

The solution of equations (8.18) – (8.20) has the form

$$\overset{(2)}{\tilde{\Phi}}{}^{00} = 4U, \quad U = \int \frac{\rho}{|x - x'|} d^3x', \tag{8.21}$$

$$\overset{(3)}{\tilde{\Phi}}{}^{0i} = -4V^i, \quad V^i = -\int \frac{\rho v^i}{|x - x'|} d^3x' \,, \tag{8.22}$$

$$\overset{(2)}{\tilde{\Phi}}{}^{ik} = 0 \,. \tag{8.23}$$

On the basis of equations (8.2) we have

$$\partial_0 \overset{(2)}{\tilde{\Phi}}{}^{00} + \partial_i \overset{(3)}{\tilde{\Phi}}{}^{0i} = 0 \,. \tag{8.24}$$

Substituting (8.21) and (8.22) we find

$$\partial_0 U - \partial_i V^i = 0 \,. \tag{8.25}$$

Hence, it is evident that differentiation of the potential U with respect to time increases the order of smallness in ϵ. We shall take advantage of this circumstance in calculating the energy-momentum tensor of the gravitational field, $\tau_g^{\epsilon\lambda}$. We note that equation (8.25) is satisfied identically by virtue of equations (8.16).

From (8.22) and (8.23) it follows that of all the density components of the tensor $\tilde{\Phi}^{\epsilon\lambda}$ only the component $\overset{(2)}{\tilde{\Phi}}{}^{00}$, determined by expression (8.21), is seen to remain in the second approximation. Precisely this circumstance significantly simplifies the method of finding the post-Newtonian approximation, when at each stage of the construction we make use of the densities of tensor quantities.

Making use of (8.21) – (8.23) with a precision up to the second order inclusively we obtain

$$\sqrt{-g}g^{00} = 1 + 4U,\ \sqrt{-g}g^{11} = \\ = \sqrt{-g}g^{22} = \sqrt{-g}g^{33} = -1\ . \quad (8.26)$$

Hence, we have

$$-g = 1 + 4U\ , \quad (8.26a)$$

consequently,

$$g_{00} = 1 - 2U,\ g_{11} = g_{22} = g_{33} = -(1 + 2U)\ . \quad (8.27)$$

We see from (8.26) that in the Newtonian approximation, when it suffices to consider only one component of the density of the tensor of matter, T^{00}, the gravitational field is described, as it was expected, by only a sole component $\tilde{\Phi}^{00}$, while the metric tensor $g_{\mu\nu}$ has in this approximation, also, several components, in accordance with (8.27). Working with the field components $\tilde{\Phi}^{\mu\nu}$, instead of the metric tensor $g_{\mu\nu}$, significantly simplifies the entire computational process of constructing the post-Newtonian approximation. Precisely for this reason, introduction of the density of the tensor of the gravitational field $\tilde{\Phi}^{\mu\nu}$ is important not only from a general theoretical point of view, but from a practical standpoint, also. Thus, the metric tensor of the effective Riemannian space is

$$g_{00} = 1 - 2U,\ g_{0i} = 4\gamma_{ik}V^k,\ g_{ik} = \gamma_{ik}(1 + 2U)\ . \quad (8.28)$$

From expression (8.21) for U it follows that the inert mass (8.17) is equal to the active gravitational mass. In RTG, as we have seen, this equality arose because the energy-momentum tensor is the source of the gravitational field.

We shall now proceed to construct the next approximation for the component of the metric tensor g_{00}. For this purpose we shall find the contribution of the energy-momentum tensor of the gravitational field. Since in expression (8.3) it is necessary under the derivative sign to take into account only $\overset{(2)}{\tilde{\Phi}}{}^{00}$, the first term in (8.3) will give a contribution equal to

$$2(\mathrm{grad}\, U)^2, \tag{8.29}$$

while the second term contributes

$$-16(\mathrm{grad}\, U)^2. \tag{8.30}$$

The contribution from all the remaining terms in this approximation will be zero. Discarded are also the terms with time derivatives of the potential U, since, owing to (8.25), they are also all of a higher order of smallness in ϵ. From (8.29) and (8.30) we have

$$-16\pi g \tau_g^{00} = -14(\mathrm{grad}\, U)^2 \,. \tag{8.31}$$

Making use of (8.31), equation (8.1) for component $\tilde{\Phi}^{00}$ in this approximation assumes the form

$$\Delta \overset{(4)}{\tilde{\Phi}}{}^{00} = 16\pi g T^{00} + 14(\mathrm{grad}\, U)^2 + 4\partial_0^2 U \,. \tag{8.32}$$

Since on the basis of (8.28) the interval equals the following in the second order in ϵ:

$$ds = dt(1 - U + \frac{1}{2} v_i v^i) \,, \tag{8.33}$$

we hence obtain

$$u^0 = \frac{dt}{ds} = 1 + U - \frac{1}{2} v_i v^i \,. \tag{8.34}$$

Substituting this expression into (8.10) we find

$$\overset{(2)}{T}{}^{00} = \rho[2U + \Pi - v_i v^i] \,. \tag{8.35}$$

On the basis of (8.26a) and (8.35) we obtain from equations (8.32) the following:

$$\Delta \overset{(4)}{\tilde{\Phi}}{}^{00} = -96\pi\rho U + 16\pi\rho v_i v^i + \\ +14(\mathrm{grad}\, U)^2 - 16\pi\rho\Pi + 4\partial_0^2 U \ . \tag{8.36}$$

Now we shall take advantage of the obvious identity

$$(\mathrm{grad}\, U)^2 = \frac{1}{2}\Delta U^2 - U\Delta U \ . \tag{8.37}$$

But, since

$$\Delta U = -4\pi\rho \ , \tag{8.38}$$

then equation (8.36), upon utilization of (8.37) and (8.38), assumes the form

$$\Delta(\overset{(4)}{\tilde{\Phi}}{}^{00} - 7U^2) = 16\pi\rho v_i v^i - 40\pi\rho U - 16\pi\rho\Pi + 4\partial_0^2 U \ . \tag{8.39}$$

Hence, we have

$$\overset{(4)}{\tilde{\Phi}}{}^{00} = 7U^2 + 4\Phi_1 + 10\Phi_2 + 4\Phi_3 - \frac{1}{\pi}\partial_0^2 \int \frac{U}{|x-x'|} d^3x' \ , \tag{8.40}$$

where

$$\Phi_1 = -\int \frac{\rho v_i v^i}{|x-x'|} d^3x', \quad , \Phi_2 = \int \frac{\rho U}{|x-x'|} d^3x', \\ \Phi_3 = \int \frac{\rho\Pi}{|x-x'|} d^3x'. \tag{8.41}$$

Thus, in the post-Newtonian approximation we find:

$$\tilde{g}^{00} = 1 + 4U + 7U^2 + 4\Phi_1 + 10\Phi_2 + \\ + 4\Phi_3 - \frac{1}{\pi}\partial_0^2 \int \frac{U}{|x-x'|} d^3x'. \tag{8.42}$$

We now have to find the determinant of g in the post-Newtonian approximation. To this end we represent $\tilde{g}^{ik}$ in the form:

$$\tilde{g}^{ik} = \tilde{\gamma}^{ik} + \overset{(4)}{\tilde{\Phi}}{}^{ik}. \qquad (8.43)$$

It must be especially underlined that calculation of the determinant of g is most readily performed if one takes advantage for this purpose of the tensor density $\tilde{g}^{\mu\nu}$ and takes into account that

$$g = det(\tilde{g}^{\mu\nu}) = det(g_{\mu\nu}) \; . \qquad (8.44)$$

From (8.42) and (8.43) we find

$$\sqrt{-g} = 1 + 2U + \frac{3}{2}U^2 + 2\Phi_1 + 5\Phi_2 +$$
$$+2\Phi_3 - \frac{1}{2}\Phi - \frac{1}{2\pi}\partial_0^2 \int \frac{U}{|x-x'|} d^3x'. \qquad (8.45)$$

Here

$$\Phi = \overset{(4)}{\tilde{\Phi}}{}^{11} + \overset{(4)}{\tilde{\Phi}}{}^{22} + \overset{(4)}{\tilde{\Phi}}{}^{33}. \qquad (8.46)$$

Since in the considered approximation $g_{00}g^{00} = 1$, from expressions (8.42) and (8.45) we obtain

$$g_{00} = 1 - 2U + \frac{5}{2}U^2 - 2\Phi_1 - 5\Phi_2 -$$
$$-2\Phi_3 - \frac{1}{2}\Phi + \frac{1}{2\pi}\partial_0^2 \int \frac{U}{|x-x'|} d^3x'. \qquad (8.47)$$

For determining g_{00} we need to calculate the quantity Φ. Since Φ, in accordance with (8.46), was derived by summation, it is possible to make use of equation (8.1) and by summation to obtain directly equations for function Φ.

From expression (8.3) by summation we derive from the first term the following expression:

$$-16\pi g\tau_g^{ii} = -2(\mathrm{grad}\, U)^2. \qquad (8.48)$$

All the remaining terms present in expression (8.3) give no contribution in this approximation. With the aid of expression (8.10) for the energy-momentum tensor we find

$$-16\pi g \overset{(2)}{\tilde{T}}{}^{ii} = -16\pi\rho v_i v^i + 48\pi p\ . \tag{8.49}$$

Taking into account (8.48) and (8.49), we write the equation for Φ as follows:

$$\Delta\Phi = 16\pi\rho v_i v^i - 48\pi p + 2(\mathrm{grad}\, U)^2. \tag{8.50}$$

Taking advantage of identity (8.37) and of equation (8.38) we obtain

$$\Delta(\Phi - U^2) = 16\pi\rho v_i v^i + 8\pi\rho U - 48\pi p\ . \tag{8.51}$$

Hence, we find

$$\Phi = U^2 + 4\Phi_1 - 2\Phi_2 + 12\Phi_4, \tag{8.52}$$

where

$$\Phi_4 = \int \frac{p}{|x - x'|} d^3x'.$$

Substituting expression (8.52) into (8.47) we have

$$g_{00} = 1 - 2U + 2U^2 - 4\Phi_1 - 4\Phi_2 - \\ -2\Phi_3 - 6\Phi_4 + \frac{1}{2\pi}\partial_0^2 \int \frac{U}{|x - x'|} d^3x'. \tag{8.53}$$

Making use of the identity

$$\frac{1}{2\pi}\int \frac{U}{|x - x'|} d^3x' = -\int \rho|x - x'| d^3x',$$

we write expression (8.53) as

$$g_{00} = 1 - 2U + 2U^2 - 4\Phi_1 - 4\Phi_2 - \\ - 2\Phi_3 - 6\Phi_4 - \partial_0^2 \int \rho|x - x'| d^3x'. \tag{8.54}$$

The solutions (8.54) and (8.28) are calculated in an inertial reference system in Galilean coordinates. The effective Riemannian metric that arises is due to the presence of the gravitational field, while the forces of inertia are totally excluded. It is quite obvious that these solutions retain their functional form in the Galilean coordinates of any inertial reference system. Since all physical quantities are independent of transformations of the time variable, then if the following transformation is applied:

$$x'^0 = x^0 + \eta^0(x),\ x'^i = x^i\ , \tag{8.55}$$

the metric coefficients will change as follows:

$$g'_{00} = g_{00} - 2\partial_0\eta^0,\ g'_{0i} = g_{0i} - \partial_i\eta^0,\ g'_{ik} = g_{ik}\ . \tag{8.56}$$

It must be noted that transformation (8.55) does not take us beyond the inertial reference system, since such a transformation is nothing but another choice of clock. All physically measurable quantities are independent of this choice.

Assuming function η^0 to be

$$\eta^0 = -\frac{1}{2}\partial_0 \int \rho|x - x'|d^3x', \tag{8.57}$$

and taking into account the identity

$$\begin{aligned} \partial_i\eta^0 &= \frac{1}{2}(\gamma_{ik}V^k - N_i), \\ N_i &= \int \frac{\rho v^k(x_k - x'_k)(x_i - x'_i)}{|x - x'|^3} d^3x' \end{aligned} \tag{8.58}$$

upon substitution into (8.56) of expressions (8.28) for g_{0i} and g_{ik} and, also, of expression (8.54) for g_{00}, and taking into account(8.57) and (8.58), we find the metric coefficients of effective Riemannian space in the so-called "canonical form":

$$\begin{aligned} g_{00} &= 1 - 2U + 2U^2 - 4\Phi_1 - 4\Phi_2 - 2\Phi_3 - 6\Phi_4, \\ g_{0i} &= \frac{7}{2}\gamma_{ik}V^k + \frac{1}{2}N_i, \\ g_{ik} &= \gamma_{ik}(1 + 2U)\ . \end{aligned} \tag{8.59}$$

These expressions coincide precisely with the formulae that are obtained on the basis of GRT. The difference only consists in that here they follow exactly from RTG, while for deriving them from GRT equations it is necessary to apply additional assumptions, that do not follow from theory, i.e. it is necessary to go beyond the limits of GRT. But we shall specially deal with this issue.

In the case of a static spherically symmetric body the post-Newtonian approximation at a distance from the body assumes, in accordance with (8.59), the form

$$\begin{aligned} g_{00} &= 1 - \frac{2MG}{r} + 2\left(\frac{MG}{r}\right)^2, \quad g_{0i} = 0, \\ g_{ik} &= \gamma_{ik}\left(1 + \frac{2MG}{r}\right), \quad M = \int \rho(x) d^3x. \end{aligned} \tag{8.59a}$$

On the basis of expressions (8.59) the post-Newtonian Nordtwedt–Will parameters in RTG assume the following values:

$$\gamma = 1,\ \beta = 1,\ \alpha_1 = \alpha_2 = \alpha_3 = \xi_1 = \xi_2 = \xi_3 = \xi_4 = \xi_W = 0.$$

We have calculated the metric coefficients (8.59) in RTG in an inertial reference system. We shall now present the expressions for the components of the energy-momentum tensor of matter in the next approximation, as compared with (8.15). Taking into account expression (8.34) for u^0 and, also, that

$$u^i = \frac{dx^i}{ds} = v^i(1 + U - \frac{1}{2}v_k v^k)\ , \tag{8.60}$$

we find from formula (8.10)

$$\overset{(3)}{T}{}^{0i} = \rho v^i(2U + \Pi - v_k v^k) + pv^i, \tag{8.61}$$

$$\overset{(2)}{T}{}^{ik} = \rho v^i v^k - p\gamma^{ik}. \tag{8.62}$$

The component $\overset{(2)}{T}{}^{00}$ is determined by expression (8.35). On the basis of expression (8.59), making use of the equations for the geodesic line, it is possible to calculate all effects in the Solar system. When gravitational effects in the Solar system are calculated in GRT on the basis of the post-Newtonian approximation, the results obtained are correct, and no ambiguity is present in the description of the effects. At the same time, if the exact solutions of GRT are applied, ambiguity arises in the description of the effects.

In conclusion we shall deal in somewhat greater detail with the comparison of RTG and GRT in analyzing effects occurring in a weak gravitational field. The set of equations (8.1) and (8.2) together with the equation of state determines all the physical quantities of one or another gravitational problem. All the calculations performed above in the post-Newtonian approximation were made in an inertial reference system. In GRT there in principle exists no inertial reference system. In this connection A. Einstein wrote: *"The starting point of theory consists in the assertion that there exists no state of motion physically singled out, i.e. not only velocity, but acceleration, also, have no absolute meaning"*[28]. But if no inertial reference system exists, to which reference system must one consider calculations performed within GRT to pertain?

In calculating gravitational effects V. A. Fock made use of the harmonicity conditions in Cartesian coordinates. He called them coordinate conditions. Thus, in a work published in 1939 [24] he wrote: *"In solving Einstein's equations we took advantage of a reference system, which we have termed harmonic, but which merits being called inertial."* Further in the same article he noted: *"It seems to us that the possibility of introducing in general relativity theory a definite inertial reference system in an unambiguous manner is noteworthy."* And, finally, in Ref. [25] he wrote: *"The relativity principle expressed by the Lorentz transformations is possible in inho-*

[28]Einstein A. Collection of scientific works, Moscow: Nauka, 1966, vol.2, art.92, p.264.

mogeneous space, also, while a general relativity principle is not possible."

All these statements of V. A. Fock were due to his aspiration to clarify the essence of GRT, freeing it of general relativity devoid of any physical meaning. However, V. A. Fock, here, actually went beyond the limits of GRT. Precisely owing to this fact he arrived at the striking conclusion on the validity of the relativity principle in inhomogeneous space, also. If one remains within Riemannian space, and no other space exists in GRT, then this assertion contradicts the correct conclusion made by V. A. Fock "*that in general relativity theory there, generally speaking, exists no relativity.*" [25]. But to realize his goal it is necessary to introduce the concept of a gravitational field in Minkowski space. Where did V. A. Fock go beyond GRT? In applying the conditions of harmonicity he actually considered Cartesian coordinates:

$$\frac{\partial \tilde{g}^{\mu\nu}}{\partial x^\mu} = 0\,, \tag{8.63}$$

where x^μ are Cartesian coordinates. In Cartesian coordinates $\gamma(x) = \det \gamma_{\mu\nu} = -1$. Therefore, in accordance with the tensor law of transformations we have

$$\tilde{g}^{\mu\nu}(x) = \frac{\partial x^\mu}{\partial y^\alpha} \cdot \frac{\partial x^\nu}{\partial y^\beta} \cdot \frac{\tilde{g}^{\alpha\beta}(y)}{\sqrt{-\gamma(y)}}\,. \tag{8.64}$$

We shall write equations (8.63) in the form

$$\partial_\mu \tilde{g}^{\mu\nu}(x) = \frac{\partial y^\tau}{\partial x^\mu} \cdot \frac{\partial \tilde{g}^{\mu\nu}(x)}{\partial y^\tau}. \tag{8.65}$$

For further calculations we present the formulae

$$\frac{\partial}{\partial y^\tau}\left(\frac{1}{\sqrt{-\gamma(y)}}\right) = -\frac{1}{\sqrt{-\gamma}}\gamma^\lambda_{\tau\lambda},\;\; \gamma^\nu_{\alpha\beta} = \frac{\partial^2 x^\sigma}{\partial y^\alpha \partial y^\beta} \cdot \frac{\partial y^\nu}{\partial x^\sigma}\,. \tag{8.66}$$

Upon substitution of (8.64) into (8.65) and taking into account (8.66) we obtain

$$\partial_\mu \tilde{g}^{\mu\nu}(x) = \frac{1}{\sqrt{-\gamma}} \frac{\partial x^\nu}{\partial y^\sigma} \cdot \frac{\partial \tilde{g}^{\alpha\sigma}(y)}{\partial y^\alpha} + \\ + \frac{1}{\sqrt{-\gamma}} \tilde{g}^{\alpha\beta}(y) \frac{\partial^2 x^\nu}{\partial y^\alpha \partial y^\beta} = 0 \; . \qquad (8.67)$$

We shall write the multiplier of the second term in the form

$$\frac{\partial^2 x^\nu}{\partial y^\alpha \partial y^\beta} = \frac{\partial x^\nu}{\partial y^\sigma} \cdot \frac{\partial y^\sigma}{\partial x^\tau} \cdot \frac{\partial^2 x^\tau}{\partial y^\alpha \partial y^\beta} = \frac{\partial x^\nu}{\partial y^\sigma} \cdot \gamma^\sigma_{\alpha\beta} \; .$$

Substituting this expression into the preceding one we find

$$\partial_\mu \tilde{g}^{\mu\nu}(x) = \frac{1}{\sqrt{-\gamma}} \cdot \frac{\partial x^\nu}{\partial y^\sigma} \left(\frac{\partial \tilde{g}^{\alpha\sigma}(y)}{\partial y^\alpha} + \gamma^\sigma_{\alpha\beta}(y) \tilde{g}^{\alpha\beta}(y) \right) = 0 \; ,$$

i.e. we have

$$\partial_\mu \tilde{g}^{\mu\nu}(x) = \frac{1}{\sqrt{-\gamma}} \cdot \frac{\partial x^\nu}{\partial y^\sigma} D_\mu \tilde{g}^{\mu\sigma}(y) = 0 \; . \qquad (8.68)$$

Thus, we have established that the density of the tensor $\tilde{g}^{\mu\sigma}(y)$ in arbitrary coordinates automatically satisfies the general covariant equation

$$D_\lambda \tilde{g}^{\lambda\sigma} = 0 \; ,$$

if the initial condition of harmonicity (8.63) is written in Cartesian coordinates. But this means that the harmonicity condition is not a coordinate condition, but a field equation in Minkowski space. Thus, application of the condition of harmonicity in Cartesian coordinates is not an innocent operation, but it implies going beyond the framework of GRT by introduction of Minkowski space.

The obtained equation coincides with equation (5.20) of RTG. In RTG it follows from the least action principle. Performing transformation from coordinates y to coordinates z we obtain (see Appendix (E.12))

$$\Box\, y^\lambda = -\gamma^\lambda_{\alpha\beta}(y) g^{\alpha\beta}(y),$$

where $\Box$ denotes the operator

$$\Box = \frac{1}{\sqrt{-g(z)}} \cdot \frac{\partial}{\partial z^\nu} \left(\tilde{g}^{\nu\sigma} \frac{\partial}{\partial z^\sigma} \right).$$

Therefore, when V. A. Fock wrote down the harmonicity conditions in the form

$$\Box\, y^\lambda = 0,$$

he actually dealt with Cartesian coordinates, for which $\gamma^\lambda_{\alpha\beta}(y) = 0$, i.e. with Minkowski space in Galilean coordinates. In choosing harmonic coordinates in the form of conditions (8.63) V. A. Fock actually made use of Minkowski space in Galilean coordinates, while equations (8.63) played the part of field equations, instead of coordinate conditions. But why it was necessary to add to the Hilbert–Einstein equations precisely equations (8.63) in Galilean coordinates, instead of some others, in order to obtain the complete set of gravity equations within V.A.Fock's approach, remained unclear. Here, V.A.Fock was most likely guided by physical intuition, and also by the mathematical simplification that arose in the course of calculations.

Did V. A. Fock attempt to consider the gravitational field in Minkowski space? No, he was far from this idea, and in this connection he wrote [24]: *"We recall it here only in connection with the sometimes observed tendency (certainly not shared by us) to pack the theory of gravity into the framework of Euclidean space."* As we have seen, application of the conditions of harmonicity in Cartesian coordinates makes us go beyond the framework of GRT. But this means that the set of equations of gravity, studied by V. A. Fock, differs from the set of equations of GRT, i.e. V. A. Fock's theory of gravity based on the conditions of harmonicity in Cartesian coordinates and Einstein's GRT are different theories. V. A. Fock's approach turns out to be closer to the ideas of RTG. Everything that V. A. Fock attempted to introduce in the theory of gravity (inertial reference systems, acceleration relative to space) is fully inherent in RTG, but this is achieved by consideration of the gravitational field, like all other physical fields, in Minkowski space. Here, all the geometric characteristics of Riemannian space are now field quantities in Minkowski space.

In analyzing gravitational effects in the Solar system V.A. Fock actually made use of Minkowski space, since he referred all the calculated gravitational effects to an inertial reference system. Precisely this circumstance permitted him to obtain correct expressions for the effects. Thus, for example, he wrote [25]: *"How should one define a straight line: as a light ray or as a straight line in that Euclidean space in which the harmonic coordinates x_1, x_2, x_3 serve as Cartesian coordinates? We think the second definition to be the only correct one. We actually made use of it in saying that a light ray has the shape of a hyperbola in the vicinity of the Sun"*, and further on, *"the argument that a straight line, like a ray of light, is more directly observable, but it has no sense: in the definitions it is not the direct observability that is decisive, but the correspondence to Nature, even though this correspondence may be established by indirect reasoning."*

In RTG gravitational effects are determined unambiguously, because in accordance with equations (8.1) and (8.2) written in the Galilean coordinates of an inertial reference system, the motion of light or of a test body, when the gravitational field is switched off, indeed proceeds along a straight line, which is a geodesic line in Minkowski space. It is absolutely clear that in a non-inertial reference system the geodesic line in Minkowski space will no longer be a straight line. But this means that in RTG, in an non-inertial reference system, for revealing a gravitational effect motion in effective Riemannian space must be compared precisely with the geodesic motion of the accelerated reference system.

In calculating gravity effects in the Solar system, when the influence of the graviton mass can be neglected, only the RTG set of equations (8.1) and (8.2) in Galilean coordinates coincides with the set of equations dealt with by V. A. Fock in harmonic (Cartesian) coordinates. If one remains within the framework of GRT, then in any other, for instance non-inertial, reference system they differ essentially. This takes place because the covariance of V. A. Fock's set of equations is not general, unlike the set of RTG equations. V. A. Fock obtained the complete set of gravitational equations (for island systems) by adding the harmonicity conditions to the Hilbert-Einstein equations. But why must precisely the harmonicity conditions be added, instead of some other conditions, remained unclear. In accordance with RTG, the complete set of gravitational equations (8.1) and (8.2) arises from the least action principle. Hence it becomes clear, why conditions (8.2), which in Cartesian coordinates coincide with the harmonicity conditions, arise, instead of some other conditions. But these equations become universal, valid not only for island systems. But if V. A. Fock had realized that in applying the harmonicity conditions he actually had to deal with Cartesian coordinates of Minkowski space, he would have readily obtained expression (8.68). As we already noted earlier, the harmonicity conditions in Cartesian coordinates successfully applied by V. A. Fock took him beyond the framework of Einstein's GRT. This fact was noted in 1957 by L. Infeld who wrote: *"Thus, for Fock the choice of the harmonicity coordinate condition becomes a certain fundamental law of Nature, which alters the very character of Einstein's general relativity theory and transforms it into a theory of the gravitational field, valid only in inertial reference systems"*[29].

If one remains within the framework of GRT, then it is absolutely incomprehensible, from the point of view of physics, why it is necessary to choose the harmonicity conditions, instead of any other conditions. While in RTG, owing to the existence of the graviton mass, these conditions arise as a consequence of the validity of the equations for matter [see (5.7)

[29] L.Infeld. Most recent problems in gravity. M.:Foreign literature publishers, 1961, p.162.

and (5.17)], i.e. they follow from the least action principle, and they therefore have universal significance. However, in GRT similar expressions for the post-Newtonian approximation are, nevertheless, obtained without application of the harmonicity conditions in Cartesian coordinates. Why is this so? The reason consists in that Minkowski space in Galilean coordinates is once again introduced and that the gravitational field is actually considered as a physical field in this space.

The metric of Minkowski space in Galilean coordinates is taken as the zero order approximation for the Riemannian metric. It is complemented with various potentials with arbitrary post-Newtonian parameters, each of which decreases like $0(\frac{1}{r})$. In this way the arbitrariness contained in GRT is discarded. Substitution of the Riemannian metric $g_{\mu\nu}$ in this form into the Hilbert-Einstein equation permits one to determine the values of the post-Newtonian parameters, and we again arrive at the same post-Newtonian approximation. Precisely here gravity is considered to be a physical field in Minkowski space, the behaviour of which is described by the introduced gravitational potentials. Such a requirement imposed on the character of the metric of Riemannian space does not follow from GRT, since in the general case the asymptotics of the metric is quite arbitrary and even depends on the choice of the three-dimensional space coordinates. Therefore it is impossible to impose physical conditions on the metric. But if it is effective and its arising is due to the physical field, then the physical conditions are imposed on the metric in a natural manner.

In RTG the gravitational equations (5.19) and (5.20) are generally covariant, but not form-invariant with respect to arbitrary transformations. They are form-invariant relative to the Lorentz transformations. But this means that in Lorentz coordinates, in case the solution $G(x)$ exists for the tensor of matter $T_{\mu\nu}(x)$, there exists, in the new Lorentz coordinates x', the solution $G'(x')$ for the tensor of matter $T'_{\mu\nu}(x')$, and, consequently, in the coordinates x the solution $G'(x)$ is possible only for the tensor of matter $T'_{\mu\nu}(x)$.

In RTG a unique correspondence is established between the Riemannian metric and the Minkowski metric, which permits one to compare motion under the influence of the gravitational field and in its absence, when calculation is performed of the gravitational effect. When the gravitational field is switched off in RTG the Riemann tensor turns to zero, and at the same time transition occurs from Riemannian metric to the Minkowski metric, previously chosen in formulating the physical problem. This is precisely what provides for the equivalence principle to be satisfied in RTG.

For calculation of the gravitational effect it is necessary to compare motion in Riemannian space with motion in absence of the gravitational field. This is precisely how the gravitational effect is determined. If in GRT one refers the set of solutions for $g_{\mu\nu}$ to a certain inertial reference system, then it is quite obvious that one will obtain a whole set of various values for the gravitational effect. Which one of them should be chosen? Since the Hilbert-Einstein equations do not contain the metric of Minkowski space, it is impossible to satisfy the equivalence principle, because it is impossible to determine, in which (inertial or non-inertial) reference system one happens to be, when the gravitational field is switched off.

To conclude this section we note that the post-Newtonian approximation (8.59) satisfies the causality principle (6.11).

9. On the equality of inert and gravitational masses

Owing to the density of the energy-momentum tensor being the source of the gravitational field, the inert and gravitational masses were shown in section 8 to be equal. In this section we shall show that the field approach to gravity permits obtaining in a trivial manner the metric of effective Riemannian space in the first approximation in the gravitational constant G. This is especially simple to establish on the basis of equations (2.2). In the case of a spherically symmetric static body, equations (2.2) have the following form in the Galilean coordinates of an inertial reference system:

$$\Delta\tilde{\Phi}^{00} - m^2\tilde{\Phi}^{00} = -16\pi t^{00}, \tag{9.1}$$

$$\Delta\tilde{\Phi}^{0i} - m^2\tilde{\Phi}^{0i} = 0, \quad \Delta\tilde{\Phi}^{ik} - m^2\tilde{\Phi}^{ik} = 0, \quad i,k = 1,2,3. \tag{9.2}$$

For a static body the sole component t^{00} differs from zero.

From equations (9.2) we have

$$\tilde{\Phi}^{0i} = 0, \quad \tilde{\Phi}^{ik} = 0\ . \tag{9.3}$$

Far away from the body, from equation (9.1) we find

$$\tilde{\Phi}^{00} \simeq \frac{4M}{r}e^{-mr}, \quad M = \int t^{00} d^3x\ , \tag{9.4}$$

M is the inert mass of the body, that creates the gravitational field. In the Solar system the exponential factor can be neglected, owing to the quantity mr being small.

$$\tilde{\Phi}^{00} \simeq \frac{4M}{r}\ . \tag{9.5}$$

We shall now find the components of the density of the metric tensor of effective Riemannian space, $\tilde{g}^{\mu\nu}$. On the basis of (2.6) we have

$$\tilde{g}^{\mu\nu} = \tilde{\gamma}^{\mu\nu} + \tilde{\Phi}^{\mu\nu}, \quad \tilde{g}^{\mu\nu} = \sqrt{-g}g^{\mu\nu}. \tag{9.6}$$

Hence, taking into account (9.3) and (9.5), we obtain the following $\tilde{g}^{\mu\nu}$ components, that differ from zero:

$$\tilde{g}^{00} = 1 + \frac{4M}{r}, \quad \tilde{g}^{11} = \tilde{g}^{22} = \tilde{g}^{33} = -1. \tag{9.7}$$

They satisfy equation (2.3) exactly. On the basis of (9.7) we find

$$g_{00} = \frac{\sqrt{-g}}{1 + \frac{4M}{r}}, \quad g_{11} = g_{22} = g_{33} = -\sqrt{-g}. \tag{9.8}$$

$$-g = -\tilde{g}^{00}\tilde{g}^{11}\tilde{g}^{22}\tilde{g}^{33} = \left(1 + \frac{4M}{r}\right). \tag{9.9}$$

Substituting the expressions for g into formulae (9.8) we obtain

$$g_{00} \simeq \left(1 - \frac{2M}{r}\right), \quad g_{11} = g_{22} = g_{33} = -\left(1 + \frac{2M}{r}\right). \tag{9.10}$$

It must be especially underlined that at the place, where in accordance with Newton's law of gravity there should be an active gravitational mass, there appears the inert mass M. Thus, the equality of the inert and active gravitational masses is a direct consequence of the density of the energy-momentum tensor being the source of the gravitational field. So the reason that the inert and gravitational masses are equal is not the local identity of the forces of inertia and of gravity (this actually does not occur in GRT), but the universality of the conserved source of the gravitational field, of the energy-momentum tensor of matter.

The interval in effective Riemannian space has the form

$$ds^2 = \left(1 - \frac{2M}{r}\right) dt^2 - \left(1 + \frac{2M}{r}\right)(dx^2 + dy^2 + dr^2). \tag{9.11}$$

Classical effects of gravity, such as the gravitational red shift of spectral lines, the deviation of a light ray by the Sun, the time delay of a radiosignal, the precession of a gyroscope on the Earth's orbit, are fully described by this interval.

From expression (9.10) it is evident that the forces of gravity are attractive, since the quantity M, being an inert mass, is always positive. As to GRT, in accordance with this theory it is not possible to prove the equality of inert and active gravitational masses. A detailed analysis of this issue is presented in joint works performed with prof. V. I. Denisov. This circumstance is dealt with in detail in the monograph [10]. The essence of the issue consists in that the expression for inert mass, determined from the pseudotensor of the gravitational field, depends on the choice of the three-dimensional coordinates, which is physically inadmissible. Precisely by a simple choice of three-dimensional space coordinates (which is always permitted) one can show that in GRT inert mass is not equal to active gravitational mass. Since the equality of physically measurable quantities in GRT depends on the choice of the three-dimensional coordinates, this means that not everything in it is alright here, also. Sometimes the opinion is voiced that within the framework of GRT it is possible to construct the energy-momentum tensor of the gravitational field by substitution of covariant derivatives in Minkowski space for the ordinary derivatives in the expression for the pseudotensor. However, here, on the one hand, it is impossible to say with definiteness which metric in Minkowski space must be taken for such a substitution, and, on the other hand, in Riemannian space no global Cartesian coordinates exist, and, consequently, no Minkowski space, so such an approach does not remove the essential difficulty of GRT: the absence of integral conservation laws of energy-momentum and of angular momentum for matter and gravitational field taken together.

10. Evolution of the homogeneous and isotropic Universe

We write the equations of RTG in the form

$$R_{\mu\nu} - \frac{m^2}{2}(g_{\mu\nu} - \gamma_{\mu\nu}) = \frac{8\pi}{\sqrt{-g}}\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right) , \qquad (10.1)$$

$$D_\mu \tilde{g}^{\mu\nu} = 0 . \qquad (10.2)$$

For convenience we have chosen the set of units $G = \hbar = c = 1$. In the final expressions we shall restore the dependence upon these constants. The density of the energy-momentum tensor has the form

$$T_{\mu\nu} = \sqrt{-g}[(\rho + p)U_\mu U_\nu - g_{\mu\nu}p], \;\; U^\nu = \frac{dx^\nu}{ds} . \qquad (10.3)$$

Here ρ is the density of matter, p is pressure, ds is the interval in effective Riemannian space. For a homogeneous and isotropic model of the Universe the interval of effective Riemannian space ds has the general form

$$ds^2 = U(t)dt^2 - V(t)\left[\frac{dr^2}{1 - kr^2} + r^2(d\Theta^2 + \sin^2\Theta d\Phi^2)\right]. \qquad (10.4)$$

Here k assumes the values $1, -1, 0$; $k = 1$ corresponds to the closed Universe, $k = -1$ — to the hyperbolic Universe, and $k = 0$ — to the "flat" Universe.

Since the set of RTG equations (10.1) and (10.2) together with the equation of state is complete, then in the case of appropriate initial conditions it can yield only a single solution describing the development of a homogeneous and isotropic model of the Universe. At the same time the equations of GRT for the same model yield three well-known scenarios for the development of the Universe. The scenario of the development of the Universe obtained on the basis of the RTG does not coincide with any of the scenarios based on GRT. We shall follow [31].

All our analysis will be made in an inertial reference system in spherical coordinates r, Θ, Φ. An interval in Minkowski space, in this case, will have the form

$$d\sigma^2 = dt^2 - dx^2 - dy^2 - dz^2 = \\ = dt^2 - dr^2 - r^2(d\Theta^2 + \sin^2\Theta d\Phi^2)\ . \qquad (10.5)$$

The determinant g, composed of components $g_{\mu\nu}$, equals

$$g = -UV^3(1 - kr^2)^{-1} r^4 \sin^2\Theta. \qquad (10.6)$$

The tensor density

$$\tilde{g}^{\mu\nu} = \sqrt{-g} g^{\mu\nu} \qquad (10.7)$$

has, in accordance with (10.4), the following components:

$$\begin{aligned} \tilde{g}^{00} &= \sqrt{\frac{r^4 V^3}{U(1 - kr^2)}} \cdot \sin\Theta, \\ \tilde{g}^{11} &= -r^2\sqrt{UV(1 - kr^2)} \cdot \sin\Theta\ , \\ \tilde{g}^{22} &= -\sqrt{\frac{UV}{1 - kr^2}} \cdot \sin\Theta, \\ \tilde{g}^{33} &= -\sqrt{\frac{UV}{1 - kr^2}} \cdot \frac{1}{\sin\Theta}. \end{aligned} \qquad (10.8)$$

The Christoffel symbols of Minkowski space are

$$\begin{aligned} &\gamma^1_{22} = -r,\ \ \gamma^1_{33} = -r\sin^2\Theta,\ \ \gamma^2_{12} = \gamma^3_{13} = \frac{1}{r}\ , \\ &\gamma^2_{33} = -\sin\Theta\cos\Theta,\ \ \gamma^3_{23} = \cot\Theta\ . \end{aligned} \qquad (10.9)$$

Equation (10.2) has the form

$$D_\mu \tilde{g}^{\mu\nu} = \partial_\mu \tilde{g}^{\mu\nu} + \gamma^\nu_{\alpha\beta}\tilde{g}^{\alpha\beta} = 0\ . \qquad (10.10)$$

Substituting (10.9) into (10.10) we obtain

$$\frac{\partial}{\partial t}\left(\frac{V^3}{U}\right) = 0 \ , \tag{10.11}$$

$$\frac{\partial}{\partial r}(r^2\sqrt{1-kr^2}) = 2r(1-kr^2)^{-1/2}. \tag{10.12}$$

From equation (10.11) it follows

$$V = aU^{1/3},$$

here a is an integration constant.

From equation (10.12) we directly find

$$k = 0 \ , \tag{10.13}$$

i.e. the space metric is Euclidean. It must be stressed that this conclusion for a homogeneous and isotropic Universe follows directly from equation (10.2) for the gravitational field and does not depend on the density of matter. Thus, equation (10.2) excludes the closed and hyperbolic models of the Universe. A homogeneous and isotropic Universe can only be "flat" according to RTG. In other words, the well-known problem of the Universe's flatness does not exist within the framework of RTG. With account of (10.11) and (10.13) the effective Riemannian metric (10.4) assumes the form

$$ds^2 = U(t)dt^2 - aU^{1/3}[dr^2 + r^2(d\Theta^2 + \sin^2\Theta d\Phi^2)] \ . \tag{10.14}$$

If one passes to the proper time $d\tau$

$$d\tau = \sqrt{U}dt \tag{10.15}$$

and introduces the notation

$$R^2 = U^{1/3}(t) \ , \tag{10.16}$$

the interval (10.14) assumes the form

$$ds^2 = d\tau^2 - aR^2(\tau)[dx^2 + dy^2 + dz^2] \ . \tag{10.17}$$

Here and further in this section R is a scaling factor. We are compelled to make use of the notation adopted in the literature for this quantity, in spite of the fact that in the previous sections R stood for the scalar curvature. For the given metric the Christoffel symbols assume the form

$$\begin{gathered}\Gamma^1_{22} = -r,\ \Gamma^1_{33} = -r\sin^2\Theta,\ \Gamma^2_{12} = \Gamma^3_{13} = \frac{1}{r},\\ \Gamma^2_{33} = -\sin\Theta\cos\Theta,\ \Gamma^3_{23} = \cot\Theta\ ,\end{gathered} \tag{10.18}$$

$$\Gamma^0_{ii} = aR\frac{dR}{d\tau},\ \ \Gamma^i_{0i} = \frac{1}{R}\frac{dR}{d\tau},\ \ i = 1, 2, 3\ . \tag{10.19}$$

Making use of expression (4.13) we have

$$\begin{gathered}R_{00} = -\frac{3}{R}\frac{d^2R}{d\tau^2},\ \ R_{11} = 2a\left(\frac{dR}{d\tau}\right)^2 + aR\frac{d^2R}{d\tau^2}\ ,\\ R_{22} = r^2R_{11},\ \ R_{33} = \sin^2\Theta\cdot R_{22},\ \ R_{0i} = 0\ ,\end{gathered} \tag{10.20}$$

$$R_{\mu\nu}g^{\mu\nu} = -\frac{6}{R}\cdot\frac{d^2R}{d\tau^2} - \frac{6}{R^2}\cdot\left(\frac{dR}{d\tau}\right)^2. \tag{10.21}$$

Since $g_{0i} = 0,\ R_{0i} = 0$, then from equation (10.1) it follows directly that

$$T_{0i} = 0\ . \tag{10.22}$$

Hence on the basis of (10.3) we have

$$U_i = 0\ . \tag{10.23}$$

This means that matter is at rest in an inertial reference system. Thus, the so-called "expansion" of the Universe, observed by the red shift, is due to the change of the gravitational field in time. Therefore, there exists no expansion of the Universe, related to the motion of objects with respect to each other. The red shift is not due to the motion of galaxies, which is absent, but to the variation of the gravitational field in time. Therefore, the red shift does not indicate that the galaxies were at a time close to each other. At the same time, in accordance with GRT *"All versions of the Friedman model have in common that at a certain moment of time in the past (ten–twenty thousand million years ago) the distance between adjacent galaxies should have been equal to zero"*[30]. We, here, have quoted S. Hawking, and below we shall establish the reason for the difference between the conclusions concerning the development of the Universe in RTG and GRT. We shall now deal in a somewhat greater detail with the nature of the red shift.

