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CONSTRUCTIVE GEOMETRY

CONSTRUCTIVE GEOMETRY

BEING

STEPS IN THE SYNTHESIS OF IDEAS REGARDING
THE PROPERTIES AND RELATIONS
OF GEOMETRICAL FIGURES

ARRANGED FOR

THE FIRST YEAR'S COURSE IN SCIENCE

BY

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"Rerum ipsarum cognitio vera e rebus ipsis"

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GLASGOW AND DUBLIN

PREFACE

The system of deductive·geometry evolved by Euclid is generally thought to be unsuitable for young minds. At the outset the definitions and the axioms present obstacles, and the formal phraseology of the arguments is not readily appreciated.

The subject-matter of these pages is virtually that of the first three books of the Elements, but there is a great departure as regards the order and the method. It was Euclid's aim to unfold from certain carefully stated first principles a body of geometrical knowledge contained implicitly in these principles. Here, on the other hand, the pupil starts with a few concrete objects. He is provided with such simple appliances as a straight-edge, a pair of compasses, a protractor, a circular disc, and a metal polygon. He is directed to make drawings and to examine them. As he proceeds with his experiments and observations he is helped to build up ideas about lines, points, triangles, circles, &c., in precisely the same way as that followed in dealing with the elements of physics and chemistry.

On account of the nature of the process by which this knowledge of the properties of plane figures is being attained the name *Constructive Geometry* has been adopted as the title of the book. But although general inferences are freely drawn from particular instances, tested of course by careful measurement, yet an endeavour is made to enable the pupil to obtain fresh results by deductive reasoning. Nor is the efficiency of the "reductio ad absurdum" disregarded.

Accordingly it is hoped that when these chapters have been mastered there will be as net result an extensive fund of available knowledge on which more advanced work can rest securely, and also such habits of enquiry and thought as will give a stimulus to further study.

In the preparation of the book I have been indebted to my colleagues, and particularly to Mr. James A. M'Bride, B.A., the senior mathematical master in Allan Glen's School.

J. G. K.

GLASGOW, *26th February, 1904.*

CONTENTS

LIST OF APPLIANCES

A straight-edged flat ruler—one edge graduated in inches and tenths, and the other in centimetres and millimetres.

Drawing-book, pencil, and rubber,

Squared paper, which may be had separately or as part of the drawing-book.

Set-squares for angles of 90°, 45°, 60°, and 30°.

A good pair of compasses.

A movable plane—the quadrilateral lamina described in Chapter V.

A thin circular metal disc about 4 cm. in diameter.

A protractor—described in Chapter IV.

Note to the Pupil.—The figures should be drawn with care, and the magnitude of the lines, angles, and areas should, whenever possible, be clearly marked on the drawings.

CONSTRUCTIVE GEOMETRY

CHAPTER I

PRELIMINARY NOTIONS—SURFACE—LINE—POINT

This sketch brings before your mind five different objects which you have frequently seen and handled. Some of them, you may remember, were made of wood

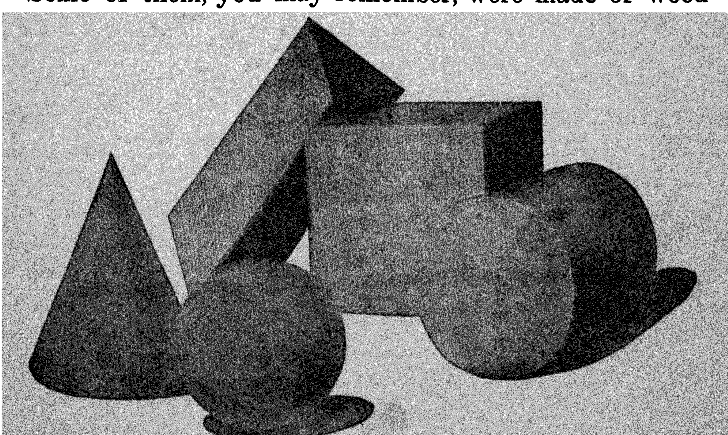


Fig. 1

and painted white, while others were made of stucco. Yet so far as the drawing is concerned, the objects suggested might be thought of as made of lead or ivory, perhaps even of gold. Then, again, the real things which you recall may have been large or small, light or heavy, while the picture gives you no information whatever about their size or their mass. Some of these things may have been solid, some hollow, while others may have been so built up that if cut with a saw they show a laminated structure. Thus the drawing tells you nothing about the colour, or size, or mass, or material, or structure of the objects pictured. How is it, then, that you seem to be on such intimate terms with the cube, prism, cylinder, cone, and sphere here delineated?

What you have so readily recognized and so unhesitatingly named in each case is not the object, but the object's shape or form. You cannot, however, think of the form without also thinking of some thing having that form, which, on account of its size, occupies a portion of space.

The particular forms you have been attending to are said to be regular, and the objects corresponding to them are spoken of as solids of regular geometric form. That you have been able to identify and name these different shapes from the drawing does not imply that each one of the figures is an accurate picture of a particular object known to you, but merely that it has the essential features of the cubes, or the prisms, or the cylinders, or the cones, or the spheres which you have seen or handled or had described to you.

It is not quite an easy matter to write accurate descriptions of such regular forms as the cube, cylinder, or cone, and it is practically impossible to convey in words a full and complete account of one of the countless irregular shapes we continually meet, as, for example, that of a branch of a tree. At best any attempt will be only partially successful, and words will convey merely a roughly-blocked-out idea of the object.

In the process, however, of considering and describing regular and irregular shapes, you reach two important notions—

(i) Bodies occupy space.

(ii) Bodies are separated from surrounding space and from other bodies by **surfaces.**

You can think, for example, of a room packed with cubes, each cube being separated from its neighbour or from the air still remaining by its six flat square surfaces. If a beaker of water is thought of, you have a mass of liquid occupying a definite space. The upper surface, which separates the water from the air, is flat and horizontal, while the side and bottom surfaces of the liquid are determined by the inner surface of the containing vessel.

The surface of a common cylinder may be said to consist of two flat circular ends and a single curved surface. The area of this curved surface is precisely that of a rectangle, whose length is equal to the circumference of the flat circular edge, and whose breadth is equal to the height of the cylinder. A coat of the above dimensions would fit the cylinder and exactly cover the curved surface.

Guard against supposing that a flat plate or lamina, such as the leaf of a book, is a surface. Every sheet of paper has two broad surfaces, an upper and a lower, and narrow surfaces at the edges.

The surfaces possessed by material objects are either flat or curved.

Examples of the former kind are: the top of a table, the face of a looking-glass, and the surface of a still pond.

Examples of the second kind are: the outside of a sphere, the curved boundary of a cylinder, and the surface of the ocean.

Consider now a white surface, *e.g.* that of a sheet of paper marred accidentally by one blot of ink. The surface is divided into two portions, that which is blotted and that which is not blotted. The edges of the blot form the boundary, within which the blotted portion of the surface lies, the white portion being without. This boundary is spoken of as a line. If, with the point of a knife, the blot is cut out so that the part removed shows no white surface and the part remaining no black surface, then the cutting edge must have travelled exactly along the line which separates the two portions of the surface of the sheet of paper. As the two portions can be fitted together, the path of the cutting edge, *i.e.* the line of separation, has no breadth.

Just as two portions of the same surface are separated by a line, so two different surfaces which meet, meet in a line. The edges of the cubes and prisms are lines formed by the meeting surfaces. The free surface of water in a beaker touches the wall of the glass vessel in a line which you may think of as being marked in some way right round the inner surface of the beaker.

Lines may be drawn on a surface by means of a pen or pencil. The finer the pen, the sharper the pencil, the more nearly is the drawing an ideal line. The lines with which you will most frequently deal will be drawn on flat surfaces, but you should be able to think of a line apart from the surface on which it is drawn, and you will do this readily after considering a piece of stiff wire of irregular shape in space, or the path of a bird on the wing.

Lines may be either straight or curved regularly or irregularly.

When two lines meet or cross we have a **point**.

Two adjoining portions of a line are separated by a point.

For the purpose of drawing straight lines on paper you are furnished with a **straight-edge.** You may assume for the present that it is what it pretends to be, but a method of testing your instrument will appear presently.

TESTING WITH THE STRAIGHT-EDGE

1. Test a flat surface. Apply the ruling edge to the surface, and note that it is wholly in contact with that surface for every possible position of the straight-edge. In geometrical language this is called a *plane surface.*

2. Test the surface of a cylinder. The circular end-surfaces are soon declared to be plane; but though you are able to find on the remaining surface an indefinite number of positions in which you have perfect contact, yet on altering the direction of the straight-edge even slightly, you observe that there are positions in which the contact is not complete, and you at once declare the surface to be *not-plane.*

3. Test a wavy or corrugated surface. Use a sheet of paper or other flexible substance, and describe the result of the test.

4. Test the surfaces and edges of a cube. Each of the six surfaces is a plane. When the straight-edge is applied to the lines of intersection of these planes there is complete contact. Hence you learn that the intersection of two planes is a straight line.

5. Test the surfaces and edges of a prism or a pyramid. You get the same result as in 5.

6. Consider the intersection of—(*a*) a plane with the surface of a sphere; (*b*) a plane with the curved surface of a conical solid; (*c*) a plane with the curved surface of a cylindrical solid; (*d*) the surface of a sphere with that of a cylinder; (*e*) the surface of a sphere with that of a cone.

7. Test your straight-edge. Mark with dots two points A and B on a sheet of paper. (*a*) Draw with pencil the line joining the two points, using different parts of the ruling edge, as in fig. 2 (*a*). It will always be found that the same line is obtained, however often the process is repeated. Now (*b*) bring the ruling edge round to the other side, as in fig. 2 (*b*), and draw the line as often

as desired. If the straight-edge is true the line will be the same in all cases.

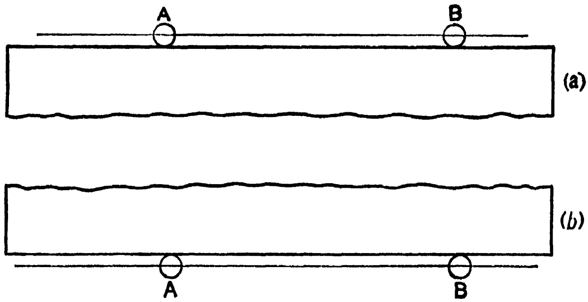


Fig. 2

8. Repeat the two parts of the test with the circular edge of your protractor. You will find that while only one line is got in each case, the two lines are different. Hence the edge is not straight. You can now confidently use your straight-edge—

(*a*) to join two points by a straight line;
(*b*) to produce a terminated straight line to any desired distance in a straight line.

You will note that the ends of a line are points, and the crossing or intersection of one line with another is also a point. Important points on a diagram should be distinguished by letters.

SUMMARY OF RESULTS

1. Bodies occupy space.
2. Surfaces separate bodies from surrounding space.
3. Surfaces may be classified as plane and not-plane.
4. Plane surfaces are tested by a straight-edge. The edge should coincide with the plane in every position.
5. Surfaces intersect in lines.
6. Plane surfaces intersect in straight lines.
7. Straight lines are tested by a straight-edge.
8. A straight-edge is used to test itself.
9. The intersections of lines are points.

QUESTIONS

1. If two edges fit exactly, are they necessarily straight edges?
2. Does a leaf of a book constitute a plane surface?
3. Does a page of a book constitute a plane surface?
4. Mention four common objects that present practically plane surfaces.
5. A straight-edge about whose accuracy you have some doubt gives complete contact when applied to a surface. Is the surface necessarily a plane? If on moving the edge into any position on the surface you still have complete contact, is the surface necessarily plane, and the straight-edge necessarily true?
6. A true straight-edge lies closely on a surface in three separate trials. Is the surface necessarily plane?
7. Write a description of the four objects of fig. 1 sufficiently full to call up in the mind of another person the figures you are describing.

EXERCISES

[Straight-edge and Pencil]

1. Mark two points on the paper: join them by a straight line: take two other points, one on each side of the line, and join every pair of points.
2. Find how many lines can be drawn so as to join—
   (*a*) three points two at a time;
   (*b*) four points two at a time;
   (*c*) five points two at a time.

[In each figure arrange to have no three points in one line.]

3. How many points are in general determined by the crossing of—
   (*a*) three lines in the same plane?
   (*b*) four lines in the same plane?
   (*c*) five lines in the same plane?

[In each figure arrange to have no three lines meeting in a point, and no two lines not meeting.]

4. Take six points, and make all the joins possible when—
   (*a*) no three points lie in one line;
   (*b*) three points lie in one line;
   (*c*) three points lie in one line, and the other three also in one line.