From (10.17) it follows that the speed of a light ray equals

$$\frac{dr}{d\tau} = \frac{1}{\sqrt{a}R(\tau)}.$$

Let us put the observation point at the origin of the reference system ($r = 0$). Consider a light signal emitted from point r during the time interval between τ and $\tau + d\tau$, and let its arrival at the point $r = 0$ take place during the time interval between τ_0 and $\tau_0 + d\tau_0$; then, for light emitted at moment τ and arriving at the point $r = 0$ at the moment τ_0 we have

$$\int\limits_{\tau}^{\tau_0} \frac{d\tau}{R(\tau)} = \sqrt{a}r,$$

similarly, for light emitted at a moment $\tau + d\tau$ and arriving at the point $r = 0$ at the moment $\tau_0 + d\tau_0$ we find

$$\int\limits_{\tau+d\tau}^{\tau_0+d\tau_0} \frac{d\tau}{R(\tau)} = \sqrt{a}r.$$

Equating these expressions we obtain

$$\frac{d\tau}{R(\tau)} = \frac{d\tau_0}{R(\tau_0)}.$$

[30]S. Hawking. From the Big Bang to Black Holes. M.:Mir, 1990, p.46.

Or, passing to the light frequency, we have

$$\omega = \frac{R(\tau_0)}{R(\tau)}\omega_0.$$

Hence, it is obvious that the light frequency ω at the point of emitting is not equal to the frequency of the light ω_0 at the point of observation.

Introducing the red shift parameter z

$$z = \frac{\omega - \omega_0}{\omega_0},$$

we have

$$z = \frac{R(\tau_0)}{R(\tau)} - 1.$$

We see that the red shift is only related to variation of the scaling factor $R(\tau)$, in the case of such variation there exists no motion of matter, in accordance with (10.23). Thus, the nature of the red shift is not related to the scattering of galaxies, which is absent, but to variation of the gravitational field with time, i.e. it is related to the fact that $R(\tau_0) > R(\tau)$.

It must be especially stressed that a given inertial reference system is singled out by Nature itself, i.e. in the considered theory the Mach principle is satisfied automatically.

Substituting (10.20) and (10.3) into equation (10.1), with account of (10.23), we have

$$\frac{1}{R}\frac{d^2R}{d\tau^2} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) - 2\omega\left(1 - \frac{1}{R^6}\right), \quad (10.24)$$

$$\left(\frac{1}{R}\frac{dR}{d\tau}\right)^2 = \frac{8\pi G}{3}\rho - \frac{\omega}{R^6}\left(1 - \frac{3R^4}{a} + 2R^6\right). \quad (10.25)$$

where

$$\omega = \frac{1}{12}\left(\frac{mc^2}{\hbar}\right)^2. \quad (10.26)$$

From (10.24) it is seen that for small values of the scaling factor R there arises an initial acceleration owing to the second term. This is precisely what "incites" the "expansion" of the Universe. The initial acceleration appears at the moment when the density of matter stops growing in the preceding cycle. From (10.25) it follows that in the region $R >> 1$ the contemporary density of matter in the Universe equals

$$\rho(\tau) = \rho_c(\tau) + \frac{1}{16\pi G}\left(\frac{mc^2}{\hbar}\right)^2, \qquad (10.27)$$

where $\rho_c(\tau)$ is the critical density determined by the Hubble "constant"

$$\rho_c = \frac{3H^2(\tau)}{8\pi G}, \quad H(\tau) = \frac{1}{R}\cdot\frac{dR}{d\tau}\,. \qquad (10.28)$$

Hence, the necessity for the existence of "dark" matter, which is in accordance with modern observational data.

From equations (10.24) and (10.25) one can obtain the expression for the deceleration parameter of the Universe, $q(\tau)$:

$$q = -\frac{\ddot{R}}{R}\frac{1}{H^2} = \frac{1}{2} + \frac{1}{4}\left(\frac{c}{H}\right)^2\left(\frac{mc}{\hbar}\right)^2. \qquad (10.29)$$

Thus, the parameter q at present is positive, i.e. "expansion" of the Universe has slowed down, instead of being accelerated. The relation (10.29) makes it possible, in principle, to determine the mass of the graviton from two observable quantities, H and q. From the causality principle (6.10), (6.11) it follows that

$$R^2(R^4 - a) \le 0\,. \qquad (10.30)$$

To satisfy the causality condition throughout the entire region of variation of $R(\tau)$ it is natural to set

$$a = R^4_{\max}. \qquad (10.31)$$

From the condition that the left-hand side of equation (10.25) is not negative it follows that the expansion should start from some minimum value $R_{\min}$, corresponding to the value $\frac{dR}{d\tau} = 0$.

On the other hand, if $R >> 1$, the expansion should stop at $R_{\max}$, when the density (10.27) reaches its minimum value

$$\rho_{\min} = \frac{1}{16\pi G}\left(\frac{mc^2}{\hbar}\right)^2\left(1 - \frac{1}{R^6_{\max}}\right). \tag{10.32}$$

and the process starts of compression down to $R_{\min}$.

Thus, in RTG there exists no cosmological singularity, and the presence of the graviton mass results in the evolution of the Universe exhibiting a cyclical character. The time required for the Universe to expand from the maximum to its minimum density is mainly determined by the stage at which nonrelativistic matter is dominant and is

$$\tau_{\max} \simeq \sqrt{\frac{2}{3}}\frac{\pi\hbar}{mc^2}\ . \tag{10.33}$$

From the covariant conservation law, that is a consequence of equations (5.19), (5.20)

$$\nabla_\mu \tilde{T}^{\mu\nu} + \Gamma^\nu_{\alpha\beta}\tilde{T}^{\alpha\beta} = 0$$

it is possible to obtain the equation

$$\frac{1}{R}\frac{dR}{d\tau} = -\frac{1}{3(\rho + \frac{p}{c^2})}\frac{d\rho}{d\tau}\ . \tag{10.34}$$

For the stage of development of the Universe dominated by radiation

$$p = \frac{1}{3}\rho c^2$$

from equation (10.34) we obtain the following expression for the radiation density ρ_r:

$$\rho_r(\tau) = \frac{A}{R^4(\tau)}\ . \tag{10.35}$$

Here A is an integration constant. At the stage of development of the Universe, when nonrelativistic matter is dominant and pressure can be neglected, from equation (10.34) we find

$$\rho_m(\tau) = \frac{B}{R^3(\tau)}\ , \tag{10.36}$$

B is an integration constant.

Consider that at a certain moment of time τ_0 the radiation density $\rho_r(\tau_0)$ becomes equal to the density of matter, $\rho_m(\tau_0)$

$$\rho_r(\tau_0) = \rho_m(\tau_0)\ , \tag{10.37}$$

then

$$A = BR(\tau_0) = BR_0.$$

Since at later stages of the development of the Universe matter is dominant, we have from formula (10.36) the following:

$$B = \rho_{\min} \cdot R^3_{\max}. \tag{10.38}$$

Thus,

$$\rho \simeq \rho_r = \frac{\rho_{\min} R_0 \cdot R^3_{\max}}{R^4}, \quad R \leq R_0, \tag{10.39}$$

$$\rho \simeq \rho_m = \rho_{\min} \left(\frac{R_{\max}}{R}\right)^3, \quad R \geq R_0. \tag{10.40}$$

According to observational data (see, for example, [33]), the present-day density of radiation (including the three sorts of neutrinos, which we for definiteness consider massless) and the critical density of matter, are

$$\rho_r(\tau_c) = 8 \cdot 10^{-34} \mathrm{g/cm}^3, \quad \rho_m(\tau_c) = 10^{-29} \mathrm{g/cm}^3. \tag{10.41}$$

The "hidden" mass must be attributed to the density of matter, $\rho_m(\tau_c)$, for our choice of the graviton mass ($m = 10^{-66}$g) it is close to the critical density ρ_c determined by the Hubble "constant". We intend matter to actually be all forms of matter, with the exception of the gravitational field.

In accordance with formulae (10.39), (10.40) and (10.41) we have

$$\rho_r(\tau_c) = \frac{\rho_{\min} R_0 \cdot R^3_{\max}}{R^4(\tau_c)} = 8 \cdot 10^{-34} \mathrm{g/cm}^3, \tag{10.42}$$

$$\rho_m(\tau_c) = \rho_{\min}\left(\frac{R_{\max}}{R(\tau_c)}\right)^3 = 10^{-29}\text{g/cm}^3. \tag{10.43}$$

Hence, we find

$$R_0 = \frac{\rho_r(\tau_c)}{\rho_m^{4/3}(\tau_c)} R_{\max}\cdot\rho_{\min}^{1/3} = 3,7\cdot 10^5 \rho_{\min}^{1/3}\cdot R_{\max}. \tag{10.44}$$

Let us introduce the notation

$$\sigma = \frac{4}{3} R_0 \cdot R_{\max}^3. \tag{10.45}$$

In accordance with (10.39), (10.44) and (10.45) we obtain

$$\rho(\tau) \simeq \rho_r(\tau) = \frac{3}{4}\cdot\frac{\sigma\cdot\rho_{\min}}{R^4(\tau)}, \quad R \leq R_0. \tag{10.46}$$

Assuming radiation to be dominant at the initial stage of expansion in the hot Universe model, from equation (10.25) by taking into account (10.31), (10.32) and (10.46), we obtain the following:

$$H^2 = \left(\frac{1}{R}\frac{dR}{d\tau}\right)^2 = \omega\left[\frac{3\sigma}{2R^4} - 2 + \frac{3}{R_{\max}^4\cdot R^2} - \frac{1}{R^6}\right]. \tag{10.47}$$

Equation (10.47) makes it possible to determine the law of expansion of the Universe at the initial stage. It is readily seen that the right-hand side of equation (10.47) turns to zero at sufficiently small values of $R = R_{\min}$. The main role, here, is due to the first term in brackets in equation (10.25), that is responsible for the graviton mass.

By introducing the variable $x = R^{-2}$ one can readily find approximate values for the roots of the equation

$$\frac{3}{2}\sigma x^2 - 2 + \frac{3}{R_{\max}^4} x - x^3 = 0\,, \tag{10.48}$$

which are the turning points

$$x_1 = \frac{3}{2}\sigma + 0\left(\frac{1}{\sigma^2}\right),\quad x_{2,3} = \pm\sqrt{\frac{4}{3}\frac{1}{\sqrt{\sigma}}} + 0\left(\frac{1}{\sigma^2}\right). \tag{10.49}$$

Hence, we find the turning point

$$R_{\min} = \sqrt{\frac{2}{3\sigma}} \ . \tag{10.50}$$

Thus, owing to the graviton mass, there exists no cosmological singularity in RTG, and expansion of the Universe starts from the finite non-zero value $R = R_{\min}$. On the basis of (10.46) we obtain

$$\rho_{\max} = \frac{3}{4} \cdot \frac{\sigma \rho_{\min}}{R^4_{\min}} = \frac{27}{16}\sigma^3 \cdot \rho_{\min}. \tag{10.51}$$

In accordance with (10.49) expression (10.47) can be written in the form

$$H^2 = \omega(x_1 - x)(x - x_2)(x - x_3) \ . \tag{10.52}$$

Within the range of variation of the scaling factor

$$R_{\min} \leq R \leq R_0 \tag{10.53}$$

the expression for H^2 is significantly simplified:

$$H^2 \simeq \omega x^2(x_1 - x) = \frac{3\sigma\omega}{2R^6}(R^2 - R^2_{\min}) \ . \tag{10.54}$$

Within this approximation equation (10.47) assumes the form

$$\frac{1}{R^2}\left(\frac{dR}{d\tau}\right)^2 = \frac{3\sigma\omega}{2R^6}(R^2 - R_{\min}) \ . \tag{10.55}$$

Upon integration we find

$$\tau = \frac{R^2_{\min}}{\sqrt{6\sigma\omega}}[Z\sqrt{Z^2 - 1} + \ln(Z + \sqrt{Z^2 - 1})] \ , \tag{10.56}$$

where

$$Z = R/R_{\min}.$$

Utilizing expressions (10.50) and (10.51) we obtain

$$\frac{R_{\min}^2}{\sqrt{6\sigma\omega}} = \frac{1}{2\sqrt{2\omega}}\left(\frac{\rho_{\min}}{\rho_{\max}}\right)^{1/2}. \tag{10.57}$$

Substituting into this expression the value $\rho_{\min}$ from (10.32) we find

$$\frac{R_{\min}^2}{\sqrt{6\sigma\omega}} = \sqrt{\frac{3}{32\pi G\rho_{\max}}}. \tag{10.58}$$

Taking into account (10.58) in (10.56) we obtain

$$\tau = \sqrt{\frac{3}{32\pi G\rho_{\max}}}[Z\sqrt{Z^2-1} + \ln(Z+\sqrt{Z^2-1})]. \tag{10.59}$$

In the vicinity of $R \simeq R_{\min}$ from (10.59) we find

$$R(\tau) = R_{\min}\left[1 + \frac{4\pi G}{3}\rho_{\max}\cdot\tau^2\right]. \tag{10.60}$$

In the region $R_{\min} << R < R_0$ we obtain

$$R(\tau) = R_{\min}\left(\frac{32\pi G}{3}\rho_{\max}\right)^{1/4}\cdot\tau^{1/2}. \tag{10.61}$$

In this region the dependence on time of the density of matter determined by equation (10.46), with account of (10.50), (10.51) and (10.61), has the form

$$\rho(\tau) = \frac{3}{32\pi G\tau^2}, \tag{10.62}$$

i.e. coincides with the known equation that yields the Friedman model in GRT for a "flat" Universe. We shall now determine the time corresponding to transition from the stage of expansion of the Universe dominated by radiation to the stage dominated by nonrelativistic matter. According to (10.61) we have

$$R_0^2 = R_{\min}^2\left(\frac{32\pi G}{3}\rho_{\max}\right)^{1/2}\cdot\tau_0. \tag{10.63}$$

Hence, taking into account (10.44), (10.45) and (10.58) we find

$$\tau_0 = \frac{\rho_r^{3/2}(\tau_c)}{\rho_m^2(\tau_c)}\sqrt{\frac{3}{32\pi G}} = $$
$$= 2,26 \cdot 10^8 \sqrt{\frac{3}{32\pi G}} \simeq 1,5 \cdot 10^{11}\text{s} \,. \tag{10.64}$$

Now consider development of the Universe, when pressure can be neglected. At this stage of evolution we write equation (10.25) as

$$\left(\frac{dx}{d\tau}\right)^2 = \frac{2\omega x^2}{\alpha}(x-1)[(\alpha - x^3)(x^2+x+1) - 3x^2] \,. \tag{10.65}$$

Here $x = R_{\max}/R$, $\alpha = 2R_{\max}^6$. Taking into account that

$$\alpha >> 3 \,, \tag{10.66}$$

we find

$$\tau = \tau_0 + \sqrt{\frac{\alpha}{2\omega}} \int\limits_x^{x_0} \frac{dy}{y\sqrt{(y^3-1)(x_1^3-y^3)}} \,. \tag{10.67}$$

Here $x_0 = R_{\max}/R_0$, $x_1 = 2^{1/3} \cdot R_{\max}^2$. Upon integration of (10.67) we obtain

$$\tau = \tau_0 + \frac{1}{3}\sqrt{\frac{\alpha}{2x_1\omega}}[\arcsin f(x_0) - \arcsin f(x)]. \tag{10.68}$$

Here

$$f(x) = \frac{(x_1^3+1)x^3 - 2x_1^3}{x^3(x_1^3-1)} \,. \tag{10.69}$$

Note that

$$f(x_0) \simeq 1 - \frac{2}{x_0^3} \,, \tag{10.70}$$

$$\arcsin f(x_0) \simeq \arccos \frac{2}{x_0^{3/2}} = \frac{\pi}{2} - \frac{2}{x_0^{3/2}} \,. \tag{10.71}$$

Taking into account (10.71) we find

$$\tau = \tau_0 - \frac{2}{3\sqrt{2\omega}x_0^{3/2}} + \frac{1}{3\sqrt{2\omega}}\left[\frac{\pi}{2} - \arcsin f(x)\right] . \qquad (10.72)$$

Taking into account the equality

$$\tau_0 = \frac{1}{2\sqrt{2\omega}x_0^{3/2}} , \qquad (10.73)$$

expression (10.72) can be written in the form

$$3\sqrt{2\omega}(\tau + \beta\tau_0) = \frac{\pi}{2} - \arcsin f(x) . \qquad (10.74)$$

Hence, we have

$$\cos\lambda(\tau + \beta\tau_0) = \frac{(\alpha+1)x^3 - 2\alpha}{x^3(\alpha-1)} . \qquad (10.75)$$

Here

$$\lambda = 3\sqrt{2\omega} = \sqrt{\frac{3}{2}}\left(\frac{mc^2}{\hbar}\right), \quad \beta = 1/3 . \qquad (10.76)$$

From expression (10.75) we find

$$R(\tau) = \left[\frac{\alpha}{2}\right]^{1/6} \cdot \left[\frac{(\alpha+1) - (\alpha-1)\cos\lambda(\tau+\beta\tau_0)}{2\alpha}\right]^{1/3} . \qquad (10.77)$$

Owing to the equality (10.40) the following relation occurs:

$$\frac{\rho_m(\tau)}{\rho_{\min}} = \left[\frac{R_{\max}}{R(\tau)}\right]^3 , \qquad (10.78)$$

taking into account (10.78) we obtain

$$\rho_m(\tau) = \frac{2\alpha\rho_{\min}}{(\alpha+1) - (\alpha-1)\cos\lambda(\tau+\beta\tau_0)} . \qquad (10.79)$$

Since $\alpha >> 1$, from (10.79) we have

$$\rho_m(\tau) = \frac{\rho_{\min}}{\sin^2 \frac{\lambda(\tau+\beta\tau_0)}{2}} \,, \tag{10.80}$$

in a similar way from formula (10.78) we have

$$R(\tau) = R_{\max} \sin^{2/3} \frac{\lambda(\tau+\beta\tau_0)}{2}. \tag{10.81}$$

In the region of values $\frac{\lambda(\tau+\beta\tau_0)}{2} \ll 1$ we have

$$\rho_m(\tau) = \frac{1}{6\pi G(\tau+\beta\tau_0)^2} \,, \tag{10.82}$$

$$R(\tau) = R_{\max} \left[\frac{\lambda(\tau+\beta\tau_0)}{2}\right]^{2/3} . \tag{10.83}$$

For $\tau >> \beta\tau_0$ formulae (10.82) and (10.83) yield for $\rho_m(\tau)$ and $R(\tau)$ time dependencies similar to those obtained within the Friedman model in GRT for a "flat" Universe.

Making use of formulae (10.44), (10.45) and (10.51) one can readily establish the following relation:

$$R_{\max} = \frac{\rho_m^{1/3}(\tau_c)}{\rho_r^{1/4}(\tau_c)} \left(\frac{\rho_{\max}}{4\rho_{\min}^2}\right)^{1/12} \simeq 3,6 \cdot 10^{-2} \left(\frac{\rho_{\max}}{\rho_{\min}^2}\right)^{1/12} . \tag{10.84}$$

In a similar manner with the aid of expression (10.57) and of (10.50) one can express, via $\rho_{\max}$, the second turning point $R_{\min}$:

$$R_{\min} = \left(\frac{\rho_{\min}}{2\rho_{\max}}\right)^{1/6} . \tag{10.85}$$

From (10.84) and (10.85) it is clear that the existence of the graviton mass not only removes the cosmological singularity, but also stops the expansion process of the Universe, which undergoes transition to the compression phase. Thus, the evolution of a homogeneous and isotropic Universe is determined by modern observational data (10.41), by the maximum density of matter and the graviton mass. The scalar curvature is the largest at the beginning of the "expansion", and on the basis of (10.21), (10.24) and (10.85) equals the following value:

$$R_{\mu\nu}g^{\mu\nu} = -16\pi G \cdot \frac{\rho_{\max}}{c^2},$$

while its minimum value at the end of "expansion" is

$$R_{\mu\nu}g^{\mu\nu} = \frac{3}{2}\left(\frac{mc}{\hbar}\right)^2 .$$

The initial acceleration, which serves as the initial "push" that led to the "expanding" Universe, is, in accordance with (10.24), (10.32) and (10.85), the following:

$$\frac{d^2R}{d\tau^2} = \frac{1}{3}\left(8\pi G\rho_{\max}\right)^{5/6}\left(\frac{mc^2}{2\hbar}\right)^{1/3} .$$

It arises at the moment when the density of matter stops growing during the preceding cycle. The maximum density of matter in the Universe in this model remains undefined. It is related to the integral of motion. The latter is readily established. We write equation (10.24) in the form

$$\frac{d^2R}{d\tau^2} = -4\pi G\left(\rho + \frac{p}{c^2}\right)R + \frac{8\pi G}{3}\rho R - 2\omega\left(R - \frac{1}{R^5}\right) . \quad (10.86)$$

Determining from equation

$$\frac{1}{R}\frac{dR}{d\tau} = -\frac{1}{3(\rho + \frac{p}{c^2})}\frac{d\rho}{d\tau} \quad (10.87)$$

the value of $\left(\rho + \frac{p}{c^2}\right)$ we find

$$\rho + \frac{p}{c^2} = -\frac{1}{3}R\frac{d\rho}{dR} . \quad (10.88)$$

Substituting this value into equation (10.86) we obtain

$$\frac{d^2R}{d\tau^2} = \frac{4\pi G}{3} \cdot \frac{d}{dR}(\rho R^2) - \omega \frac{d}{dR}\left(R^2 + \frac{1}{2R^4}\right) . \qquad (10.89)$$

Introducing the notation

$$V = -\frac{4\pi G}{3}\rho R^2 + \omega\left(R^2 + \frac{1}{2R^4}\right) , \qquad (10.90)$$

one can write equation (10.89) in the form of the Newton equation of motion

$$\frac{d^2R}{d\tau^2} = -\frac{dV}{dR} , \qquad (10.91)$$

where V plays the role of the potential. Multiplying (10.91) by $\frac{dR}{d\tau}$ we obtain

$$\frac{d}{d\tau}\left[\frac{1}{2}\left(\frac{dR}{d\tau}\right)^2 + V\right] = 0 . \qquad (10.92)$$

Hence, we have

$$\frac{1}{2}\left(\frac{dR}{d\tau}\right)^2 + V = E , \qquad (10.93)$$

where E is an intergal of motion, the analog of energy in classical mechanics. Comparing (10.93) with (10.25) and taking into account (10.31) we obtain

$$R^4_{\max} = \frac{1}{8E}\left(\frac{mc^2}{\hbar}\right)^2 . \qquad (10.94)$$

Substituting into (10.94) expression (10.84) we find

$$E = 7,4 \cdot 10^4 \left[\frac{\left(\frac{mc^2}{\hbar}\right)^{10}}{(16\pi G)^2 \rho_{\max}}\right]^{1/3} . \qquad (10.95)$$

This quantity is extremely small.

Thus, $\rho_{\max}$ is actually an integral of motion, determined by the initial conditions of the dynamic system. The analysis performed reveals that the model of a homogeneous and isotropic Universe develops, in accordance with RTG, cyclically starting from a certain finite maximum density $\rho_{\max}$ down to the minimum density, and so on. The Universe can only be "flat". Theory predicts the existence in the Universe of a large "hidden" mass of matter. The Universe is infinite and exists for an indefinite time, during which an intense exchange of information took place between its regions, which resulted in the Universe being homogeneous and isotropic, with a certain inhomogeneity structure. In the model of a homogeneous and isotropic Universe this inhomogeneity is not taken into account, for simplifying studies. The information obtained is considered a zeroth approximation, that usually serves as a background in considering the development of inhomogeneities caused by gravitational instability. "Expansion" in a homogeneous and isotropic Universe, as we are convinced, is due to variation of the gravitational field, and no motion of matter occurs, here. The existence of a certain inhomogeneity structure in the distribution of matter in space introduces a significant change, especially in the period after the recombination of hydrogen, when the Universe becomes transparent and the pressure of radiation no longer hinders the collection of matter in various parts of the Universe.

This circumstance results in the motion of matter relative to the inertial reference system. Thus arise the peculiar velocities of galaxies with respect to the inertial reference system. A reference system related to the relic radiation can, with a great precision, be considered inertial. Naturally, a reference system related to the relic gravitational radiation would to an extremely high degree be close to an inertial system. What was the maximum density of matter, $\rho_{\max}$, earlier in the Universe? An attractive possibility is reflected in the hypothesis that $\rho_{\max}$ is determined by the world constants. In this case, the Planck density is usually considered to be $\rho_{\max}$.

Here, however, there exists the problem of overproduction of monopoles arising in Grand Unification theories. To overcome this problem, one usually applies the "burning out" mechanism of monopoles during the inflational expansion process due to the Higgs bosons. Our model provides another, alternative, possibility. The quantity $\rho_{\max}$ may even be significantly smaller, than the Planck density. In this case the temperature of the early Universe may turn out to be insufficient for the production of monopoles, and the problem of their overproduction is removed in a trivial manner. This, naturally, does not exclude the possibility of inflational expansion of the Universe, if it turns out to be that at a certain stage of its development the equation of state is $p = -\rho$.

Thus, in accordance with RTG, no pointlike Big Bang occurred, and, consequently, no situation took place, when the distance between galaxies were extremely small. Instead of the explosion, at each point of the space there occurred a state of matter of high density and temperature, and it further developed till the present moment, as described above. The difference between the development of a homogeneous and isotropic Universe in RTG and GRT arose owing to the scaling factor $R(\tau)$ in RTG not turning into zero, while in GRT it becomes zero at a certain moment in the past.

In GRT with the cosmological term λ the homogeneous and isotropic model of the Universe is also possible in the absence of matter. The solution of the GRT equations for this case was found by de Sitter. This solution corresponds to curved four-dimensional space-time. This signifies the existence of gravity without matter. What is the source of this gravity? Usually, it is vacuum energy identified with the cosmological constant λ that is considered to be this source. In RTG, when matter is absent ($\rho = 0$), in accordance with equations (10.25) and (10.31), the right-hand side should turn to zero, which is possible only if $R_{\max} = 1$. Hence it follows that $R \equiv 1$, and, consequently, the geometry of space-time in the absence of matter will be pseudo-Euclidean. Thus, in accordance with

RTG, when matter is absent in the Universe, there also exists no gravitational field, and, consequently, vacuum possesses no energy, as it should be. According to RTG, the Universe cannot exist without matter.

In conclusion let us determine the horizon of particles and the horizon of events. For a light ray, in accordance with the interval (10.17), we have

$$\frac{dr}{d\tau} = \frac{1}{\sqrt{a}R(\tau)} \ . \tag{10.96}$$

The distance covered by the light by the moment τ is

$$d_r(\tau) = \sqrt{a}R(\tau) \int\limits_0^{r(\tau)} dr = R(\tau) \int\limits_0^{\tau} \frac{d\sigma}{R(\sigma)} \ . \tag{10.97}$$

If the gravitational field were absent, the distance covered by the light would be $c\tau$. As R we should have substituted expression (10.59) for the interval $(0, \tau_0)$, and expression (10.81) for the time interval (τ_0, τ). We shall estimate $d_r(\tau)$ approximately by the expression

$$R(\tau) = R_{\max} \sin^{2/3} \frac{\pi\tau}{2\tau_{\max}} \tag{10.98}$$

Throughout the entire integration interval

$$d_r(\tau) = \frac{2\tau_{\max}}{\pi} \left[\sin \frac{\pi\tau}{2\tau_{\max}}\right]^{2/3} \cdot \int\limits_0^{\sqrt{y}} \frac{dx}{x^{2/3}\sqrt{1-x^2}} =$$

$$= \frac{6\tau_{\max}}{\pi} \sqrt{y} F\left(\frac{1}{2}, \frac{1}{6}, \frac{7}{6}, y\right) \ . \tag{10.99}$$

Here $y = \sin^2 \frac{\pi\tau}{2\tau_{\max}}$, $F(a, b, c, y)$ is a hypergeometric function.

We shall give the values for some quantities determining the evolution of a homogeneous and isotropic Universe. We set the graviton mass to $m = 10^{-66}$g, while the present-day Hubble "constant" is

$$H_c \simeq 74 \frac{\text{km}}{\text{s Mpc}} \ . \tag{10.100}$$

Then for the present-day moment of time τ_c, q_c will be equal to

$$\tau_c \simeq 3 \cdot 10^{17}\text{s}, \quad q_c = 0,59, \quad \rho_c = 10^{-28}\frac{\text{g}}{\text{cm}^3} \ . \qquad (10.101)$$

According to formula (10.33) the half-period of cyclical development is

$$\tau_{\max} = 9\pi \cdot 10^{17}\text{s} \ . \qquad (10.102)$$

It must be stressed that the parameters τ_c, q_c determining the evolution of the Universe are practically independent of the maximum density of matter $\rho_{\max}$. The maximum temperature (and, hence the maximum density), that could occur in the Universe, may be determined by such phenomena, that took place in these extreme conditions, and the consequences of which may be observed today. A special role, here, is played by the gravitational field, which contains the most complete information on the extreme conditions in the Universe. In the model, considered above, of an isotropic and homogeneous Universe the known problems of singularities, of causality, of flatness, that are present in GRT, do not arise.

Making use of (10.99) and (10.101) we find the size of the observable part of the Universe at the moment τ_c:

$$d_r(\tau_c) \simeq 3c\tau_c = 2,7 \cdot 10^{28}\text{cm} \ .$$

We see that the path covered by light in the gravitational field of the Universe during the time τ_c is three times larger than the corresponding distance in absence of the gravitational field, $c\tau_c$. During the half-period of evolution, $\tau_{\max}$, the horizon of particles be

$$d_r(\tau_{\max}) = \frac{c\tau_{\max}}{\sqrt{\pi}} \cdot \frac{\Gamma(1/6)}{\Gamma(2/3)} \ . \qquad (10.103)$$

The horizon of events is determined by the expression

$$d_c = R(\tau)\int\limits_{\tau}^{\infty}\frac{d\sigma}{R(\sigma)} \ . \qquad (10.104)$$

Since the integral (10.104) turns to infinity, the horizon of events in our case does not exist. This means that information on events taking place in any region of the Universe at the moment of time τ will reach us. This information can be obtained with the aid of gravitational waves, since they are capable of passing through periods, when the density of matter was high.

We shall especially note that within the framework of RTG a homogeneous and isotropic Universe can exist only if the graviton mass differs from zero. Indeed, in accordance with (10.46) and (10.51), the constant A in expression (10.35) equals $A = \rho_{\max}^{1/3}\left(\frac{\rho_{\min}}{2}\right)^{2/3}$. Therefore, if $\rho_{\max}$ is fixed, then the constant A turns to zero, when $m = 0$.

11. The gravitational field of a spherically symmetric static body

The issue of what takes place in the vicinity of the Schwarzschild sphere, when the graviton has a rest mass, was first dealt with in relativistic theory of gravity in ref. [2], in which the following conclusion was made: in vacuum the metric coefficient of effective Riemannian space, g_{00}, on the Schwarzschild sphere differs from zero, while g_{11} has a pole. These changes, that in the theory are due to the graviton mass, result in the "rebounding" effect of incident particles and of light from the singularity on the Schwarzschild sphere, and consequently, in the absence of "black holes".

Further, in ref. [14] a detailed analysis of this problem in RTG was performed, which clarified a number of issues, but which at the same time revealed that the "rebounding" takes place near the Schwarzschild sphere. In the present work we follow the article [13], in which it was shown in a most simple and clear manner that at the point in vacuum, where the metric coefficient of effective Riemannian space g_{11} has a pole, the other metric coefficient g_{00} does not turn to zero. The resulting singularity cannot be removed by a choice of the reference system, so the solution inside a body cannot be made to match the external solution. In this case, if transition is performed to the reference system related to a falling test body, then it turns out to be that the test body will never reach the surface of the body, that is the source of the gravitational field. Precisely this circumstance leads to the conclusion that the radius of a body cannot be inferior to the Schwarzschild radius. All this issue will be dealt with in detail in this section.

We now write equations (5.19), (5.20) in the form

$$R^\mu_\nu - \frac{1}{2}\delta^\mu_\nu R + \frac{1}{2}\left(\frac{mc}{\hbar}\right)^2 \Big(\delta^\mu_\nu + g^{\mu\alpha}\gamma_{\alpha\nu} - \\ -\frac{1}{2}\delta^\mu_\nu g^{\alpha\beta}\gamma_{\alpha\beta}\Big) = \kappa T^\mu_\nu, \tag{11.1}$$

$$D_\mu \tilde{g}^{\mu\nu} = 0\ . \tag{11.2}$$

Here $\tilde{g}^{\mu\nu} = \sqrt{-g}g^{\mu\nu}$, $g = \det\, g_{\mu\nu}$, R^μ_ν is the Ricci tensor, $\kappa = \frac{8\pi G}{c^2}$, G is the gravitational constant, D_μ is the covariant derivative in Minkowski space, $\gamma_{\mu\nu}(x)$ is the metric tensor of Minkowski space in arbitrary curvilinear coordinates.

Let us now determine the gravitational field created by a spherically symmetric static source. The general form of an interval of effective Riemannian space for such a source has the form

$$ds^2 = g_{00}dt^2 + 2g_{01}dtdr + g_{11}dr^2 + g_{22}d\Theta^2 + g_{33}d\Phi^2, \tag{11.3}$$

We introduce the notation

$$\begin{aligned} &g_{00}(r) = U(r),\ g_{01}(r) = B(r), \\ &g_{11}(r) = -\left[V(r) - \frac{B^2(r)}{U(r)}\right]\ , \\ &g_{22}(r) = -W^2(r), \quad g_{33}(r,\Theta) = -W^2(r)\sin^2\Theta. \end{aligned} \tag{11.4}$$

The components of the contravariant metric tensor are

$$\begin{aligned} &g^{00}(r) = \frac{1}{U}\left(1 - \frac{B^2}{UV}\right),\ g^{01}(r) = -\frac{B}{UV},\ g^{11}(r) = -\frac{1}{V}, \\ &g^{22}(r) = -\frac{1}{W^2},\ g^{33}(r,\Theta) = -\frac{1}{W^2\sin^2\Theta}. \end{aligned} \tag{11.5}$$

The determinant of the metric tensor $g_{\mu\nu}$ is

$$g = \det g_{\mu\nu} = -UVW^4\sin^2\Theta\ . \tag{11.6}$$

For a solution to have physical meaning the following condition must be satisfied:

$$g < 0 \; . \tag{11.7}$$

In the case of spherical coordinates g can turn to zero only at the point $r = 0$. On the basis of (11.5) and (11.6) we find the density components of the metric tensor

$$\tilde{g}^{\mu\nu} = \sqrt{-g} g^{\mu\nu}. \tag{11.8}$$

One have the form

$$\begin{aligned}
&\tilde{g}^{00} = \frac{W^2}{\sqrt{UV}}\left(V - \frac{B^2}{U}\right)\sin\Theta,\ \tilde{g}^{01} = -\frac{BW^2}{\sqrt{UV}}\sin\Theta,\\
&\tilde{g}^{11} = -\sqrt{\frac{U}{V}}W^2\sin\Theta, \\
&\tilde{g}^{22} = -\sqrt{UV}\sin\Theta,\ \ \tilde{g}^{22} = -\sqrt{UV}\sin\Theta,\ \tilde{g}^{33} = -\frac{\sqrt{UV}}{\sin\Theta} \; .
\end{aligned} \tag{11.9}$$

We shall carry out all reasoning in an inertial reference system in spherical coordinates. An interval in Minkowski space has the form

$$d\sigma^2 = dt^2 - dr^2 - r^2(d\Theta^2 + \sin^2\Theta d\Phi^2) \; . \tag{11.10}$$

The Christoffel symbols in Minkowski space that differ from zero and that are determined by the formula

$$\gamma^\lambda_{\mu\nu} = \frac{1}{2}\gamma^{\lambda\sigma}(\partial_\mu\gamma_{\sigma\nu} + \partial_\nu\gamma_{\sigma\mu} - \partial_\sigma\gamma_{\mu\nu}) \tag{11.11}$$

are equal to

$$\begin{aligned}
&\gamma^1_{22} = -r\ \gamma^1_{33} = -r\sin^2\Theta,\ \gamma^2_{12} = \gamma^3_{13} = \frac{1}{r},\\
&\gamma^2_{33} = -\sin\Theta\cos\Theta,\ \gamma^3_{23} = \cot\Theta \; .
\end{aligned} \tag{11.12}$$

We write equation (11.2) in an expanded form,

$$D_\mu\tilde{g}^{\mu\nu} = \partial_\mu\tilde{g}^{\mu\nu} + \gamma^\nu_{\lambda\sigma}\tilde{g}^{\lambda\sigma} = 0 \; . \tag{11.13}$$

In Galilean coordinates of Minkowski space they have the form

$$\partial_\mu \tilde{g}^{\mu\nu} = 0\ . \tag{11.14}$$

In the case of a static gravitational field we have from (11.14)

$$\partial_i \tilde{g}^{i\nu} = 0,\ i = 1, 2, 3\ . \tag{11.15}$$

Applying the tensor transformation law, it is possible to express the components $\tilde{g}^{i0}$ in Cartesian coordinates in terms of the components in spherical coordinates

$$\tilde{g}^{i0} = -\frac{BW^2}{\sqrt{UV}} \cdot \frac{x^i}{r^3}\ , \sqrt{-g} = \sqrt{UV} W^2 r^{-2}. \tag{11.16}$$

Here x^i are spatial Cartesian coordinates. Assuming $\nu = 0$ in (11.15) and integrating over the spherical volume upon application of the Gauss-Ostrogradsky theorem, we obtain the following integral over the spherical surface:

$$\oint \tilde{g}^{i0} ds_i = -\frac{BW^2}{r^3\sqrt{UV}} \oint (\vec{x} d\vec{s}) = 0\ . \tag{11.17}$$

Taking into account the equality

$$\oint (\vec{x} d\vec{s}) = 4\pi r^3, \tag{11.18}$$

we obtain

$$\frac{BW^2}{\sqrt{UV}} = 0\ . \tag{11.19}$$

Since equation (11.14) holds valid both within matter and outside it, (11.19) should be valid for any value of r. But since, owing to (11.7), U, V and W cannot be equal to zero, then from (11.19) it follows that

$$B = 0\ . \tag{11.20}$$

The interval (11.3) of effective Riemannian space assumes the form

$$ds^2 = Udt^2 - Vdr^2 - W^2(d\Theta^2 + \sin^2\Theta d\Phi^2) \ . \tag{11.21}$$

From (11.20) it follows that no static solution exists of the Hilbert–Einstein equations in harmonic coordinates, that contains in the interval a term of the form

$$B(r)dtdr \ . \tag{11.22}$$

The energy-momentum tensor of matter has the form

$$T_\nu^\mu = \left(\rho + \frac{p}{c^2}\right) v^\mu v_\nu - \delta_\nu^\mu \cdot \frac{p}{c^2} \ . \tag{11.23}$$

In expression (11.23) ρ is the mass density of matter, p is the isotropic pressure, and

$$v^\mu = \frac{dx^\mu}{ds} \tag{11.24}$$

is the four-velocity satisfying the condition

$$g_{\mu\nu} v^\mu v^\nu = 1 \ . \tag{11.25}$$

From equations (11.1) and (11.2) follows

$$\nabla_\mu T_\nu^\mu = 0 \ , \tag{11.26}$$

where ∇_μ is the covariant derivative in effective Riemannian space with the metric tensor $g_{\mu\nu}$. In the case of a static body

$$v^i = 0, \ \ i = 1, 2, 3; \ \ v^0 = \frac{1}{\sqrt{U}} \ , \tag{11.27}$$

and therefore

$$\begin{aligned} &T_0^0 = \rho(r), \ \ T_1^1 = T_2^2 = T_3^3 = -\frac{p(r)}{c^2} \ , \\ &T_\nu^\mu = 0, \ \ \mu \neq \nu \ . \end{aligned} \tag{11.28}$$

For the interval (11.21) the Christoffel symbols, differing from zero, are

$$\Gamma^0_{01} = \frac{1}{2U}\frac{dU}{dr} \ , \ \Gamma^1_{00} = \frac{1}{2V}\frac{dU}{dr} \ , \ \Gamma^1_{11} = \frac{1}{2V}\frac{dV}{dr} \ ,$$
$$\Gamma^1_{22} = -\frac{W}{V}\frac{dW}{dr}, \Gamma^1_{33} = \sin^2\Theta \cdot \Gamma^1_{22} \ , \tag{11.29}$$
$$\Gamma^2_{12} = \Gamma^3_{13} = \frac{1}{W}\frac{dW}{dr} \ , \ \Gamma^2_{33} = -\sin\Theta\cos\Theta \ , \quad \Gamma^3_{23} = \cot\Theta.$$

Applying the following expression for the Ricci tensor:

$$R_{\mu\nu} = \partial_\sigma \Gamma^\sigma_{\mu\nu} - \partial_\nu \Gamma^\sigma_{\mu\sigma} +$$
$$+\Gamma^\sigma_{\mu\nu}\Gamma^\lambda_{\sigma\lambda} - \Gamma^\sigma_{\mu\lambda}\Gamma^\lambda_{\sigma\nu} \ , \ R^\mu_\nu = g^{\mu\lambda}R_{\lambda\nu} \tag{11.30}$$

and substituting into it expressions for the Christoffel symbols from (11.29), it is possible to reduce equations (11.1) for functions U, V and W to the form

$$\frac{1}{W^2} - \frac{1}{VW^2}\left(\frac{dW}{dr}\right)^2 - \frac{2}{VW}\frac{d^2W}{dr^2} - \frac{1}{W}\frac{dW}{dr}\frac{d}{dr}\left(\frac{1}{V}\right) +$$
$$+\frac{1}{2}\left(\frac{mc}{\hbar}\right)^2\left[1 + \frac{1}{2}\left(\frac{1}{U} - \frac{1}{V}\right) - \frac{r^2}{W^2}\right] = \kappa\rho \ , \tag{11.31}$$

$$\frac{1}{W^2} - \frac{1}{VW^2}\left(\frac{dW}{dr}\right)^2 - \frac{1}{UVW}\frac{dW}{dr}\ \frac{dU}{dr} +$$
$$+\frac{1}{2}\left(\frac{mc}{\hbar}\right)^2\left[1 - \frac{1}{2}\left(\frac{1}{U} - \frac{1}{V}\right) - \frac{r^2}{W^2}\right] = -\kappa\frac{p}{c^2} \ , \tag{11.32}$$

$$-\frac{1}{VW}W'' - \frac{1}{2UV}U'' + \frac{1}{2WV^2}W'V' + \frac{1}{4VU^2}(U')^2 +$$
$$+\frac{1}{4UV^2}U'V' - \frac{1}{2UVW}W'U' +$$
$$+\frac{1}{2}\left(\frac{mc}{\hbar}\right)^2\left[1 - \frac{1}{2}\left(\frac{1}{U} + \frac{1}{V}\right)\right] = -\kappa\frac{p}{c^2} \ . \tag{11.33}$$

Equation (11.13), with account of (11.12), (11.9) and (11.20), can be reduced to the form

$$\frac{d}{dr}\left(\sqrt{\frac{U}{V}}W^2\right) = 2r\sqrt{UV}\ . \tag{11.34}$$

We note that by virtue of the Bianchi identity and of equation (11.2) one of the equations (11.31) – (11.33) is a consequence of the other ones. We shall further take equations (11.31), (11.32) and (11.34) to be independent.