5. From a point O draw the lines OL, OM, ON; from another point P draw two lines cutting OL, OM in A, A′ and B, B′ respectively; on ON take two points C and C′; join CA and C′A′, and let

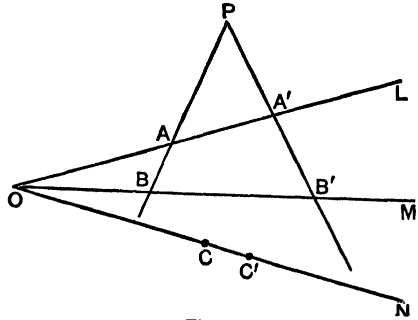


Fig. 3

them meet in Q: join CB and C′B′, and let them meet in R. What do you note regarding the points P, Q, and R?

Commence your drawing as in fig. 3.

CHAPTER II

ANGLES—RIGHT ANGLES

Formation of Angles.—Think of the minute hand of a clock. It is fixed to a spindle and moves steadily round the dial, taking one hour to make a complete revolution. Let us imagine a straight line drawn from the centre of rotation to the hour-mark III, and let us commence observations when the minute hand is just over this line. Five minutes later it points to the hour-mark IIII, and in moving to its new position from its former one it rotates through a definite **angle.**

Fifteen minutes from the start the hand points to VI, vertically downwards. As time goes on the angle turned through gets greater and greater, and when half-an-hour has elapsed the hand is continuous with the fixed line, and points in the opposite direction. After one hour it has returned to its starting-point.

This is an example of rotation, and we say that the radial moving hand is *describing an angle at the rate of one revolution per hour.*

Again, if two straight lines O A and O B meet in O (fig. 4) we may regard the angle between O A and O B as being measured by the amount of rotation necessary to bring a line from coincidence with O A to coincidence with O B.

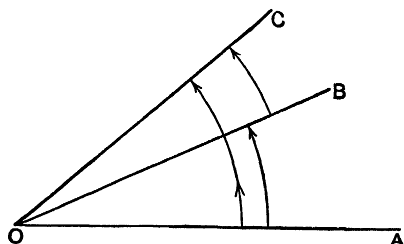


Fig. 4

It has been agreed to speak of the rotation of a line as *positive* when the motion is against the hands of a clock.

In short, *positive* rotation is *counter-clockwise,*
negative " *clockwise.*

In fig. 4 we have three lines O A, O B, O C, meeting in O. Clearly the angle A O C is equal to the sum of the angles A O B and B O C, for the angle A O C can be formed by the rotation of a line first from the position O A to the position O B, and afterwards to the further position O C.

If a line O P hinged at O rotate positively through a certain angle from the position O A (fig. 5), then the same amount of negative rotation will bring it back to coincidence with O A, the result being the same as if it had never moved at all. Thus positive and negative signs attached to angles have algebraic significance ($+a - a = 0$). An angle may consist of more than half a revolu-

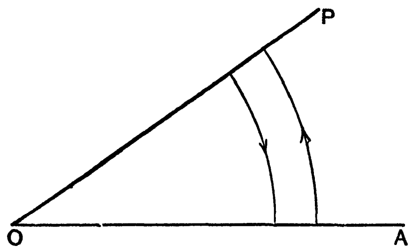


Fig. 5

tion, and for that matter of more than a hundred or a thousand revolutions.

The Right Angle.—The revolution is divided in the

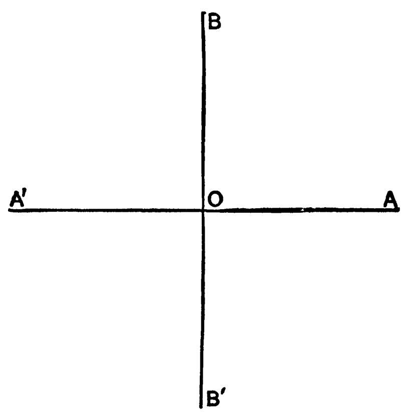


Fig. 6

first instance into four equal sections called right angles. When the generating line O P has moved from the fixed line O A through one right angle it comes into the position O B (fig. 6). The line O B is said to be *perpendicular* to the line O A. When the generating line has moved through a second right angle it coincides with O A′, which is in line with A O. A further rotation through a right angle brings the generating line into line with B O, and the remaining right angle completes the revolution.

The **set-square** is an instrument used in drawing right angles. Like the straight-edge, it can be used to test itself. The method is as follows:—Place the set-square in the position shown in fig. 7 (*a*), and draw O P at right angles to A A′; turn the instrument into the position shown in fig. 7 (*b*), and through the same point O in A A′ draw another line at right angles to A A′. The two lines so drawn should be coincident, otherwise the set-square is untrue, and must be corrected in the workshop before being used. The set-square is used along with a straight-edge. If one edge of the square be placed in contact with a clamped straight-edge, and the square caused to slide, the other edges of the square move sideways without altering in direction. Suppose, now, it is desired to draw a perpendicular to a line X Y through any point P (fig. 8). The set-square *a b c* is laid so that the edge *a b*, the right angle being at *b*, lies against the scale, and the opposite edge *a c* coincides with X Y. The set-square is then caused to rotate about *b* until the edge

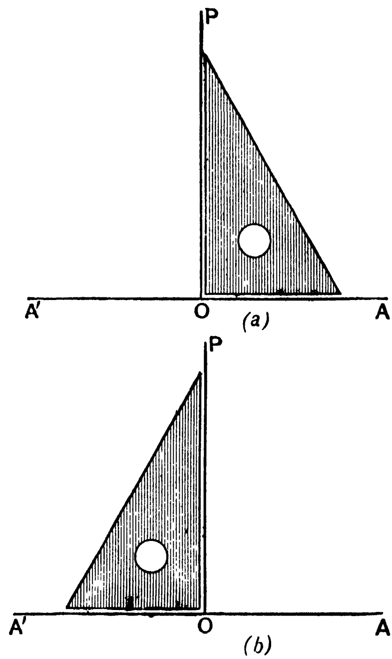


Fig. 7

b c touches the straight-edge. The instrument has been turned through a right angle, and so *a c* is perpendicular

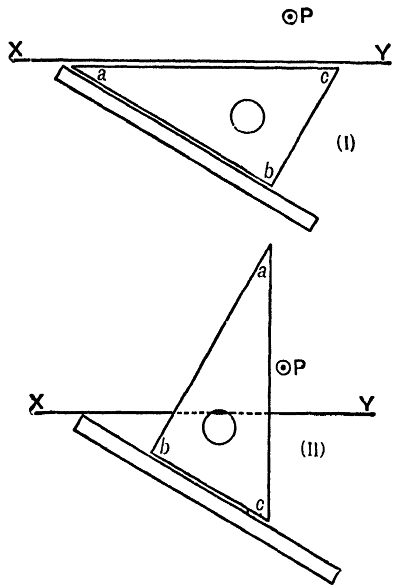


Fig. 8

to X Y. After sliding the square until *ac* is nearly over P, the perpendicular can be drawn.[1]

It is clear that only one perpendicular can be drawn to a straight line from a given point.