We write equation (11.26) in an expanded form

$$\nabla_\mu T^\mu_\nu \equiv \partial_\mu T^\mu_\nu + \Gamma^\mu_{\alpha\mu} T^\alpha_\nu - \Gamma^\alpha_{\mu\nu} T^\mu_\alpha = 0\ . \tag{11.35}$$

Making use of expressions (11.28) and (11.29) we obtain

$$\frac{1}{c^2}\cdot\frac{dp}{dr} = -\frac{\rho + \frac{p}{c^2}}{2U}\cdot\frac{dU}{dr}\ . \tag{11.36}$$

Taking into account the identity

$$\frac{1}{W^2\left(\frac{dW}{dr}\right)}\cdot\frac{d}{dr}\left[\frac{W}{V}\left(\frac{dW}{dr}\right)^2\right] = \frac{1}{VW^2}\left(\frac{dW}{dr}\right)^2 +$$
$$+\frac{2}{VW}\,\frac{d^2W}{dr^2} + \frac{1}{W}\frac{dW}{dr}\frac{d}{dr}\left(\frac{1}{V}\right)\ , \tag{11.37}$$

equation (11.31) can be written as

$$1 - \frac{d}{dW}\left[\frac{W}{V\left(\frac{dr}{dW}\right)^2}\right] + \frac{1}{2}\left(\frac{mc}{\hbar}\right)^2\Big[W^2 - r^2 +$$
$$+\frac{W^2}{2}\left(\frac{1}{U} - \frac{1}{V}\right)\Big] = \kappa W^2\rho\ . \tag{11.38}$$

In a similar manner we transform equation (11.32):

$$1 - \frac{W}{V\left(\frac{dr}{dW}\right)^2}\,\frac{d}{dW}\ln(UW) + \frac{1}{2}\left(\frac{mc}{\hbar}\right)^2\Big[W^2 - r^2 -$$
$$-\frac{1}{2}\left(\frac{1}{U} - \frac{1}{V}\right)\Big] = -\kappa\frac{W^2 p}{c^2}\ . \tag{11.39}$$

We write equations (11.34) and (11.36) in the form

$$\frac{d}{dW}\left(W^2\sqrt{\frac{U}{V}}\right) = 2r\sqrt{UV}\frac{dr}{dW}\ . \tag{11.40}$$

$$\frac{1}{c^2}\cdot\frac{dp}{dW} = -\left(\rho + \frac{p}{c^2}\right)\frac{1}{2U}\cdot\frac{dU}{dW}\ . \tag{11.41}$$

In equations (11.38) – (11.41) we pass to dimensionless variables. Let l be the Schwarzschild radius of the source, the mass of which equals M, then

$$l = \frac{2GM}{c^2}\ . \tag{11.42}$$

We introduce new variables x and z, equal to

$$W = lx,\quad r = lz. \tag{11.43}$$

Equations (11.38) – (11.41) assume the form

$$1 - \frac{d}{dx}\left(\frac{x}{V\left(\frac{dz}{dx}\right)^2}\right) + \epsilon\Big[x^2 - z^2 +$$

$$+\frac{1}{2}x^2\left(\frac{1}{U} - \frac{1}{V}\right)\Big] = \tilde{\kappa}x^2\rho(x), \tag{11.38a}$$

$$1 - \frac{x}{V\left(\frac{dz}{dx}\right)^2}\frac{d}{dx}\ln(xU) + \epsilon\Big[x^2 - z^2 -$$

$$-\frac{x^2}{2}\left(\frac{1}{U} - \frac{1}{V}\right)\Big] = -\tilde{\kappa}\frac{x^2 p(x)}{c^2}\ , \tag{11.39a}$$

$$\frac{d}{dx}\left(x^2\sqrt{\frac{U}{V}}\right) = 2z\frac{dz}{dx}\sqrt{UV}\ , \tag{11.40a}$$

$$\frac{1}{c^2}\frac{dp}{dx} = -\left(\rho + \frac{p}{c^2}\right)\frac{1}{2U}\frac{dU}{dx}\ . \tag{11.41a}$$

Here ϵ is a dimensionless constant equal to

$$\epsilon = \frac{1}{2}\left(\frac{2GMm}{\hbar c}\right)^2, \quad \tilde{\kappa} = \kappa l^2. \tag{11.44}$$

The sum of and the difference between equations (11.38a) and (11.39a) are

$$2 - \frac{d}{dx}\left[\frac{x}{V\left(\frac{dz}{dx}\right)^2}\right] - \frac{x}{V\left(\frac{dz}{dx}\right)^2}\frac{d}{dx}\ln(xU) + \\ + 2\epsilon(x^2 - z^2) = \tilde{\kappa}x^2\left(\rho - \frac{p}{c^2}\right) , \tag{11.45}$$

$$\frac{d}{dx}\left[\frac{x}{V\left(\frac{dz}{dx}\right)^2}\right] - \frac{x}{V\left(\frac{dz}{dx}\right)^2}\frac{d}{dx}\ln(xU) - \\ - \epsilon x^2\left(\frac{1}{U} - \frac{1}{V}\right) = -\tilde{\kappa}x^2\left(\rho + \frac{p}{c^2}\right) . \tag{11.46}$$

We introduce the new functions A and η:

$$U = \frac{1}{x\eta A}, \quad V = \frac{x}{A\left(\frac{dz}{dx}\right)^2} . \tag{11.47}$$

In these new variables equation (11.45) assumes the form

$$A\frac{d\ln\eta}{dx} + 2 + 2\epsilon(x^2 - z^2) = \tilde{\kappa}x^2\left(\rho - \frac{p}{c^2}\right) . \tag{11.48}$$

Equation (11.38a) is written in the form

$$\frac{dA}{dx} = 1 + \epsilon(x^2 - z^2) + \epsilon\frac{x^2}{2}\left(\frac{1}{U} - \frac{1}{V}\right) - \tilde{\kappa}\cdot x^2\rho(x) . \tag{11.49}$$

In accordance with the causality condition (see Addendum)

$$\gamma_{\mu\nu}U^\mu U^\nu = 0 , \tag{11.50}$$

$$g_{\mu\nu}U^{\mu}U^{\nu} \leq 0 \; , \tag{11.50a}$$

it is easy to establish the inequality

$$U \leq V \; . \tag{11.51}$$

For our problem it suffices to consider only the values of x and z from the interval

$$0 \leq x \ll \frac{1}{\sqrt{2\epsilon}}, \quad 0 \leq z \ll \frac{1}{\sqrt{2\epsilon}}. \tag{11.52}$$

These inequalities impose an upper limit on r, W:

$$r, W \ll \frac{\hbar}{mc} \; . \tag{11.53}$$

In the case of such a restriction equation (11.49) assumes the form

$$\frac{dA}{dx} = 1 + \epsilon \frac{x^2}{2}\left(\frac{1}{U} - \frac{1}{V}\right) - \tilde{\kappa} x^2 \rho(x) \; . \tag{11.54}$$

Outside matter we have

$$\frac{dA}{dx} = 1 + \epsilon \frac{x^2}{2}\left(\frac{1}{U} - \frac{1}{V}\right) \; . \tag{11.55}$$

By virtue of causality (11.51) the following inequality holds valid beyond matter

$$\frac{dA}{dx} \geq 1. \tag{11.56}$$

Integrating (11.54) over the interval $(0, x)$ we obtain

$$A(x) = x + \frac{\epsilon}{2}\int\limits_0^x x'^2 \left(\frac{1}{U} - \frac{1}{V}\right) dx' - \tilde{\kappa}\int\limits_0^x x'^2 \rho(x') dx'. \tag{11.57}$$

$A(0)$ in (11.57) is set equal to zero, since if it were different from zero, function $V(x)$ would turn to zero as x tends toward zero, which is inadmissible from a physical standpoint. On the basis of (11.56) function $A(x)$ beyond matter grows in monotonic way with x, and it therefore can only have the sole root

$$A(x_1) = 0, \quad x_1 > x_0. \tag{11.58}$$

On the basis of (11.57) we have

$$x_1 = 1 - \frac{\epsilon}{2} \int\limits_0^{x_1} x'^2 \left(\frac{1}{U} - \frac{1}{V}\right) dx'. \tag{11.59}$$

We have here taken into account that when l is chosen to be equal to (11.42)

$$\tilde{\kappa} \int\limits_0^{x_0} x'^2 \rho(x') dx' = 1.$$

The matter is concentrated inside the sphere $0 \leq x \leq x_0$. We shall further consider the case, when the radius of the body, x_0, is less than x_1. Precisely in this case in vacuum, i.e. outside the body, there will exist a singularity which cannot be removed by a choice of reference system.

Owing to the graviton mass the zero of function A is shifted inward the Schwarzschild sphere. Since as x tends toward x_1, $V(x)$ tends toward infinity, owing to $A(x)$ tending toward zero, there will exist such a vicinity about the point x_1

$$x_1(1 - \lambda_1) \leq x \leq x_1(1 + \lambda_2), \quad \lambda_1 > 0, \lambda_2 > 0 \;, \tag{11.60}$$

(λ_1 and λ_2 assume small fixed values), inside which the following inequality holds valid:

$$\frac{1}{U} \gg \frac{1}{V} \;. \tag{11.61}$$

In this approximation we obtain

$$A(x) = x - x_1 + \frac{\epsilon}{2} \int\limits_{x_1}^{x} dx' x'^2 \frac{1}{U} \;. \tag{11.62}$$

Substituting into this expression U in the form (11.47) we find

$$A(x) = x - x_1 + \frac{\epsilon}{2} \int\limits_{x_1}^{x} dx' x'^3 \eta(x') A(x'). \tag{11.63}$$

In the region of variation of x one can substitute x_1^3 for x^3 within the interval (11.60):

$$A(x) = x - x_1 + \frac{\epsilon}{2} x_1^3 \int\limits_{x_1}^{x} \eta(x') A(x') dx'. \tag{11.64}$$

Hence, we obtain

$$\frac{dA}{dx} = 1 + \frac{\epsilon}{2} x_1^3 \eta(x) A(x). \tag{11.65}$$

In the considered approximation (11.52) equation (11.48) assumes the form

$$A \frac{d \ln \eta}{dx} + 2 = 0 \ . \tag{11.66}$$

We now introduce a new function

$$f(x) = \frac{x_1^3}{2} \eta(x) A(x) \ . \tag{11.67}$$

Equation (11.65) assumes the form

$$\frac{dA}{dx} = 1 + \epsilon f(x) \ , \tag{11.68}$$

and equation (11.66) assumes the form

$$\frac{A}{f} \cdot \frac{df}{dx} - \frac{dA}{dx} = -2 \ . \tag{11.69}$$

From equations (11.68) and (11.69) we find

$$A(x) = -\frac{(1 - \epsilon f) f}{\left(\frac{df}{dx}\right)} \ . \tag{11.70}$$

From expression (11.67) we obtain

$$\eta(x) = -\frac{2\frac{df}{dx}}{x_1^3(1-\epsilon f)} \ . \tag{11.71}$$

Substituting (11.70) and (11.71) into (11.47) we find

$$U = \frac{x_1^3}{2xf}, \ \ V = -\frac{x\frac{df}{dx}}{f(1-\epsilon f)\left(\frac{dz}{dx}\right)^2} \ . \tag{11.72}$$

Making use of these expressions we can rewrite the determinant g as

$$g = \frac{x_1^3\frac{df}{dx}x^4}{2f^2\left(\frac{dz}{dx}\right)^2(1-\epsilon f)}\sin^2\Theta < 0 \ . \tag{11.73}$$

For condition (11.7) to be satisfied, it is necessary for expressions $\frac{df}{dx}$ and $(1-\epsilon f)$ to have opposite signs. Substituting (11.70) into (11.68) we obtain

$$\frac{d}{dx}\ln\left|\frac{df}{dx}\right| - \frac{d}{dx}\ln|f(1-\epsilon f)| = \frac{1+\epsilon f}{f(1-\epsilon f)}\cdot\frac{df}{dx} \ . \tag{11.74}$$

Hence, we find

$$\frac{d}{dx}\ln\left|\frac{(1-\epsilon f)\frac{df}{dx}}{f^2}\right| = 0 \ . \tag{11.75}$$

Thus,

$$\left|\frac{(1-\epsilon f)\frac{df}{dx}}{f^2}\right| = C_0 > 0 \ . \tag{11.76}$$

Taking into account that the quantities $(1-\epsilon f)$ and $\frac{df}{dx}$ must have opposite signs we find

$$\frac{df}{dx} = -\frac{C_0 f^2}{(1-\epsilon f)} \ . \tag{11.77}$$

Substituting this expression into (11.70) we find

$$A(x) = \frac{(1-\epsilon f)^2}{C_0 f}, \quad A(x_1) = 0 \quad \text{at} \quad f = \frac{1}{\epsilon}\,. \tag{11.78}$$

With account of (11.78), expression (11.47) for the function V assumes the form

$$V = \frac{C_0 x f}{(1-\epsilon f)^2 \left(\frac{dz}{dx}\right)^2}\,. \tag{11.79}$$

Integrating (11.77) and taking into account (11.78) we obtain

$$C_0 \cdot (x - x_1) = \frac{1}{f} + \epsilon \ln \epsilon |f| - \epsilon\,. \tag{11.80}$$

Relation (11.80) has been obtained for the domain of x values determined by equalities (11.60), however, it is also valid in the region where the graviton mass can be neglected.

In accordance with (11.60) the domain $C_0(x-x_1)$ is within the limits

$$-\,C_0 x_1 \lambda_1 \leq C_0(x - x_1) \leq C_0 x_1 \lambda_2, \tag{11.81}$$

when f is positive, it satisfies the inequalities

$$\tilde{C} \leq f \leq \frac{1}{\epsilon}\,. \tag{11.82}$$

Making use of (11.80), in accordance with (11.81), we have

$$\frac{1}{f} + \epsilon \ln \epsilon f - \epsilon \leq C_0 x_1 \lambda_2.$$

Hence, we can find $\tilde{C}$:

$$\frac{1}{\tilde{C}} + \epsilon \ln \epsilon \tilde{C} - \epsilon = C_0 x_1 \lambda_2. \tag{11.83}$$

From expression (11.83) we find the approximate value for $\tilde{C}$:

$$\tilde{C} = \frac{1}{C_0 x_1 \lambda_2} \ . \tag{11.84}$$

For negative values of f, the value $|f|$, determined from the following equation, corresponds to the point $x = x_1$:

$$-\frac{1}{|f|} + \epsilon \ln \epsilon |f| - \epsilon = 0 \ . \tag{11.85}$$

Hence, we find

$$|f| = \frac{a}{\epsilon}, \ \ \ln a = \frac{1+a}{a} \ . \tag{11.86}$$

In accordance with (11.81), the following inequality should be satisfied:

$$-C_0 x_1 \lambda_1 \leq -\frac{1}{|f|} + \epsilon \ln \epsilon |f| - \epsilon \ . \tag{11.87}$$

Hence it is possible to find the lower boundary for $|f| = D$:

$$-C_0 x_1 \lambda_1 = -\frac{1}{D} + \epsilon \ln \epsilon D - \epsilon \ . \tag{11.88}$$

From expression (11.88) we find the approximate value for D:

$$D = \frac{1}{C_0 x_1 \lambda_1} \ . \tag{11.89}$$

This means that the quantity $|f|$ satisfies the inequality

$$|f| \geq D = \frac{1}{C_0 x_1 \lambda_1} \ . \tag{11.89a}$$

Let us now establish the dependence of variable z upon x. Substituting (11.47) into (11.40a) and taking into account (11.48) we obtain

$$A\frac{d}{dx}\left(x\frac{dz}{dx}\right) = 2z - x\frac{dz}{dx}\Big[1 + \epsilon(x^2 - z^2) - \\ -\frac{1}{2}\tilde{\kappa}x^2\left(\rho - \frac{p}{c^2}\right)\Big] \ . \tag{11.90}$$

In the approximation (11.52), outside matter, equation (11.90) assumes the form

$$A\frac{d}{dx}\left(x\frac{dz}{dx}\right)+x\frac{dz}{dx}-2z=0\ . \qquad (11.91)$$

We have to find the regular solution $z(x)$ of equation (11.91). In equation (11.91) we pass from variable x to f. Applying relation (11.80)

$$x=\frac{1}{C_0 f}[C_0 x_1 f+1-\epsilon f+\epsilon f\ln\epsilon|f|]\ , \qquad (11.92)$$

and taking into account (11.65), (11.66) and (11.83), equation (11.91) can be represented in the form

$$\frac{d^2 z}{df^2}+\frac{C_0 x f+\epsilon f-1}{C_0 f^2 x}\cdot\frac{dz}{df}-\frac{2z}{C_0 f^3 x}=0\ . \qquad (11.93)$$

By direct substitution one can establish that the expression

$$z=\frac{x_1}{2}+\frac{1}{C_0 f}[1-\epsilon f+\epsilon f\ln\epsilon|f|] \qquad (11.94)$$

satisfies equation (11.93) with an accuracy up to the quantity

$$\epsilon\frac{(1-\epsilon f+\ln\epsilon|f|)}{C_0^2 x f^3}\ , \qquad (11.95)$$

that is extremely small in the vicinity of the point x_1. From expressions (11.92) and (11.94) we find

$$z=x-\frac{x_1}{2}\ . \qquad (11.96)$$

Taking this relation, as well as (11.79) and (11.72), into account we obtain

$$U=\frac{x_1^3}{2xf},\quad V=\frac{C_0 x f}{(1-\epsilon f)^2}. \qquad (11.97)$$

For negative values of f the causality condition (11.51) assumes the form

$$|f|^2(2x^2C_0 - \epsilon^2 x_1^3) - 2\epsilon x_1^3|f| - x_1^3 \leq 0. \tag{11.98}$$

Inequality (11.98) is not satisfied, since it does not comply with inequality (11.89a). Thus, the causality principle is violated in the domain of negative values of f. This means that in the region $x_1(1-\lambda_1) \leq x < x_1$ the solution has no physical sense. If $x_0 < x_1(1-\lambda_1)$, the situation arises, when the physical solution inside the body, $0 \leq x \leq x_0$, cannot be made to match the physical solution in the region $x > x_1$, since there exists an intermediate region $x_1(1-\lambda_1) \leq x < x_1$, within which the solution does not satisfy the causality principle. Hence it necessarily follows that $x_0 \geq x_1$. From a physical point of view, it is necessary to exclude the equality $x_0 = x_1$, since the solution inside the body should undergo smooth transition to the external solution. Consequently, the variable f only assumes positive values. For values from the region $x \geq x_1(1+\lambda_2)$ one may, in equations (11.38a) and (11.39a), drop the terms containing the small parameter ϵ. Thus, we arrive at Schwarzschild's external solution

$$z_s = (x-\omega)\left[1 + \frac{b}{2\omega}\ln\frac{x-2\omega}{x}\right], \tag{11.99}$$

$$V_s = \frac{x}{\left(\frac{dz}{dx}\right)^2(x-2\omega)}, U_s = \frac{x-2\omega}{x}. \tag{11.100}$$

Here, "ω" and "b" are certain constants that are determined from the condition that solution (11.96), (11.97) is made to match solution (11.99), (11.100). At point $x = x_1(1+\lambda_2)$ the function z from (11.96) is

$$z = x_1\left(\frac{1}{2} + \lambda_2\right), \tag{11.101}$$

At the same point z_s equals

$$z_s = [x_1(1+\lambda_2) - \omega]\left[1 + \frac{b}{2\omega}\ln\frac{x_1(1+\lambda_2) - 2\omega}{x_1(1+\lambda_2)}\right]. \quad (11.102)$$

From the condition that (11.101) and (11.102) match we find

$$\omega = \frac{x_1}{2}, \quad b = 0 \ . \quad (11.103)$$

At point $x = x_1(1+\lambda_2)$ the function U from (11.97) equals

$$U = \frac{x_1^3}{2x_1(1+\lambda_2)\tilde{C}} \ , \quad (11.104)$$

since $\tilde{C}$, in accordance with (11.84), is

$$\tilde{C} = \frac{1}{C_0 x_1 \lambda_2} \ . \quad (11.105)$$

Substituting (11.105) into (11.104) we obtain

$$U = \frac{C_0 x_1^3 \lambda_2}{2(1+\lambda_2)} \ , \quad (11.106)$$

At the same point, with account of (11.103), U_s is

$$U_s = \frac{\lambda_2}{1+\lambda_2} \ . \quad (11.107)$$

From the condition that (11.106) and (11.107) match we find

$$C_0 = \frac{2}{x_1^3} \ . \quad (11.108)$$

At point $x = x_1(1+\lambda_2)$ the function V from (11.97) equals

$$V = C_0 x_1 (1+\lambda_1)\tilde{C} \ . \quad (11.109)$$

Substituting into (11.109) the value $\tilde{C}$ from (11.105) we obtain

$$V = \frac{1+\lambda_2}{\lambda_2} \quad (11.110)$$

at the same point, with account of (11.99) and (11.103), V_s equals

$$V_s = \frac{1+\lambda_2}{\lambda_2}\ , \tag{11.111}$$

i.e. the solution for V matches the solution for V_s.

Thus, if the radius of a body exceeds the Schwarzschild radius, then the graviton mass can be neglected, and the interval of effective Riemannian space in an inertial reference system in spherical coordinates outside the body in the region (11.53) has the form:

$$ds^2 = \frac{r-GM}{r+GM}dt^2 - \frac{r+GM}{r-GM}dr^2 -$$
$$-(r+GM)^2[(d\Theta)^2 + \sin^2\Theta(d\varphi)^2].$$

This expression is determined unambiguously from the complete set of equations (11.1) and (11.2), and, here, there exists no arbitrariness. When the solution inside the body is made to match the solution outside the body it is also necessary, as first shown by R. Avakian, to take into account the logarithmic term (11.99) which arises when the solution of equations (11.2) is sought. However, since the radius of the Sun exceeds the Schwarzschild radius significantly, we can do not take it into account in calculations of gravitational effects in the Solar system.

Now consider (11.92) for values of ϵf close to unity:

$$f = \frac{1}{\epsilon\left(1+\frac{y}{\epsilon}\right)}, \quad \frac{y}{\epsilon} \ll 1. \tag{11.112}$$

Substituting this expression into (11.92) and expanding in $\frac{y}{\epsilon}$ we obtain

$$y^2 = 2\epsilon C_0(x-x_1). \tag{11.113}$$

Inequality (11.112) signifies, that the quantity $(x - x_1) = \delta \ll \epsilon$, i.e.

$$\frac{y}{\epsilon} = \sqrt{2C_0} \cdot \sqrt{\frac{x - x_1}{\epsilon}} \ll 1 \,. \tag{11.114}$$

Substituting (11.113) into (11.112), and then f into (11.97), we obtain for U and V the following expressions:

$$U = \frac{x_1^3[\epsilon + \sqrt{2\epsilon C_0(x - x_1)}]}{2x} \,,$$
$$V = \frac{x[\epsilon + \sqrt{2\epsilon C_0(x - x_1)}]}{2\epsilon(x - x_1)} \,. \tag{11.115}$$

Hence, within the domain of variable x satisfying inequality (11.114), we have

$$U = \frac{\epsilon x_1^3}{2x}, \quad V = \frac{x}{2(x - x_1)} \,. \tag{11.116}$$

We see that the presence of the graviton mass essentially alters the character of the solution in the region close to the gravitational radius. At the point, where the function V, in accordance with (11.116), has a pole, the function U differs from zero, while in general relativity theory it equals zero. It is precisely owing to this circumstance, that an irreversible gravitational collapse arises in GRT, during which there appear "black holes" (objects that have no material boundaries and that are "cut off" from the external world). In RTG "black holes" are impossible.

If one takes into account (11.42), (11.43), (11.96) and neglects the second term in (11.59), then expressions (11.116) for U and V assume the form:

$$U = \left(\frac{GMm}{\hbar c}\right)^2, \quad V = \frac{1}{2} \cdot \frac{r + \frac{GM}{c^2}}{r - \frac{GM}{c^2}} \,, \tag{11.117}$$

which coincides with the formulae of [2]. We note that the residue at the pole of function V at $\epsilon \neq 0$ equals $\frac{GM}{c^2}$, while at $\epsilon = 0$ it equals $\frac{2GM}{c^2}$. This is so, because, when $\epsilon = 0$, the pole of function V at point $x = x_1$ is due to function f, which at this point has a pole, while, if $\epsilon \neq 0$, it is due to function $(1 - \epsilon f)$, which, in accordance with (11.92), turns to zero at the point $x = x_1$.

We shall now compare the character of motion of test bodies in effective Riemannian space with the metric (11.117) and with the Schwarzschild metric. We write the interval (11.21) of Riemannian space in the form

$$ds^2 = Udt^2 - \tilde{V}dW^2 - W^2(d\Theta^2 + \sin^2\Theta d\Phi^2) \ . \qquad (11.118)$$

Here $\tilde{V}$ is

$$\tilde{V}(W) = V\left(\frac{dr}{dW}\right)^2 . \qquad (11.119)$$

The motion of a test body proceeds along a geodesic line of Riemannian space

$$\frac{dv^\mu}{ds} + \Gamma^\mu_{\alpha\beta}v^\alpha v^\beta = 0 \ , \qquad (11.120)$$

where

$$v^\mu = \frac{dx^\mu}{ds} \ , \qquad (11.121)$$

the velocity four-vector v^μ satisfies the condition

$$g_{\mu\nu}v^\mu v^\nu = 1 \ . \qquad (11.122)$$

Now consider radial motion, when

$$v^\Theta = v^\Phi = 0 \ . \qquad (11.123)$$

Taking into account (11.29), from equation (11.120) we find

$$\frac{dv^0}{ds} + \frac{1}{U}\cdot\frac{dU}{dW}v^0v^1 = 0 \ , \qquad (11.124)$$

where

$$v^1 = \frac{dW}{ds} \ . \tag{11.125}$$

From equation (11.124) we find

$$\frac{d}{dW} \ln(v^0 U) = 0 \ . \tag{11.126}$$

Hence, we have

$$v^0 = \frac{dx^0}{ds} = \frac{U_0}{U} \ , \tag{11.127}$$

where U_0 is the integration constant.

Taking into account (11.127), condition (11.122) for radial motion assumes the form

$$\frac{U_0^2}{U} - 1 = \tilde{V} \cdot \left(\frac{dW}{ds}\right)^2 \ . \tag{11.128}$$

If we assume the velocity of a falling test body to be zero at infinity, then we obtain $U_0 = 1$. From (11.128) we find

$$\frac{dW}{ds} = -\sqrt{\frac{1-U}{U\tilde{V}}} \ . \tag{11.129}$$

Taking into account (11.79), (11.96), (11.97) and (11.108) we have

$$U = \frac{x_1^3}{2xf}, \quad \tilde{V} = \frac{2xf}{x_1^3(1-\epsilon f)^2} \ .$$

Substituting these expressions into (11.129) we obtain

$$\frac{dW}{ds} = -\sqrt{1-U}(1-\epsilon f) \ . \tag{11.130}$$

Applying (11.108), (11.112) and (11.113), in the vicinity of the point x_1 we have

$$\frac{dW}{ds} = -\frac{2}{x_1}\sqrt{\frac{x-x_1}{\epsilon x_1}}. \tag{11.131}$$

Passing from the variable x to W, in accordance with (11.43) and taking into account (11.44), we obtain

$$\frac{dW}{ds} = -\frac{\hbar c^2}{mGM}\sqrt{\frac{W}{GM}\left(1 - \frac{2GM}{c^2 W}\right)}\,. \tag{11.132}$$

Hence there evidently arises a turning point. Differentiating (11.132) with respect to s we find

$$\frac{d^2W}{ds^2} = \frac{1}{2GM}\left(\frac{\hbar c^2}{mGM}\right)^2. \tag{11.133}$$

At the turning point, the acceleration (11.133) is very large, and it is positive, i.e. there occurs repulsion. Integrating (11.132), we obtain

$$W = \frac{2GM}{c^2} + \left(\frac{\hbar c^2}{2mGM}\right)^2 \cdot \frac{1}{GM}(s - s_0)^2. \tag{11.134}$$

Formulae (11.132) – (11.134) coincide with the formulae of ref. [2]. The presence of the Planck constant in formula (11.132) is due to the wave nature of matter, in our case, of gravitons exhibiting rest mass. From formula (11.134) it is evident that a test body cannot cross the Schwarzschild sphere. In GRT the situation is totally different. From the Schwarzschild solution and expression (11.129) it follows that a test body will cross the Schwarzschild sphere, and that a "black hole" will form. Test bodies or light can only cross the Schwarzschild sphere inwards, and then they can never leave the Schwarzschild sphere any more. The same result is obtained, if we pass to a synchronous set of freely falling test bodies with the aid of the transformations

$$\tau = t + \int dW \left[\frac{\tilde{V}(1-U)}{U}\right]^{1/2}. \tag{11.135}$$

$$R = t + \int dW \left[\frac{\tilde{V}}{U(1-U)}\right]^{1/2} . \qquad (11.136)$$

In this case the interval (11.118) assumes the form

$$ds^2 = d\tau^2 - (1-U)dR^2 - W^2(d\Theta^2 + \sin^2\Theta d\Phi^2) . \quad (11.137)$$

In this form, the singularities of the metric coefficients disappear both for the Schwarzschild solution, when $\epsilon = 0$, and for the solution in our case, when $\epsilon \neq 0$. However, while in GRT the variable W may turn to zero, in RTG, by virtue of expression (11.134), it is always larger, than the Schwarzschild radius.

Subtracting from expression (11.136) expression (11.135) we obtain

$$R - \tau = \int dW \sqrt{\frac{U\tilde{V}}{(1-U)}} . \qquad (11.138)$$

Differentiating equality (11.138) with respect to τ, we find

$$\frac{dW}{d\tau} = -\sqrt{\frac{(1-U)}{U\tilde{V}}} . \qquad (11.139)$$

Thus, we arrive at the same initial equation (11.129). Taking into account that $r = W - \frac{GM}{c^2}$, on the basis of expressions (11.117), we obtain from equation (11.139) the following:

$$W = \frac{2GM}{c^2} + \frac{1}{4}\left(\frac{\hbar c^2}{GMm}\right)^2 \cdot \frac{(R - c\tau)^2}{GM} \qquad (11.134a)$$

Hence, it is also evident that, if $\epsilon \neq 0$, then a falling test body can never cross the Schwarzschild sphere. In that case, when $\epsilon = 0$, the Schwarzschild singularity in the metric does not influence the motion of the test body in a falling synchronous reference system. In GRT the following expression will occur, instead of formula (11.134a):

$$W = \left[\frac{3}{2}(R - c\tau)\right]^{2/3} \left(\frac{2GM}{c^2}\right)^{1/3} ,$$

which testifies that a test body will reach the point $W = 0$ in a finite interval of proper time. The falling particles, here, will only cross the Schwarzschild sphere in one direction, inward. We shall now calculate the propagation time of a light signal from a certain point W_0 to the point $W_1 = \frac{2GM}{c^2}$, given by the clock of a distant observer. From the expression $ds^2 = 0$ we have the following for the Schwarzschild solution:

$$\frac{dW}{dt} = -c\left(1 - \frac{2GM}{c^2 W}\right) . \tag{11.140}$$

Integrating this equation we obtain

$$W_0 - W + \frac{2GM}{c^2} \ln \frac{W_0 - \frac{2GM}{c^2}}{W - \frac{2GM}{c^2}} = c(t - t_0) . \tag{11.141}$$

Hence, it is obvious that an infinite time, by the clock of a distant observer, is required in GRT in order to reach the gravitational radius $W_1 = \frac{2GM}{c^2}$. In RTG, as we established earlier, the Schwarzschild solution is valid up to the point $W = W_1(1 + \lambda_2)$, so the time required to reach this point is

$$c(t - t_0) = W_0 - W_1(1 + \lambda_2) + \frac{2GM}{c^2} \ln \frac{W_0 - \frac{2GM}{c^2}}{\lambda_2 \frac{2GM}{c^2}} . \tag{11.142}$$

The propagation time of a light ray from the point $W = W_1(1 + \lambda_2)$ to the point W_1 can be calculated making use of formulae (11.97) and (11.108). Within this interval we have

$$\frac{dW}{dt} = -c\frac{x_1^3}{2xf}(1 - \epsilon f). \tag{11.143}$$

Hence, upon integration and a change of variable, we obtain

$$\frac{2MG}{c^2} \int\limits_f^{1/\epsilon} \frac{xdf}{f} = c(t_1 - t) . \tag{11.144}$$

In accordance with (11.84) and (11.108) the lower integration limit is

$$f = \tilde{C} = \frac{x_1^2}{2\lambda_2}\,. \tag{11.145}$$

The integral (11.144) is readily calculated and with a good accuracy leads to the following relation:

$$c(t_1 - t) = W_1\lambda_2 + \frac{2GM}{c^2}\ln\frac{2\lambda_2}{\epsilon}\,. \tag{11.146}$$

On the basis of (11.142) and (11.146), the time required for a light signal to cover the distance between the points W_0 and $W_1 = \frac{2GM}{c^2}$, is equal to the sum of expressions (11.142) and (11.146),

$$c(t_1 - t_0) = W_0 - W_1 + \frac{2GM}{c^2}\ln\frac{W_0 - \frac{2GM}{c^2}}{\epsilon\frac{GM}{c^2}}\,. \tag{11.147}$$

Hence it is seen, that in RTG, unlike GRT, the propagation time of a light signal to the Schwarzschild sphere is finite, even if measured by the clock of a distant observer. From formula (11.147) it is evident that the propagation time is not enhanced significantly by the influence of the gravitational field.

On the basis of the above presentation it is clear that, if the graviton mass exists, $\epsilon \neq 0$, then the solution in RTG differs essentially from the Schwarzschild solution owing to the presence on the Schwarzschild sphere of a singularity, that cannot be removed by a choice of reference system. Thus, in the case we have considered, when the radius of a body is smaller than the Schwarzschild radius, or to be more precise, when $x_0 < x_1$, a test particle can never reach the surface of the body, by virtue of (11.134). Owing to the presence of a singularity, the physical condition $g < 0$ is violated outside the body, and precisely for this reason a physical solution for a static spherically symmetric body is possible only in the case, when the point x_1 is inside the body. This conclusion is conserved, also, for a synchronous reference system, when the metric coefficients (see (11.134a)) are functions of time.

Thus, in accordance with RTG, no Schwarzschild singularity for a body of arbitrary mass exists, since the radius of the body is greater than the Schwarzschild radius, and so the formation of "black holes" (objects without material boundaries and "cut off" from the external world) is impossible. This conclusion complies with the conclusion made by A. Einstein in 1939, most likely based on his physical intuition, than on GRT logic. He wrote: "*Schwarzschild's singularity does not exist, since matter cannot be concentrated in an arbitrary manner; otherwise clustering particles would achieve the velocity of light*" [31]. A. Einstein, naturally, saw that the existence of the Schwarzschild singularity violated his main principle: "*to acknowledge all conceivable* (we shall not, here, deal with certain restrictions, following from the requirement of uniqueness and continuity) *reference systems to be essentially equivalent for describing nature*" [32]. Precisely for this reason, he considered, from a physical point of view, that no Schwarzschild singularity in the metric coefficients should exist in a reference system related to a distant observer, also. All this, however, is realized in RTG, but not in GRT.

In accordance with RTG, as a field theory of gravity, a body of arbitrary mass cannot undergo compression indefinitely, and therefore no gravitational collapse involving the formation of a "black hole" is possible. **This means that a collapsing star cannot go beneath its gravitational radius.** Spherically symmetric accretion of matter onto such a body, at its final

[31]Einstein A. Collection of scientific works, M.: Nauka, 1966, vol.2, art.119, p.531.

[32]Einstein A. Collection of scientific works, M.: Nauka, 1965, vol.1, art.38, p.459.

stage of evolution (when nuclear resources are exhausted), will be accompanied by a great release of energy, owing to matter falling onto the surface of the body. According to RTG, gravitational absorption of light is impossible. In GRT, when spherically symmetric accretion of matter onto a "black hole" takes place, the energy release is quite low, since the falling matter brings energy into the "black hole". Gravitational absorption of light takes place. Gravitational self-closure of the object occurs. Observational data on such objects could provide the answer, as to what happens with stars of large mass at their final stage of evolution, when all nuclear resources are exhausted.

Addendum

In spherical coordinates of Minkowski space the intervals of Minkowski space and of effective Riemannian space have the form

$$d\sigma^2 = dt^2 - dr^2 - r^2(d\Theta^2 + \sin^2\Theta d\Phi^2) \; , \qquad (1)$$

$$ds^2 = U(r)dt^2 - V(r)dr^2 - W^2(r)(d\Theta^2 + \sin^2\Theta d\Phi^2) \; . \qquad (2)$$

We now introduce the velocity vector

$$v^i = \frac{dx^i}{dt}, \;\; v^i = ve^i, \;\; (x^i = r, \Theta, \Phi) \; . \qquad (3)$$

e^i represents the unit vector with respect to the metric of the spatial part of Minkowski space

$$\kappa_{ik} e^i e^k = 1 \; . \qquad (4)$$

In the general case κ_{ik} is

$$\kappa_{ik} = -\gamma_{ik} + \frac{\gamma_{0i}\gamma_{0k}}{\gamma_{00}} \; . \qquad (5)$$

In case (1)

$$\kappa_{ik} = -\gamma_{ik}. \qquad (6)$$

Condition (4) for metric (1) has the form

$$(e^1)^2 + r^2[(e^2)^2 + \sin^2\Theta \cdot (e^3)^2] = 1 \; . \qquad (7)$$

We define the velocity four-vector by the equality

$$v^\mu = (1, ve^i) \qquad (8)$$

and require that it be isotropic in Minkowski space

$$\gamma_{\mu\nu} v^\mu v^\nu = 0 \; . \qquad (9)$$

Substituting (8) into (9) and taking into account (7) we find

$$v = 1 \ . \tag{10}$$

Thus, the isotropic four-vector v^μ is equal to

$$v^\mu = (1, e^i) \ . \tag{11}$$

Since, in accordance with special relativity theory, motion always proceeds inside or on the boundary of the Minkowski causality cone, then, in the case of a gravitational field, the causality principle

$$g_{\mu\nu} v^\mu v^\nu \leq 0 \tag{12}$$

will be valid, i.e.

$$U - V(e^1)^2 - W^2[(e^2)^2 + (e^3)^2 \sin^2\Theta] \leq 0 \ . \tag{13}$$

Taking into account (7), expression (13) may be written as

$$U - \frac{W^2}{r^2} - \left(V - \frac{W^2}{r^2}\right)(e^1)^2 \leq 0 \ . \tag{14}$$

Let

$$V - \frac{W^2}{r^2} \geq 0 \ . \tag{15}$$

Owing to arbitrariness, $0 \leq (e^1)^2 \leq 1$, inequality (14) will be satisfied only if

$$U - \frac{W^2}{r^2} \leq 0 \ . \tag{16}$$

From inequalities (15) and (16) follows

$$U \leq V \ . \tag{17}$$

In the case, when

$$V - \frac{W^2}{r^2} < 0 \ , \tag{18}$$

we write inequality (14) as

$$U - V - \left(\frac{W^2}{r^2} - V\right)(1 - (e^1)^2) \leq 0 \,. \tag{19}$$

Owing to the arbitrariness of e^1, expression (19) will hold valid for any values $0 \leq (e^1)^2 \leq 1$, only if

$$U \leq V \,. \tag{20}$$

Thus, the causality principle in RTG always results in the inequality

$$U(r) \leq V(r) \,. \tag{21}$$

12. Gravitational effects in the Solar system

Before proceeding to examine such effects, we shall first dwell upon certain general assertions of RTG and GRT, which are explicitly manifested in calculations of gravitational effects. The RTG equations (5.19) and (5.20) are universally covariant under arbitrary transformations of coordinates and form-invariant under the Lorentz transformations. In other words, the situation in RTG is the same as in electrodynamics. If in two inertial reference systems the respective distributions of matter in Galilean coordinates, $T_{\mu\nu}[x, g_{\alpha\beta}(x)]$ and $T_{\mu\nu}[x', g_{\alpha\beta}(x')]$, are identical, then by virtue of the form-invariance of the equations relative to the Lorentz transformations we obtain identical equations, which in identical conditions of the problem provide for the relativity principle to be satisfied. On the other hand, if in a certain inertial reference system and for a given distribution of matter $T_{\mu\nu}(x)$ the solution we have is $g_{\mu\nu}(x)$, then applying the Lorentz transformations to another inertial reference system we obtain the metric $g'_{\mu\nu}(x')$, but it corresponds to the distribution of matter $T'_{\mu\nu}(x')$. Owing to the equations being form-invariant under the Lorentz transformations we can return to the initial variables x and obtain a new solution $g'_{\mu\nu}(x)$ corresponding to the distribution of matter $T'_{\mu\nu}(x)$. This means that a unique correspondence exists between the distribution of matter and the metric. When the distribution of matter changes, the metric changes also. An essential point in RTG is the presence of the metric of Minkowski space in the equations. Precisely this circumstance permits performing a comparison of the motion of matter in a gravitational field with the motion of matter in the absence of any gravitational field.

In GRT the situation is quite different. The GRT equations outside matter are form-invariant relative to arbitrary transformations of coordinates, and therefore, if for the distribution of matter $T_{\mu\nu}(x)$ the solution we have is $g_{\mu\nu}(x)$, then by transforming coordinates, so that in the region of matter they coincide with the initial ones, and outside matter differ from them, our solution in the new coordinates will have the form $g'_{\mu\nu}(x')$. Owing to the equations being form-invariant outside matter, we can go back to the initial variables x, and, consequently, obtain a new solution $g'_{\mu\nu}(x)$ for the same distribution of matter, $T_{\mu\nu}(x)$. To these two metrics (any amount of metrics can be constructed) there correspond differing intervals:

$$ds_1^2 = g_{\mu\nu}(x)dx^\mu dx^\nu$$

$$ds_2^2 = g'_{\mu\nu}(x)dx^\mu dx^\nu.$$

Which interval must be chosen? The point is that the geodesic lines of these intervals differ from each other. In this connection, attempts are made to identify the gravitational field in GRT with the class of equivalent diffeomorphic metrics $g_{\mu\nu}(x), g'_{\mu\nu}(x)$..., obtained with the aid of transformations of coordinates. From the point of view of mathematics this is obvious, but what about the physical interpretation?

Thus, there exists a fundamental difference between the conclusions in RTG and GRT, and its essence consists in that the RTG equations are not form-invariant with respect to arbitrary coordinate transformations, while the GRT equations outside matter are form-invariant relative to such transformations. The RTG equations are only form-invariant relative to such transformations of coordinates that leave the Minkowski metric $\gamma_{\mu\nu}(x)$ form-invariant. Hence, for example, follows the form-invariance of equations with respect to the Lorentz transformations.

The issue of the multiplicity of solutions $g_{\mu\nu}(x), g'_{\mu\nu}(x)$... worried A. Einstein seriously, and he discussed this issue in detail in 1913–1914 in four articles [29] and arrived at the conclusion that the choice of coordinate reference systems is limited, since he considered that from the universal covariance for one and the same distribution of matter $T_{\mu\nu}(x)$ there arises a whole set of metrics, which is physically inadmissible. However, the main reason for ambiguity is not related to the general covariance, it is related to the form-invariance of GRT equations outside matter with respect to arbitrary transformations of coordinates. To remove this ambiguity, there is no need to renounce general covariance, since it is not the cause, but it is necessary to restrict the form-invariance of equations in accordance with the relativity principle. Precisely this is done in RTG on the basis of the field approach. A simple example from electrodynamics can be presented. Assume that for a current $j_\mu(x)$ we have the solution $A_\mu(x)$. Upon performing transformations to the new variables x' coinciding with the initial variables x in the region of the distribution of current $j_\mu(x)$ and differing from them in the region outside the current, our solution assumes the form $A'_\mu(x')$. But it is absolutely obvious that $A'_\mu(x)$ will not be a solution of the equations of electrodynamics in the coordinates x, since the equations of electrodynamics are not form-invariant with respect to arbitrary transformations of coordinates.

This means that in electrodynamics for one and the same distribution of current $j_\mu(x)$ in identical conditions there exists only one distribution of the electromagnetic field $\vec{E}, \vec{H}$. If in GRT the distribution of matter, determined by the tensors $T_{\mu\nu}(x)$ and $T_{\mu\nu}(x')$, is the same in two arbitrary reference systems, then the GRT equations being form-invariant outside matter, we can, in identical conditions, for example, have identical metric coefficients $g_{\mu\nu}(x)$ and $g_{\mu\nu}(x')$. Precisely this circumstance permitted A. Einstein to put forward the general relativity principle for all physical processes. However, the requirement that the metric coefficients be identical results in a strong restriction being imposed on the structure of Riemannian space, it turns out to be a space with constant curvature.

Since Riemannian space in GRT does not have this property, in the general case, then the general relativity principle, as a physical principle, is not realized in Nature. This also follows from the equations of electrodynamics, for example, not being form-invariant with respect to **arbitrary** transformations of coordinates. The relativity principle, as a physical principle, is not related to universal covariance, but to the form-invariance of equations and of the **metric** relative to the transformations of coordinates. V.A. Fock was right, when he wrote: *"a general relativity principle, as a physical principle, that could be valid with respect to arbitrary reference systems, is not possible"* [25]. In GRT for one and the same distribution of matter $T_{\mu\nu}(x)$ there exists a whole range of solutions of the GRT equations for the metric coefficients $g_{\mu\nu}(x), g'_{\mu\nu}(x), ...$ Outside matter the geodesic lines for these solutions will be different.

The issue of the multiplicity of metrics in GRT in one coordinate system was widely discussed in 1921-1922 by P.Painlevé, M.Chazy, J.Becquerel, A.Gullstrand, E.Kretschmann. The essence of the polemic actually reduced to the question: with which radial variable in the GRT equations is it necessary to identify the astronomically determined distance between the Sun and a planet? It must be noted that this arbitrariness in the first order in the gravitational constant does not influence certain gravitational effects: the deflection of a light ray, the shift of the perihelion of Mercury, the precession of a gyroscope. However, it does, already in the first order in G, influence the delay effect of a radiosignal.

Thus, depending on the choice of solutions in the Schwarzschild form or in harmonic coordinates we will obtain different values for the delay time. We will further see that there exists no such arbitrariness in RTG, and that effects are determined unambiguously.

The reason for the multiplicity of metrics is not general covariance, but the form-invariance of equations with respect to arbitrary transformations of coordinates. There exists no such ambiguity in RTG, since the metric $g_{\mu\nu}(x)$ is unambiguously determined by the distribution of matter $T_{\mu\nu}(x)$. In section 11 it was shown that since the radius of a static spherically symmetric body exceeds the Schwarzschild radius, then the external solution of RTG equations in an inertial reference system in spherical coordinates in the region (11.53) has the form

$$ds^2 = \frac{r - MG}{r + MG}(dx^0)^2 - \frac{r + MG}{r - MG}(dr^2) - \\ -(r + MG)^2[(d\theta)^2 + \sin^2\Theta(d\varphi)^2]. \qquad (\alpha)$$

Precisely such a solution in the post-Newtonian approximation yields expressions for the metric coefficients of effective Riemannian space that coincide with the previously obtained formulae (8.59a) applied for explanation of gravitational effects in the Solar system.

An essential point is that, when the gravitational field is switched off (for instance, the body is removed), we necessarily turn out to be in Minkowski space in an inertial reference system with the metric

$$d\sigma^2 = (dx^0)^2 - (dr)^2 - r^2[(d\theta)^2 + \sin^2\Theta(d\varphi)^2].$$

In calculating gravitational effects in the Solar system we have to calculate the trajectory of motion in effective Riemannian space, that is determined by the interval ds, and to compare it with the corresponding trajectory determined by the interval $d\sigma$. The metric of Minkowski space is present in the RTG equations. This is precisely how the deflection angle of a light ray and the delay time of a radiosignal, due to the influence of the gravitational field of the Sun, are determined. As to calculating the shift in the perihelion of a planet, here it is necessary to compare the trajectory of motion of a test body around the Sun calculated within RTG with the trajectory obtained from Newton's theory of gravity. Precisely in these calculations there exists a difference between the RTG and GRT conclusions, since within GRT one cannot say in which reference system (inertial or non-inertial) of Minkowski space one happens to be, when the gravitational field is switched off. For calculating the gravitational effect it is necessary to compare in one coordinate system the motion along a geodesic line in Riemannian space with motion along the geodesic line in Minkowski space with gravity switched off. But to this end it is necessary to know exactly both the metric $g_{\mu\nu}(x)$ and the metric $\gamma_{\mu\nu}(x)$.

However, in GRT, owing to the multiplicity of solutions both for $g_{\mu\nu}(x)$ and for $\gamma_{\mu\nu}(x)$ we cannot with definiteness say which Riemannian metric $g_{\mu\nu}(x)$ it is necessary to take for the chosen metric $\gamma_{\mu\nu}(x)$ in order to find the geodesic lines in Riemannian space and in Minkowski space. This is actually the essence of the ambiguity in predictions of gravitational effects in GRT. Sometimes errors are avoided in GRT by considering the initial reference system to be an inertial reference system in Cartesian coordinates (but no such coordinates exist in

GRT) and then dealing with a weak gravitational field against this background. No such difficulty exists in RTG, since for the chosen metric $\gamma_{\mu\nu}(x)$, with the aid of equations (5.19) and (5.20), under appropriate conditions, the metric $g_{\mu\nu}(x)$ of effective Riemannian space is determined unambiguously, which permits to determine unambiguously the gravitational effect.

In calculations of effects in the gravitational field of the Sun one usually takes as the idealized model of the Sun a static spherically symmetric body of radius $R_\odot$. The general form of the metric of Riemannian space in an inertial reference system in spherical coordinates is

$$\begin{aligned} ds^2 = U(r)(dx^0)^2 - V(r)(dr)^2 - \\ -W^2(r)[(d\theta)^2 + \sin^2\Theta(d\varphi)^2]. \end{aligned} \tag{12.1}$$

In the absence of a gravitational field the metric has the form

$$d\sigma^2 = (dx^0)^2 - (dr)^2 - r^2[(d\theta)^2 + \sin^2\Theta(d\varphi)^2]. \tag{12.1a}$$

Substituting (12.1) and (12.1a) into equations (5.19) and (5.20) we precisely obtain the external solution for the Sun (α).

In section 5 it was shown that from the RTG equations (5.19) and (5.20) follow directly the equations of motion for matter,

$$\nabla_\nu T^{\mu\nu} = 0. \tag{12.2}$$

Hence it is easy to obtain the equations of motion for a test body in a static gravitational field. The energy-momentum tensor for matter, $T^{\mu\nu}$, in this case assumes the form

$$T^{\mu\nu} = \rho U^\mu U^\nu, \quad U^\mu = \frac{dx^\mu}{ds}. \tag{12.3}$$

Substituting (12.3) into (12.2) we obtain

$$U^\mu \nabla_\nu(\rho U^\nu) + \rho U^\nu \nabla_\nu U^\mu = 0. \tag{12.4}$$

Multiplying this equation by U_μ and taking into account $U_\mu U^\mu = 1$ we obtain

$$\nabla_\nu(\rho U^\nu) + \rho U^\nu U_\mu \nabla_\nu U^\mu = 0. \quad (12.4a)$$

Since

$$\nabla_\nu(U_\mu U^\mu) = 2U_\mu \nabla_\nu U^\mu = 0.$$

from equation (12.4a) we have

$$\nabla_\nu(\rho U^\nu) = 0. \quad (12.5)$$

Substituting (12.5) into (12.4) we find

$$U^\nu \nabla_\nu U^\mu = 0. \quad (12.6)$$

Applying the definition of a covariant derivative, equations (12.6) may be written as

$$\left[\frac{\partial U^\mu}{\partial x^\nu} + \Gamma^\mu_{\nu\sigma} U^\sigma\right] \frac{dx^\nu}{ds} = 0. \quad (12.7)$$

Taking into account the definition of a total differential we have

$$dU^\mu = \frac{\partial U^\mu}{\partial x^\nu} dx^\nu. \quad (12.8)$$

On the basis of (12.8) equation (12.7) assumes the form

$$\frac{dU^\mu}{ds} + \Gamma^\mu_{\nu\sigma} U^\nu U^\sigma = 0, \quad U^\mu = \frac{dx^\mu}{ds}. \quad (12.9)$$

The equation of motion of a test body, (12.9), is an equation of geodesic lines in the space with the metric $g_{\mu\nu}$. The Christoffel symbols are determined by the formula

$$\Gamma^\mu_{\nu\sigma} = \frac{1}{2} g^{\mu\lambda}(\partial_\nu g_{\sigma\lambda} + \partial_\sigma g_{\nu\lambda} - \partial_\lambda g_{\nu\sigma}). \quad (12.10)$$

On the basis of (12.1) and (12.10) it is easy to obtain the Christoffel symbols of interest to us:

$$\Gamma^2_{12} = \frac{1}{W}\frac{dW}{dr}, \quad \Gamma^2_{33} = -\sin\Theta\cos\Theta,$$
$$\Gamma^3_{13} = \frac{1}{W}\frac{dW}{dr}, \quad \Gamma^3_{23} = \cot\Theta, \quad \Gamma^0_{01} = \frac{1}{2U}\frac{dU}{dr}. \tag{12.11}$$

Of the four equations (12.9) only three are independent, since the following relation is valid:

$$g_{\mu\nu}U^\mu U^\nu = 1. \tag{12.12}$$

We shall further make use of this circumstance by choosing the three simplest equations from (12.9). From equations (12.9) we shall take equations

$$\frac{d^2x^0}{ds^2} + \frac{1}{U}\frac{dU}{dr}\frac{dx^0}{ds}\frac{dr}{ds} = 0, \tag{12.13}$$

$$\frac{d^2\Theta}{ds^2} + \frac{2}{W}\frac{dW}{dr}\frac{dr}{ds}\frac{d\Theta}{ds} - \sin\Theta\cos\Theta\left(\frac{d\varphi}{ds}\right)^2 = 0, \tag{12.14}$$

$$\frac{d^2\varphi}{ds^2} + \frac{2}{W}\frac{dW}{dr}\frac{dr}{ds}\frac{d\varphi}{ds} + 2\cot\Theta\frac{d\Theta}{ds}\frac{d\varphi}{ds} = 0 \tag{12.15}$$

and supplement them with equation (12.12) in the form

$$U\left(\frac{dx^0}{ds}\right)^2 - V\left(\frac{dr}{ds}\right)^2 - W^2\left(\frac{d\Theta}{ds}\right)^2 -$$
$$-W^2\sin^2\Theta\left(\frac{d\varphi}{ds}\right)^2 = 1. \tag{12.16}$$

Since the gravitational field is spherically symmetric, it is natural to choose the reference system so as to make the motion take place in the equatorial plane, i.e.

$$\Theta = \frac{\pi}{2}. \tag{12.17}$$

In the case of our choice equation (12.14) is identically satisfied.

Equations (12.13) and (12.15) can be respectively written as

$$\frac{d}{ds}\left[\ln\frac{dx^0}{ds}U\right]=0, \tag{12.18}$$

$$\frac{d}{ds}\left[\ln\frac{d\varphi}{ds}W^2\right]=0. \tag{12.19}$$

Hence, we find the first integrals of motion E, J:

$$\frac{dx^0}{ds}U=\frac{1}{\sqrt{E}},\quad \frac{d\varphi}{ds}W^2=\frac{J}{\sqrt{E}}. \tag{12.20}$$

Substituting these expressions into (12.16) we obtain

$$\frac{1}{EU}-V\left(\frac{dr}{ds}\right)^2-\frac{J^2}{W^2E}=1. \tag{12.21}$$

From the second relation (12.20) we find

$$ds=d\varphi\frac{\sqrt{E}W^2}{J}. \tag{12.22}$$

Passing in (12.21), with the aid of (12.22), to the variable φ we obtain

$$\frac{V}{W^4}\left(\frac{dr}{d\varphi}\right)^2+\frac{1}{W^2}-\frac{1}{J^2U}+\frac{E}{J^2}=0. \tag{12.23}$$

From the first relation (12.20) we find

$$(ds)^2=EU^2(dx^0)^2. \tag{12.24}$$

Hence it follows that $E>0$ for test bodies and $E=0$ for light.

Gravitational effects in the Solar system have been calculated within GRT by various methods. Here, we shall follow S. Weinberg's method of calculation [1].

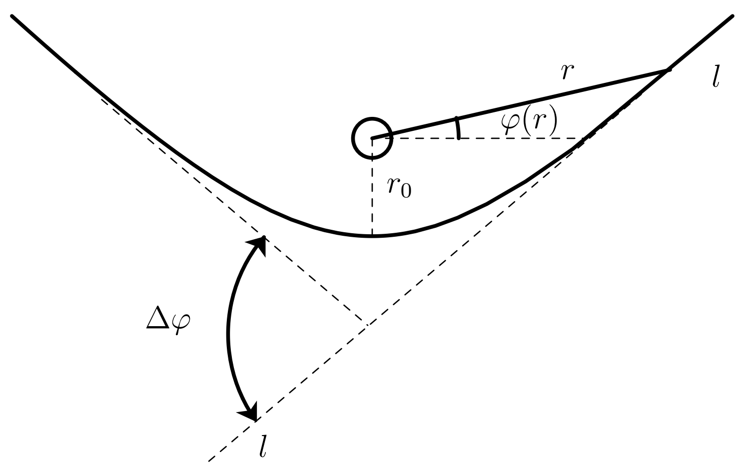
12.1. Deflection of light rays by the Sun

Consider a photon from a distant region passing near the Sun. What is the trajectory of the light ray? It is determined from equation (12.23), when $E = 0$, and it has the form

$$d\varphi = dr\sqrt{\frac{UV}{W^2\left(\frac{W^2}{J^2} - U\right)}}. \tag{12.25}$$

At the point of the light ray's trajectory (see the figure) closest to the Sun

$$\frac{dr}{d\varphi}\Big|_{r_0} = 0. \tag{12.26}$$



Deflection of the light ray

The integral of motion J is expressed via the metric parameters U_0 and W_0:

$$J^2 = \frac{W^2(r_0)}{U(r_0)} = \frac{W_0^2}{U_0}. \tag{12.27}$$

Integrating (12.25) we obtain

$$\varphi(r) = \varphi(\infty) + \int\limits_r^\infty dr \left[\frac{V}{W^2 \left(\frac{W^2}{W_0^2} \frac{U_0}{U} - 1 \right)} \right]^{1/2}. \qquad (12.28)$$

The deflection angle of a light ray is

$$\Delta\varphi = 2|\varphi(r_0) - \varphi(\infty)| - \pi. \qquad (12.29)$$

Here, we have taken into account that in the absence of a gravitational field a light ray propagates along a straight line l, and precisely for this reason π has appeared in (12.29). For the Sun, from the RTG equations we have

$$U(r) = \frac{r - GM}{r + GM}, \quad V = \frac{r + GM}{r - GM}, \quad W^2 = (r + GM)^2. \qquad (12.30)$$

For calculations it is sometimes convenient to make use of an independent variable W:

$$U(W) = 1 - \frac{2GM}{W}, \quad V(W) = \frac{1}{1 - \frac{2GM}{W}}. \qquad (12.31)$$

In the first approximation in the gravitational constant G the metric coefficients are

$$\begin{aligned} &U(r) = 1 - \frac{2GM}{r}, \quad V(r) = 1 + \frac{2GM}{r}, \\ &W^2 = r^2 \left(1 + \frac{2GM}{r} \right). \end{aligned} \qquad (12.32)$$

Substituting these expressions into the integral (12.28) we obtain for it the following expression:

$$I = r_0 \int\limits_{r_0}^\infty \frac{dr}{r \left[r^2 \left(1 - \frac{4MG}{r_0} \right) + 4MGr - r_0^2 \right]^{1/2}}. \qquad (12.33)$$

Performing a change of variables, $r = \frac{1}{t}$, we obtain

$$I = \int\limits_0^{1/r_0} \frac{dt}{\sqrt{r_0^{-2}\left(1 - \frac{2MG}{r_0}\right)^2 - \left(t - \frac{2MG}{r_0^2}\right)^2}}. \tag{12.34}$$

Making use of the tabular integral

$$\int \frac{dx}{\sqrt{m^2 - \left(x - \frac{b}{2}\right)^2}} = \arcsin \frac{x - \frac{b}{2}}{m} + c,$$

we find

$$I = \frac{\pi}{2} + \frac{2MG}{r_0}. \tag{12.35}$$

On the basis of (12.29) we obtain

$$\Delta\varphi = \frac{4M_\odot G}{c^2 r_0}, \tag{12.36}$$

taking into account

$$\frac{M_\odot G}{c^2} = 1,475 \cdot 10^5 \text{cm}, \quad R_\odot G = 6,95 \cdot 10^{10} \text{cm}, \tag{12.37}$$

we find

$$\Delta\varphi = \frac{R_\odot}{r_0} \sum_\odot, \quad \sum_\odot = \frac{4M_\odot G}{R_\odot c^2} = 1,75''. \tag{12.38}$$

Thus, the deflection of a light ray by the gravitational field of the Sun is equal to

$$\Delta\varphi = 1,75'' \cdot \frac{R_\odot}{r_0}. \tag{12.39}$$

In calculating the deflection angle of a light ray we took into account that in the absence of a field in an inertial reference system, a light ray travels, by virtue of the metric (12.1a), along a straight line l. Precisely the deviation from this straight line is the gravitational effect.

12.2. The delay of a radiosignal

I.I. Shapiro [43] proposed and implemented an experiment for measurement of the time required for a radiosignal to reach the planet Mercury and, upon reflection, to return to the Earth. We shall calculate this time on the basis of RTG equations.

We shall pass from the independent variable φ to the independent variable x^0. To this end, making use of (12.22) and (12.24), we shall obtain

$$(d\varphi)^2 = \frac{J^2 U^2}{W^4}(dx^0)^2. \qquad (12.40)$$

With the aid of (12.40), equation (12.23) assumes the form

$$ct(r, r_0) = \int\limits_{r_0}^{r} dr \left[\frac{V}{\left(1 - \frac{W_0^2}{W^2} \cdot \frac{U}{U_0}\right) U}\right]^{1/2}. \qquad (12.41)$$

Substituting expressions (12.32) into the integral (12.41), we find

$$ct(r, r_0) = \int\limits_{r_0}^{r} \frac{r dr}{\sqrt{r^2 - r_0^2}}\left[1 + \frac{2MG}{r} + \frac{2MG}{r}\frac{r_0}{r + r_0}\right]. \qquad (12.42)$$

Applying the tabular integrals

$$\int\limits_{r_0}^{r} \frac{dr}{\sqrt{r^2 - r_0^2}} = \ln\frac{r + \sqrt{r^2 - r_0^2}}{r_0},$$
$$\int\limits_{r_0}^{r} \frac{dr}{\sqrt{r^2 - r_0^2}}\frac{r_0}{r + r_0} = \left[\frac{r - r_0}{r + r_0}\right]^{1/2}, \qquad (12.43)$$

we obtain

$$ct(r, r_0) = \sqrt{r^2 - r_0^2} + 2MG \ln\frac{r + \sqrt{r^2 - r_0^2}}{r_0} +$$
$$+ 2MG\left[\frac{r - r_0}{r + r_0}\right]^{1/2}. \qquad (12.44)$$

Let r_e, r_p be the heliocentric coordinates of the Earth and of Mercury. Since $r_e, r_p >> r_0$, then in the summands of expressions (12.44), that contain the gravitational constant, the influence of r_0 present under the square root sign can be neglected, which will result in

$$ct(r_p, r_e) = \sqrt{r_e^2 - r_0^2} + \sqrt{r_p^2 - r_0^2} + \\ + \frac{2MG}{c^2} \ln \frac{4r_e r_p}{r_0^2} + \frac{4MG}{c^2}. \qquad (12.45)$$

We shall drop a perpendicular $r_\perp$ from the center of the source of the gravitational field onto the straight line connecting points r_e and r_p. Then, according to Pythagoras' theorem, we have

$$r_e^2 = R_e^2 + r_\perp^2, \quad r_p^2 = R_p^2 + r_\perp^2. \qquad (12.46)$$

In the first order in G

$$r_0 \simeq r_\perp + R_e \frac{\Delta\varphi}{2}, \quad r_0^2 - r_\perp^2 \simeq R_e r_0 \Delta\varphi, \qquad (12.47)$$

$\Delta\varphi$ is the deflection angle of a light ray due to the influence of a source of the gravitational field (see (12.36)).

$$\sqrt{r_e^2 - r_0^2} = R_e \sqrt{1 - \frac{r_0}{R_e}\Delta\varphi} \simeq \\ \simeq R_e - r_0 \frac{\Delta\varphi}{2} = R_e - \frac{2MG}{c^2}, \qquad (12.48)$$

similarly

$$\sqrt{r_p^2 - r_0^2} \simeq R_p - r_0 \frac{\Delta\varphi}{2} = R_p - \frac{2MG}{c^2}. \qquad (12.49)$$

With account of (12.48) and (12.49) expression (12.45) assumes the form

$$ct(r_p, r_e) - R = \frac{2MG}{c^2} \ln \frac{4r_e r_p}{r_0^2}, \qquad (12.50)$$

here $R = R_e + R_p$ is the distance between the planets.

The delay time of a radiosignal propagating from the Earth to Mercury and back is

$$\begin{gathered}\Delta\tau = 2[t(r_p, r_e) - R/c] = \frac{4MG}{c^3}\ln\frac{4r_e r_p}{r_0}, \\ r_e = r_\oplus = 15\cdot 10^{12}\text{cm}, \quad r_p = r_{☿} = 5,8\cdot 10^{12}\text{cm},\end{gathered} \tag{12.51}$$

as r_0 it is possible to take the radius of the Sun, $R_\odot$:

$$R_\odot = 6,95\cdot 10^{10}\text{cm}.$$

Substituting these values into (12.51) and taking into account that

$$\frac{4M_\odot G}{c^2} = 5,9\cdot 10^5\text{cm},$$

we obtain

$$\Delta\tau = \frac{4M_\odot G}{c^3}\ln\frac{4r_\oplus r_{☿}}{R_\odot^2} = 219,9\,\mu\text{s}. \tag{12.52}$$

In calculating the delay effect of a radiosignal we have taken into account that in the absence of a gravitational field a light ray, by virtue of (12.1a), travels from point e to point p along a straight line in an inertial reference system. Comparison with such motion is precisely how the gravitational effect is determined. It is precisely for this reason, that the summand $\frac{2R}{c}$ has appeared to the left in (12.51). In observations, the time $\frac{2R}{c}$ is determined during a period, when the Sun moves away from the trajectory of the light ray, so its influence is significantly reduced.

In GRT, if the solution is sought of the Hilbert–Einstein equations for a static spherically symmetric body of mass M, then within one and the same coordinate system it is possible to obtain the external solution for the metric, that involves two arbitrary functions

$$ds^2 = g_{00}dt^2 + 2g_{01}dtdr +$$

$$+g_{11}dr^2 + g_{22}(d\Theta)^2 + g_{33}(d\varphi)^2, \qquad (12.31a)$$

where

$$g_{00} = 1 - \frac{2GM}{W(r)}, \quad g_{01} = -B(r),$$

$$g_{11} = -\left(1 - \frac{2GM}{W}\right)^{-1} \cdot \left[\left(\frac{dW}{dr}\right)^2 - B^2\right],$$

$$g_{22} = -W^2(r), \quad g_{33} = -W^2 \sin\Theta.$$

Thus, within one and the same coordinate system for a body of mass M there exists an infinite number of solutions. Here, functions $B(r)$ and $W(r)$ are, generally speaking, arbitrary, they are not determined by GRT. P. Painlevé wrote about all this some 80 years ago and stressed that the choice of initial formulae is purely arbitrary. Hence it is obvious that within GRT neither Newton's law nor the post-Newtonian approximation (8.59a), used in GRT for explaining gravitational effects in the Solar system, follow unambiguously from the exact external solution of equations. Hence, also, it follows, for example, that an infinitesimal period of true physical time in GRT,

$$d\tau = \left[dt\sqrt{1 - \frac{2GM}{W(r)}} - \frac{B(r)dr}{\sqrt{1 - \frac{2GM}{W(r)}}}\right]$$

will differ depending on the choice of the arbitrary functions $W(r)$ and $B(r)$. This means that for a static spherically symmetric body of mass M the course of physical time for one or another process is not determined unambiguously.

As we can see, the situation in GRT differs totally from the situation in electrodynamics, where the Coulomb law unambiguously follows from the equations of theory. We shall clarify the situation making use of the example of Minkowski space. From the point of view of geometry, Minkowski space both in an inertial and in a noninertial reference system remains essentially the same, since the tensor of Riemann curvature equals zero. By virtue of the form-invariance of the Riemann tensor we will have in one and the same coordinate system an infinite amount of metrics $\gamma_{\mu\nu}(x)$, $\gamma'_{\mu\nu}(x)$... and so on, that reduce the Riemann tensor to zero. But, depending on the choice of metric we will have different geodesic lines in one and the same coordinate system, i.e. the physics will vary. All this is obvious and well known, since the dynamics in an inertial reference system differs from the dynamics in a non-inertial reference system owing to the appearance of forces of inertia. Precisely for this reason the choice of the non-inertial reference system in four-dimensional Minkowski space alters the physics. In Minkowski space, however, there exist inertial reference systems, and observational astronomical data are referred precisely to an inertial reference system. The choice of reference system in physical equations is based precisely on this circumstance. In GRT Riemannian geometry there exist no such reference systems, so it is absolutely unclear, which coordinates it is necessary to choose so as to compare theoretical calculations and observational data. The geometry does not depend on the choice of reference system (or, in other words, in our particular case, on the choice of functions $B(r)$ and $W(r)$)), it remains, like before, Riemannian geometry, however, the physics changes. Naturally, it is possible, in our case, to select arbitrary functions $B(r)$ and $W(r)$ so as to provide for Newton's law of gravity to be satisfied, and for the post-Newtonian approximation to have the form (8.59a). However, such a choice in GRT is, regrettably, arbitrary, since it is not imposed by any physical conditions. It is not possible to formulate physical requirements to be imposed on the behaviour of Riemannian metric, if it is not of a field origin, because such behaviour even depends on the choice of three-dimensional space coordinates. V.A. Fock resolved the issue of the choice of coordinates for island systems, with the aid of the harmonicity conditions. But why precisely they have to be chosen, instead of some other ones, remained unclear.

Now, let us go back to the analysis of a concrete example demonstrating the ambiguity of GRT in calculations of the gravitational delay effect of a radiosignal traveling from the Earth to Mercury and back. The predictions of theory depend on the choice of solution. For simplicity we shall advantage of the simplest partial case

$$B(r) = 0, \quad W(r) = r + (\lambda + 1)M, \quad r > R_0,$$

λ is an arbitrary parameter, R_0 is the radius of the body dealt with. If an appropriate choice is made of function $W(r)$ in the vicinity of the body, then this solution can be made to match the solution inside the body. If the previous calculations are repeated, then for this metric we find

$$ct(r, r_0) = \sqrt{r^2 - r_0^2} + 2MG \ln \frac{r + \sqrt{r^2 - r_0^2}}{r_0} +$$

$$+2MG\left(1 + \frac{\lambda}{2}\right)\left[\frac{r - r_0}{r + r_0}\right]^{1/2}. \qquad (A)$$

Comparing this expression with (12.44) we see that already in the first order in G an ambiguity due to the influence of the source of the gravitational field arises in GRT in the description of the delay effect of a radiosignal. On the basis of (A) the delay time of a radiosignal traveling from the Earth to Mercury and back, in accordance with GRT, equals the following:

$$\Delta\tau = 2\left[t(r_e, r_p) - \frac{R}{c}\right] = \frac{4MG}{c^3}\ln\frac{4r_e r_p}{r_0^2} + \frac{2MG}{c^3}\lambda. \qquad (B)$$

We note that in the first order in G the physical distance l, determined by the expression

$$l = \int\limits_{R_0}^{r}\sqrt{g_{11}}dr \simeq r - R_0 + \frac{GM}{c^2}\ln\frac{r}{R_0}$$

does not depend on the parameter λ, and precisely for this reason the first term in (B) will also not depend on λ, and, consequently, owing to the presence of the second term, we will have different predictions for the delay time of a radiosignal, depending on the choice of the constant λ. Expression (B) differs essentially from the result (12.51), which follows exactly from RTG and complies with experimental data [43]. For the Schwarzschild solution $\lambda = -1$ the difference from (12.51) will be $\frac{2GM}{c^3}$, which for the Sun amounts to about 10 microseconds. In calculations of the following gravitational effects: of the deflection of a light ray by the Sun, of the shift of a planet's perihelion, of the precession of a gyroscope, of the shift of spectral lines, an ambiguity also arises, but in the second order in the gravitational constant G. All the above has been discussed in detail with prof. Yu.M. Loskutov (see, also, monograph [10]).

Thus, GRT cannot, in principle, provide definite predictions on gravitational effects, which is still another of its fundamental defects. Certain attempts to relate the gravitational field in GRT to the equivalence class of diffeomorphic metrics do not remove this ambiguity, since they do not discard the fundamental defect of GRT — the form-invariance of the Hilbert–Einstein equations outside matter with respect to arbitrary transformations of coordinates[33]. Precisely this circumstance results in the entire set of diffeomorphic metrics arising within one coordinate system for one and the same distribution of matter, while this, according to the Weyl–Lorentz–Petrov theorem (see end of section 14), leads to different geodesic lines in identical conditions of the problem, which is physically inadmissible. **The essence of the issue does not consist in the general covariance, which must always exist, but in whether the form-invariance of physical equations relative to arbitrary transformations of coordinates is admissible?**

Since reference systems are not equivalent in the presence of forces of inertia, no sense whatsoever can be attributed to the form-invariance of physical equations with respect to arbitrary transformations of coordinates. General covariance is a mathematical requirement, while form-invariance has a profound physical content. Actually, in all physical theories the form-invariance of equations and of **metrics** holds valid relative to the Lorentz transformations — precisely this is the essence of the relativity principle.

The non-equivalence of various reference systems is especially evident, if one considers pseudo-Euclidean geometry,

[33]See, for instance: J.Stachel. Conference "Jena-1980", 1981 (DDR).

for which the curvature tensor is zero. From the equality of the curvature tensor to zero it is possible, by virtue of form-invariance, to obtain a set of solutions for the metric tensor in one and the same coordinate system. But it is quite obvious that they are physically not equivalent, since some of them are inertial reference systems, while others are non-inertial reference systems. All the above, involving significant complications, also takes place in the case of Riemannian geometry.

It is important to stress once more that in GRT, owing to the equations being, outside the distribution of matter, form-invariant relative to arbitrary transformations of coordinates, there arises a situation when for one and the same distribution of matter there exists, in one and the same coordinate system, an indefinite amount of metrics. No such situation exists in any other physical theory, since form-invariance within them is admissible only with respect to transformations of coordinates leaving the metric $\gamma_{\mu\nu}(x)$ form-invariant. From this fact, for example, the form-invariance of equations relative to the Lorentz transformations follows.