When the sum of two angles is one right angle each is said to be the *complement* of the other; when the sum of two angles is two right angles, each is called the *supplement* of the other.

Let us take a fixed line X Y (fig. 9), and draw to it a perpendicular P A from a fixed point P without it, and let us describe a number of circles with P as centre. Let the radius of the first circle be P A: it will be seen that

[1] The principle here used will be discussed later.

the circle just touches the line X Y. All circles with larger radius will cut X Y in two points. Thus with

P B as radius the circle cuts X Y in B, B′;
with P C „ „ „ X Y in C, C′; and so on.
Now join P B, P B′; P C, P C′, &c.

From careful drawings made in this manner the following results may be obtained:—

(i) The perpendicular is the shortest line that can be drawn from a given point to a given straight line.

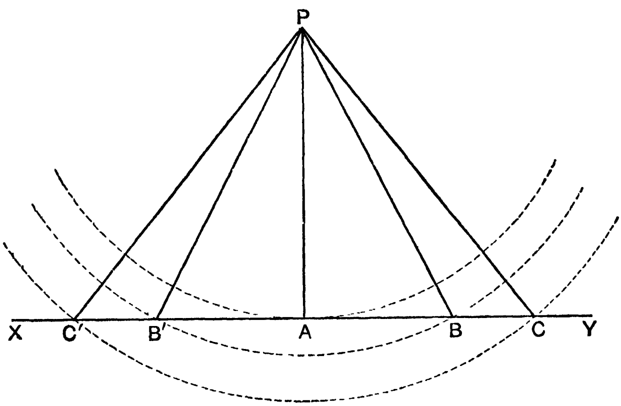


Fig. 9

(ii) Any number of pairs of equal lines, *e.g.* P B, P B′; P C, P C′, can be drawn from a point to a line.

(iii) No two equal obliques can be drawn on the same side of the perpendicular.

(iv) The feet of (*e.g.* B, B′) any pair of equal obliques are equidistant from the foot of the perpendicular.

NOTE.—Although it is sufficient in the meantime to test Result iv by means of compasses, it might be well to show that this result follows if we assume those that precede it. A conclusion reached in such a manner is spoken of as a **deduction**.

In fig. 9 we have P A perpendicular to X Y. So P A B is a right-angled triangle.

Suppose this triangle to turn round the line P A, as if it were hinged along that line, until it comes again into the plane of the paper.

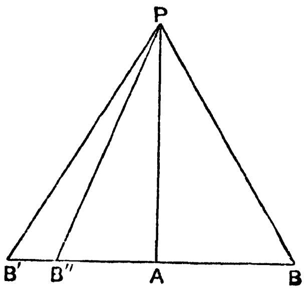


Fig 10

As angle P A B is a right angle, and as angle P A B′ is also a right angle, it follows that the line A B will now lie along the line A B′.

The point B will now lie on the point B′, for if it does not it will take up another position, B″ (fig. 10).

The line P B″ is just the line P B. But P B = P B′ by construction, and so it follows that if B does not coincide with B′ there will be two equal obliques, P B′ and P B″, on the same side of the perpendicular.

This is impossible by Result iii, and therefore B coincides with B′. So A B = A B′.

(v) The angles at the base of an isosceles triangle are equal.

NOTE.—An **isosceles** triangle has two of its sides equal, and so in fig. 9 P B B′, P C C′, &c., are isosceles triangles. In the demonstration of Result iv we have considered that the right-angled triangle P A B could be turned round the line P A until it came again into the plane of the paper, when it would fit into, *i.e.* coincide with, the right-angled triangle P A B′. Therefore the angles P B A and P B′ A are equal.

(vi) Of two obliques that one is the longer the foot of which is the farther from the perpendicular.

(vii) Any point equidistant from other two points is on the **right bisector** of the line joining these two points.

NOTE.—In fig. 9 the point P is equidistant from B and B′, and P A, which bisects B B′ at right angles, is its right bisector.

The right bisector of a line A B can readily be drawn by means of compasses and straight-edge.

With centre at A and any radius greater than half A B make a circle (fig. 11). With centre B and the same radius make another circle. These circles intersect at only two points, C and D, one on each side of A B.

The line that joins C D bisects A B at right angles.

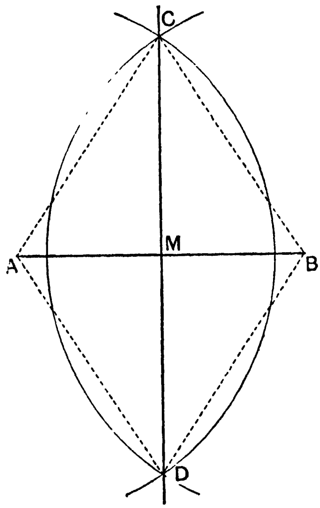


Fig. 11

(viii) The right bisector of the base of an isosceles triangle bisects the vertical angle.

NOTE.—In fig. 9 triangle P B B′ is isosceles, and P A is the right bisector of base B B′. It has been shown that if the right-angled triangle A P B be turned round A P until it comes again into the plane of the paper, it will coincide with triangle A P B′. Therefore the angles A P B and A P B′ are equal, and so the line P A bisects the angle B P B′.

The following construction is used to *bisect a given angle* B A C (fig. 12). With centre A and any radius draw an arc to cut A B and A C in B and C. With centres at B and C and any radius greater than half the straight line B C make two circles intersecting in such a point as D.

The line A D bisects the given angle.

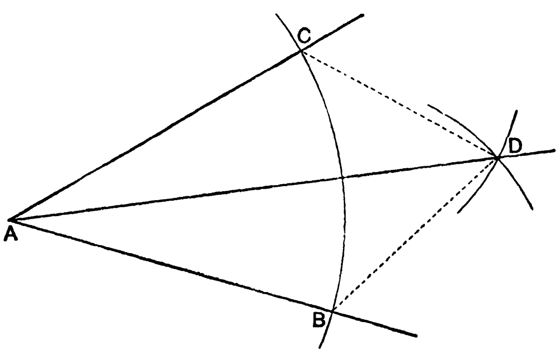


Fig. 12

SUMMARY OF RESULTS

1. A line rotating in a plane describes an angle. Angles are positive and negative according to the direction of rotation.
2. An angle is an amount of rotation.
3. A right angle is one quarter of a revolution.
4. A set-square has one of its angles a right angle and is used for drawing perpendiculars to straight lines. It can be used to test itself.
5. The equality of the lengths of two straight lines may be tested by means of a pair of compasses.
6. The perpendicular is the shortest line that can be drawn from a point to a straight line.
7. Any number of pairs of equal straight lines can be drawn from a point to a straight line.
8. Two equal obliques drawn from the same point have their end-points equidistant from the foot of the perpendicular.
9. If two unequal obliques be drawn from the same point to a line, the end-point of the greater is farther from the foot of the perpendicular than the end-point of the less.
10. The angles at the base of an isosceles triangle are equal.
11. Any point which is equidistant from the ends of a line is on the right bisector of that line.
12. The right bisector of the base of an isosceles triangle bisects the vertical angle.