12.3. The shift of a planet's perihelion

Consider the motion of a test body on a solar orbit. At the perihelion the heliocentric distance of the test body is at its minimum and is r_-, while at the aphelion it reaches its maximum and equals r_+. Since at the perihelion and at the aphelion $\frac{dr}{d\varphi} = 0$, from equation (12.23) we obtain

$$\begin{aligned} \frac{1}{W(r_+)} - \frac{1}{J^2 U(r_+)} &= -\frac{E}{J^2}, \\ \frac{1}{W(r_-)} - \frac{1}{J^2 U(r_-)} &= -\frac{E}{J^2}. \end{aligned} \tag{12.53}$$

Hence, we find

$$J^2 = \frac{\frac{1}{U_+} - \frac{1}{U_-}}{\frac{1}{W_+^2} - \frac{1}{W_-^2}}. \tag{12.54}$$

Now we write equations (12.53) in another form:

$$J^2 = W_+^2 \left(\frac{1}{U_+} - E \right), \quad J^2 = W_-^2 \left(\frac{1}{U_-} - E \right). \tag{12.55}$$

Hence, we obtain

$$E = \frac{\frac{W_+^2}{U_+} - \frac{W_-^2}{U_-}}{W_+^2 - W_-^2}. \tag{12.56}$$

By integration of equation (12.23) we find

$$\varphi(r) = \varphi(r_-) + \int\limits_{r_-}^{r} \sqrt{V} \left[\frac{1}{J^2 U} - \frac{1}{W^2} - \frac{E}{J^2} \right]^{-1/2} \frac{dr}{W^2}. \tag{12.57}$$

For convenience of calculations we introduce a new independent variable

$$W = r + GM. \tag{12.58}$$

Substituting (12.54) and (12.56) into (12.57) and passing to the new independent variable W we find

$$\begin{aligned} &\varphi(W) = \varphi(W_-) + \\ &+ \int\limits_{W_-}^{W} \left\{ \frac{W_-^2 [U^{-1}(W) - U_-^{-1}] - W_+^2 [U^{-1}(W) - U_+^{-1}]}{W_-^2 W_+^2 [U_+^{-1} - U_-^{-1}]} \right\}^{-1/2} \times \\ &\times \frac{\sqrt{V} dW}{W^2}. \end{aligned} \tag{12.59}$$

On the basis of (12.31) we have for function $U^{-1}(W)$ in the second order in the gravitational constant G the following:

$$U^{-1}(W) = 1 + \frac{2GM}{W} + \frac{(2GM)^2}{W^2}. \tag{12.60}$$

We must take into account the second order in $U^{-1}(W)$, since, if we only consider the first order in G, then the expression under the root sign in figure brackets will be independent of the gravitational constant. But this means that in calculations this circumstance results in our losing the term containing G in the first order. We will have taken into account only the term of the first order in G entering into function V. For the metric coefficient of V it suffices to take into account only the first order in G,

$$V(W) = 1 + \frac{2GM}{W}. \tag{12.61}$$

In the approximation (12.60) the numerator of the expression under the root sign in the figure brackets of (12.59) is a quadratic function of the variable $\frac{1}{W}$ of the following form:

$$2GMW_-W_+(W_+ - W_-)\left[\frac{1}{W^2} - \frac{1}{W}\left(\frac{1}{W_-} + \frac{1}{W_+}\right) + \right.$$
$$\left. + \frac{1}{W_-W_+}\right] = 2GMW_-W_+(W_+ - W_-) \times$$
$$\times \left(\frac{1}{W} - \frac{1}{W_-}\right)\left(\frac{1}{W} - \frac{1}{W_+}\right). \tag{12.62}$$

The denominator of the expression under the root sign in (12.59) is

$$W_-^2 W_+^2 \left[U_+^{-1} - U_-^{-1}\right] = 2GMW_-W_+(W_- - W_+) \times$$
$$\times \left[1 + 2GM\left(\frac{1}{W_-} + \frac{1}{W_+}\right)\right]. \tag{12.63}$$

Taking (12.62) and (12.63) we find the expression under the root sign in the figure brackets (12.59),

$$\frac{1}{\left[1+2GM\left(\frac{1}{W_-}+\frac{1}{W_+}\right)\right]}\left(\frac{1}{W_-}-\frac{1}{W}\right)\left(\frac{1}{W}-\frac{1}{W_+}\right). \quad (12.64)$$

Substituting (12.61) and (12.64) into (12.59) and considering only terms of the first order in the gravitational constant G, we obtain

$$\varphi(W)=\varphi(W_-)+\left[1+\frac{1}{2}\left(\frac{1}{W_-}+\frac{1}{W_+}\right)2GM\right]\times$$
$$\times\int\limits_{W_-}^{W}\frac{\left(1+\frac{MG}{W}\right)dW}{W^2\left[\left(\frac{1}{W_-}-\frac{1}{W}\right)\left(\frac{1}{W}-\frac{1}{W_+}\right)\right]^{1/2}}. \quad (12.65)$$

For calculating the integral in (12.65) we introduce a new variable ψ

$$\frac{1}{W}=\frac{1}{2}\left(\frac{1}{W_+}+\frac{1}{W}\right)+\frac{1}{2}\left(\frac{1}{W_+}-\frac{1}{W_-}\right)\sin\psi. \quad (12.66)$$

Applying (12.66) we obtain

$$\left(\frac{1}{W_-}-\frac{1}{W}\right)\left(\frac{1}{W}-\frac{1}{W_+}\right)=$$
$$=\frac{1}{4}\left(\frac{1}{W_-}-\frac{1}{W_+}\right)^2\cos^2\psi. \quad (12.67)$$

Upon substitution of (12.66) and (12.67) we find

$$I(W)=\int\limits_{W_-}^{W}\frac{\left(1+\frac{MG}{W}\right)dW}{W^2\left[\left(\frac{1}{W_-}-\frac{1}{W}\right)\left(\frac{1}{W}-\frac{1}{W_+}\right)\right]^{1/2}}=$$
$$=\psi+GM\left\{\frac{1}{2}\left(\frac{1}{W_-}+\frac{1}{W_+}\right)\psi+\right.$$
$$\left.+\frac{1}{2}\left(\frac{1}{W_-}-\frac{1}{W_+}\right)\cos\psi\ \right\}\Big|_{-\pi/2}^{\psi}.$$

Hence we obtain

$$I(W_+) = \pi + GM\frac{1}{2}\left(\frac{1}{W_-} + \frac{1}{W_+}\right)\pi. \tag{12.68}$$

Making use of (12.68), from (12.65) we have

$$\varphi(W_+) - \varphi(W_-) = \pi + \frac{3}{2}\pi GM\left(\frac{1}{W_-} + \frac{1}{W_+}\right). \tag{12.69}$$

Hence the change in angle φ during one revolution is

$$2|\varphi(W_+) - \varphi(W_-)| = 2\pi + 6\pi GM\frac{1}{2}\left(\frac{1}{W_-} + \frac{1}{W_+}\right). \tag{12.70}$$

The shift of the perihelion during one revolution will amount to

$$\begin{aligned} \delta\varphi &= 2|\varphi(W_+) - \varphi(W_-)| - 2\pi = \\ &= 6\pi GM\frac{1}{2}\left(\frac{1}{W_+} + \frac{1}{W_-}\right), \end{aligned} \tag{12.71}$$

or, going back to the variable determined from (12.58), we obtain the following in the same approximation in G:

$$\delta\varphi = 6\pi GM\frac{1}{2}\left(\frac{1}{r_+} + \frac{1}{r_-}\right). \tag{12.72}$$

The quantities r_- and r_+ are expressed via the large semiaxis a and the eccentricity e e

$$r_\pm = (1 \pm e)a. \tag{12.73}$$

Usually a focal parameter is introduced:

$$\frac{1}{L} = \frac{1}{2}\left(\frac{1}{r_+} + \frac{1}{r_-}\right). \tag{12.74}$$

Using (12.73) we find

$$L = (1 - e^2)a. \tag{12.75}$$

Substituting (12.75) into (12.74) we obtain

$$\frac{1}{2}\left(\frac{1}{r_+} + \frac{1}{r_-}\right) = \frac{1}{(1 - e^2)a}. \tag{12.76}$$

Taking into account (12.76) in (12.72) we find

$$\delta\varphi = \frac{6\pi GM}{c^2(1 - e^2)a}. \tag{12.77}$$

In formula (12.77) we have restored the dependence upon the velocity of light. For Mercury

$$e = 0,2056, \ a = 57,91 \cdot 10^{11}\text{cm}. \tag{12.78}$$

Substituting these values into formula (12.77) we obtain the shift of Mercury's perihelion during one revolution:

$$\delta\varphi_{☿} = 0.1037''. \tag{12.79}$$

In one century Mercury undergoes 415 revolutions, therefore the shift of Mercury's perihelion in 100 years amounts to

$$\Delta\varphi = 43.03''. \tag{12.80}$$

Modern data confirm this result with an accuracy up to 1%. Astronomers have been studying the shift of Mercury's perihelion for several centuries. In 1882 S. Newcomb established the difference between observations and theoretical calculations, that turned out to be 43″ per century. At present optical observations, that have been under way for over 200 years, yield an uncertainty in the determination of the precession velocity of approximately $0,4''$ per century.

To conclude this section we shall write equation (12.23) in the variables $u = \frac{1}{W}$, $W = r + GM$.

$$\left(\frac{du}{d\varphi}\right)^2 + \frac{u^2}{V} - \frac{1}{J^2UV} + \frac{E}{J^2V} = 0. \tag{12.81}$$

For a static spherically symmetric field, by virtue of (12.31) we have

$$U = V^{-1} = (1 - 2GMu). \tag{12.82}$$

Differentiating (12.81) with respect to φ and taking into account (12.82) we obtain

$$\frac{d^2u}{d\varphi^2} + u = \frac{EGM}{J^2c^2} + \frac{3GM}{c^2}u^2. \tag{12.83}$$

Here, we have restored the dependence upon the velocity of light. This equation differs from the equation obtained on the basis of Newton's theory of gravity by an additional term $\frac{3GM}{c^2}u^2$. As we see, this term is relativistic. Precisely this term leads to a shift in a planet's perihelion.

Expressing the integrals of motion E and J^2 in terms of the eccentricity and the superior semiaxis in the nonrelativistic approximation, we have

$$\frac{GME}{c^2J^2} = \frac{1}{a(1-\epsilon^2)}. \tag{12.84}$$

Thus, in the nonrelativistic approximation we obtain the equation of Newton's theory of gravity

$$\frac{d^2\sigma}{d\varphi^2} + \sigma = \frac{1}{a(1-\epsilon^2)}, \sigma = \frac{1}{r}. \tag{12.85}$$

Precisely such an expression is found in classical mechanics, if the initial Newton equations are referred to an inertial reference system. In our calculation this is natural, since the initial RTG equations are also written in an inertial reference system.

Comparing the motion complying with (12.83) with the motion (12.85), we precisely determine the shift effect of the perihelion for a single revolution of the body about the Sun. In calculating the shift of Mercury's perihelion and the deflection of a light ray by the Sun, A. Einstein intuitively considered gravity to be a weak physical field against the background of Minkowski space. Precisely such an approach brought him to the well-known formulae for these gravitational effects. However, these formulae are not unambiguous consequences of the GRT equations. In deriving them A. Einstein rather followed his physical intuition, than the logic of his theory. However, upon finding these effects in 1915 he anyhow noted: "*Consider a material point (the Sun) at the origin of the reference system. The gravitational field created by this material point can be calculated from equations by successive approximations. However, it can be assumed that for a given mass of the Sun,* $g_{\mu\nu}$ *are not quite fully determined by equations (1) and (3).* (Here the equations $R_{\mu\nu} = 0$, given the restriction $|g_{\mu\nu}| = -1$, are intended. – A.L.) *This follows from these equations being covariant with respect to any transformations with a determinant 1. Nevertheless, we, most likely, are justified in assuming that by such transformations all these solutions can transform into each other and that, consequently, (for given boundary conditions) they differ from each other only formally, but not physically. Following this conviction, I shall first restrict my-*

self, here, to obtaining **one** *of the solutions, without going into the issue of whether it is the* **sole** *possible solution"*[34].

Later the issue of other possible external solutions arose in the twenties, when the French mathematician P. Painlevé criticized A. Einstein's results. Following P. Painlevé, we shall consider this issue from the point of view of the exact external solution (12.31a) of the GRT equations for a static spherically symmetric body.

In GRT, calculation of the shift of Mercury's perihelion on the basis of the exact external solution (12.31a), for a choice of the arbitrary functions $B(r)$ and $W(r)$ in the simplest form

$$B(r) = 0, \quad W(r) = r + (\lambda + 1)GM,$$

would lead to the following result:

$$\delta\varphi = \frac{6\pi GM}{L}\left[-\frac{(1+\lambda)GM}{L}(1+e^2)) + \right.$$
$$\left. + \frac{9GM}{2L}\left(1 + \frac{e^2}{18}\right)\right]. \qquad (12.72a)$$

This expression is presented in monograph [10], therein, also, references to original articles can be found. From formula (12.72a) it can be seen that in GRT, also, ambiguity exists in predicting the shift effect of Mercury's perihelion, but it manifests itself in the second order in G, instead of the first, and therefore is beyond the accuracy limits of modern observational data, if one is restricted to small values of the arbitrary parameter λ. However, from the point of view of principle it is seen that the ambiguity is also present in the case of such

[34]Einstein A. Collection of scientific works, Moscow: Nauka, 1965, vol.1, art.36, p.440.

a choice of λ, when Newton's law of gravity holds valid. But from GRT it does not follow that the parameter λ should be small. Since in the solution (12.31a) the arbitrary functions $B(r)$ and $W(r)$ are not determined in GRT, therefore for a chosen partial case the parameter λ can assume any values. If it is chosen sufficiently large, so that the second term in the brackets in expression (12.72a) is of the order of 10^{-1}, we would arrive at a contradiction both with observational data on the shift of Mercury's perihelion and with Newton's universal law of gravity. But this means, owing to the arbitrariness of λ, that Newton's law is not the only possible consequence of GRT. If it were unknown to us, then from GRT, as a theoretical scheme, we would never obtain neither it, nor any corrections to it. The maximum, that we could establish, would be the asymptotics at infinity. **All this reveals that although GRT happened to be an important landmark in gravity after the works of I. Newton, it nevertheless turned out to be an incomplete scheme, from the point of view of both its physical aspects and its main equations, applied for explaining and predicting gravitational phenomena.**

After the sharp criticism of GRT (in the twenties) by P. Painlevé and A. Gullstrand concerning the ambiguity in determining gravitational effects, V.A. Fock (in the thirties) clearly understood the essence of GRT and its not complete definiteness. While studying island systems in the distribution of matter in GRT, V.A. Fock added to the Hilbert–Einstein equations harmonic coordinate conditions (actually, certain equations were taken in Galilean coordinates of Minkowski space, and thus departure beyond the limits of GRT was performed) and obtained a complete set of gravitational equations. In RTG, in studying island systems of the distribution of matter precisely such a set of equations arises in an inertial reference system (in Galilean coordinates) from the least action principle. Thus, it becomes clear, why the harmonic conditions in Galilean (Cartesian) coordinates are universal equations.

In studying island systems, A. Einstein and L. Infeld applied other coordinate conditions, however, in the post-Newtonian approximation they are close to the harmonic conditions, and therefore within this approximation they yield the same result. Thus, V.A. Fock's theory of gravity permitted to unambiguously determine all the effects in the Solar system. But his approach was not consistent.

The RTG way consists in total renunciation of A. Einstein's ideas on inertia and gravity and returning to the physical gravitational field in the spirit of Faraday–Maxwell, exact conservation of special relativity theory, proclaiming a universal conserved quantity — the energy-momentum tensor of all matter, including the gravitational field, the source of the gravitational field. Precisely such an approach leads to a new set of equations of the theory of gravity, removes the fundamental difficulties of GRT, discards ambiguity in the determination of gravitational effects, predicts an another (unlike GRT) development of the collapse and of the Universe, and at the same time retains what is most valuable in GRT: the tensor character of gravity and Riemannian space. But now it already stops being the starting point and fundamental, but becomes only effective, that arises because the energy-momentum tensor of all matter, including the gravitational field, is the source of the gravitational field.

All this is reflected in the complete set of gravitational equations (5.19) and (5.20), that differ from the GRT equations. The effective Riemannian space, that arises in RTG owing to the influence of the gravitational field, only has a simple topology. This means that, in principle, no “miracles”, that are possible in GRT owing to the complex topology of Riemannian space, can take place in RTG.

12.4. The precession of a gyroscope

In the works of Pugh [42] and Schiff [44] a proposal was made to put a gyroscope on an orbit around the Earth and to examine its precession for studying the Earth's gravitational field and for testing general relativity theory. Precisely in this effect, was the existence to be revealed of an inertial reference system connected with distant stars. For simplicity, we shall consider the gyroscope to be a pointlike test body. The equation for the angular momentum of the gyroscope, S_μ, has the following form:

$$\frac{dS_\mu}{ds} = \Gamma^\lambda_{\mu\nu} S_\lambda \frac{dx^\nu}{ds}. \qquad (12.86)$$

In a reference system connected with the gyroscope it undergoes no precession, which is reflected in equation (12.86). The angular momentum of the gyroscope, $\vec{J}$, does not change in value, below it will be shown to be expressed via the angular momentum $\vec{S}$ and the velocity $\vec{v}$. In the rest frame of the test body $S_\mu = (0, \vec{S})$, and therefore

$$S_\mu U^\mu = 0, \;\; U^\mu = \frac{dx^\mu}{ds}. \qquad (12.87)$$

From equality (12.87) we obtain

$$S_0 = -\frac{1}{c} S_i v^i, \quad v^i = \frac{dx^i}{dt}. \tag{12.88}$$

Equation (12.86) for $\mu = i$ assumes the form

$$\frac{dS_i}{dt} = c\Gamma^j_{i0} S_j - \Gamma^0_{i0} v^j S_j + \Gamma^j_{ik} v^k S_j - \Gamma^0_{ik} v^j S_j \frac{1}{c}. \tag{12.89}$$

For a static spherically symmetric source of the gravitational field in the linear approximation in the gravitational constant we have

$$g_{00} = 1+2\Phi, \quad g_{11} = g_{22} = g_{33} = 1-2\Phi, \quad \Phi = -\frac{GM}{c^2 r}. \tag{12.90}$$

Applying these expressions we calculate the Christoffel symbols

$$\begin{gathered} \Gamma^j_{i0} = 0, \quad \Gamma^0_{ik} = 0, \quad \Gamma^0_{i0} = \frac{\partial \Phi}{\partial x^i}, \\ \Gamma^j_{ik} = \frac{\partial \Phi}{\partial x^k} \delta_{ij} - \frac{\partial \Phi}{\partial x^i} \delta_{jk} + \frac{\partial \Phi}{\partial x^j} \delta_{ik}. \end{gathered} \tag{12.91}$$

Substituting these expressions into equation (12.89) we obtain

$$\frac{d\vec{S}}{dt} = -2(\vec{v}\vec{S})\nabla\Phi - (\vec{v}\nabla\Phi)\vec{S} + (\vec{S}\nabla\Phi)\vec{v}. \tag{12.92}$$

The following expression will be the integral of motion of this equation:

$$\vec{J}^2 = \vec{S}^2 + 2\Phi\vec{S}^2 - (\vec{v}\vec{S})^2. \tag{12.93}$$

This is readily verified by differentiating it with respect to time:

$$\begin{gathered} 2\vec{J}\frac{d\vec{J}}{dt} = 2\vec{S}\frac{d\vec{S}}{dt} + 4\Phi\vec{S}\frac{d\vec{S}}{dt} + \\ +2(\vec{v}\vec{S}) \left\{ (\vec{S}\nabla\Phi) + \vec{v}\left(\frac{d\vec{S}}{dt}\right) \right\}. \end{gathered} \tag{12.94}$$

Retaining the principal terms in this expression, we have

$$2\vec{J}\frac{d\vec{J}}{dt} = 2\vec{S}\frac{d\vec{S}}{dt} + 2(\vec{v}\vec{S})(\vec{S}\nabla\Phi). \qquad (12.95)$$

Multiplying equation (12.92) by $\vec{S}$ and retaining the principal terms, we obtain

$$\vec{S}\frac{d\vec{S}}{dt} = -(\vec{v}\vec{S})(\vec{S}\nabla\Phi). \qquad (12.96)$$

Substituting this expression into (12.95) we find

$$\vec{J}\frac{d\vec{J}}{dt} = 0. \qquad (12.97)$$

Thus, we have established that expression (12.93) is an integral of motion of equation (12.92). On the basis of (12.93) it is possible to construct the vector $\vec{J}$. Within the limits of accuracy it has the form

$$\vec{J} = (1+\Phi)\vec{S} - \frac{1}{2}\vec{v}(\vec{v}\vec{S}). \qquad (12.98)$$

Differentiating (12.98) with respect to time, within the limits of our accuracy, we obtain

$$\frac{d\vec{J}}{dt} = [\vec{\Omega}, \vec{J}], \quad \vec{\Omega} = -\frac{3}{2}[\vec{v}, \nabla\Phi]. \qquad (12.99)$$

The vector $\vec{J}$, while remaining the same in absolute value, undergoes precession with a velocity $|\vec{\Omega}|$ about the direction of vector $\vec{\Omega}$. At present such an experiment is at the stage of preparation. The precession of a gyroscope, determined by formula (12.99), shows that a reference system, connected with a gyroscope undergoing free motion, is not inertial. From the point of view of RTG this is obvious, since the motion of a gyroscope in the gravitational field represents an accelerated motion with respect to the inertial reference system related to distant stars. Precisely for this reason, the reference system connected with the gyroscope will be non-inertial, which is what causes precession of the gyroscope. In GRT a reference system connected with a gyroscope, undergoing free motion, is considered inertial. But then it is absolutely unclear, why this inertial reference system rotates with an angular velocity of $|\vec{\Omega}|$ relative to distant stars.

12.5. The gravitational shift of spectral lines

Consider a stationary gravitational field, i.e. when the metric coefficients are independent of time. Let radiation be emitted from point e by the source, and let it be received at point p by a receiver. If the source emits radiation during a time interval $(dt)_e$, then the receiver will also perceive it during an identical time interval, since the gravitational field is stationary.

The proper time at point e is

$$(d\tau)_e = (\sqrt{g_{00}}dt)_e, \tag{12.100}$$

and at point p it will be

$$(d\tau)_p = (\sqrt{g_{00}}dt)_p. \tag{12.101}$$

But, since the time $(dt)_e = (dt)_p$, from formulae (12.100) and (12.101) we obtain

$$\frac{(d\tau)_e}{(d\tau)_p} = \sqrt{\frac{(g_{00})_e}{(g_{00})_p}}. \tag{12.102}$$

Thus, the proper time interval, during which the source emits the signal, is not equal to the proper time interval, during which the signal is received, since the gravitational field differs from point e to p.

If we pass to the light frequency ω, then we obtain

$$\frac{\omega_e}{\omega_p} = \sqrt{\frac{(g_{00})_p}{(g_{00})_e}}. \tag{12.103}$$

Here ω_e is the frequency of light measured at the source point e, and ω_p is the frequency of the light that arrives from point e and is measured at the receiver point p. The change in frequency is characterized by the quantity

$$\frac{\delta\omega}{\omega} = \frac{\omega_e - \omega_p}{\omega_p}. \tag{12.104}$$

On the basis of (12.103) and (12.104) we find

$$\frac{\delta\omega}{\omega} = \sqrt{\frac{(g_{00})_p}{(g_{00})_e}} - 1. \tag{12.105}$$

For a weak gravitational field we have

$$g_{00} = 1 - 2U. \tag{12.106}$$

Substituting this expression into (12.105) we obtain

$$\frac{\delta\omega}{\omega} = U_e - U_p. \tag{12.107}$$

If the source (for example, an atom) is in a strong gravitational field, and the receiver is in a weaker field, then a red shift is observed, and the quantity $\delta\omega/\omega$ will be positive.

13. Some other physical conclusions of RTG

At large distances r from a static spherically symmetric body the metric coefficients have the form

$$U(r) = 1 - \frac{2M}{r}e^{-mr}, \quad V(r) = 1 + \frac{2M}{r}e^{-mr},$$
$$W = r\left(1 + \frac{M}{r}e^{-mr}\right).$$

We shall now deal with the problem of radiation of weak gravitational waves, when the graviton has mass. It has long been well known that in linear tensor theory introduction of the graviton mass is always accompanied by "ghosts". However, in refs. [15, 16, 39] it is shown that the intensity of the gravitational radiation of massive gravitons in nonlinear theory is a positive definite quantity, equal to

$$\frac{dI}{d\Omega} = \frac{2}{\pi}\int\limits_{\omega_{\min}}^{\infty} d\omega\omega^2 q\{|T_2^1|^2 + \frac{1}{4}|T_1^1 -_2^2|^2 +$$
$$+\frac{m^2}{\omega^2}(|T_3^1|^2 + |T_3^2|^2) + \frac{3m^4}{4\omega^4}|T_3^3|^2\}\ , \qquad (13.1)$$

here $q = \left(1 - \frac{m^2}{\omega^2}\right)^{1/2}$.

In RTG, like in GRT, outside matter the density of the energy-momentum tensor of the gravitational field in Riemannian space equals zero:

$$T_g^{\mu\nu} = -2\frac{\delta L_g}{\delta g_{\mu\nu}} = 0\ . \qquad (13.2)$$

However, from expression (13.2) no absence of the gravitational field follows. It is precisely in this expression, that the difference between the gravitational field and other physical fields is especially revealed. But this means that the energy flux of the gravitational field in the theory of gravity is not determined by the density components of the tensor T_g^{0i}, calculated with the aid of the solutions of equations (13.2), since they are equal to zero. The problem of determining the energy flux in the theory of gravity, unlike other theories, requires a different approach.

Yu.M. Loskutov [15, 16, 39] finds the solution of (13.2) in the form

$$\tilde{\Phi}^{\mu\nu} = \chi^{\mu\nu} + \psi^{\mu\nu} \ , \tag{13.3}$$

where the quantities $\chi^{\mu\nu}$ and $\psi^{\mu\nu}$ are of the same order of smallness, and $\psi^{\mu\nu}$ describes waves diverging from the source, while $\chi^{\mu\nu}$ characterizes the background. Energy transport is only realized by divergent waves. In ref. [15] it is shown that the flux of gravitational energy is actually determined by the quantity $T_g^{0i}(\psi)$ calculated not on the solutions of equations (13.2) themselves, but only on that part of solutions, that describes divergent waves $\psi^{\mu\nu}$. Here it is taken into account that gravitons do not travel in Minkowski space, like in linear theory, but in effective Riemannian space. Therefore, in the linear approximation the following equality is satisfied:

$$\begin{gathered}7\gamma_{\mu\nu}\frac{dx^\mu}{ds}\cdot\frac{dx^\nu}{ds} - 1 = \frac{d\sigma^2 - ds^2}{ds^2} \simeq \\ \simeq -\frac{1}{2}\gamma_{\mu\nu}\Phi^{\mu\nu} + \Phi^{\mu\nu}\frac{dx^\alpha}{d\sigma}\cdot\frac{dx^\beta}{d\sigma}\gamma_{\mu\alpha}\gamma_{\nu\beta} \ .\end{gathered} \tag{13.4}$$

Precisely taking this circumstance into account consistently in the course of finding the intensity has led the author of ref. [15] to the positively definite energy flux, determined by formula (13.1), the obtained result is of fundamental importance, since it alters conventional ideas and, therefore, it necessarily requires further analysis.

It must be noted that the set of gravitational equations (5.19) and (5.20) is hyperbolic, and precisely the causality principle provides for existence, throughout the entire space, of a spacelike surface, which is crossed by each nonspacelike curve in Riemannian space only once, i.e., in other words, there exists a global Cauchy surface, precisely on which the initial physical conditions are given for one or another problem. R. Penrose and S. Hawking [32] proved, for certain general conditions, singularity existence theorems in GRT. On the basis of equations (5.21a) outside matter, for isotropic vectors of Riemannian space, by virtue of the causality conditions (6.12a), the following inequality holds valid:

$$R_{\mu\nu}v^{\mu}v^{\nu} \leq 0 \ . \qquad (13.5)$$

The conditions of the aforementioned theorems are contrary to inequality (13.5), so they are not applicable in RTG.

In RTG spacelike events in the absence of a gravitational field can never become timelike under the influence of the gravitational field. On the basis of the causality principle effective Riemannian space in RTG will exhibit isotropic and timelike geodesic completeness. In accordance with RTG, an inertial reference system is determined by the distribution of matter and of the gravitational field in the Universe (Mach's principle).

In GRT the fields of inertia and of gravity are inseparable. A. Einstein wrote about this: "*... there exists no real division into inertia and gravity, since the answer to the question, of whether a body at a certain moment is exclusively under the influence of inertia or under the combined influence* *of inertia and gravity, depends on the reference system, i.e. on the method of dealing with it"*[35]. Fields of inertia satisfy the Hilbert-Einstein equations. In RTG the gravitational field and the fields of inertia, determined by the metric tensor of Minkowski space, are separated, they have nothing in common. They are of different natures. The fields of inertia are not solutions of RTG equations (5.19) and (5.20). In RTG the fields of inertia are given by the metric tensor $\gamma_{\mu\nu}$, while the gravitational field $\tilde{\Phi}^{\mu\nu}$ is determined from the equations of gravity (5.19) and (5.20).

In conclusion it must be noted that the idea that one can arbitrarily choose both the geometry (G) and the physics (P), since the sum (G+P) apparently seems to be the sole test object in the experiment, is not quite correct. The choice of pseudo-Euclidean geometry with the metric tensor $\gamma_{\mu\nu}$ is dictated both by fundamental physical principles — the integral conservation laws of energy-momentum and of angular momentum, and by other physical phenomena. Thus, physics (at the present-day stage) unambiguously determines the structure of the space-time geometry, within which all physical fields, including the gravitational field, develop. In accordance with RTG, the universal gravitational field creates effective Riemannian space with a simple topology, and Minkowski space does not vanish, here, but is manifested in the equations of theory and reflects a fundamental principle — the relativity principle. Effective Riemannian space is of a field nature.

On the basis of RTG one can draw the following general

[35]Einstein A. Collection of scientific works, M.: Nauka, 1965, vol.1, art.33, p.422.

conclusions:

Representation of the gravitational field as a physical field possessing the energy-momentum tensor, has drastically altered the general picture of gravity, earlier worked out on the basis of GRT. First, the theory of gravity now occupies its place in the same row as other physical theories based on the relativity principle, i.e. the primary space is Minkowski space. Hence, it immediately follows that for all natural phenomena, including gravitational phenomena, there exist fundamental physical laws of energy-momentum and angular momentum conservation. Since a universal quantity — the conserved energy-momentum tensor of matter (including the gravitational field) is the source of the gravitational field, then there arises effective Riemannian space-time, which is of a field nature. Since the formation of effective Riemannian space-time is due to the influence of the gravitational field, it automatically has a simple topology and is described in a sole coordinate system. The forces of inertia, unlike GRT, have nothing in common with the forces of gravity, since they differ in nature, the first arise owing to the choice of reference system in Minkowski space, while the latter are due to the presence of matter. The theory of the gravitation, like all other physical theories (unlike GRT) satisfy the equivalence principle.

Second, the complete set of equations of the theory of gravity permits to determine unambiguously gravitational effects in the Solar system and leads to other (differing from those of GRT) predictions both on the evolution of objects of large mass and on the development of a homogeneous and isotropic Universe. Theory reveals the formation of “black holes” (objects without material boundaries that are "cut off" from the external world) to be impossible and predicts the existence in the Universe of a large hidden mass of "dark" matter. From the theory it follows that there was no Big Bang, while some time in the past (about ten-fifteen billion years ago) there existed a state of high density and temperature, and the so-called "expansion" of the Universe, observed by the red shift, is not related to the relative motion of matter, but to variation in time of the gravitational field. Matter is at rest in an inertial reference system. The peculiar velocities of galaxies relative to inertial reference systems are due to inhomogeneities in the density distribution of matter, which is precisely what led to the accumulation of matter during the period, when the Universe became transparent.

The universal integral conservation laws of energy-momentum and such universal properties of matter, as, for example, gravitational interactions, are reflected in the metric properties of space-time. While the first are embodied in the pseudo-Euclidean geometry of space-time, the latter are reflected in effective Riemannian geometry of space-time, that arose owing to the presence of the gravitational field in Minkowski space. Everything that has a character common to all matter can be considered as a part of the structure of the effective geometry. But, here, Minkowski space will be present for certain, which is precisely what leads to the integral conservation laws of energy-momentum and angular momentum, and, also, provides for the equivalence principle to be satisfied, when the gravitational field, as well as other fields, are switched off.

Appendix A

Let us establish the relation

$$\frac{\delta L}{\delta \gamma_{\mu\nu}} = \frac{\delta L}{\delta g_{\alpha\beta}} \cdot \frac{\partial g_{\alpha\beta}}{\partial \gamma_{\mu\nu}} + \frac{\delta^\star L}{\delta \gamma_{\mu\nu}} \ , \tag{A.1}$$

here

$$\frac{\delta L}{\delta \gamma_{\mu\nu}} = \frac{\partial L}{\partial \gamma_{\mu\nu}} - \partial_\sigma \left(\frac{\partial L}{\partial \gamma_{\mu\nu,\sigma}} \right) \ , \tag{A.2}$$

$$\frac{\delta L}{\delta g_{\mu\nu}} = \frac{\partial L}{\partial g_{\alpha\beta}} - \partial_\sigma \left(\frac{\partial L}{\partial g_{\alpha\beta,\sigma}} \right), \tag{A.3}$$

the asterisk in the upper formula (A.1) indicates the variational derivative of the density of the Lagrangian with respect to the metric $\gamma_{\mu\nu}$ explicitly occurring in L. Upon differentiation we obtain

$$\frac{\partial L}{\partial \gamma_{\mu\nu}} = \frac{\partial^\star L}{\partial \gamma_{\mu\nu}} + \frac{\partial L}{\partial g_{\alpha\beta,\sigma}} \cdot \frac{\partial g_{\alpha\beta,\sigma}}{\partial \gamma_{\mu\nu}} + \frac{\partial L}{\partial g_{\alpha\beta}} \cdot \frac{\partial g_{\alpha\beta}}{\partial \gamma_{\mu\nu}} \ , \tag{A.4}$$

$$\frac{\partial L}{\partial \gamma_{\mu\nu,\sigma}} = \frac{\partial^\star L}{\partial \gamma_{\mu\nu,\sigma}} + \frac{\partial L}{\partial g_{\alpha\beta,\tau}} \cdot \frac{\partial g_{\alpha\beta,\tau}}{\partial \gamma_{\mu\nu,\sigma}} \ . \tag{A.5}$$

We substitute these expressions into formula (A.2):

$$\begin{aligned}
&\frac{\partial L}{\partial \gamma_{\mu\nu}} - \partial_\sigma \left(\frac{\partial L}{\partial \gamma_{\mu\nu,\sigma}} \right) = \frac{\delta^\star L}{\delta \gamma_{\mu\nu}} + \frac{\partial L}{\partial g_{\alpha\beta,\sigma}} \cdot \frac{\partial g_{\alpha\beta,\sigma}}{\partial \gamma_{\mu\nu}} + \\
&+\frac{\partial L}{\partial g_{\alpha\beta}} \cdot \frac{\partial g_{\alpha\beta}}{\partial \gamma_{\mu\nu}} - \partial_\sigma \left(\frac{\partial L}{\partial g_{\alpha\beta,\tau}} \cdot \frac{\partial g_{\alpha\beta,\tau}}{\partial \gamma_{\mu\nu,\sigma}} \right) = \frac{\delta^\star L}{\delta \gamma_{\mu\nu}} + \\
&+\frac{\partial L}{\partial g_{\alpha\beta}} \cdot \frac{\partial g_{\alpha\beta}}{\partial \gamma_{\mu\nu}} - \partial_\sigma \left(\frac{\partial L}{\partial g_{\alpha\beta,\tau}} \right) \cdot \frac{\partial g_{\alpha\beta,\tau}}{\partial \gamma_{\mu\nu,\sigma}} + \\
&+\frac{\partial L}{\partial g_{\alpha\beta,\sigma}} \left[\frac{\partial g_{\alpha\beta,\sigma}}{\partial \gamma_{\mu\nu}} - \partial_\rho \left(\frac{\partial g_{\alpha\beta,\sigma}}{\partial \gamma_{\mu\nu,\rho}} \right) \right].
\end{aligned} \tag{A.6}$$

Now, consider expression

$$\frac{\partial g_{\alpha\beta,\sigma}}{\partial \gamma_{\mu\nu}} - \partial_\rho \left(\frac{\partial g_{\alpha\beta,\sigma}}{\partial \gamma_{\mu\nu,\rho}} \right). \tag{A.7}$$

For this purpose we shall write the derivative $g_{\alpha\beta,\sigma}$ in the form

$$g_{\alpha\beta,\sigma} = \frac{\partial g_{\alpha\beta}}{\partial \gamma_{\lambda\omega}} \partial_\sigma \gamma_{\lambda\omega} + \frac{\partial g_{\alpha\beta}}{\partial \Phi_{\lambda\omega}} \partial_\sigma \Phi_{\lambda\omega} \ , \tag{A.8}$$

hence it is easy to find

$$\frac{\partial g_{\alpha\beta,\sigma}}{\partial \gamma_{\mu\nu,\rho}} = \frac{\partial g_{\alpha\beta}}{\partial \gamma_{\mu\nu}} \cdot \delta_\sigma^\rho \ . \tag{A.9}$$

Upon differentiating this expression we have

$$\partial_\rho \left(\frac{\partial g_{\alpha\beta,\sigma}}{\partial \gamma_{\mu\nu,\rho}} \right) = \frac{\partial^2 g_{\alpha\beta}}{\partial \gamma_{\mu\nu} \partial \gamma_{\lambda\omega}} \partial_\sigma \gamma_{\lambda\omega} + \frac{\partial^2 g_{\alpha\beta}}{\partial \gamma_{\mu\nu} \partial \Phi_{\lambda\omega}} \partial_\sigma \Phi_{\lambda\omega} \ . \tag{A.10}$$

On the other hand, differentiating (A.8) with respect to $\gamma_{\mu\nu}$ we have

$$\frac{\partial g_{\alpha\beta,\sigma}}{\partial \gamma_{\mu\nu}} = \frac{\partial^2 g_{\alpha\beta}}{\partial \gamma_{\mu\nu} \partial \gamma_{\lambda\omega}} \partial_\sigma \gamma_{\lambda\omega} + \frac{\partial^2 g_{\alpha\beta}}{\partial \gamma_{\mu\nu} \partial \Phi_{\lambda\omega}} \partial_\sigma \Phi_{\lambda\omega} \ . \tag{A.11}$$

Comparing (A.10) and (A.11) we find

$$\frac{\partial g_{\alpha\beta,\sigma}}{\partial \gamma_{\mu\nu}} - \partial_\rho \left(\frac{\partial g_{\alpha\beta,\sigma}}{\partial \gamma_{\mu\nu,\rho}} \right) = 0 \ . \tag{A.12}$$

Taking this relation into account, we obtain in (A.6)

$$\frac{\delta L}{\delta \gamma_{\mu\nu}} = \frac{\delta^\star L}{\delta \gamma_{\mu\nu}} + \frac{\partial L}{\partial g_{\alpha\beta}} \cdot \frac{\partial g_{\alpha\beta}}{\partial \gamma_{\mu\nu}} - \partial_\sigma \left(\frac{\partial L}{\partial g_{\alpha\beta,\tau}} \right) \cdot \frac{\partial g_{\alpha\beta,\tau}}{\partial \gamma_{\mu\nu,\sigma}} \ . \tag{A.13}$$

Substituting (A.9) into (A.13) we find

$$\frac{\delta L}{\delta \gamma_{\mu\nu}} = \frac{\delta^\star L}{\delta \gamma_{\mu\nu}} + \left[\frac{\partial L}{\partial g_{\alpha\beta}} - \partial_\sigma \left(\frac{\partial L}{\partial g_{\alpha\beta,\sigma}} \right) \right] \frac{\partial g_{\alpha\beta}}{\partial \gamma_{\mu\nu}} \ , \tag{A.14}$$

i.e.,

$$\frac{\delta L}{\delta \gamma_{\mu\nu}} = \frac{\delta^\star L}{\delta \gamma_{\mu\nu}} + \frac{\delta L}{\delta g_{\alpha\beta}} \cdot \frac{\partial g_{\alpha\beta}}{\partial \gamma_{\mu\nu}}. \tag{A.15}$$

The following is calculated in a similar manner:

$$\frac{\delta L}{\delta g_{\alpha\beta}} = \frac{\delta L}{\delta \tilde{g}^{\lambda\rho}} \cdot \frac{\partial \tilde{g}^{\lambda\rho}}{\partial g_{\alpha\beta}} \,. \tag{A.16}$$

Making use of (A.16), one can write expression (A.15) as follows:

$$\frac{\delta L}{\delta \gamma_{\mu\nu}} = \frac{\delta^{\star} L}{\delta \gamma_{\mu\nu}} + \frac{\delta L}{\delta \tilde{g}^{\lambda\rho}} \cdot \frac{\partial \tilde{g}^{\lambda\rho}}{\partial \gamma_{\mu\nu}} \,. \tag{A.17}$$

Appendix B

The density of the Lagrangian of the gravitational field proper has the form

$$L_g = L_{g0} + L_{gm} \ , \tag{B.1}$$

$$L_{g0} = -\frac{1}{16\pi}\tilde{g}^{\alpha\beta}\left(G^\tau_{\lambda\alpha}G^\lambda_{\tau\beta} - G^\tau_{\alpha\beta}G^\lambda_{\tau\lambda}\right), \tag{B.2}$$

$$L_{gm} = -\frac{m^2}{16\pi}\left(\frac{1}{2}\gamma_{\alpha\beta}\tilde{g}^{\alpha\beta} - \sqrt{-g} - \sqrt{-\gamma}\right). \tag{B.3}$$

The third-rank tensor $G^\tau_{\alpha\beta}$ is

$$G^\tau_{\alpha\beta} = \frac{1}{2}g^{\tau\lambda}(D_\alpha g_{\beta\lambda} + D_\beta g_{\alpha\lambda} - D_\lambda g_{\alpha\beta}) \ , \tag{B.4}$$

it is expressed via the Christoffel symbols of Riemannian space and of Minkowski space:

$$G^\tau_{\alpha\beta} = \Gamma^\tau_{\alpha\beta} - \gamma^\tau_{\alpha\beta} \ . \tag{B.5}$$

Let us calculate the variational derivative of L_g with respect to the explicitly present metric of Minkowski space, $\gamma_{\mu\nu}$:

$$\frac{\delta^\star L_{g0}}{\delta\gamma_{\mu\nu}} = \frac{\partial L_{g0}}{\partial\gamma_{\mu\nu}} - \partial_\sigma\left(\frac{\partial L_{g0}}{\partial\gamma_{\mu\nu,\sigma}}\right). \tag{B.6}$$

For this purpose we perform certain preparatory calculations:

$$\begin{aligned}\frac{\partial G^\lambda_{\alpha\beta}}{\partial\gamma_{\mu\nu}} &= -\frac{\partial\gamma^\lambda_{\alpha\beta}}{\partial\gamma_{\mu\nu}} = \frac{1}{2}(\gamma^{\lambda\mu}\gamma^\nu_{\alpha\beta} + \gamma^{\lambda\nu}\gamma^\mu_{\alpha\beta}) \ , \\ \frac{\partial G^\lambda_{\alpha\lambda}}{\partial\gamma_{\mu\nu}} &= -\frac{\partial\gamma^\lambda_{\alpha\lambda}}{\partial\gamma_{\mu\nu}} = \frac{1}{2}(\gamma^{\lambda\mu}\gamma^\nu_{\lambda\alpha} + \gamma^{\lambda\nu}\gamma^\mu_{\alpha\lambda}) \ ,\end{aligned} \tag{B.7}$$

$$\begin{aligned}&\frac{\partial G^\lambda_{\alpha\beta}}{\partial\gamma_{\mu\nu,\sigma}} = -\frac{\partial\gamma^\lambda_{\alpha\beta}}{\partial\gamma_{\mu\nu,\sigma}} = -\frac{1}{4}\Big[\gamma^{\lambda\mu}(\delta^\nu_\alpha\delta^\sigma_\beta + \delta^\sigma_\alpha\delta^\nu_\beta) + \\ &+\gamma^{\lambda\nu}(\delta^\mu_\alpha\delta^\sigma_\beta + \delta^\sigma_\alpha\delta^\mu_\beta) - \gamma^{\lambda\sigma}(\delta^\mu_\alpha\delta^\nu_\beta + \delta^\nu_\alpha\delta^\mu_\beta)\Big] \ , \\ &\frac{\partial G^\lambda_{\alpha\lambda}}{\partial\gamma_{\mu\nu,\sigma}} = -\frac{\partial\gamma^\lambda_{\alpha\lambda}}{\partial\gamma_{\mu\nu,\sigma}} = -\frac{1}{2}\gamma^{\mu\nu}\delta^\sigma_\alpha \ .\end{aligned} \tag{B.8}$$

Differentiating (B.2) we obtain

$$\frac{\partial L_{g0}}{\partial \gamma_{\mu\nu}} = -\frac{1}{16\pi}\tilde{g}^{\alpha\beta}\left[\frac{\partial G^{\tau}_{\alpha\lambda}}{\partial \gamma_{\mu\nu}}G^{\lambda}_{\tau\beta} + G^{\tau}_{\lambda\alpha}\frac{\partial G^{\lambda}_{\tau\beta}}{\partial \gamma_{\mu\nu}} - \right.$$
$$\left. -\frac{\partial G^{\tau}_{\alpha\beta}}{\partial \gamma_{\mu\nu}}G^{\lambda}_{\tau\lambda} - G^{\tau}_{\alpha\beta}\frac{\partial G^{\lambda}_{\tau\lambda}}{\partial \gamma_{\mu\nu}}\right].$$

Substituting into this expression formulae (B.7) we find

$$\frac{\partial L_{g0}}{\partial \gamma_{\mu\nu}} = -\frac{1}{16\pi}\tilde{g}^{\alpha\beta}\Big\{G^{\tau}_{\lambda\alpha}\gamma^{\lambda\mu}\gamma^{\nu}_{\tau\beta} + G^{\tau}_{\lambda\alpha}\gamma^{\lambda\nu}\gamma^{\mu}_{\tau\beta} - \frac{1}{2}G^{\lambda}_{\tau\lambda}\gamma^{\tau\mu}\gamma^{\nu}_{\alpha\beta} -$$
$$-\frac{1}{2}G^{\lambda}_{\tau\lambda}\gamma^{\tau\nu}\gamma^{\mu}_{\alpha\beta} - \frac{1}{2}G^{\tau}_{\alpha\beta}\gamma^{\lambda\mu}\gamma^{\nu}_{\tau\lambda} - \frac{1}{2}G^{\tau}_{\alpha\beta}\gamma^{\lambda\nu}\gamma^{\mu}_{\tau\lambda}\Big\} = \frac{1}{32\pi}B^{\mu\nu}. \quad \text{(B.9)}$$

With the aid of the derivatives (B.8) we obtain

$$\frac{\partial L_{g0}}{\partial \gamma_{\mu\nu,\sigma}} = \frac{1}{32\pi}A^{\sigma\mu\nu}, \qquad \text{(B.10)}$$

where

$$A^{\sigma\mu\nu} = \gamma^{\tau\mu}(G^{\sigma}_{\tau\beta}\tilde{g}^{\nu\beta} + G^{\nu}_{\tau\beta}\tilde{g}^{\sigma\beta} - G^{\lambda}_{\tau\lambda}\tilde{g}^{\sigma\nu}) -$$
$$-\gamma^{\mu\nu}G^{\sigma}_{\alpha\beta}\tilde{g}^{\alpha\beta} + \gamma^{\tau\nu}(G^{\sigma}_{\tau\beta}\tilde{g}^{\mu\beta} + G^{\mu}_{\tau\beta}\tilde{g}^{\sigma\beta} - G^{\lambda}_{\tau\lambda}\tilde{g}^{\sigma\mu}) +$$
$$+\gamma^{\tau\sigma}(G^{\lambda}_{\tau\lambda}\tilde{g}^{\mu\nu} - G^{\mu}_{\tau\beta}\tilde{g}^{\nu\beta} - G^{\nu}_{\tau\beta}\tilde{g}^{\mu\beta}). \qquad \text{(B.10}'\text{)}$$

The density of the tensor $A^{\sigma\mu\nu}$ is symmetric in indices μ and ν. The ordinary derivative of this density can be represented in the form

$$\partial_\sigma A^{\sigma\mu\nu} = D_\sigma A^{\sigma\mu\nu} - \gamma^{\mu}_{\sigma\rho}A^{\sigma\rho\nu} - \gamma^{\nu}_{\sigma\rho}A^{\sigma\mu\rho}.$$

Substituting into (B.6) expressions (B.9) and (B.10) we find

$$\frac{\delta^\star L_{g0}}{\delta \gamma_{\mu\nu}} = \frac{1}{32\pi}B^{\mu\nu} - \frac{1}{32\pi}D_\sigma A^{\sigma\mu\nu} +$$
$$+\frac{1}{32\pi}\gamma^{\mu}_{\sigma\rho}A^{\sigma\rho\nu} + \frac{1}{32\pi}\gamma^{\nu}_{\sigma\rho}A^{\sigma\mu\rho}. \qquad \text{(B.11)}$$

Now, we write the density of the tensor $A^{\sigma\rho\nu}$ in the form

$$A^{\sigma\rho\nu} = (G^{\sigma}_{\tau\beta}\gamma^{\tau\rho}\tilde{g}^{\nu\beta} - G^{\rho}_{\tau\beta}\gamma^{\tau\sigma}\tilde{g}^{\nu\beta}) + (G^{\nu}_{\tau\beta}\gamma^{\tau\rho}\tilde{g}^{\sigma\beta} - G^{\nu}_{\tau\beta}\gamma^{\tau\sigma}\tilde{g}^{\rho\beta}) -$$
$$-(G^{\lambda}_{\tau\lambda}\gamma^{\tau\rho}\tilde{g}^{\sigma\nu} - G^{\lambda}_{\tau\lambda}\gamma^{\tau\sigma}\tilde{g}^{\rho\nu}) + G^{\sigma}_{\tau\beta}\gamma^{\tau\nu}\tilde{g}^{\rho\beta} + G^{\rho}_{\tau\beta}\gamma^{\tau\nu}\tilde{g}^{\sigma\beta} -$$
$$-G^{\lambda}_{\tau\lambda}\gamma^{\tau\nu}\tilde{g}^{\sigma\rho} - G^{\sigma}_{\alpha\beta}\gamma^{\rho\nu}\tilde{g}^{\alpha\beta},$$

in the brackets, terms antisymmetric in indices σ and ρ have been formed. Writing in such a way facilitates finding the expression for the quantity $\gamma^{\mu}_{\sigma\rho}A^{\sigma\rho\nu}$, since the terms antisymmetric in indices σ and ρ vanish automatically, here.

$$\gamma^{\mu}_{\sigma\rho}A^{\sigma\rho\nu} = 2G^{\sigma}_{\tau\beta}\gamma^{\mu}_{\sigma\rho}\gamma^{\tau\nu}\tilde{g}^{\rho\beta} -$$
$$-G^{\lambda}_{\tau\lambda}\gamma^{\mu}_{\sigma\rho}\gamma^{\tau\nu}\tilde{g}^{\sigma\rho} - G^{\sigma}_{\alpha\beta}\gamma^{\mu}_{\sigma\rho}\gamma^{\nu\rho}\tilde{g}^{\alpha\beta}. \quad \text{(B.12)}$$

Representing in a similar manner $A^{\sigma\mu\rho}$ as

$$A^{\sigma\mu\rho} = (G^{\sigma}_{\tau\beta}\gamma^{\tau\rho}\tilde{g}^{\mu\beta} - G^{\rho}_{\tau\beta}\gamma^{\tau\sigma}\tilde{g}^{\mu\beta}) + (G^{\mu}_{\tau\beta}\gamma^{\tau\rho}\tilde{g}^{\sigma\beta} - G^{\mu}_{\tau\beta}\gamma^{\tau\sigma}\tilde{g}^{\rho\beta}) +$$
$$+(G^{\lambda}_{\tau\lambda}\gamma^{\tau\sigma}\tilde{g}^{\mu\rho} - G^{\lambda}_{\tau\lambda}\gamma^{\tau\rho}\tilde{g}^{\sigma\mu}) + G^{\sigma}_{\tau\beta}\gamma^{\tau\mu}\tilde{g}^{\rho\beta} + G^{\rho}_{\tau\beta}\gamma^{\tau\mu}\tilde{g}^{\sigma\beta} -$$
$$-G^{\lambda}_{\tau\lambda}\gamma^{\tau\mu}\tilde{g}^{\sigma\rho} - G^{\sigma}_{\alpha\beta}\gamma^{\mu\rho}\tilde{g}^{\alpha\beta},$$

where again in the brackets terms antisymmetric in the indices σ and ρ are formed, we obtain

$$\gamma^{\nu}_{\sigma\rho}A^{\sigma\mu\rho} = 2G^{\sigma}_{\tau\beta}\gamma^{\nu}_{\sigma\rho}\gamma^{\tau\mu}\tilde{g}^{\rho\beta} -$$
$$-G^{\lambda}_{\tau\lambda}\gamma^{\nu}_{\sigma\rho}\gamma^{\tau\mu}\tilde{g}^{\sigma\rho} - G^{\sigma}_{\alpha\beta}\gamma^{\nu}_{\sigma\rho}\gamma^{\mu\rho}\tilde{g}^{\alpha\beta}. \quad \text{(B.13)}$$

Summing (B.12) and (B.13) one readily verifies the following equality:

$$\gamma^{\mu}_{\sigma\rho}A^{\sigma\rho\nu} + \gamma^{\nu}_{\sigma\rho}A^{\sigma\mu\rho} = -B^{\mu\nu}. \quad \text{(B.14)}$$

Taking this equality into account we write expression (B.11) in the form

$$\frac{\delta L_{g0}}{\delta\gamma_{\mu\nu}} = -\frac{1}{32\pi}D_{\sigma}A^{\sigma\mu\nu}. \quad \text{(B.15)}$$

Taking into account the equalities

$$G^{\lambda}_{\tau\lambda} = \frac{1}{2} g^{\lambda\rho} D_\tau g_{\lambda\rho}, \;\; D_\tau\sqrt{-g} = \sqrt{-g}G^{\lambda}_{\tau\lambda} \; ,$$

we find

$$\begin{aligned} G^{\sigma}_{\tau\beta}\tilde{g}^{\nu\beta} + G^{\nu}_{\tau\beta}\tilde{g}^{\sigma\beta} - G^{\lambda}_{\tau\lambda}\tilde{g}^{\sigma\nu} &= -D_\tau \tilde{g}^{\nu\sigma}, \\ G^{\sigma}_{\tau\beta}\tilde{g}^{\mu\beta} + G^{\mu}_{\tau\beta}\tilde{g}^{\sigma\beta} - G^{\lambda}_{\tau\lambda}\tilde{g}^{\sigma\mu} &= -D_\tau \tilde{g}^{\mu\sigma} \\ G^{\nu}_{\tau\beta}\tilde{g}^{\mu\beta} + G^{\mu}_{\tau\beta}\tilde{g}^{\nu\beta} - G^{\lambda}_{\tau\lambda}\tilde{g}^{\mu\nu} &= -D_\tau \tilde{g}^{\mu\nu}. \end{aligned} \tag{B.16}$$

Substituting these expressions into (B.10′) we obtain

$$A^{\sigma\mu\nu} = \gamma^{\tau\sigma} D_\tau \tilde{g}^{\mu\nu} + \gamma^{\mu\nu} D_\tau \tilde{g}^{\tau\sigma} - \gamma^{\tau\mu} D_\tau \tilde{g}^{\nu\sigma} - \gamma^{\tau\nu} D_\tau \tilde{g}^{\mu\sigma}.$$

Substituting this expression into (B.15) we find

$$\frac{\delta^\star L_{g0}}{\delta\gamma_{\mu\nu}} = \frac{1}{32\pi} J^{\mu\nu}, \tag{B.17}$$

where $J^{\mu\nu} = -D_\sigma D_\tau(\gamma^{\tau\sigma}\tilde{g}^{\mu\nu} + \gamma^{\mu\nu}\tilde{g}^{\tau\sigma} - \gamma^{\tau\mu}\tilde{g}^{\nu\sigma} - \gamma^{\tau\nu}\tilde{g}^{\mu\sigma})$.

On the basis of (B.3) we have

$$\frac{\delta^\star L_{gm}}{\delta\gamma_{\mu\nu}} = -\frac{m^2}{32\pi}(\tilde{g}^{\mu\nu} - \tilde{\gamma}^{\mu\nu}) = -\frac{m^2}{32\pi}\tilde{\Phi}^{\mu\nu}. \tag{B.18}$$

Thus, taking into account (B.1) and applying (B.17) and (B.18) we find

$$\frac{\delta^\star L_{g}}{\delta\gamma_{\mu\nu}} = \frac{1}{32\pi}(J^{\mu\nu} - m^2\tilde{\Phi}^{\mu\nu}) \; , \tag{B.19}$$

and, consequently,

$$-2\frac{\delta^\star L_{g}}{\delta\gamma_{\mu\nu}} = \frac{1}{16\pi}(-J^{\mu\nu} + m^2\tilde{\Phi}^{\mu\nu}) \; . \tag{B.20}$$

Appendix B*

In this Appendix we shall make use of expressions (B.2) and (B.3) for the density of the Lagrangian L_{g0} and L_{gm} in order to establish the following equalities:

$$\frac{\delta L_{g0}}{\delta \tilde{g}^{\alpha\beta}} = -\frac{1}{16\pi} R_{\alpha\beta}, \quad \frac{\delta L_{gm}}{\delta \tilde{g}^{\alpha\beta}} = \frac{m^2}{32\pi}(g_{\alpha\beta} - \gamma_{\alpha\beta}), \qquad \text{(B*.1)}$$

here, by definition, the tensors $\frac{\delta L_{g0}}{\delta \tilde{g}^{\alpha\beta}}$, $\frac{\delta L_{gm}}{\delta \tilde{g}^{\alpha\beta}}$ are equal to

$$\frac{\delta L_{g0}}{\delta \tilde{g}^{\alpha\beta}} = \frac{\partial L_{g0}}{\partial \tilde{g}^{\alpha\beta}} - \partial_\sigma \frac{\partial L_{g0}}{\partial \tilde{g}^{\alpha\beta}_{,\sigma}}, \quad \frac{\delta L_{gm}}{\delta \tilde{g}^{\alpha\beta}} = \frac{\partial L_{gm}}{\partial \tilde{g}^{\alpha\beta}} - \partial_\sigma \frac{\partial L_{gm}}{\partial \tilde{g}^{\alpha\beta}_{,\sigma}}. \quad \text{(B*.2)}$$

The tensor relations (B*.1) are most readily established in a local Riemann reference system, where the derivatives of the components of the metric tensor $g_{\mu\nu}$ with respect to the coordinates are zero and, consequently, the Christoffel symbols $\Gamma^\lambda_{\mu\nu}$ are also zero.

On the basis of the formula

$$\frac{\partial \Gamma^\tau_{\lambda\alpha}}{\partial g_{\mu\nu}} = -\frac{1}{2}\left(g^{\mu\tau}\Gamma^\nu_{\alpha\lambda} + g^{\nu\tau}\Gamma^\mu_{\alpha\lambda}\right). \qquad \text{(B*.3)}$$

it is easy to establish that in the indicated reference system the following equality holds valid:

$$\frac{\partial L_{g0}}{\partial g_{\mu\nu}} = \frac{\sqrt{-g}}{16\pi}\left(g^{\alpha\mu}g^{\beta\nu} - \frac{1}{2}g^{\mu\nu}g^{\alpha\beta}\right)) \times$$
$$\times \left(\gamma^\tau_{\lambda\alpha}\gamma^\lambda_{\tau\beta} - \gamma^\tau_{\alpha\beta}\gamma^\lambda_{\tau\lambda}\right). \qquad \text{(B*.4)}$$

Here, $\gamma^\lambda_{\tau\beta}$ are the Christoffel symbols of Minkowski space. Making use of expression

$$\frac{\partial \Gamma^\tau_{\alpha\lambda}}{\partial g_{\mu\nu,\sigma}} = \frac{1}{4}\{g^{\tau\mu}(\delta^\nu_\alpha\delta^\sigma_\lambda + \delta^\nu_\lambda\delta^\sigma_\alpha) + g^{\tau\nu}(\delta^\mu_\alpha\delta^\sigma_\lambda +$$
$$+\delta^\mu_\lambda\delta^\sigma_\alpha) - g^{\tau\sigma}(\delta^\mu_\alpha\delta^\nu_\lambda + \delta^\mu_\lambda\delta^\nu_\alpha)\}, \qquad \text{(B*.5)}$$

we obtain

$$-\frac{\partial L_{g0}}{\partial g_{\mu\nu,\sigma}} = \frac{\sqrt{-g}}{16\pi}\Big[(g^{\alpha\mu}g^{\beta\nu} - \frac{1}{2}g^{\mu\nu}g^{\alpha\beta})(\Gamma^\sigma_{\alpha\beta} - \gamma^\sigma_{\alpha\beta}) - \\ - (g^{\alpha\mu}g^{\sigma\nu} - \frac{1}{2}g^{\mu\nu}g^{\alpha\sigma})(\Gamma^\lambda_{\alpha\lambda} - \gamma^\lambda_{\alpha\lambda})\Big]. \tag{B*.6}$$

Hence in a local Riemann reference system we find

$$-\partial_\sigma \frac{\partial L_{g0}}{\partial g_{\mu\nu,\sigma}} = \frac{\sqrt{-g}}{16\pi}\left(g^{\alpha\mu}g^{\beta\nu} - \frac{1}{2}g^{\mu\nu}g^{\alpha\beta}\right)\cdot \\ \times\Big[(\partial_\sigma\Gamma^\sigma_{\alpha\beta} - \partial_\beta\Gamma^\lambda_{\alpha\lambda}) - (\partial_\sigma\gamma^\sigma_{\alpha\beta} - \partial_\beta\gamma^\lambda_{\alpha\lambda})\Big]. \tag{B*.7}$$

On the basis of (B*.4) and (B*.7) the tensor (B*.2) in the local Riemann reference system is

$$\frac{\delta L_{g0}}{\delta g_{\mu\nu}} = \frac{\partial L_{g0}}{\partial g_{\mu\nu}} - \partial_\sigma\frac{\partial L_{g0}}{\partial g_{\mu\nu,\sigma}} = \frac{\sqrt{-g}}{16\pi}\Big(g^{\alpha\mu}g^{\beta\nu} - \\ -\frac{1}{2}g^{\mu\nu}g^{\alpha\beta}\Big)\left(\partial_\sigma\Gamma^\sigma_{\alpha\beta} - \partial_\beta\Gamma^\lambda_{\alpha\lambda} - R_{\alpha\beta}(\gamma)\right). \tag{B*.8}$$

In a local Riemann reference system the second-rank curvature tensor of Riemann space, $R_{\alpha\beta}(g)$, has the form

$$R_{\alpha\beta}(g) = \partial_\sigma\Gamma^\sigma_{\alpha\beta} - \partial_\beta\Gamma^\lambda_{\alpha\lambda}. \tag{B*.9}$$

In expression (B*.8), the second-rank tensor $R_{\alpha\beta}(\gamma)$ is

$$R_{\alpha\beta}(\gamma) = \partial_\sigma\gamma^\sigma_{\alpha\beta} - \partial_\beta\gamma^\lambda_{\alpha\lambda} + \gamma^\tau_{\alpha\beta}\gamma^\lambda_{\tau\lambda} - \gamma^\tau_{\lambda\alpha}\gamma^\lambda_{\tau\beta}.$$

In Minkowski space with the metric $\gamma_{\mu\nu}$ and Christoffel symbols $\gamma^\sigma_{\mu\nu}$ the tensor $R_{\alpha\beta}(\gamma)$ equals zero. Taking into account (B*.9) and, also, that the tensor $R_{\alpha\beta}(\gamma)$ equals zero, the tensor relation (B*.8) assumes the form

$$\frac{\partial L_{g0}}{\partial g_{\mu\nu}} = \frac{\sqrt{-g}}{16\pi}\left(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R\right). \tag{B*.10}$$

This equality has been established in a local Riemann reference system, but by virtue of its tensor character it holds valid in any reference system.

Applying relation

$$dg = -g g_{\alpha\beta} dg^{\alpha\beta}, \tag{B*.11}$$

we find

$$\frac{\partial g}{\partial \tilde{g}^{\alpha\beta}} = g\tilde{g}_{\alpha\beta}. \tag{B*.12}$$

On the basis of equality

$$\tilde{g}^{\mu\sigma} g_{\sigma\nu} = \delta^{\mu}_{\nu}\sqrt{-g} \tag{B*.13}$$

it is easy to obtain the following relation:

$$\frac{\partial g_{\lambda\nu}}{\partial \tilde{g}^{\alpha\beta}} = \frac{1}{\sqrt{-g}}\left\{-\frac{1}{2}(g_{\lambda\alpha}g_{\nu\beta} + g_{\lambda\beta}g_{\nu\alpha}) + \frac{1}{2}g_{\alpha\beta}g_{\nu\lambda}\right\}. \tag{B*.14}$$

Since on the basis of Appendix A the following equality is valid:

$$\frac{\delta L_{g0}}{\delta \tilde{g}^{\alpha\beta}} = \frac{\delta L_{g0}}{\delta g_{\lambda\nu}} \cdot \frac{\partial g_{\lambda\nu}}{\partial \tilde{g}^{\alpha\beta}}, \tag{B*.15}$$

then, upon applying expressions (B*.10) and (B*.14) we find

$$\frac{\delta L_{g0}}{\delta \tilde{g}^{\alpha\beta}} = -\frac{1}{16\pi} R_{\alpha\beta}. \tag{B*.16}$$

In a similar manner we have

$$\frac{\delta L_{gm}}{\delta \tilde{g}^{\alpha\beta}} = \frac{\partial L_{gm}}{\partial \tilde{g}^{\alpha\beta}} = \frac{m^2}{32\pi}(g_{\alpha\beta} - \gamma_{\alpha\beta}). \tag{B*.17}$$

Adding up expressions (B*.16) and (B*.17) we obtain

$$\frac{\delta L_g}{\delta \tilde{g}^{\alpha\beta}} = -\frac{1}{16\pi} R_{\alpha\beta} + \frac{m^2}{32\pi}(g_{\alpha\beta} - \gamma_{\alpha\beta}). \tag{B*.18}$$

It is also easy to obtain the following relation:

$$\frac{\delta L_M}{\delta \tilde{g}^{\alpha\beta}} = \frac{\delta L_M}{\delta g_{\lambda\nu}} \cdot \frac{\partial g_{\lambda\nu}}{\partial \tilde{g}^{\alpha\beta}} = -\frac{1}{2} T^{\lambda\nu} \frac{\partial g_{\lambda\nu}}{\partial \tilde{g}^{\alpha\beta}} =$$
$$= \frac{1}{2\sqrt{-g}} (T_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} T). \qquad (\mathrm{B}^*.19)$$

Here $T^{\lambda\nu} = -2\frac{\delta L_M}{\delta g_{\lambda\nu}}$ is the density of the energy-momentum tensor of matter in effective Riemannian space. At last we present the following relations:

$$D_\nu \sqrt{-g} = \partial_\nu \left(\sqrt{\frac{g}{\gamma}} \right) = \sqrt{-g} G^\lambda_{\nu\lambda},$$
$$\partial_\nu \sqrt{-g} = \sqrt{-g} \Gamma^\lambda_{\nu\lambda}, \quad \partial_\nu \sqrt{-\gamma} = \sqrt{-\gamma} \gamma^\lambda_{\nu\lambda}, \qquad (\mathrm{B}^*.20)$$

that are applied in obtaining equality (5.15).

Appendix C

For any given density of the Lagrangian L, in the case of an infinitesimal change in the coordinates, the variation of the action

$$S = \int L d^4 x$$

will be zero. We shall calculate the variation of the action of the density of the Lagrangian L_M

$$S_M = \int L_M(\tilde{g}^{\mu\nu}, \Phi_A) d^4 x$$

of matter and establish a strong identity. In the case of the following transformation of the coordinates:

$$x'^{\mu} = x^{\mu} + \xi^{\mu}(x) \ , \tag{C.1}$$

where $\xi^{\mu}(x)$ is an infinitesimal displacement four-vector, the variation of the action under transformation of the coordinates equals

$$\delta_c S_M = \int d^4 x \left(\frac{\delta L_M}{\delta \tilde{g}^{\mu\nu}} \delta_L \tilde{g}^{\mu\nu} + \frac{\delta L_M}{\delta \Phi_A} \delta_L \Phi_A + \mathrm{div} \right) = 0. \tag{C.2}$$

In this expression div stands for the divergence terms, which are not essential for our purposes.

The Euler variation is defined as usual:

$$\frac{\delta L}{\delta \Phi} \equiv \frac{\partial L}{\partial \Phi} - \partial_{\mu} \frac{\partial L}{\partial(\partial_{\mu} \Phi)} + \partial_{\mu} \partial_{\nu} \frac{\partial L}{\partial(\partial_{\mu} \partial_{\nu} \Phi)} \ .$$

The Lie variations $\delta_L \tilde{g}^{\mu\nu}, \delta_L \Phi_A$ are readily calculated under changes of the coordinates, if the transformation law of the quantities $g^{\mu\nu}, \Phi_A$ is applied:

$$\begin{aligned} \delta_L \tilde{g}^{\mu\nu} &= \tilde{g}^{\lambda\mu} D_{\lambda} \xi^{\nu} + \tilde{g}^{\lambda\nu} D_{\lambda} \xi^{\mu} - D_{\lambda}(\xi^{\lambda} \tilde{g}^{\mu\nu}), \\ \delta_L \Phi_A &= -\xi^{\lambda} D_{\lambda} \Phi_A + F_{A;\sigma}^{B;\lambda} \Phi_B D_{\lambda} \xi^{\sigma}, \end{aligned} \tag{C.3}$$

D_λ are covariant derivatives in Minkowski space. Substituting these expressions into (C.2) and integrating by parts we obtain

$$\delta S_M = \int d^4x \left\{ -\xi^\lambda \left[D_\alpha \left(2\frac{\delta L_M}{\delta \tilde{g}^{\lambda\nu}} \tilde{g}^{\alpha\nu} \right) - D_\lambda \left(\frac{\delta L_M}{\delta \tilde{g}^{\alpha\beta}} \right) \tilde{g}^{\alpha\beta} + \right.\right.$$
$$\left.\left. + \, D_\sigma \left(\frac{\delta L_M}{\delta \Phi_A} F^{B;\sigma}_{A;\lambda} \Phi_B \right) + \frac{\delta L_M}{\delta \Phi_A} D_\lambda \Phi_A \right] + \mathrm{div} \right\} = 0 \, . \qquad (C.4)$$

Owing to the arbitrariness of vector ξ^λ, we derive from this equality a strong identity, which is valid independently of whether the equations of motion for the fields are satisfied or not. It has the form

$$D_\alpha \left(2\frac{\delta L_M}{\delta \tilde{g}^{\lambda\nu}} \tilde{g}^{\alpha\nu} \right) - D_\lambda \left(\frac{\delta L_M}{\delta \tilde{g}^{\alpha\beta}} \right) \tilde{g}^{\alpha\beta} =$$
$$= -D_\sigma \left(\frac{\delta L_M}{\delta \Phi_A} F^{B;\sigma}_{A;\lambda} \Phi_B \right) - \frac{\delta L_M}{\delta \Phi_A} D_\lambda \Phi_A. \qquad (C.5)$$

We now introduce the notation

$$T_{\mu\nu} = 2\frac{\delta L_M}{\delta g^{\mu\nu}}, \;\; T^{\mu\nu} = -2\frac{\delta L_M}{\delta g_{\mu\nu}} = g^{\mu\alpha} g^{\nu\beta} T_{\alpha\beta},$$
$$T = T^{\mu\nu} g_{\mu\nu}, \;\; \tilde{T}_{\mu\nu} = 2\frac{\delta L_M}{\delta \tilde{g}^{\mu\nu}}, \qquad (C.6)$$
$$\tilde{T}^{\mu\nu} = -2\frac{\delta L_M}{\delta \tilde{g}_{\mu\nu}} = \tilde{g}^{\mu\alpha} \tilde{g}^{\nu\beta} \tilde{T}_{\alpha\beta}, \;\; \tilde{T} = \tilde{T}^{\alpha\beta} \tilde{g}_{\alpha\beta}.$$

Taking into account this notation, one can write the left-hand side of identity (C.5) as

$$D_\alpha(\tilde{T}_{\lambda\nu} \tilde{g}^{\alpha\nu}) - \frac{1}{2} \tilde{g}^{\alpha\beta} D_\lambda \tilde{T}_{\alpha\beta} = \partial_\alpha(\tilde{T}_{\lambda\nu} \tilde{g}^{\alpha\nu}) - \frac{1}{2} \tilde{g}^{\alpha\beta} \partial_\lambda \tilde{T}_{\alpha\beta}.$$

The right-hand side of this equation is readily reduced to the form

$$\partial_\alpha(\tilde{T}_{\lambda\nu} \tilde{g}^{\alpha\nu}) - \frac{1}{2} \tilde{g}^{\alpha\beta} \partial_\lambda \tilde{T}_{\alpha\beta} = \tilde{g}_{\lambda\nu} \nabla_\alpha \left(\tilde{T}^{\alpha\nu} - \frac{1}{2} \tilde{g}^{\alpha\nu} \tilde{T} \right) \, , \quad (C.7)$$

where ∇_α is the covariant derivative in Riemannian space. We shall now represent the expression under the sign of the covariant derivative ∇_α in terms of the density of the tensor $T^{\alpha\nu}$. To this end we take advantage of formula (A.16):

$$\frac{\delta L_M}{\delta g_{\mu\nu}} = \frac{\delta L_M}{\delta \tilde{g}^{\alpha\beta}} \cdot \frac{\partial \tilde{g}^{\alpha\beta}}{\partial g_{\mu\nu}}, \tag{C.8}$$

where

$$\frac{\partial \tilde{g}^{\alpha\beta}}{\partial g_{\mu\nu}} = \sqrt{-g}\frac{\partial g^{\alpha\beta}}{\partial g_{\mu\nu}} - \frac{1}{2\sqrt{-g}} \cdot \frac{\partial g}{\partial g_{\mu\nu}} g^{\alpha\beta}. \tag{C.9}$$

Using the relations

$$g^{\alpha\beta} g_{\beta\sigma} = \delta^\alpha_\sigma,$$

we find

$$\frac{\partial g^{\alpha\beta}}{\partial g_{\mu\nu}} = -\frac{1}{2}(g^{\alpha\mu} g^{\nu\beta} + g^{\alpha\nu} g^{\mu\beta}) \ . \tag{C.10}$$

Applying the rule for differentiating determinants we find

$$dg = g g^{\mu\nu} dg_{\mu\nu}, \tag{C.11}$$

from which we find

$$\frac{\partial g}{\partial g_{\mu\nu}} = g g^{\mu\nu}. \tag{C.12}$$

Substituting expressions (C.10) and (C.12) into (C.9) we obtain

$$\frac{\partial \tilde{g}^{\alpha\beta}}{\partial g_{\mu\nu}} = -\frac{1}{2}\sqrt{-g}[g^{\alpha\mu} g^{\beta\nu} + g^{\alpha\nu} g^{\beta\mu} - g^{\mu\nu} g^{\alpha\beta}] \ . \tag{C.13}$$

Using this relation in (C.8) we find

$$\frac{\delta L_M}{\delta g_{\mu\nu}} = \sqrt{-g}\left(\frac{\delta L_M}{\delta \tilde{g}^{\alpha\beta}} g^{\alpha\mu} g^{\beta\nu} - \frac{1}{2}\frac{\delta L_M}{\delta \tilde{g}^{\alpha\beta}} g^{\alpha\beta} g^{\mu\nu}\right) \ . \tag{C.14}$$

With account of notation (C.6) this expression can be written in the form

$$\sqrt{-g} T^{\mu\nu} = \tilde{T}^{\mu\nu} - \frac{1}{2}\tilde{g}^{\mu\nu}\tilde{T}. \tag{C.15}$$

On the basis of equality (C.15), the strong identity (C.5) assumes, with account of (C.7), the following form

$$g_{\lambda\nu}\nabla_\alpha T^{\alpha\nu} = -D_\sigma\left(\frac{\delta L_M}{\delta\Phi_A}F^{B;\sigma}_{A;\lambda}\Phi_B\right) - \frac{\delta L_M}{\delta\Phi_A}D_\lambda\Phi_A, \quad \text{or}$$

$$\nabla_\alpha T^\alpha_\lambda = -D_\sigma\left(\frac{\delta L_M}{\delta\Phi_A}F^{B;\sigma}_{A;\lambda}\Phi_B\right) - \frac{\delta L_M}{\delta\Phi_A}D_\lambda\Phi_A. \tag{C.16}$$

Appendix D

A second-rank curvature tensor $R_{\mu\nu}$ can be written in the form

$$R_{\mu\nu} = \frac{1}{2}[\tilde{g}^{\alpha\beta}(\tilde{g}_{\mu\kappa}\tilde{g}_{\nu\rho} - \frac{1}{2}\tilde{g}_{\mu\nu}\tilde{g}_{\kappa\rho})D_\alpha D_\beta \tilde{g}^{\kappa\rho} -$$
$$-\tilde{g}_{\nu\rho}D_\kappa D_\mu \tilde{g}^{\kappa\rho} - \tilde{g}_{\mu\kappa}D_\nu D_\rho \tilde{g}^{\kappa\rho}] + \frac{1}{2}\tilde{g}_{\nu\omega}\tilde{g}_{\rho\tau}D_\mu \tilde{g}^{\kappa\rho}D_\kappa \tilde{g}^{\omega\tau} +$$
$$+\frac{1}{2}\tilde{g}_{\mu\omega}\tilde{g}_{\rho\tau}D_\nu \tilde{g}^{\kappa\rho}D_\kappa \tilde{g}^{\omega\tau} - \frac{1}{2}\tilde{g}_{\mu\omega}\tilde{g}_{\nu\rho}D_\tau \tilde{g}^{\omega\kappa}D_\kappa \tilde{g}^{\rho\tau} -$$
$$-\frac{1}{4}(\tilde{g}_{\omega\rho}\tilde{g}_{\kappa\tau} - \frac{1}{2}\tilde{g}_{\omega\tau}\tilde{g}_{\kappa\rho})D_\mu \tilde{g}^{\kappa\rho}D_\nu \tilde{g}^{\omega\tau} -$$
$$-\frac{1}{2}\tilde{g}^{\alpha\beta}\tilde{g}_{\rho\tau}(\tilde{g}_{\mu\kappa}\tilde{g}_{\nu\omega} - \frac{1}{2}\tilde{g}_{\mu\nu}\tilde{g}_{\kappa\omega})D_\alpha \tilde{g}^{\kappa\rho}D_\beta \tilde{g}^{\omega\tau}. \quad \text{(D.1)}$$

Raising the indices by multiplying by $g^{\epsilon\mu}g^{\lambda\nu}$ and taking into account the equation

$$D_\mu \tilde{g}^{\mu\nu} = 0\,, \quad \text{(D.2)}$$

we obtain

$$-gR^{\epsilon\lambda} = \frac{1}{2}\tilde{g}^{\alpha\beta}D_\alpha D_\beta \tilde{g}^{\epsilon\lambda} - \frac{1}{4}\tilde{g}^{\epsilon\lambda}\tilde{g}_{\kappa\rho}\tilde{g}^{\alpha\beta}D_\alpha D_\beta \tilde{g}^{\kappa\rho} +$$
$$+\frac{1}{2}\tilde{g}_{\rho\tau}\tilde{g}^{\epsilon\mu}D_\mu \tilde{g}^{\kappa\rho}D_\kappa \tilde{g}^{\lambda\tau} +$$
$$+\frac{1}{2}\tilde{g}_{\rho\tau}\tilde{g}^{\lambda\nu}D_\nu \tilde{g}^{\kappa\rho}D_\kappa \tilde{g}^{\epsilon\tau} - \frac{1}{2}D_\tau \tilde{g}^{\epsilon\kappa}D_\kappa \tilde{g}^{\lambda\tau} -$$
$$-\frac{1}{4}(\tilde{g}_{\omega\rho}\tilde{g}_{\kappa\tau} - \frac{1}{2}\tilde{g}_{\omega\tau}\tilde{g}_{\kappa\rho})\tilde{g}^{\epsilon\mu}\tilde{g}^{\lambda\nu}D_\mu \tilde{g}^{\kappa\rho}D_\nu \tilde{g}^{\omega\tau} -$$
$$-\frac{1}{2}\tilde{g}_{\rho\tau}\tilde{g}^{\alpha\beta}D_\alpha \tilde{g}^{\epsilon\rho}D_\beta \tilde{g}^{\lambda\tau} + \frac{1}{4}\tilde{g}_{\rho\tau}\tilde{g}^{\epsilon\lambda}\tilde{g}_{\kappa\omega}\tilde{g}^{\alpha\beta}D_\alpha \tilde{g}^{\kappa\rho}D_\beta \tilde{g}^{\omega\tau}. \quad \text{(D.3)}$$