QUESTIONS

1. Define the terms "angle" and "right angle".
2. How many right angles are there in seven and a half revolutions?
3. Justify the phrase "a negative angle". A wheel turns through three right angles clockwise and seven counter-clockwise. What is its resultant angular displacement?
4. If the sum of the angles AOC and COB on opposite sides of the line OC be two right angles, show that AO and BO must be in the same straight line.
5. If AB and CD intersect in O, prove by the method of rotation that the angles COA and DOB are equal and that the angles BOC and AOD are equal.
6. If from a point O in the straight line AOA′ two straight lines OP, OP′ be drawn on opposite sides of AA′ such that the angle

A O P is equal to the angle A′ O P′, show that O P and O P′ are in the same straight line.

7. What do you mean by "distance of a point from a line"?

8. How do you show that the perpendicular is the shortest line that can be drawn from a given point to a given straight line? What do you assume?

9. How would you prove that three equal straight lines cannot be drawn from a point to a given straight line?

10. If in the triangle A B C you have C A equal to C B, and if D be the middle point of A B, prove that C D is perpendicular to A B.

11. How do you prove that if two angles of a triangle are equal the triangle is isosceles?

12. If two circles, centres A and B, intersect in C and D, show that A B is the right bisector of C D.

EXERCISES

1. Test your set-squares after the manner described in the text.

2. Use a set-square in drawing a perpendicular to a given straight line from a given point without it.

3. From each of three points not in the same straight line, draw perpendiculars to the line joining the other two. Note the result.

4. Construct an isosceles triangle A B C where C A = C B = 5 cm. and A B = 6 cm. Draw C D perpendicular to A B. Measure C D, A D, D B.

5. Construct an isosceles triangle A B C where C A = C B = 5 cm. and the angle B C A = 90°. Draw from C a perpendicular C D on A B. Write down the lengths of A B, D A, D B, D C.

6. Bisect a straight line, using compasses and straight-edge as indicated on page 23.

7. Bisect a given straight line, using only set-square, straight-edge, and pencil.

8. Draw an angle of 270° (three right angles) and bisect it, using compasses and straight-edge.

9. Draw a triangle of considerable size. Bisect each of the angles. Note an important result.

10. Draw a straight line 7·5 cm. long. Construct upon this line as base a series of isosceles triangles, the sides being 4, 5, 6, &c., cms. in length. Show that the vertices of these triangles lie on a certain straight line.

11. Mark two points A and B 7·5 cm. apart. Through A draw

a series of straight lines. To each of these lines draw a perpendicular from B, and mark each point where there is intersection at right angles. Show that these points lie on a certain circle.

12. Upon a line AB 5 cm. long draw an isosceles triangle ABC with sides 7·5 cm. long. Using set-square, draw from A a perpendicular AD to CB, and from B a perpendicular BE to CA. Through O, the crossing point of these two perpendiculars, draw a line CO cutting AB in F. Measure AE and BD. Test angle BFC with set-square.

CHAPTER III

EQUALITY OF ANGLES—ARCS AND CHORDS

Two angles are said to be equal if the same amount of rotation is required in each case to bring one containing line into coincidence with the other. If it can be said, in any given instance, that one-third, or two-fifths, or any other definite fraction of a revolution forms each of the angles, then they are equal. But the estimation of fractions of a complete revolution as a test of equality is rarely convenient, and other modes have to be adopted. We have seen that the equality of straight lines may be tested by compasses: it will now appear that compasses are also serviceable in testing the equality of angles. In using the instrument for describing circles, the needle-point is kept fixed at the centre of the circle, while the pencil-point describes the circumference.

Any limited portion of the circumference is called an **arc** of the circle, while the straight line joining any two points on the circumference is called the **chord** of the intervening arc.

A **diameter** of a circle is any straight line drawn through the centre and terminated by the circumference.

A **radius** is any straight line drawn from the centre to a point on the circumference. It is evidently the distance or stretch maintained between the pencil and needle points when the circle is being described.

It is clear that all circles described with the same radius or "stretch" are of the same size, and are, in fact, exact copies of each other.

Now let us describe a circle A B P with any convenient radius O A (fig. 13), and consider how we shall place in it a chord equal in length to a given straight line X Y. The compasses are opened until the stretch is exactly equal to X Y, and the needle-point being kept fixed at A on the circumference of the circle, the pencil is made to describe an arc crossing the circumference in P and P′. These points are now joined to A; the lines A P and A P′ are each equal to the given line X Y. If now the construction be repeated, using different stretches between the compass-points, it will be found that—

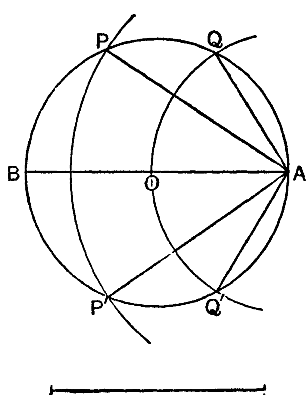


Fig. 13

(i) So long as the stretch is less than the diameter of the circle, two, and only two, chords can be drawn from A equal to a given straight line.

(ii) These equal chords are on opposite sides of the diameter through A, chords drawn from A on the same side of the diameter through A being unequal.

(iii) The greater the chord, the greater will be the arc cut off.

Let us now draw (fig. 14) two circles with centres O and O′ and equal radii O A and O′ A′, and let us place in them equal chords A P and A′ P′. The two figures are necessarily equal in every respect. The moving pencil-point (in describing the arcs A P and A′ P′) has traced out

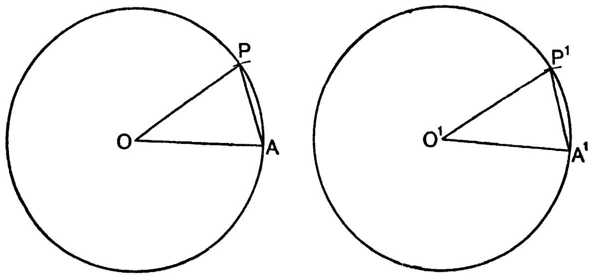


Fig. 14

exactly equal paths, and the amount of rotation necessary to bring O A into the position O P is exactly

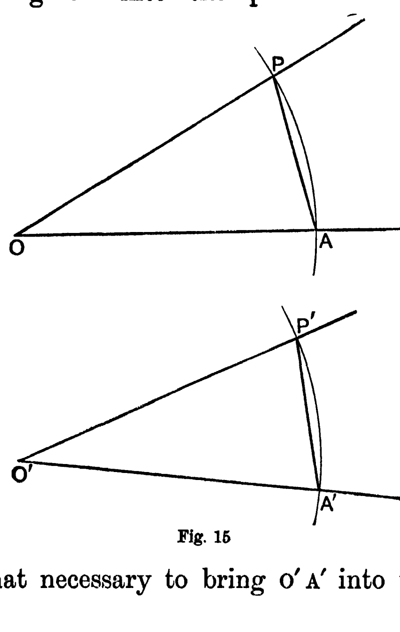


Fig. 15

equal to that necessary to bring O′ A′ into the position O′ P′.

In other words, the angles A O P and A′ O′ P′ are equal.

The conclusion arrived at is that—

(iv) In equal circles equal chords cut off equal arcs, and

(v) Equal chords subtend equal angles at the centres of equal circles.

We are now in a position to **construct an angle equal to any given angle.**

Let O (fig. 15) be the vertex of the given angle. With centre O and any convenient radius describe an arc cutting the lines containing the angle in A and P. Join A P. With any chosen point O′ as centre, and a radius equal to the former, describe another arc, and in it place a chord A′ P′ equal to A P. Join A′ O′, P′ O′. It is clear that the angle A′ O′ P′ is equal to the angle A O P. For

∵ the radius O A is equal to the radius O′ A′,

∴ the circle A P (centre O) is equal to the circle A′ P′ (centre O′),

and ∵ the chord A P is equal to the chord A′ P′,

∴ the rotation from O A to O P is equal to the rotation from O′ A′ to O′ P′,

∴ the angle A O P is equal to the angle A′ O′ P′.[1]

Let us now (fig. 16) describe a circle with centre O and radius O A, and cut off equal arcs A B, B C, C D, &c. It is evident that (joining A B, B C, C D, and A D) the arc A D is equal to the sum of the arcs A B, B C, and C D; the angle A O D is equal to the sum of the angles A O B, B O C, C O D; but that the chord A D is less than the sum of the chords A B, B C, and C D.

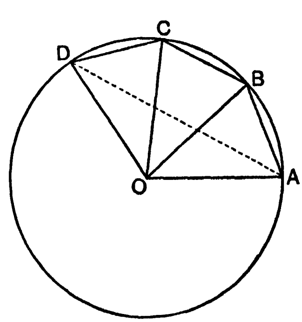


Fig. 16

[1] It may be noted that the method of determining the equality of angles by equality of chords is in common use, and all draughtsmen have a "scale of chords".

SUMMARY OF RESULTS

1. A circle is the path described on a plane by a point which moves so that it is always at a fixed distance from a given point in the plane.

2. Two circles are equal if their radii are equal.

3. From a point on the circumference of a circle only one chord of given length can be drawn on one side of the diameter through that point.

4. If in each of two equal circles a chord of definite length is placed, its end-points mark off equal arcs, and the angles subtended at the centres of the circles by these equal arcs are equal.

5. The equality of two angles is tested by describing two equal circles with the vertices of the angles for centres, and finding whether the chords of the intercepted arcs are equal or unequal.

6. An angle can be described which shall be equal to a given angle.

QUESTIONS

1. When are two angles equal? Discuss a test.

2. If the circumference of a circle be 50 cm. long, what fraction of a revolution and of a right angle is the angle subtended at the centre by an arc—

(*a*) $12\frac{1}{2}$ cm. long?
(*b*) $6\frac{1}{4}$ cm. long?
(*c*) 25 cm. long?

3. A bicycle wheel makes 8 revolutions per minute. What fraction of a right angle does a given spoke describe in one-third of a second?

4. What is meant by saying that the angle at the centre is directly proportional to the arc which subtends it? Make a drawing to illustrate your answer.

5. Explain how the magnitude of an angle subtended at the centre of a circle is determined by the length of the chord. Does it follow that the size of the angle is proportional to the length of the chord?

EXERCISES

1. Draw the smallest angle you find in your set-squares. With the apex of this angle as centre and a radius of 5 cm., describe a circle. Step the chord of the intercepted arc round the circum-

ference. What fraction of a complete revolution is this angle? Test the other angles of the set-squares in the same way.

2. Draw a circle of 5 cm. radius. Place in order in the circle chords A B, B C, C D, each 2·5 cm. long. Join O the centre of the circle to A, B, C, and D. Join also A C, A D, B D. What information does the drawing give regarding the relations between arcs, chords, and angles?

3. Draw two intersecting lines. From any chosen point draw two lines perpendicular to them. Test the equality of the angles between the lines, and the angles between the perpendiculars. What conclusion do you draw?

4. Describe a circle of radius 5 cm. From several points P, Q, R, &c., on the circumference draw chords to A and B, the ends of a diameter A O B. Test the equality of the angles A P B, A Q B, A R B, &c. What conclusions do you draw?

5. Describe a circle of 7·5 cm. radius. In it place a chord A B 10 cm. long. Take points P, Q, R on one of the arcs cut off by chord A B. Join each of the points P, Q, R to A and B, and test the equality between the angles made by these several pairs of lines.

6. Let A, B, C, D be points taken in order on the circumference of a circle. Join A B, B C, C D, D A. Produce one of these four lines, say C D, and test the equality of the angle which this line thus produced makes with D A and the angle A B C. What conclusion do you draw?

CHAPTER IV

MEASUREMENT OF ANGLES—CIRCULAR MEASURE

The first natural measure of an angle is the revolution. From this we derive at once the right angle. Any angle might be expressed in terms of a right angle. The right angle is too large a unit for most practical purposes, and the revolution has been divided into 360 parts called degrees. Thus there are 90 degrees in a right angle.