Hence, we find

$$-gR = \frac{1}{2}g_{\epsilon\lambda}\tilde{g}^{\alpha\beta}D_\alpha D_\beta \tilde{g}^{\epsilon\lambda} - g_{\kappa\rho}\tilde{g}^{\alpha\beta}D_\alpha D_\beta \tilde{g}^{\kappa\rho} + \frac{1}{2}g_{\rho\tau}D_\mu \tilde{g}^{\kappa\rho}D_\kappa \tilde{g}^{\mu\tau} -$$
$$-\frac{1}{4}(\tilde{g}_{\omega\rho}\tilde{g}_{\kappa\tau} - \frac{1}{2}\tilde{g}_{\omega\tau}\tilde{g}_{\kappa\rho})\sqrt{-g}\tilde{g}^{\mu\nu}D_\mu \tilde{g}^{\kappa\rho}D_\nu \tilde{g}^{\omega\tau} -$$
$$-\frac{1}{2}\tilde{g}_{\rho\tau}\tilde{g}^{\alpha\beta}g_{\epsilon\lambda}D_\alpha \tilde{g}^{\epsilon\rho}D_\beta \tilde{g}^{\lambda\tau} + \tilde{g}_{\rho\tau}g_{\kappa\omega}\tilde{g}^{\alpha\beta}D_\alpha \tilde{g}^{\kappa\rho}D_\beta \tilde{g}^{\omega\tau}. \quad \text{(D.4)}$$

With the aid of expressions (D.3) and (D.4) we find

$$
\begin{aligned}
&-g(\left(R^{\epsilon\lambda}-\frac{1}{2}g^{\epsilon\lambda}R\right)=\\
&=-\frac{1}{2}\left\{\frac{1}{2}\left(\tilde{g}_{\nu\sigma}\tilde{g}_{\tau\kappa}\frac{1}{2}\tilde{g}_{\nu\kappa}\tilde{g}_{\tau\sigma}\right)\tilde{g}^{\epsilon\alpha}\tilde{g}^{\lambda\beta}D_\alpha\tilde{g}^{\sigma\tau}D_\beta\tilde{g}^{\nu\kappa}-\right.\\
&-\frac{1}{4}\tilde{g}^{\epsilon\lambda}\tilde{g}^{\alpha\beta}\left(\tilde{g}_{\nu\sigma}\tilde{g}_{\tau\kappa}-\frac{1}{2}\tilde{g}_{\nu\kappa}\tilde{g}_{\tau\sigma}\right)D_\alpha\tilde{g}^{\tau\sigma}D_\beta\tilde{g}^{\nu\kappa}+\\
&+\tilde{g}^{\alpha\beta}\tilde{g}_{\sigma\tau}D_\alpha\tilde{g}^{\epsilon\tau}D_\beta\tilde{g}^{\lambda\sigma}-\tilde{g}^{\epsilon\beta}\tilde{g}_{\tau\sigma}D_\alpha\tilde{g}^{\lambda\sigma}D_\beta\tilde{g}^{\alpha\tau}-\\
&-\tilde{g}^{\lambda\alpha}\tilde{g}_{\tau\sigma}D_\alpha\tilde{g}^{\beta\sigma}D_\beta\tilde{g}^{\epsilon\tau}+\frac{1}{2}\tilde{g}^{\epsilon\lambda}\tilde{g}_{\tau\sigma}D_\alpha\tilde{g}^{\beta\sigma}D_\beta\tilde{g}^{\alpha\tau}+\\
&\left.+D_\alpha\tilde{g}^{\epsilon\beta}D_\beta\tilde{g}^{\lambda\alpha}-\tilde{g}^{\alpha\beta}D_\alpha D_\beta\tilde{g}^{\epsilon\lambda}\right\}. \qquad \text{(D.5)}
\end{aligned}
$$

It must be especially stressed that in finding expression (D.5) we made use of equation (D.2). By substituting expression (D.5) into equation (5.19) and writing the thus obtained equation in the form (8.1) we find the expression for the quantity $-16\pi g\tau_g^{\epsilon\lambda}$:

$$
\begin{aligned}
&-16\pi g\tau_g^{\epsilon\lambda}=\frac{1}{2}(\tilde{g}^{\epsilon\alpha}\tilde{g}^{\lambda\beta}-\frac{1}{2}\tilde{g}^{\epsilon\lambda}\tilde{g}^{\alpha\beta})(\tilde{g}_{\nu\sigma}\tilde{g}_{\tau\mu}-\frac{1}{2}\tilde{g}_{\tau\sigma}\tilde{g}_{\nu\mu})\times\\
&\times D_\alpha\tilde{\Phi}^{\tau\sigma}D_\beta\tilde{\Phi}^{\mu\nu}++\tilde{g}^{\alpha\beta}\tilde{g}_{\tau\sigma}D_\alpha\tilde{\Phi}^{\epsilon\tau}D_\beta\tilde{\Phi}^{\lambda\sigma}-\tilde{g}^{\epsilon\beta}\tilde{g}_{\tau\sigma}D_\alpha\tilde{\Phi}^{\lambda\sigma}D_\beta\tilde{\Phi}^{\alpha\tau}-\\
&-\tilde{g}^{\lambda\alpha}\tilde{g}_{\tau\sigma}D_\alpha\tilde{\Phi}^{\beta\sigma}D_\beta\tilde{\Phi}^{\epsilon\tau}+\frac{1}{2}\tilde{g}^{\epsilon\lambda}\tilde{g}_{\tau\sigma}D_\alpha\tilde{\Phi}^{\sigma\beta}D_\beta\tilde{\Phi}^{\alpha\tau}+D_\alpha\tilde{\Phi}^{\epsilon\beta}D_\beta\tilde{\Phi}^{\lambda\alpha}-\\
&-\tilde{\Phi}^{\alpha\beta}D_\alpha D_\beta\tilde{\Phi}^{\epsilon\lambda}--m^2\Big((\sqrt{-g}\tilde{g}^{\epsilon\lambda}-\sqrt{-\gamma}\tilde{\Phi}^{\epsilon\lambda}+\tilde{g}^{\epsilon\alpha}\tilde{g}^{\lambda\beta}\gamma_{\alpha\beta}-\\
&-\frac{1}{2}\tilde{g}^{\epsilon\lambda}\tilde{g}^{\alpha\beta}\gamma_{\alpha\beta})\Big). \qquad \text{(D.6)}
\end{aligned}
$$

Appendix E

Let us write the RTG equation (5.20)

$$D_\sigma \tilde{g}^{\sigma\nu}(y) = \partial_\sigma \tilde{g}^{\sigma\nu}(y) + \gamma^\nu_{\alpha\beta}(y)\tilde{g}^{\alpha\beta}(y) = 0 \qquad (\Sigma)$$

in a somewhat different form. For this purpose, making use of the definition of a Christoffel symbol,

$$\Gamma^\nu_{\alpha\beta}(y) = \frac{1}{2} g^{\nu\sigma}(\partial_\alpha g_{\sigma\beta} + \partial_\beta g_{\sigma\alpha} - \partial_\sigma g_{\alpha\beta}), \qquad \text{(E.1)}$$

we find

$$\Gamma^\nu_{\alpha\beta}\tilde{g}^{\alpha\beta}(y) = \sqrt{-g}\left(g^{\nu\sigma} g^{\alpha\beta}\partial_\alpha g_{\sigma\beta} - \frac{1}{2} g^{\nu\sigma} g^{\alpha\beta}\partial_\sigma g_{\alpha\beta}\right). \qquad \text{(E.2)}$$

Taking into account the equalities

$$\Gamma^\lambda_{\sigma\lambda} = \frac{1}{2} g^{\alpha\beta}\partial_\sigma g_{\alpha\beta} = \frac{1}{\sqrt{-g}}\partial_\sigma\sqrt{-g(y)},$$
$$\partial_\alpha g^{\alpha\nu} = -g^{\nu\sigma} g^{\alpha\beta}\partial_\alpha g_{\sigma\beta} \qquad \text{(E.3)}$$

we rewrite (E.2) as

$$\Gamma^\nu_{\alpha\beta}(y)\tilde{g}^{\alpha\beta}(y) = -\sqrt{-g}\partial_\sigma g^{\sigma\nu} - g^{\nu\sigma}\partial_\sigma\sqrt{-g} = -\frac{\partial\tilde{g}^{\sigma\nu}}{\partial y^\sigma}. \qquad \text{(E.4)}$$

With account of this equality the initial equation (Σ) assumes the form

$$(\Gamma^\nu_{\alpha\beta}(y) - \gamma^\nu_{\alpha\beta}(y))g^{\alpha\beta}(y) = 0. \qquad \text{(E.5)}$$

If we pass from coordinates "y" to other curvilinear coordinates "z", then the Christoffel symbols assume the form

$$\Gamma^\lambda_{\mu\nu}(y) = \frac{\partial y^\lambda}{\partial z^\sigma}\cdot\frac{\partial z^\alpha}{\partial y^\mu}\frac{\partial z^\beta}{\partial y^\nu}\Gamma^\sigma_{\alpha\beta}(z) +$$
$$+\frac{\partial^2 z^\sigma}{\partial y^\mu \partial y^\nu}\cdot\frac{\partial y^\lambda}{\partial z^\sigma}. \qquad \text{(E.6)}$$

Applying this expression we find

$$\Gamma^\lambda_{\mu\nu}(y)g^{\mu\nu}(y) = \frac{\partial y^\lambda}{\partial z^\sigma}\Bigg[\Gamma^\sigma_{\alpha\beta}(z)g^{\alpha\beta}(z)+$$
$$+\frac{\partial^2 z^\sigma}{\partial y^\mu \partial y^\nu}\frac{\partial y^\mu}{\partial z^\alpha}\cdot\frac{\partial y^\nu}{\partial z^\beta}g^{\alpha\beta}(z).\Bigg] \tag{E.7}$$

On the basis of (E.4) we write expression (E.7) in the form

$$\Gamma^\lambda_{\mu\nu}(y)g^{\mu\nu}(y) = -\frac{1}{\sqrt{-g}}\frac{\partial}{\partial z^\mu}\left(\tilde{g}^{\mu\sigma}\frac{\partial y^\lambda}{\partial z^\sigma}\right)+$$
$$+g^{\mu\sigma}\frac{\partial^2 y^\lambda}{\partial z^\mu \partial z^\sigma}+\frac{\partial y^\lambda}{\partial z^\sigma}\cdot\frac{\partial^2 z^\sigma}{\partial y^\mu \partial y^\nu}\frac{\partial y^\mu}{\partial z^\alpha}\frac{\partial y^\nu}{\partial z^\beta}g^{\alpha\beta}(z). \tag{E.8}$$

Upon differentiating equality

$$\frac{\partial z^\sigma}{\partial y^\mu}\cdot\frac{\partial y^\mu}{\partial z^\alpha} = \delta^\sigma_\alpha \tag{E.9}$$

with respect to the variable z^β we obtain

$$\frac{\partial^2 z^\sigma}{\partial y^\mu \partial y^\nu}\frac{\partial y^\mu}{\partial z^\alpha}\cdot\frac{\partial y^\nu}{\partial z^\beta} = -\frac{\partial z^\sigma}{\partial y^\mu}\cdot\frac{\partial^2 y^\mu}{\partial z^\alpha \partial z^\beta}. \tag{E.10}$$

Taking into account this equality, in the third term of (E.8) we find

$$\Gamma^\lambda_{\mu\nu}(y)g^{\mu\nu}(y) = -\frac{1}{\sqrt{-g(z)}}\frac{\partial}{\partial z^\nu}\left(\tilde{g}^{\nu\sigma}\frac{\partial y^\lambda}{\partial z^\sigma}\right). \tag{E.11}$$

Substituting this expression into (E.5) we obtain

$$\Box\, y^\lambda = -\gamma^\lambda_{\alpha\beta}(y)g^{\alpha\beta}(y), \tag{E.12}$$

where $\Box$ denotes the operator

$$\Box = \frac{1}{\sqrt{-g(z)}}\frac{\partial}{\partial z^\nu}\left(\tilde{g}^{\nu\sigma}\frac{\partial}{\partial z^\sigma}\right). \tag{E.13}$$

14. Elements of tensor analysis and of Riemannian geometry

Consider a certain coordinate system $x^\alpha, i = 1, ...n$ to be defined in n-dimensional space. Instead of this coordinate system one may also choose another one defined by expression

$$x'^\alpha = f(x^\alpha), \quad \alpha = 1, ...n. \tag{14.1}$$

These functions must be continuous and have continuous partial derivatives of order N. If the transformation Jacobian at each point,

$$J = \det \left| \frac{\partial f^\alpha}{\partial x^\beta} \right|, \tag{14.2}$$

differs from zero, then in this condition the variables x'^α will be independent, and, consequently, the initial variables x^α can be unambiguously expressed in terms of the new ones, x'^α:

$$x^\alpha = \varphi(x'^\alpha). \tag{14.3}$$

Physical quantities must not depend on the choice of coordinate system, and therefore they must be expressed in terms of geometrical objects. The simplest geometrical object is a scalar, that transforms in transition to the new coordinates as follows:

$$\Phi'(x') = \Phi(x(x')). \tag{14.4}$$

The gradient of a scalar function $\Phi(x)$ transforms in accordance with the rule for the differentiation of composite functions,

$$\frac{\partial \Phi'(x')}{\partial x'^\alpha} = \frac{\partial \Phi}{\partial x^\beta} \cdot \frac{\partial x^\beta}{\partial x'^\alpha}. \tag{14.5}$$

Here, summation is performed over identical indices β. The set of functions transforming under coordinate transformations by the rule (14.5) is termed the covariant vector

$$A'_\alpha(x') = A_\beta(x)\frac{\partial x^\beta}{\partial x'^\alpha}. \tag{14.6}$$

Correspondingly the quantity $B_{\mu\nu}$ is a covariant second-rank tensor, that transforms by the rule

$$B'_{\mu\nu}(x') = B_{\alpha\beta}(x)\frac{\partial x^\alpha}{\partial x'^\mu}\cdot\frac{\partial x^\beta}{\partial x'^\nu} \tag{14.7}$$

and so on.

We shall now pass to another group of geometric objects. Consider transformation of the differential of coordinates

$$dx'^\mu = \frac{\partial x'^\mu}{\partial x^\alpha}dx^\alpha. \tag{14.8}$$

A set of functions transforming under coordinate transformations by the rule (14.8) has been termed a contravariant vector,

$$A'^\mu(x') = \frac{\partial x'^\mu}{\partial x^\alpha}A^\alpha(x), \tag{14.9}$$

correspondingly, the quantity $B^{\mu\nu}$ a contravariant second-rank tensor transforming by the rule

$$B'^{\mu\nu}(x') = \frac{\partial x'^\mu}{\partial x^\alpha}\frac{\partial x'^\nu}{\partial x^\beta}B^{\alpha\beta}(x) \tag{14.10}$$

and so on. Expressions (14.6), (14.7), (14.9) and (14.10) permit to write the transformation law of tensors of any form. For example,

$$B'^\mu_\nu(x') = \frac{\partial x'^\mu}{\partial x^\alpha}\cdot\frac{\partial x^\beta}{\partial x'^\nu}B^\alpha_\beta(x) \tag{14.11}$$

From the transformational properties of a tensor it follows that, if all its components are equal to zero in one coordinate system, then they equal zero in another coordinate system, also. It is readily verified that the transformations of covariant and contravariant quantities exhibit the group property. For example:

$$A'^{\mu} = \frac{\partial x'^{\mu}}{\partial x^{\alpha}} A^{\alpha}(x), A''^{\nu}(x'') = \frac{\partial x''^{\nu}}{\partial x'^{\mu}} A'^{\mu}(x'),$$
$$A''^{\nu}(x'') = \frac{\partial x''^{\nu}}{\partial x'^{\mu}} \cdot \frac{\partial x'^{\mu}}{\partial x^{\alpha}} A^{\alpha}(x) = \frac{\partial x''^{\nu}}{\partial x^{\alpha}} A^{\alpha}(x). \tag{14.12}$$

Now, let us pass to tensor algebra. Here, four operations are possible: addition, multiplication, convolution, and permutation of indices.

Addition and subtraction of tensors

If we have tensors of identical structure, i.e. that have the same number of contravariant indices and the same number of covariant indices, for example,

$$A^{\alpha\beta}_{\mu\nu\sigma}, \quad B^{\alpha\beta}_{\mu\nu\sigma},$$

then it is possible to form the tensor

$$C^{\alpha\beta}_{\mu\nu\sigma} = A^{\alpha\beta}_{\mu\nu\sigma} + B^{\alpha\beta}_{\mu\nu\sigma}. \tag{14.13}$$

Multiplication of tensors

Tensors can be multiplied independently of their structure. For example,

$$C^{\alpha\beta\lambda}_{\mu\nu\sigma\rho} = A^{\alpha\beta}_{\mu\nu\sigma} \cdot B^{\lambda}_{\rho}. \tag{14.14}$$

Here, both the order of multipliers and the order of indices must be observed.

The convolution operation of tensors

With the aid of the Kronecker symbol

$$\delta^{\mu}_{\nu} = \begin{cases} 0 & \text{at} \quad \mu \neq \nu \\ 1 & \text{at} \quad \mu = \nu, \end{cases} \tag{14.15}$$

which is a tensor, it is possible to perform the convolution operation of indices, for example,

$$A^{\alpha\beta}_{\mu\nu} \cdot \delta^{\nu}_{\sigma} = A^{\alpha\beta}_{\mu\sigma}. \tag{14.16}$$

Here, on the left, summation is performed over identical indices.

The permutation operation of indices

By permutation of indices of the tensor we obtain another tensor, if the initial tensor was not symmetric over these indices, for example,

$$B^{\mu\nu}_{\lambda\sigma} = A^{\mu\nu}_{\sigma\lambda}. \tag{14.17}$$

With the aid of this operation, as well as addition, it is possible to construct a tensor that is symmetric over several indices. For example,

$$A_{(\mu\nu)} = \frac{1}{2}(A_{\mu\nu} + A_{\nu\mu}). \tag{14.18}$$

It is also possible to construct a tensor, that is antisymmetric over several indices. For example,

$$A_{[\mu\nu]} = \frac{1}{2}(A_{\mu\nu} - A_{\nu\mu}). \tag{14.19}$$

Such an operation is called antisymmetrization.

Riemannian geometry

A Riemannian space V_n is a real differentiable manifold, at each point of which there is given the field of a tensor

$$g_{\mu\nu}(x) = g_{\mu\nu}(x^1, \ldots, x^n) \tag{14.20}$$

twice covariant, symmetric and nondegenerate

$$g_{\mu\nu} = g_{\nu\mu}, \quad g = \det|g_{\mu\nu}| \neq 0. \tag{14.21}$$

The tensor $g_{\mu\nu}$ is called a metric tensor of Riemannian space. The functions $g_{\mu\nu}$ are continuous and differentiable with respect to all variables $x^1, \dots, x^n$ up to the n-th order.

With the aid of the metric tensor in Riemannian space it is possible to introduce an invariant differential form termed an interval

$$(ds)^2 = g_{\mu\nu}(x)dx^\mu dx^\nu. \tag{14.22}$$

With the aid of coordinate transformations this form at any fixed point can be reduced to a diagonal form. Here, in the general case, the diagonal components of the matrix $g_{\mu\nu}$ will not all be positive. But, by virtue of the law of inertia for quadratic forms the difference between the amounts of positive and of negative diagonal components will be constant. This difference is called the signature of a metric tensor. In an arbitrary Riemannian space V_n the interval will exhibit alternating signs. We shall further call it timelike, if $ds^2 > 0$, spacelike, if $ds^2 < 0$, isotropic, if $ds^2 = 0$. These terms originated within special relativity theory, where space and time form a unique manifold, while the interval in Cartesian (Galilean) coordinates has the form

$$d\sigma^2 = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2. \tag{14.23}$$

In arbitrary coordinates it assumes the form

$$d\sigma^2 = \gamma_{\mu\nu}(x)dx^\mu dx^\nu. \tag{14.24}$$

Since the determinant $|g_{\mu\nu}| \neq 0$, we can construct a contravariant metric tensor with the aid of equations

$$g_{\mu\sigma}g^{\sigma\nu} = \delta^\nu_\mu. \tag{14.25}$$

With the aid of tensors $g_{\mu\nu}$ and $g^{\lambda\sigma}$ it is possible to raise and to lower indices

$$A^{\nu} = g^{\nu\sigma} A_{\sigma}, \ \ A_{\nu} = g_{\nu\sigma} A^{\sigma}. \tag{14.26}$$

Geodesic lines in Riemannian space

Geodesic lines in Riemannian space play the same role as straight lines in Euclidean space. They are called extremal lines. For defining an extremal we shall take advantage of variational calculus. The essence of variational calculus consists in generalization of the concepts of maximum and minimum. The issue is not finding the extremum of a function, but finding the extremum of a functional, i.e. finding such functions, that make it an extremum. The distance between close points in Riemannian space is determined by the interval ds. The quantity ds is not a total differential. The interval between points a and b is

$$S = \int_a^b ds = \int_a^b \sqrt{g_{\mu\nu}(x) dx^{\mu} dx^{\nu}}. \tag{14.27}$$

The extremum is determined by the relation

$$\delta \int_a^b ds = \int_a^b \delta(ds) = 0. \tag{14.28}$$

Thus, such functions $g_{\mu\nu}(x)$ are sought, that provide for the functional (integral) achieving its extremum:

$$\begin{aligned} &\delta(ds^2) = 2ds\delta(ds) = \delta(g_{\mu\nu}(x) dx^{\mu} dx^{\nu}) = \\ &= \frac{\partial g_{\mu\nu}}{\partial x^{\sigma}} \delta x^{\sigma} dx^{\mu} dx^{\nu} + 2 g_{\mu\nu}(x) dx^{\mu} \delta(dx^{\nu}). \end{aligned} \tag{14.29}$$

We note that

$$\delta(dx^\nu) = d(\delta x^\nu). \tag{14.30}$$

On the basis of (14.29) and (14.30) we have

$$\delta(ds) = \frac{1}{2}\frac{\partial g_{\mu\nu}}{\partial x^\sigma} U^\mu dx^\nu \delta x^\sigma + g_{\mu\nu} U^\mu d(\delta x^\nu), \quad U^\mu = \frac{dx^\mu}{ds}. \tag{14.31}$$

Substituting (14.31) into (14.28) we obtain

$$\delta S = \int\limits_a^b \left[\frac{1}{2}\frac{\partial g_{\mu\nu}}{\partial x^\sigma} U^\mu U^\nu \delta x^\sigma + g_{\mu\nu} U^\mu \frac{d(\delta x^\nu)}{ds}\right] ds = 0. \tag{14.32}$$

Since

$$g_{\mu\nu} U^\mu \frac{d(\delta x^\nu)}{ds} = \frac{d}{ds}(g_{\mu\nu} U^\mu \delta x^\nu) - \delta x^\nu \frac{d}{ds}(g_{\mu\nu} U^\mu), \tag{14.33}$$

and at the integration limits $\delta x^\nu = 0$, from (14.32) we obtain

$$\delta S = \int\limits_a^b \left[\frac{1}{2}\frac{\partial g_{\mu\nu}}{\partial x^\sigma} U^\mu U^\nu - g_{\mu\sigma}\frac{dU^\mu}{ds} - \right.$$

$$\left. - \frac{\partial g_{\mu\sigma}}{\partial x^\lambda} U^\mu U^\lambda \right] ds \delta x^\sigma = 0. \tag{14.34}$$

We now represent the last term in (14.34) as

$$U^\mu U^\lambda \frac{\partial g_{\mu\sigma}}{dx^\lambda} = \frac{1}{2}\left(\frac{\partial g_{\mu\sigma}}{\partial x^\lambda} + \frac{\partial g_{\lambda\sigma}}{\partial x^\mu}\right) U^\mu U^\lambda. \tag{14.35}$$

Substituting (14.35) into (14.34) we find

$$\delta S = \int\limits_a^b \left[U^\mu U^\lambda \frac{1}{2}\left(\frac{\partial g_{\mu\sigma}}{\partial x^\lambda} + \frac{\partial g_{\lambda\sigma}}{\partial x^\mu} - \frac{\partial g_{\mu\lambda}}{\partial x^\sigma}\right) + \right.$$

$$\left. + g_{\mu\sigma}\frac{dU^\mu}{ds}\right] ds \delta x^\sigma = 0. \tag{14.36}$$

Since the variation δx^σ is arbitrary, the integral (14.36) turns to zero, only if

$$g_{\mu\sigma}\frac{dU^\mu}{ds}+\frac{1}{2}\left(\frac{\partial g_{\mu\sigma}}{\partial x^\lambda}+\frac{\partial g_{\lambda\sigma}}{\partial x^\mu}-\frac{\partial g_{\mu\lambda}}{\partial x^\sigma}\right)U^\mu U^\lambda=0. \qquad (14.37)$$

Multiplying (14.37) by $g^{\sigma\alpha}$ we obtain

$$\frac{dU^\alpha}{ds}+\Gamma^\alpha_{\mu\lambda}U^\mu U^\lambda=0, \qquad (14.38)$$

where the Christoffel symbols $\Gamma^\alpha_{\mu\nu}$ are

$$\Gamma^\alpha_{\mu\lambda}=\frac{1}{2}g^{\alpha\sigma}(\partial_\lambda g_{\mu\sigma}+\partial_\mu g_{\lambda\sigma}-\partial_\sigma g_{\mu\lambda}). \qquad (14.39)$$

The Christoffel symbols are not tensor quantities. Precisely equations (14.38) are the equations for a geodesic line. There are four of them, but not all are independent, since the following condition takes place:

$$g_{\mu\nu}(x)U^\mu U^\nu=1. \qquad (14.40)$$

By transformations of coordinates x^μ it is possible to equate the Christoffel symbols to zero along any not self-intersecting chosen line [22].

<u>Covariant differentiation</u>

We now take an arbitrary covariant vector A_λ and form its convolution with the vector U^λ, and thus obtain the scalar

$$A_\lambda U^\lambda, \qquad (14.41)$$

upon differentiating it with respect to ds we also have a scalar:

$$\frac{d}{ds}(A_\lambda U^\lambda)=\frac{dA_\lambda}{ds}U^\lambda+A_\nu\frac{dU^\nu}{ds}=$$
$$=\frac{\partial A_\lambda}{\partial x^\sigma}U^\sigma U^\lambda+A_\nu\frac{dU^\nu}{ds}. \qquad (14.42)$$

Substituting into the right-hand side expression (14.38) we obtain

$$\frac{d}{ds}(A_\lambda U^\lambda) = \left[\frac{\partial A_\lambda}{\partial x^\sigma} - \Gamma^\nu_{\sigma\lambda} A_\nu\right] U^\sigma U^\lambda. \tag{14.43}$$

Since (14.43) is a scalar, and U^σ is a vector, we hence have a second-rank tensor

$$A_{\lambda;\sigma} = \frac{DA_\lambda}{dx^\sigma} = \frac{\partial A_\lambda}{\partial x^\sigma} - \Gamma^\nu_{\sigma\lambda} A_\nu. \tag{14.44}$$

Here and further a semicolon denotes covariant differentiation. Thus, we have defined the covariant derivative of the covariant vector A_λ. We shall now define the covariant derivative of the contravariant vector A^λ.

To this end we write the same scalar in the form

$$\frac{d}{ds}(A^\mu U^\nu g_{\mu\nu}) = \frac{\partial A^\mu}{\partial x^\sigma} U^\sigma U^\nu g_{\mu\nu} +$$
$$+ A^\mu g_{\mu\lambda} \frac{dU^\lambda}{ds} + A^\mu U^\nu U^\sigma \partial_\sigma g_{\mu\nu}. \tag{14.45}$$

Substituting into the right-hand side expression (14.38) we obtain

$$\frac{d}{ds}(A^\mu U^\nu g_{\mu\nu}) = U^\nu U^\sigma \Bigg[g_{\mu\nu} \frac{\partial A^\mu}{\partial x^\sigma} -$$
$$- A^\mu g_{\mu\lambda} \Gamma^\lambda_{\sigma\nu} + A^\mu \partial_\sigma g_{\mu\nu}\Bigg]. \tag{14.46}$$

Taking into account expression (14.39) we have

$$\frac{d}{ds}(A^\mu U^\nu g_{\mu\nu}) = \Bigg[g_{\mu\nu} \frac{\partial A^\mu}{\partial x^\sigma} +$$
$$+ \frac{1}{2}(\partial_\sigma g_{\mu\nu} + \partial_\mu g_{\sigma\nu} - \partial_\nu g_{\sigma\mu}) A^\mu\Bigg] U^\nu U^\sigma. \tag{14.47}$$

Representing U^ν in the form

$$U^\nu = U_\lambda g^{\lambda\nu} \tag{14.48}$$

and substituting it into relation (14.47) we obtain

$$\frac{d}{ds}(A^\mu U^\nu g_{\mu\nu}) = \left[\frac{\partial A^\lambda}{\partial x^\sigma} + \Gamma^\lambda_{\sigma\mu} A^\mu\right] U^\sigma U_\lambda. \tag{14.49}$$

Since this expression is a scalar, hence it follows that the contravariant derivative is a tensor,

$$A^\lambda_{;\sigma} = \frac{DA^\lambda}{dx^\sigma} = \frac{\partial A^\lambda}{\partial x^\sigma} + \Gamma^\lambda_{\sigma\mu} A^\mu. \tag{14.50}$$

Thus, we have defined the covariant derivative of the contravariant vector A^λ.

Applying formulae (14.44) and (14.50), it is also possible to obtain covariant derivatives of a second-rank tensor:

$$A_{\mu\nu;\sigma} = \frac{\partial A_{\mu\nu}}{\partial x^\sigma} - \Gamma^\lambda_{\sigma\mu} A_{\lambda\nu} - \Gamma^\lambda_{\sigma\nu} A_{\lambda\mu}, \tag{14.51}$$

$$A^{\mu\nu}_{;\sigma} = \frac{\partial A^{\mu\nu}}{\partial x^\sigma} + \Gamma^\mu_{\sigma\lambda} A^{\nu\lambda} + \Gamma^\nu_{\sigma\lambda} A^{\mu\lambda}. \tag{14.52}$$

$$A^\nu_{\rho;\sigma} = \frac{\partial A^\nu_\rho}{\partial x^\sigma} - \Gamma^\lambda_{\rho\sigma} A^\nu_\lambda + \Gamma^\nu_{\sigma_\lambda} A^\lambda_\rho. \tag{14.53}$$

Making use of expression (14.51) it is easy to show, that

$$g_{\mu\nu;\sigma} \equiv 0,$$

i.e. the covariant derivative of a metric tensor is equal to zero.

The Riemann-Christoffel curvature tensor

In Riemannian space the operation of covariant differentiation is noncommutative. Covariant differentiation of vector A_λ, first, with respect to the variable x^μ and, then, with respect to x^ν leads to the following expression:

$$A_{\lambda;\mu\nu} = \frac{\partial A_{\lambda;\mu}}{\partial x^\nu} - \Gamma^\tau_{\nu\lambda} A_{\tau;\mu} - \Gamma^\tau_{\mu\nu} A_{\lambda;\tau}, \tag{14.54}$$

but since

$$\begin{aligned} A_{\lambda;\mu} &= \frac{\partial A_\lambda}{\partial x^\mu} - \Gamma^\tau_{\lambda\mu} A_\tau, \quad A_{\tau;\mu} = \frac{\partial A_\tau}{\partial x^\mu} - \Gamma^\sigma_{\mu\tau} A_\sigma, \\ A_{\lambda;\tau} &= \frac{\partial A_\lambda}{\partial x^\tau} - \Gamma^\sigma_{\lambda\tau} A_\sigma, \end{aligned} \tag{14.55}$$

upon substitution of these expressions into (14.54) we have

$$\begin{aligned} A_{\lambda;\mu\nu} &= \frac{\partial^2 A_\lambda}{\partial x^\mu \partial x^\nu} - \Gamma^\tau_{\lambda\mu} \frac{\partial A_\tau}{\partial x^\nu} - \Gamma^\tau_{\nu\lambda} \frac{\partial A_\tau}{\partial x^\mu} - -\Gamma^\tau_{\mu\nu} \frac{\partial A_\lambda}{\partial x^\tau} - \\ &-A_\tau \frac{\partial \Gamma^\tau_{\lambda\mu}}{\partial x^\nu} + \Gamma^\tau_{\nu\lambda} \Gamma^\sigma_{\mu\tau} A_\sigma + \Gamma^\tau_{\mu\nu} \Gamma^\sigma_{\lambda\tau} A_\sigma. \end{aligned} \tag{14.56}$$

We shall now calculate the quantity $A_{\lambda;\nu\mu}$:

$$A_{\lambda;\nu\mu} = \frac{\partial A_{\lambda;\nu}}{\partial x^\mu} - \Gamma^\tau_{\mu\lambda} A_{\tau;\nu} - \Gamma^\tau_{\mu\nu} A_{\lambda;\tau}, \tag{14.57}$$

with account of the expression

$$\begin{aligned} A_{\lambda;\nu} &= \frac{\partial A_\lambda}{\partial x^\nu} - \Gamma^\tau_{\lambda\nu} A_\tau, \quad A_{\tau;\nu} = \frac{\partial A_\tau}{\partial x^\nu} - \Gamma^\sigma_{\tau\nu} A_\sigma, \\ A_{\lambda;\tau} &= \frac{\partial A_\lambda}{\partial x^\tau} - \Gamma^\sigma_{\lambda\tau} A_\sigma, \end{aligned} \tag{14.58}$$

relation (14.57) assumes the form

$$\begin{aligned} A_{\lambda;\nu\mu} &= \frac{\partial^2 A_\lambda}{\partial x^\mu \partial x^\nu} - \Gamma^\tau_{\lambda\nu} \frac{\partial A_\tau}{\partial x^\mu} - \Gamma^\tau_{\mu\lambda} \frac{\partial A_\tau}{\partial x^\nu} - \Gamma^\tau_{\mu\nu} \frac{\partial A_\lambda}{\partial x^\tau} - \\ &-A_\tau \frac{\partial \Gamma^\tau_{\lambda\nu}}{\partial x^\mu} + \Gamma^\tau_{\mu\lambda} \Gamma^\sigma_{\nu\tau} A_\sigma + \Gamma^\tau_{\mu\nu} \Gamma^\sigma_{\lambda\tau} A_\sigma. \end{aligned} \tag{14.59}$$

On the basis of (14.56) and (14.59), only the following terms are retained in the difference:

$$A_{\lambda;\mu\nu} - A_{\lambda;\nu\mu} = A_\sigma \left[\frac{\partial \Gamma^\sigma_{\lambda\nu}}{\partial x^\mu} - \frac{\partial \Gamma^\sigma_{\lambda\mu}}{\partial x^\nu} + \right.$$
$$\left. + \Gamma^\tau_{\nu\lambda}\Gamma^\sigma_{\mu\tau} - \Gamma^\tau_{\mu\lambda}\Gamma^\sigma_{\nu\tau} \right]. \tag{14.60}$$

The quantity $R^\sigma_{\lambda\mu\nu}$ is termed the Riemann curvature tensor

$$R^\sigma_{\lambda\mu\nu} = \frac{\partial \Gamma^\sigma_{\lambda\nu}}{\partial x^\mu} - \frac{\partial \Gamma^\sigma_{\lambda\mu}}{\partial x^\nu} + \Gamma^\tau_{\nu\lambda}\Gamma^\sigma_{\mu\tau} - \Gamma^\tau_{\mu\lambda}\Gamma^\sigma_{\nu\tau}. \tag{14.61}$$

From this tensor it is possible, by convolution, to obtain a second-rank tensor, the Ricci tensor:

$$R_{\lambda\nu} = R^\sigma_{\lambda\sigma\nu} = \frac{\partial \Gamma^\sigma_{\lambda\nu}}{\partial x^\sigma} - \frac{\partial \Gamma^\sigma_{\lambda\sigma}}{\partial x^\nu} + \Gamma^\tau_{\nu\lambda}\Gamma^\sigma_{\sigma\tau} - \Gamma^\tau_{\sigma\lambda}\Gamma^\sigma_{\nu\tau}. \tag{14.62}$$

We note that for an interval of the form (14.23) or (14.24) the curvature tensor equals zero.

From expression (14.61) it is obvious that the curvature tensor is antisymmetric with respect to the two last indices μ, ν:

$$R^\sigma_{\lambda\mu\nu} = -R^\sigma_{\lambda\nu\mu}$$

It is possible to construct a curvature tensor with lower indices:

$$R_{\rho\lambda\mu\nu} = g_{\rho\sigma} R^\sigma_{\lambda\mu\nu}.$$

It possesses the following symmetry properties:

$$R_{\rho\lambda\mu\nu} = -R_{\lambda\rho\mu\nu} = -R_{\rho\lambda\nu\mu}, \quad R_{\rho\lambda\mu\nu} = R_{\mu\nu\rho\lambda}.$$

We see that the curvature tensor is antisymmetric both with respect to the first pair of indices and with respect to the second. It is also symmetric with respect to permutation of index pairs, without any change of their order.

In Riemannian space there exists a local coordinate system, within which the first derivatives of the components of the metric tensor $g_{\mu\nu}$ are equal to zero. Here, the Christoffel symbols are, naturally, also equal to zero. Such coordinates are called Riemann coordinates. They are convenient for finding tensor identities, since if it has been established, that in this coordinate system a certain tensor is zero, then, by virtue of tensor transformations, it will also be zero in any coordinate system.

The curvature tensor in a Riemann coordinate system is

$$R^{\sigma}_{\lambda\mu\nu} = \partial_\mu \Gamma^{\sigma}_{\lambda\nu} - \partial_\nu \Gamma^{\sigma}_{\lambda\mu}. \tag{14.63}$$

The covariant derivative of the curvature tensor has the form

$$R^{\sigma}_{\lambda\mu\nu;\rho} = \partial_\rho \partial_\mu \Gamma^{\sigma}_{\lambda\nu} - \partial_\rho \partial_\nu \Gamma^{\sigma}_{\lambda\mu}. \tag{14.64}$$

Cyclically transposing indices μ, ν, ρ and adding up the obtained expressions we obtain the Bianchi identity

$$R^{\sigma}_{\lambda\mu\nu;\rho} + R^{\sigma}_{\lambda\rho\mu;\nu} + R^{\sigma}_{\lambda\nu\rho;\mu} \equiv 0. \tag{14.65}$$

Performing convolution of indices σ and ν we obtain

$$- R_{\lambda\mu;\rho} + R^{\sigma}_{\lambda\rho\mu;\sigma} + R_{\lambda\rho;\mu} = 0. \tag{14.66}$$

We multiply this expression by $g^{\lambda\alpha}$:

$$-R^{\alpha}_{\mu;\rho} + (g^{\lambda\alpha} R^{\sigma}_{\lambda\rho\mu})_{;\sigma} + R^{\alpha}_{\rho;\mu} = 0.$$

We have, here, taken into account the previously established property of metric coefficients consisting in that they can, in case of covariant differentiation, be freely brought or taken out from under the derivative sign.

Performing convolution of indices ρ and α we obtain

$$-R^{\rho}_{\mu;\rho}+(g^{\lambda\rho}R^{\sigma}_{\lambda\rho\mu})_{;\sigma}+\partial_\mu R\equiv 0, \tag{14.67}$$

where

$$R=R^{\rho}_{\rho}=R_{\mu\nu}g^{\mu\nu}$$

is the scalar curvature.

Let us consider under the derivative sign the second term in identity (14.67):

$$g^{\lambda\rho}R^{\sigma}_{\lambda\rho\mu}=g^{\lambda\rho}g^{\nu\sigma}R_{\nu\lambda\rho\mu}=g^{\nu\sigma}g^{\lambda\rho}R_{\lambda\nu\mu\rho}=g^{\nu\sigma}R^{\rho}_{\nu\mu\rho}=-R^{\sigma}_{\mu}.$$

We have, here, applied the symmetry properties of the curvature tensor and the definition of the tensor $R_{\mu\nu}$. Substituting this expression into (14.67) we obtain

$$(R^{\rho}_{\mu}-\frac{1}{2}\delta^{\rho}_{\mu}R)_{;\rho}=\nabla_\rho(R^{\rho}_{\mu}-\frac{1}{2}\delta^{\rho}_{\mu}R)\equiv 0. \tag{14.68}$$

We now introduce the notation

$$G^{\rho}_{\mu}=R^{\rho}_{\mu}-\frac{1}{2}\delta^{\rho}_{\mu}R. \tag{14.69}$$

On the basis of (14.53), identity (14.68) can be written in the expanded form

$$\nabla_\nu G^{\nu}_{\rho}=G^{\nu}_{\rho;\nu}=\frac{\partial G^{\nu}_{\rho}}{\partial x^\nu}-\Gamma^{\lambda}_{\rho\nu}G^{\nu}_{\lambda}+\Gamma^{\nu}_{\nu\lambda}G^{\lambda}_{\rho}\equiv 0, \tag{14.70}$$

taking into account that

$$\Gamma^{\nu}_{\nu\lambda}=\frac{1}{2}g^{\mu\nu}\frac{\partial g_{\mu\nu}}{\partial x^\lambda} \tag{14.71}$$

and differentiating the determinant g,

$$\frac{\partial g}{\partial x^\lambda} = g g^{\mu\nu} \frac{\partial g_{\mu\nu}}{\partial x^\lambda}, \tag{14.72}$$

we find, by comparison of (14.71) and (14.72), the following:

$$\Gamma^\nu_{\nu\lambda} = \frac{1}{2} \cdot \frac{1}{g} \frac{\partial g}{\partial x^\lambda} = \frac{1}{\sqrt{-g}} \partial_\lambda(\sqrt{-g}). \tag{14.73}$$

Substituting this expression into (14.70) we obtain

$$\nabla_\nu(\sqrt{-g} G^\nu_\rho) = \partial_\nu(\sqrt{-g} G^\nu_\rho) - \sqrt{-g} \Gamma^\lambda_{\rho\nu} G^\nu_\lambda \equiv 0. \tag{14.74}$$

Making use of expression (14.39), we find for the Christoffel symbol

$$\partial_\nu(\sqrt{-g} G^\nu_\rho) + \frac{1}{2} \cdot \frac{\partial g^{\lambda\sigma}}{\partial x^\rho} \sqrt{-g} G_{\lambda\sigma} \equiv 0. \tag{14.75}$$

Such an identity was first obtained by D. Hilbert. It was necessary for constructing the equations of general relativity theory.