The **protractor** (fig. 17) is a carefully graduated instrument of great service in measuring angles. Protractors may be had of various forms and sizes, but that recommended is a semicircular one made of transparent celluloid. The curved edge is divided into 180 equal parts, degrees, and the line is shown joining each point of division to

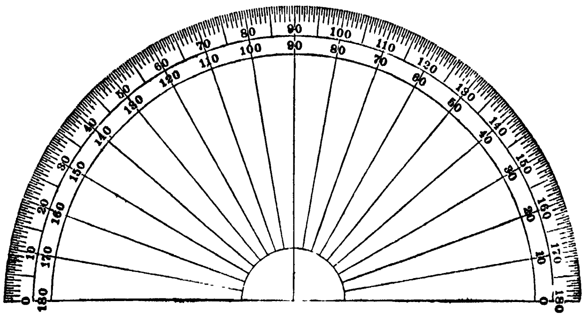


Fig. 17

the centre. There are usually two scales giving readings from both ends of the diameter of the semicircle, thus enabling both positive and negative angles to be directly measured. The centre-point is placed over the vertex of the angle, the zero line of the protractor along the line from which the rotation starts, and the size of the angle is read off on the scale at the generating line.

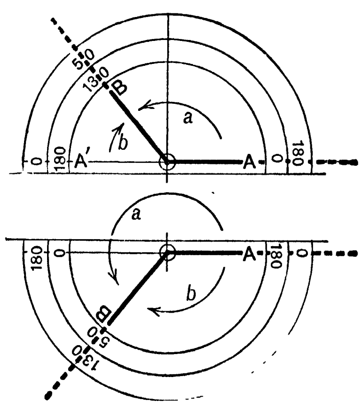


Fig. 18

Examples of the use of the protractor in measuring amount of rotation of a line—

(i) If a line has moved positively, as in fig. 18 (a), from initial position O A to position O B, the angle described is + 130°.

(ii) If a line has moved negatively, as in fig. 18 (*a*), from initial point in O A′ to position O B, the angle described is $-50°$.

(iii) If a line has moved positively, as in fig. 18 (*b*), from initial position O A to position O B, the angle described is $(180 + 50)°$, *i.e.* 230°.

(iv) If a line has moved negatively, as in fig. 18 (*b*), from initial position O A to position O B, the angle described is $-130°$.

Circular Measure.—The choice of the degree was purely arbitrary. For instance, it might have been better to divide the revolution into 400 parts, when the right angle would have contained 100 degrees. Whenever a "natural" scale appears it is employed in a pure science: a "natural" scale is one suggested by the nature of what has to be measured; quantities measured on a natural scale are most easily dealt with mathematically.

In fig. 16, page 29, we know that if the arc A B equals the arc B C, then the angle A O B equals the angle B O C. Proceeding, we see that if the arc A D is three times the arc A B, the angle A O D is three times the angle A O B. This suggests "circular measure", or a scale in which angles are measured by lengths of arcs. Clearly, for any given angle A O P the length of the arc increases with the radius; yet the ratio of the arc to the radius ought to be a constant depending on and measuring the angle.

Draw two lines (fig. 19) O X, O Y at right angles. With centre O and with various radii describe circles intersecting O X and O Y in A, Q; A′, Q′; A″, Q″, &c. Then each of the arcs A Q, A′ Q′, A″ Q″ is equal to one-fourth of the corresponding circumference.

Draw a third line O Z intersecting these circles in P P′ P″. The angle X O Z is regarded as formed by the rotation of the straight line O Z from a position of coincidence with O X to its present position. The points

P, P′, P″ have thus described arcs A P, A′ P′, A″ P″, &c., and each of these arcs is the same fraction of the corre-

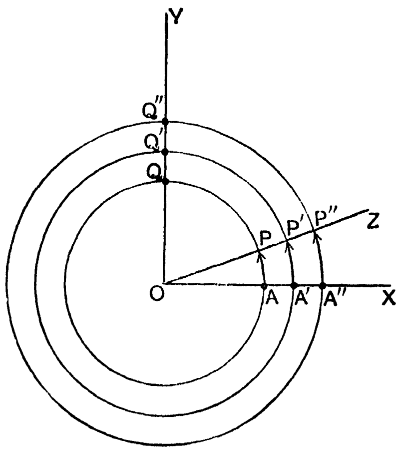


Fig. 19

sponding circumference as the angle X O Z is of a complete revolution.

This important property can be established experimentally. Further, as the length of each circumference bears a constant ratio to the length of the corresponding radius, the theorem may be stated thus—If a series of concentric circles be described, then the length of an arc intercepted on any of these circles by two fixed lines drawn from the common centre has a constant ratio to the length of the radius of that circle.

Consider first the case in which the two fixed lines are in the same straight line. Each circle is divided into two equal parts, and the exercise is to determine the ratio of the length of the semi-circumference to the length of the radius.

Draw as large a semicircle as the paper will allow, and draw the diameter. Now take the circular metal disc, and, placing the point marked on its edge on one end of the arc, roll the disc without slipping right round the semicircle, noting the number of whole turns, and mark on the edge of the disc the point at which it meets the second end-point of the semicircle. Roll the disc in a similar manner along a straight line, and measure the length of the semicircle in centimetres.

Repeat the exercise a number of times, using different radii, and tabulate your results as follows, calculating the ratio of the length of the semicircular arc to that of the radius of each circle:—

Number of cm. in radius	Number of cm in semi-circumference.	Ratio of semi-circumference to radius.
10	31·4...	3·14...
12	37·7...	3·14...
15	47·2...	3·14...
8	25·1...	3·14...

$$\text{Thus we have } \frac{\text{semi-circumference}}{\text{radius}} = 3{\cdot}14 \text{ (nearly).}$$

This ratio 3·14... is represented by the symbol π.

If r represents the length of the radius of a circle,

the length of the circumference $= 2\pi r$.

Take now such an angle as XOZ (fig. 19), and, using a rolling disc in the manner described, show that the length of the arc subtending at the common centre the angle XOZ bears a constant ratio to the length of the circumference of which it is a part. Tabulate your results for several circles and several angles. For each angle, whatever be the size of the circle, a constant ratio between the length of the arc and the length of the radius is obtained. The angle **in circular measure** is given by this ratio.

Thus, when the ratio is		1	the angle is	unit angle or a **radian**,
„	„	$\frac{2}{3}$	„	two-thirds of a radian,
„	„	3·14	„	3·14 radians,
„	„	θ	„	θ radians.

When the length of the arc is equal to that of the radius the ratio $\frac{\text{arc}}{\text{radius}} = 1$, and so the unit angle, or radian, is the angle subtended at the centre of a circle by an arc the length of which is that of the radius (fig. 20).