In conclusion we shall show that the quantity determining volume

$$v' = \int \sqrt{-g'} dx^{0\,\prime} dx^{1\,\prime} dx^{2\,\prime} dx^{3\,\prime} \tag{14.76}$$

is an invariant under arbitrary transformations of coordinates. Under coordinate transformations we have

$$g'_{\mu\nu}(x') = g_{\lambda\sigma}(x) \frac{\partial x^\lambda}{\partial x'^\mu} \cdot \frac{\partial x^\sigma}{\partial x'^\nu}.$$

We write this expression in the form

$$g'_{\mu\nu}(x') = \frac{\partial x^\lambda}{\partial x'^\mu} \cdot g_{\lambda\sigma} \frac{\partial x^\sigma}{\partial x'^\nu}.$$

We shall, now, calculate the determinant $g' = \det g'_{\mu\nu}$

$$g' = \det\left(g_{\sigma\lambda} \frac{\partial x^\lambda}{\partial x'^\mu}\right) \det\left(\frac{\partial x^\sigma}{\partial x'^\nu}\right) =$$

$$= \det(g_{\lambda\sigma}) \det\left(\frac{\partial x^\lambda}{\partial x'^\mu}\right) \det\left(\frac{\partial x^\sigma}{\partial x'^\nu}\right).$$

Hence, we have

$$g' = gJ^2. \tag{14.77}$$

Here J is the transformation Jacobian,

$$J = \frac{\partial(x^0.x^1, x^2, x^3)}{\partial(x^{0\prime}, x^{1\prime}, x^{2\prime}, x^{3\prime})}. \tag{14.78}$$

Thus,

$$\sqrt{-g'} = \sqrt{-g}J. \tag{14.79}$$

Substituting this expression into (14.76) we obtain

$$v' = \int \sqrt{-g}\frac{\partial(x^0.x^1, x^2, x^3)}{\partial(x^{0\prime}, x^{1\prime}, x^{2\prime}, x^{3\prime})}dx^0{}'dx^1{}'dx^2{}'dx^3{}' =$$
$$= \int \sqrt{-g}dx^0dx^1dx^2dx^3. \tag{14.80}$$

But the right-hand side represents volume

$$v = \int \sqrt{-g}dx^0dx^1dx^2dx^3. \tag{14.81}$$

Thus, we have established the equality

$$v' = v. \tag{14.82}$$

Hence it follows, that the quantity

$$\sqrt{-g}d^4x \tag{14.83}$$

is also an invariant relative to arbitrary coordinate transformations.

Certain special features of Riemannian geometry should be noted. In the general case, Riemannian space cannot be described in a sole coordinate system. For its description an atlas of maps is necessary. Precisely for this reason, the topology of

Riemannian space differs essentially from the topology of Euclidean space. In the general case, no group of motion exists in Riemannian space. In pseudo-Euclidean space, described by the interval (14.23) or (14.24), there exists a ten-parameter group of space motions.

The main characteristic of Riemannian geometry — the curvature tensor $R^{\sigma}_{\lambda\mu\nu}$ — is a form-invariant quantity relative to coordinate transformations. The tensor $R_{\lambda\nu}$ is also a form-invariant quantity. Here, form-invariance is not understood as one and the same functional dependence of the curvature tensor upon the choice of coordinate system, but identity in constructing the curvature tensor for a given expression $g_{\mu\nu}(x)$, similarly to how expression

$$\Box A^{\nu}(x)$$

is written in the same way in Galilean coordinates in differing inertial reference systems for a given expression A^{ν}. There exists an essential difference between invariance and form-invariance. for example, the operator $\gamma^{\mu\nu}(x)D_{\mu}D_{\nu}$ (where $\gamma^{\mu\nu}(x)$ is the metric tensor of Minkowski space) for arbitrary coordinate transformations is an invariant, i.e. a scalar, but it is not form-invariant. It will be form-invariant only in the case of such coordinate transformations, under which the tensor $\gamma^{\mu\nu}(x)$ remains form-invariant, i.e.

$$\delta\gamma^{\mu\nu}(x) = 0.$$

The curvature tensor varies under gauge transformations (3.16) according to the following rule:

$$\delta_{\epsilon}R_{\mu\nu\alpha\beta} = -R_{\sigma\nu\alpha\beta}D_{\mu}\epsilon^{\sigma} - R_{\mu\sigma\alpha\beta}D_{\nu}\epsilon^{\sigma} - \\ -R_{\mu\nu\sigma\beta}D_{\alpha}\epsilon^{\sigma} - R_{\mu\nu\alpha\sigma}D_{\beta}\epsilon^{\sigma} - \epsilon^{\sigma}D_{\sigma}R_{\mu\nu\alpha\beta}.$$

This variation is due to arbitrary coordinate systems not being physically equivalent.

Within the text of this book there are encountered, together with covariant derivatives in Riemannian space, ∇_λ, covariant derivatives in Minkowski space, D_λ. The difference consists in that in constructing the covariant derivatives D_λ it is necessary to substitute into formulae (14.50 - 14.53) the Christoffel symbols of Minkowski space, $\gamma^\nu_{\alpha\beta}$, instead of the Christoffel symbols of Riemannian space, $\Gamma^\nu_{\alpha\beta}$,

In conclusion, we present the Weyl–Lorentz–Petrov theorem [21]. Coincidence of the respective equations of isotropic and timelike geodesic lines for two Riemannian spaces with the metrics $g_{\mu\nu}(x)$ and $g'_{\mu\nu}(x)$ and with the same signature –2 leads to their metric tensors only differing by a constant factor. From this theorem it follows that, if in one and the same coordinate system x we have different metric tensors $g_{\mu\nu}(x)$ and $g'_{\mu\nu}(x)$, then, in identical conditions, different geodesic lines, and, consequently, different physics, will correspond to them. Precisely for this reason, the situation, that arises in GRT with the appearance of a multiplicity of metrics within one coordinate system, leads to ambiguity in the description of gravitational effects.

ADDENDUM

On the gravitational force

The expression for the gravitational force is presented at page 66. We shall now derive this expression from the equation of a geodesic in effective Riemannian space. The equation for the geodesic line has the form

$$\frac{dp^\nu}{ds} + \Gamma^\nu_{\alpha\beta} p^\alpha p^\beta = 0, \;\; p^\nu = \frac{dx^\nu}{ds}, \;\; ds^2 = g_{\mu\nu} dx^\mu dx^\nu > 0. \quad (1)$$

In accordance with the definition of a covariant derivative in Minkowski space we have

$$\frac{Dp^\nu}{ds} = \frac{dp^\nu}{ds} + \gamma^\nu_{\alpha\beta} p^\alpha p^\beta. \quad (2)$$

Applying (1) and (2) we obtain

$$\frac{Dp^\nu}{ds} = -G^\nu_{\alpha\beta} p^\alpha p^\beta. \quad (3)$$

Here

$$G^\nu_{\alpha\beta} = \Gamma^\nu_{\alpha\beta} - \gamma^\nu_{\alpha\beta} \quad (4)$$

We shall write the left-hand side of relation (3) as

$$\frac{Dp^\nu}{ds} = \left(\frac{d\sigma}{ds}\right)^2 \left[\frac{DV^\nu}{d\sigma} + V^\nu \frac{\frac{d^2\sigma}{ds^2}}{\left(\frac{d\sigma}{ds}\right)^2}\right], \;\; V^\nu = \frac{dx^\nu}{d\sigma}. \quad (5)$$

Here V^ν is the timelike velocity four-vector in Minkowski space, that satisfies the condition

$$\gamma_{\mu\nu} V^\mu V^\nu = 1, \;\; d\sigma^2 > 0. \quad (6)$$

Substituting (5) into (3) we obtain

$$\frac{DV^\nu}{d\sigma} = -G^\nu_{\alpha\beta} V^\alpha V^\beta - V^\nu \frac{\frac{d^2\sigma}{ds^2}}{\left(\frac{d\sigma}{ds}\right)^2} \quad (7)$$

From (6) we have

$$\left(\frac{d\sigma}{ds}\right)^2 = \gamma_{\alpha\beta} p^\alpha p^\beta. \tag{8}$$

Differentiating this expression with respect to ds we obtain

$$\frac{\frac{d^2\sigma}{ds^2}}{\left(\frac{d\sigma}{ds}\right)^2} = -\gamma_{\lambda\mu} G^\mu_{\alpha\beta} V^\lambda V^\alpha V^\beta. \tag{9}$$

Substituting this expression into (7) we find [4]

$$\frac{DV^\nu}{d\sigma} = -G^\mu_{\alpha\beta} V^\alpha V^\beta (\delta^\nu_\mu - V^\nu V_\mu). \tag{10}$$

Hence it is evident that the motion of a test body in Minkowski space is due to the action of the force four-vector F^ν:

$$F^\nu = -G^\mu_{\alpha\beta} V^\alpha V^\beta (\delta^\nu_\mu - V^\nu V_\mu), \quad V_\mu = \gamma_{\mu\sigma} V^\sigma. \tag{11}$$

One can readily verify that

$$F^\nu V_\nu = 0. \tag{12}$$

By definition, the left-hand side of equation (10) is

$$\frac{DV^\nu}{d\sigma} = \frac{dV^\nu}{d\sigma} + \gamma^\nu_{\alpha\beta} V^\alpha V^\beta. \tag{13}$$

It must be especially noted that the motion of a test body along a geodesic of effective Riemannian space can be represented as motion in Minkowski space due to the action of the force F^ν, only if the causality principle is satisfied. The force of gravity and the Riemann curvature tensor, arising from the gravitational equations (5.19) and (5.20), are correlated. Thus, if the curvature tensor is zero, then, by virtue of equations (5.19) and (5.20), the gravitational force will also be equal to zero. When the curvature tensor differs from zero and $R_{\mu\nu} \neq 0$, the force of gravity will also not be zero. And, on the contrary, if the force of gravity F^ν, arising from equations (5.19) and (5.20), differs from zero, then the Riemann curvature is also not zero. Equating the gravitational force F^ν to zero results in the Riemann curvature tensor being equal to zero.

Is the metric field of a non-inertial reference system a special case of the physical gravitational field?

From the causality conditions (6.10) and (6.11) it follows that, if the vector L^ν satisfies the condition

$$\gamma_{\mu\nu} L^\mu L^\nu < 0, \tag{1}$$

then the inequality

$$g_{\mu\nu} L^\mu L^\nu < 0. \tag{2}$$

should also be fulfilled. We now form the convolution of equation (10.1) with the aid of the vector L^ν defined by inequality (1),

$$m^2 \gamma_{\mu\nu} L^\mu L^\nu = 16\pi \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right) - \\ -2 R_{\mu\nu} L^\mu L^\nu + m^2 g_{\mu\nu} L^\mu L^\nu. \tag{3}$$

Since we are only considering metric fields of Minkowski space, equation (3) is simplified:

$$m^2 \gamma_{\mu\nu} L^\mu L^\nu = 16\pi \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right) + m^2 g_{\mu\nu} L^\mu L^\nu. \tag{4}$$

In the case of an ideal fluid, the energy-momentum tensor of matter has the form

$$T_{\mu\nu} = (\rho + p)U_\mu U_\nu - pg_{\mu\nu},$$
$$T = T_{\mu\nu}g^{\mu\nu} = \rho - 3p, \quad U^\nu = \frac{dx^\mu}{ds}. \tag{5}$$

Substituting (5) into (4) we obtain

$$m^2\gamma_{\mu\nu}L^\mu L^\nu = 16\pi(\rho + p)(U_\mu L^\mu)^2 -$$
$$-8\pi g_{\mu\nu}L^\mu L^\nu \left(\rho - p - \frac{m^2}{8\pi}\right). \tag{6}$$

From conditions (1) and (2) it follows that the right-hand side of equation (6) is strictly positive, since

$$\rho > p + \frac{m^2}{8\pi}, \tag{7}$$

while the left-hand side of equation (6) is strictly negative. Hence it follows that in the presence of matter no metric field of Minkowski space satisfies the gravitational equations, and therefore the metric fields arising in non-inertial reference systems of Minkowski space cannot be considered gravitational fields. In the absence of matter, $\rho = p = 0$, equation (6) has the sole solution

$$g_{\mu\nu}(x) = \gamma_{\mu\nu}(x). \tag{8}$$

On the covariant conservation law

The covariant conservation law of matter energy-momentum tensor density T_μ^ν in General Relativity (GRT) takes in Riemannian space the following form

$$\nabla_\nu T_\mu^\nu = \partial_\nu T_\mu^\nu - \frac{1}{2}T^{\sigma\lambda}\partial_\mu g_{\sigma\lambda}\,, \qquad T^{\sigma\lambda} = -2\frac{\delta L_M}{\delta g_{\sigma\lambda}}\,. \tag{1}$$

This equation is a straightforward consequence of Gilbert-Einstein equations. Though the equation has a covariant form, nevertheless the energy-momentum conservation law of matter and gravitational field taken together has in GRT a noncovariant appearance

$$\partial_\nu (T^\nu_\mu + \tau^\nu_\mu) = 0. \tag{2}$$

Just by this way the gravitational field pseudotensor τ^ν_μ, which is not a covariant quantity, arises in GRT. It is impossible in principle to write conservation equations of the energy-momentum of matter and gravitational field in the generally covariant form. The idea that the gravitational energy cannot be localized in GRT has arisen from this fact.

If we will not use Eq. (8.2) in the derivation of Eqs. (8.1), then gravitational equations will take the following form

$$\sqrt{-\gamma}(-J^{\varepsilon\lambda} + m^2\tilde{\phi}^{\varepsilon\lambda}) = 16\pi\sqrt{-g}(T^{\varepsilon\lambda} + t_g^{\varepsilon\lambda})\,. \tag{3}$$

Here $t_g^{\varepsilon\lambda}$ is the energy-momentum tensor density for the gravitational field.

$$\begin{aligned}
&16\pi\sqrt{-g}t^{\varepsilon\lambda} = -D_\mu D_\sigma(\tilde{\phi}^{\varepsilon\lambda}\tilde{\phi}^{\mu\sigma} - \tilde{\phi}^{\varepsilon\mu}\tilde{\phi}^{\lambda\sigma}) + \\
&+D_\sigma\tilde{\phi}^{\varepsilon\lambda}D_\mu\tilde{\phi}^{\mu\sigma} - D_\mu\tilde{\phi}^{\varepsilon\mu}D_\sigma\tilde{\phi}^{\lambda\sigma} + \\
&+\tfrac{1}{2}g^{\varepsilon\lambda}g_{\rho\tau}D_\mu\tilde{\phi}^{\alpha\rho}D_\alpha\tilde{\phi}^{\mu\tau} - g_{\rho\tau}g^{\varepsilon\mu}D_\mu\tilde{\phi}^{\alpha\rho}D_\alpha\tilde{\phi}^{\lambda\tau} - \\
&-g_{\rho\tau}g^{\lambda\nu}D_\nu\tilde{\phi}^{\alpha\rho}D_\alpha\tilde{\phi}^{\varepsilon\tau} + g_{\rho\tau}g^{\alpha\beta}D_\alpha\tilde{\phi}^{\varepsilon\rho}D_\beta\tilde{\phi}^{\lambda\tau} + \\
&+\tfrac{1}{2}\left(g_{\beta\rho}g_{\alpha\tau} - \tfrac{1}{2}g_{\beta\tau}g_{\alpha\rho}\right)\times \\
&\times\left(g^{\varepsilon\mu}g^{\lambda\nu} - \tfrac{1}{2}g^{\varepsilon\lambda}g^{\mu\nu}\right)D_\mu\tilde{\phi}^{\lambda\rho}D_\nu\tilde{\phi}^{\beta\tau} - \\
&-m^2\left(\sqrt{-g}\tilde{g}^{\varepsilon\lambda} - \sqrt{-\gamma}\tilde{\phi}^{\varepsilon\lambda} + \tilde{g}^{\varepsilon\alpha}\tilde{g}^{\lambda\beta}\gamma_{\alpha\beta} - \tfrac{1}{2}\tilde{g}^{\varepsilon\lambda}\tilde{g}^{\alpha\beta}\gamma_{\alpha\beta}\right).
\end{aligned} \tag{4}$$

$$J^{\varepsilon\lambda} = -D_\mu D_\nu(\gamma^{\mu\nu}\tilde{g}^{\varepsilon\lambda} + \gamma^{\varepsilon\lambda}\tilde{g}^{\mu\nu} - \gamma^{\varepsilon\nu}\tilde{g}^{\mu\lambda} - \gamma^{\varepsilon\mu}\tilde{g}^{\lambda\nu}). \tag{5}$$

Let us mention that expression

$$D_\sigma(\tilde{\phi}^{\varepsilon\lambda}\tilde{\phi}^{\mu\sigma} - \tilde{\phi}^{\varepsilon\mu}\tilde{\phi}^{\lambda\sigma})$$

is antisymmetric under permutation of indicies $\lambda \quad \mu$, and so the following identity takes place:

$$D_\lambda D_\mu D_\sigma(\tilde{\phi}^{\varepsilon\lambda}\tilde{\phi}^{\mu\sigma} - \tilde{\phi}^{\varepsilon\mu}\tilde{\phi}^{\lambda\sigma}) = 0\,.$$

It is also easy to get convinced, that the following equation is valid

$$D_\lambda J^{\sigma\lambda} = 0\,. \tag{6}$$

Gravitational field equation (3) can be presented also in other form (5.19)

$$\begin{aligned}&\sqrt{-g}\left(R^{\mu\nu} - \tfrac{1}{2}g^{\mu\nu}R\right) + \\ &+\tfrac{m^2}{2}\left[\tilde{g}^{\mu\nu} + \left(\tilde{g}^{\mu\alpha}g^{\nu\beta} - \tfrac{1}{2}\tilde{g}^{\mu\nu}g^{\alpha\beta}\right)\gamma_{\alpha\beta}\right] = 8\pi T^{\mu\nu}\,.\end{aligned} \tag{7}$$

From Eqs. (7) it follows

$$m^2\sqrt{-g}\left(g^{\mu\alpha}g^{\nu\beta} - \frac{1}{2}g^{\mu\nu}g^{\alpha\beta}\right)\nabla_\mu\gamma_{\alpha\beta} = 16\pi\nabla_\mu T^{\mu\nu}\,. \tag{8}$$

By taking into account relation

$$\nabla_\mu\gamma_{\alpha\beta} = -G^\sigma_{\mu\alpha}\gamma_{\sigma\beta} - G^\sigma_{\mu\beta}\gamma_{\sigma\alpha}\,, \tag{9}$$

we find

$$\begin{aligned}&\left(g^{\mu\alpha}g^{\nu\beta} - \tfrac{1}{2}g^{\mu\nu}g^{\alpha\beta}\right)\nabla_\mu\gamma_{\alpha\beta} = \\ &= -g^{\mu\alpha}g^{\nu\beta}G^\sigma_{\mu\alpha}\gamma_{\sigma\beta} - g^{\mu\alpha}g^{\nu\beta}G^\sigma_{\mu\beta}\gamma_{\sigma\alpha} + g^{\mu\nu}g^{\alpha\beta}G^\sigma_{\mu\alpha}\gamma_{\sigma\beta}\,.\end{aligned} \tag{10}$$

It is easy to see that the following identity takes place:

$$-g^{\mu\alpha}g^{\nu\beta}G^\sigma_{\mu\beta}\gamma_{\sigma\alpha} + g^{\mu\nu}g^{\alpha\beta}G^\sigma_{\mu\alpha}\gamma_{\sigma\beta}\,. \tag{11}$$

Therefore we have

$$\left(g^{\mu\alpha}g^{\nu\beta} - \frac{1}{2}g^{\mu\nu}g^{\alpha\beta}\right)\nabla_\mu\gamma_{\alpha\beta} = -g^{\mu\alpha}g^{\nu\beta}G^\sigma_{\mu\alpha}\gamma_{\sigma\beta}\,. \tag{12}$$

By substituting expression (4.5) instead of $G^{\sigma}_{\mu\alpha}$ we get

$$\begin{aligned}&\left(g^{\mu\alpha}g^{\nu\beta}-\tfrac{1}{2}g^{\mu\nu}g^{\alpha\beta}\right)\nabla_{\mu}\gamma_{\alpha\beta}=\\&=\gamma_{\mu\lambda}g^{\mu\nu}\left(D_{\sigma}g^{\sigma\lambda}+G^{\sigma}_{\alpha\sigma}g^{\alpha\lambda}\right).\end{aligned} \tag{13}$$

After taking into account Eq. (13), Eq. (8) takes the following form

$$16\pi\nabla_{\mu}T^{\mu\nu}=m^{2}\sqrt{-g}\,\gamma_{\mu\lambda}\,g^{\mu\nu}\left(D_{\sigma}g^{\sigma\lambda}+G^{\sigma}_{\alpha\sigma}g^{\alpha\lambda}\right). \tag{14}$$

Applying equation

$$\sqrt{-g}\left(D_{\sigma}g^{\sigma\lambda}+G^{\sigma}_{\alpha\sigma}g^{\alpha\lambda}\right)=D_{\sigma}\tilde{\phi}^{\sigma\lambda}, \tag{15}$$

we find

$$m^{2}\gamma_{\nu\lambda}D_{\sigma}\tilde{\phi}^{\sigma\lambda}=16\pi\nabla_{\mu}T^{\mu}_{\nu}. \tag{16}$$

According to Eq. (3) we have

$$m^{2}\tilde{\phi}^{\sigma\lambda}=J^{\sigma\lambda}+16\pi\sqrt{\frac{g}{\gamma}}\left(T^{\sigma\lambda}+t^{\sigma\lambda}_{g}\right). \tag{17}$$

Substituting Eq. (17) into Eq. (16) we get

$$D_{\sigma}\left[\sqrt{\frac{g}{\gamma}}\left(T^{\sigma\lambda}+t^{\sigma\lambda}_{g}\right)\right]=\gamma^{\nu\lambda}\nabla_{\mu}T^{\mu}_{\nu}. \tag{18}$$

When matter equations of motion are valid we have

$$\frac{\delta L_{M}}{\delta\phi_{A}}=0. \tag{19}$$

According to strong identity (.16) the following equality is valid

$$\nabla_{\mu}T^{\mu}_{\nu}=0, \tag{20}$$

and therefore, according to Eq. (18), the covariant conservation law for energy-momentum of matter and gravitational field taken together is as follows

$$D_\sigma \left[\sqrt{-g}\left(T^{\sigma\lambda} + t_g^{\sigma\lambda}\right)\right] = 0\,. \qquad (21)$$

So, if in GRT covariant law (1) leads to a noncovariant conservation law for energy-momentum of matter and gravitational field (2), and also to arising of a noncovariant quantity – pseudotensor τ_μ^ν of the gravitational field, then in RTG covariant law (1) together with gravitational equations written in a form (3) or (7) exactly leads to the covariant conservation law for energy-momentum of matter and gravitational field written in a form (21). In Eq. (21) the gravitational component $\sqrt{-g}\,t_g^{\sigma\lambda}$ enters in additive form under the Minkowski space covariant derivative symbol, whereas the gravitational component disappears from Eq. (20), it is used for generating the effective Riemannian space and so only the energy-momentum tensor density of matter in Riemannian space stays under the covariant derivative symbol. The presence of covariant conservation laws (21) is just the point to see that the gravitational energy as also all other forms of energy is localizable.

H.Poincare On the dynamics of the electron

(5 June 1905)[36]. (The comments are italicized and indicated by an asterisk).

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[36]Poincare H. Sur la dynamique de l'electron // Comptes rendus hebdomadaires des seances de l'Akademie des sciences. – Paris, 1905. – V.140. – P.1504–1508.

It seems at first sight that the aberration of light and the related optical and electrical phenomena would provide us with a means of determining the absolute motion of the Earth, or rather its motion not with respect to the other stars, but with respect to the ether. Actually, this is not so: experiments in which only terms of the first order in the aberration were taken into account first yielded negative results, which was soon given an explanation; but Michelson also, who proposed an experiment in which terms depending on the square aberration were noticeable, also met with no luck. The impossibility to disclose experimentally the absolute motion of the Earth seems to be a general law of Nature.

* *"Experiment has provided numerous facts justifying the following generalization: absolute motion of matter, or, to be more precise, the relative motion of weighable matter and ether cannot be disclosed. All that can be done is to reveal the motion of weighable matter with respect to weighable matter"* [37].

The above words written by Poincare ten years earlier quite clearly demonstrate that his vision of a general law determining the impossibility of absolute motion of matter had been maturing since long ago.

In development of his idea on the total impossibility of defining absolute motion in relation to the new hypothesis, put forward by Lorentz and according to which all bodies should experience a decrease in length by $1/2 \cdot 10^{-9}$ *in the direction of motion of the Earth, Poincare wrote:*

"Such a strange property seems to be a real **coup de pouce** *presented by Nature itself, for avoiding the disclosure of ab-*

[37]Poincare H. On Larmor's theory // The relativity principle: Collection of works on special relativity theory. — Moscow, 1973. — P.7.

solute motion with the aid of optical phenomena. I can't be satisfied and I must here voice my opinion: I consider quite probable that optical phenomena depend only on the relative motion of the material bodies present, of the sources of light or optical instruments, and this dependence is not accurate up to orders of magnitude of the square or cubic aberration, but rigorous. This principle will be confirmed with increasing precision, as measurements become more and more accurate.

Will a new **coup de pouce** *or a new hypothesis be necessary for each approximation? Clearly this is not so: a well formulated theory should permit proving a principle at once with all rigour. The theory of Lorentz does not permit this yet. But, of all theories proposed it is the one nearest to achieving this goal"* [40].

In a report to the Congress of art and science held in Saint Louis in 1904 Poincare among the main principles of theoretical physics formulates the relativity principle, in accordance with which, in the words of Poincare, "the laws governing physical phenomena should be the same for a motionless observer and for an observer experiencing uniform motion, so there is no way and cannot be any way of determining whether one experiences such motion or not"[41].

An explanation has been proposed by Lorentz, who has introduced the hypothesis of a contraction experienced by all bodies in the direction of the motion of the Earth; this contraction should account for Michelson's experiment and for all

[40] Poincare H. Electricite et optique: La lumiere et les theories electrodynamiques. — 2 ed., rev. et complettee par Jules Blondin, Eugene Neculcea. Paris: Gauthier–Villars, 1901.

[41] Poincare H. L'etat et l'avenir de la Physique mathematique // Bulletin des Sciences Mathematiques. — Janvier 1904 — V.28. Ser.2 — P.302–324; The Monist. — 1905. V.XV, N1.

other relevant experiments performed to date. It would, however, leave place for other even more subtle experiments, more simple to be contemplated than to be implemented, aimed at revealing absolute motion of the Earth. But considering the impossibility of such a claim to be highly probable one may foresee that these experiments, if ever they will be performed, to once again provide a negative result. Lorentz has attempted to complement and alter the hypothesis so as to establish a correspondence between it and the postulate of total impossibility of determining absolute motion. He has succeeded in doing so in his article entitled Electromagnetic phenomena in a system moving with any velocity smaller than that of light (Proceedings de l'Academie d'Amsterdam, 27 May 1904).

The importance of this issue has induced me to consider it once again; the results I have obtained are in agreeement with those obtained by Lorentz in what concerns all the main points; I have only attempted to modify them somewhat and to complement them with some details.

The essential idea of Lorentz consists in that the equations of the electromagnetic field will not be altered by a certain transformation (which I shall further term the Lorentz transformation) of the following form:

$$x' = \gamma l(x - \beta t), \quad y' = ly,\ z' = lz, \quad t' = \gamma l(t - \beta x), \tag{1}$$

where x, y, z are the coordinates and t is the time before the transformation; and x', y', z' and t' are the same after the transformation. The quantity β is a constant determined by the transformation

$$\gamma = \frac{1}{\sqrt{1 - \beta^2}},$$

while l is a certain function of β:

** Poincare writes: "The idea of Lorentz", but Lorentz never wrote such words before Poincare. Here, Poincare formulated his own fundamental idea, but attributed it fully to Lorentz. Probably more than any other person, did he always highly esteem and note each person, who gave his thought an impetus and presented him with the happiness of creativity. He was totally alien to issues of his own personal priority.*

From formulae (1) it is immediately seen that the condition $x = \beta t$ *corresponds to the origin* $(x' = y' = z' = 0)$ *of the new reference system. In other words, the new origin is shifted in the reference system* x, y, z *with a speed* β *along the* x*-axis.*

Thus, the Lorentz tranformations relate the variables (x, y, z, t) *referred to one reference system and the variables* (x', y', z', t') *in another system moving uniformly in a straight line along the x-axis with the velocity* β *relative to the first system.*

Proof of the statement that the equations of the electromagnetic field do not alter under the Lorentz transformations signifies that electromagnetic phenomena are described in both reference systems by identical equations and that, consequently, no electromagnetic processes can be utilized to distinguish between the (x, y, z, t) *reference system and the* (x', y', z', t') *reference system moving uniformly in a straight line with respect to the first system.*

We see that invariance of the equations of the electromagnetic field under transformations of the Lorentz group results in the relativity principle being fulfilled in electromagnetic phenomena. In other words, the relativity principle for electromagnetic phenomena follows from the Maxwell–Lorentz equations in the form of a rigorous mathematical truth.

In this transformation the x-axis plays a particular role, but it is clearly possible to construct such a transformation, in which this role will be assumed by a certain straight line passing through the origin. The set of all such transformations together with all spatial rotations should form a group; but for this to take place it is necessary that $l = 1$; hence one is led to assume $l = 1$, which is precisely the consequence obtained by Lorentz in another way.

* *It must be underlined that, by having established the group nature of the set of all purely spatial transformations together with the Lorentz transformations, that leave the equations of electrodynamics invariant, Poincare thus discovered the existence in physics of an essentially new type of symmetry related to the group of linear space-time transformations, which he called the Lorentz group.*

Supplemented with transformations of space coordinate and time translations, the Lorentz group forms a maximum group of space-time transformations, under which all equations of motion for particles and fields remain invariant and which now is called the Poincare group, the name given to it subsequently by E. Wigner. Richard Feynman wrote about this fact as follows: "Precisely Poincare proposed to find out what one can do with equations without altering their form. He was the person who had the idea to examine the symmetry properties of physical laws".

Let ρ be the charge density of the electron, and v_x, v_y, and v_z the components of the electron velocity before the transformation; then, after applying the transformation one has for these same quantities ρ', ${v_x}'$, ${v_y}'$, and ${v_z}'$ the following:

$$\begin{aligned} &\rho' = \gamma l^{-3}\rho(1-\beta v_x), \qquad \rho' {v_x}' = \gamma l^{-3}\rho(v_x - \beta), \\ &\rho' {v_y}' = l^{-3}\rho v_y, \qquad \rho' {v_z}' = l^{-3}\rho v_z. \end{aligned} \tag{2}$$

These formulas differ somewhat from the ones found by Lorentz.

Now, let $\vec{f}$ and $\vec{f'}$ be the three-force components before and after application of the transformation (the force is referred to unit volume); then

$$f'_x = \gamma l^{-5}(f_x - \beta \vec{f}\vec{v}); f'_y = l^{-5} f_y; f'_z = l^{-5} f_z. \tag{3}$$

These formulas also differ somewhat from the ones proposed by Lorentz; the additional term in $\vec{f}\cdot\vec{v}$ reminds the result earlier obtained by Lienard.

If we now denote by $\vec{F}$ and $\vec{F'}$ the force components referred to the electron mass unit, instead of unit volume, we obtain

$$F'_x = \gamma l^{-5}\frac{\rho}{\rho'}(F_x - \beta \vec{F}\cdot\vec{v}); F'_y = \frac{\rho}{\rho'} l^{-5} F_y; F'_y = \frac{\rho}{\rho'} l^{-5} F_x. \tag{4}$$

* *Formulae (2), (3), and (4) comprise the relativistic transformation laws, first established by Poincare, for the charge density and velocity of motion of an electron, referred both to unit charge and unit volume.*

Thus, Poincare, as compared with Lorentz, made the decisive step in this work and laid down the foundations of relativity theory.

It is notable that, while developing totally new ideas in articles on the dynamics of the electron and correcting and complementing Lorentz, Poincare is careful to paid maximum tribute to Lorentz as the discoverer and leaves it to others to judge about his own personal contribution to the creation of relativity theory.

The testimony of Lorentz himself is extremely important in this respect, since it permits filling in the gap, sometimes left by certain authors writing about the history of the creation of relativity theory, and to do justice to Poincare as the creator of relativistic mechanics and special relativity.

Thus, in discussing the relativistic transformation formulae for the velocities, charge densities, and current of the electron, Lorentz wrote:[46] *"Formulae (4) and (7) are absent in my article published in 1904, since I didn't even think of a direct path leading to them, because I thought an essential difference existed between the systems* x, y, z, t *and* x', y', z', t'*. In one of them – such was my reasoning – coordinate axes were used that had a fixed position in ether and what could be called true time; in the other system, on the contrary, one dealt with simply auxiliary quantities introduced only with the aid of a mathematical trick. Thus, for instance, the variable* t' *could not be considered to be time in the same sense as the variable* t*.*

Given such reasoning, I had no intention of describing phenomena in the system x', y', z', t' *in precisely the same manner as in the system* x, y, z, t*..." And further, in the same work: "... I was unable to achieve total invariance of the equations; my formulae remained cumbersome owing to additional terms, that should have disappeared. These terms were too small to exert noticeable influence on the phenomena, and this supplied me with an explanation for their being independent of the Earth's motion revealed by obs and universal truth.*

[46]Lorentz H.A. Two articles by Henri Poincare on mathematical physics // The relativity principle: Collection of works on special relativity theory. – Moscow, 1973. – P.189–196.
Formulae (4) and (7) dealt with by Lorentz are the transformation formulae for the electron velocities and charge densities, respectively.

Contrariwise, Poincare achieved total invariance of the equations of electrodynamics and formulated the ¡¡relativity postulate¿¿ — a term introduced by him. Indeed, adopting the point of view, that I had failed to take into account, he derived formulae (4) and (7). We should add, that in correcting the defects of my work he never reproached me for them".

Lorentz also arrived at the necessity of assuming a moving electron to have the shape of a compressed ellipsoid; the same hypothesis was made by Langevin, but while Lorentz assumed the two axes of the ellipsoid to be constant, in agreement with his hypothesis that $l = 1$, Langevin assumed, contrariwise, the volume of the ellipsoid to be constant. Both authors showed the two hypotheses to be in the same good agreement with the experiments performed by Kaufmann, as the initial hypothesis of Abraham (the spherical electron). The advantage of Langevin's hypothesis consists in its being sufficient, i.e. it suffices to consider the electron to be deformable and incompressible for explaining why it assumes an ellipsoidal shape in motion. But I can show, without contradicting Lorentz, that this hypothesis cannot be consistent with the impossibility of revealing absolute motion. As I have already said, this occurs because $l = 1$ is the only hypothesis for which the Lorentz transformations form a group.

But in the Lorentz hypothesis, also, the agreement between the formulas does not occur just by itself; it is obtained together with a possible explanation of the compression of the electron under the assumption that **the deformed and compressed electron is subject to constant external pressure, the work done by which is proportional to the variation of volume of this electron.**

Applying the principle of least action, I can demonstrate the compensation under these conditions to be complete, if inertia is assumed to be of a totally electromagnetic origin, as generally acknowledged after Kaufmann's experiments, and if all forces are of an electromagnetic origin, with the exception of the constant pressure of which I just spoke and which acts on the electron. Thus, it is possible to explain the impossibility of revealing the absolute motion of the Earth and the contraction of all bodies in the direction of the Earth's motion.

But this is not all. In the quoted work Lorentz considers it necessary to complement his hypothesis with the assumption that in the case of uniform motion all forces, of whatever origin, behave exactly like electromagnetic forces, and that, consequently, the influence of the Lorentz transformation on the force components is determined by equations (4).

* *Here, Poincare in development of the assumption expressed by Lorentz extends the Lorentz transformations to all forces, including, for instance, gravitational forces.*

He was the first to point out that the relativity postulate requires such a modification of the laws of gravity, according to which the propagation of forces of gravity is not instantaneous, but proceeds with the speed of light.

It has turned out necessary to consider more carefully this hypothesis and, in particular, to clarify which changes it compels us to introduce into the laws of gravity. This is just what I attempted to determine: I was first induced to assume the propagation of gravity forces to proceed with the speed of light, and not instantaneously. This seems to contradict the result obtained by Laplace who claims that although this propagation may not be instantaneous, it is at least more rapid than the propagation of light. However, the issue actually raised by Laplace differs significantly from the issue dealt with here by us. According to Laplace, a finite propagation velocity was **the sole** alteration, introduced by him to Newton's law. Here, also, a similar change is accompanied by many others; hence, partial compensation between them is possible, and it actually does take place.

Consequently, if we speak about the position or velocity of a body exerting attraction, we shall bear in mind its position or velocity at the moment, when **the gravitational wave** departs from this body; if we speak about the position or velocity of a body being attracted, we shall intend its position or velocity at the moment, when this body being attracted is overcome by the gravitational wave emitted by another body: the first moment clearly precedes the second.

Hence, if x, y, z are the projections onto three axes of the vector $\vec{r}$ connecting the two positions and if $\vec{v} = (v_x, v_y, v_z)$ are the velocity components of the body attracted and $\vec{v_1} = (v_{1x}, v_{1y}, v_{1z})$ are the velocity components of the attracting body, then the $1z$ three components of the attraction (which I may also call $\vec{F}$) will be functions of $\vec{r}, \vec{v}, \vec{v_1}$. The question is whether these functions can be defined in such a way that they behave under the Lorentz transformation in accordance with equations (4) and that the conventional law of gravity be valid in all cases of the velocities $\vec{v}, \vec{v_1}$ being sufficiently small to allow neglecting their square values as compared with the square speed of light?

The answer to this question must be affirmative. It has been revealed that the attraction, taking into account the correction, consists of two forces, one of which is parallel to the components of the vector $\vec{r}$, and the other to the components of the velocity $\vec{v_1}$.

The disagreement with the conventional law of gravity, as I just pointed out, is of the order of v^2; if, on the other hand, one assumes, as Laplace did, the propagation velocity to be equal to the speed of light, this divergence will be of the order of v, i.e. 10000 times greater. Consequently, at first sight, it does not seem absurd to assume astronomical observations to be insufficiently precise for revealing the smallest imaginable divergence. Only a profound investigation can resolve this issue.

* *Poincare thus introduces the physical concept of gravitational waves, the exchange of which generates gravitational forces, and supplies an estimation of the contribution of relativistic corrections to Newton's law of gravity.*

For example, he shows that the terms of first order in v/c cancel out exactly and so the relativistic corrections to Newton's law are quantities of the order of $(v/c)^2$.

These results remove the difficulty noted previously by Laplace and permit making the conclusion that the hypothesis equating the speeds of light and of gravitational influence is not in contradiction with observational data.

Thus, in this first work Poincare already gave a general and precise formulation of the main points of relativity theory. It is here that such concepts as the following first appeared: the Lorentz group, invariance of the equations of the electromagnetic field with respect to the Lorentz transformations, the transformation laws for charge and current, the addition formulae of velocities, the transformation laws of force. Here, *also, Poincare extends the transformation laws to all the forces of Nature, whatever their origin might be.*

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