We have seen that $\frac{\text{semi-circumference}}{\text{radius}} = 3{\cdot}14$ (nearly) $= \pi$

$\therefore$ the angle subtended at centre by semi-circumference $= \pi$ radians,

But „ „ „ „ $= 180°$

$$\therefore 180° = \pi \text{ radians},$$

$$\text{A revolution} = 2\pi \text{ radians} = 360°.$$

SUMMARY OF RESULTS

1. Equal arcs in a circle are marked off by stepping round the circumference a pair of compasses with the points at a fixed distance apart. Each of these arcs is greater than its own chord.

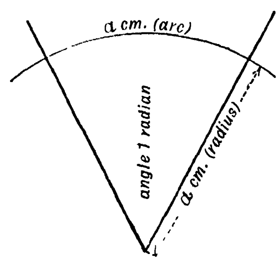


Fig. 20

2. If the chord is equal in length to the radius, the circumference can be divided into exactly six such arcs. The length of the circumferences is greater than six times the length of the radius.

3. The length of a curved line may be measured by means of a rolling disc and a centimetre scale.

4. The ratio of the circumference to the diameter of the same circle is nearly 3·14. The symbol for the exact value of the ratio is π. If r be the radius, then the circumference is equal to $2\pi r$.

5. If the length of the path described by any point on a line revolving round a fixed point in itself be a definite fraction of the whole path which the moving point would describe in a complete revolution, the angle formed is always of the same size.

6. If an angle be of a fixed size, and a circle be described with the apex as centre, the length of the intercepted arc is the same fraction of the complete circumference whatever radius be used.

7. The protractor can be used for measuring angles.

QUESTIONS

1. If the distance between the points of the compasses is equal to the radius of a given circle, how many steps will the compasses take to go round the complete circumference?

2. What angle at the centre is marked off by each successive step?

3. On what grounds do you state that the ratio of the circumference of a circle to its diameter is constant?

4. A circular disc 5 cm. in diameter has a point marked on its edge. The disc is rolled along a metre-stick. If the marked point on the disc is just over the zero of the scale at the start, where will it be after six whole turns?

5. What is the length of the path travelled in fifty minutes by the end-point of the 5-foot minute hand of a large clock?

6. What is the length of the path travelled by a point on the minute hand (5) two feet from the end-point.

7. Show that 60° is about 5 per cent greater than a radian.

8. Should two protractors of different size give the same or different readings for the same angle?

EXERCISES

1. Describe a circle of diameter not less than 12 cm. Step round the complete circumference with the compasses at a stretch of 1 cm. Join the consecutive points on the circumference.

Assuming that the sum of the lengths of the chords so drawn is not far short of the length of the circumference, find approximately the value of the ratio π.

2. Construct by means of your protractor angles AOP, AOQ, AOR, and AOS, equal respectively to 25°, 137°, $-25°$, $-137°$.

3. Construct angles AOP, AOQ, AOR, and AOS, equal respectively to $\frac{\pi}{4}, \frac{2\pi}{3}, -\frac{3\pi}{4}, \frac{5\pi}{16}$.

4. A′OA is a straight line, and OP is another line inclined to it. Measure the angles and arrange results thus—

Angle AOP = ...° Angle POA′ = ...°
Angle AOP + angle POA′ = ...°

5. Draw two intersecting lines and measure the vertically opposite angles.

6. Form a triangle by joining three points A B and C as in fig. 21 (*a*). Measure by protractor the angles of the triangle, and set down results thus—

Angle B A C = ...°
C B A = ...°
A C B = ...°

Sum of the angles of the triangle = ...°

7. Now produce B C to X as in fig. 21 (*b*) and show that the angle X C A is equal to the sum of the angles B A C and C B A.

8. Produce B C to X, C A to Y, and A B to Z, and show that the sum of the exterior angles so formed is equal to four right angles. Tabulate results thus—

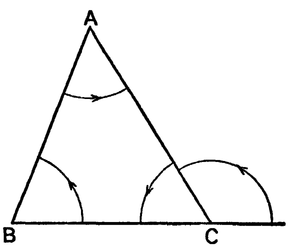


Fig 21 (*a*)

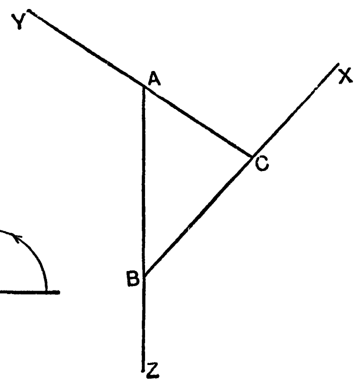


Fig. 21 (*b*)

Angle X C A = ...°
Y A B = ...°
Z B C = ...°

Sum of the three exterior angles = ...°

9. Make two unequal circles with centres O and O′. Using protractor, make the angle A O P in one circle equal to the angle A′ O′ P′ in the other.

Find by measurement the lengths of the arcs, radii, and chords, and calculate the ratios $\frac{\text{arc AP}}{\text{radius OA}}$, $\frac{\text{arc A}'\text{P}'}{\text{radius O}'\text{A}'}$, $\frac{\text{chord AP}}{\text{radius OA}}$, and $\frac{\text{chord A}'\text{P}'}{\text{radius O}'\text{A}'}$. Draw inferences.

10. Draw two quadrants, one of 10 cm., the other of 5 cm. radius. Mark off in each angles of 15°, 30°, 45°, 60°, and 75°. Measure the chords of these angles in the two quadrants.

Fill up the following table:—Express lengths and ratios as decimals.

Angle.	Chord (*a*) when radius is 10 cm.	Chord (*b*) when radius is 5 cm.	Ratio of (*a*) to (*b*).
15°	....	....	....
30°	....	....	....
45°	....	....	....
60°	....	....	....
75°	....	....	....
90°	....	....	....

State the general theorem illustrated by these results.

Calculate the ratios between the chord of 90° and the chords of 75°, 60°, 45°, 30°, and 15°, in the case of (*a*) a circle of radius 10 cm., and (*b*) a circle of radius 5 cm.

Give a clear statement of the inferences that may be drawn from a consideration of each of these two sets of ratios and from a comparison of the corresponding results in the two sets.

CHAPTER V

ROTATION OF A PLANE FIGURE IN ITS OWN PLANE

Up to the present an angle has been considered as being formed by the rotation in a plane of a straight line round a point in itself.

In this chapter we shall deal with the **rotation of a plane figure** in its own plane, and therefore with the rotation of straight lines in that figure round points in that plane.

By a plane figure is meant a portion of a plane surface having a definite shape and a definite boundary. When the figure is moved its boundary, of course, moves with it, and its shape remains unaltered. For practical purposes we associate these figures with such pieces of rigid material as a plate of metal, a lamina of wood, or a piece of stiff cardboard.

The sketch (fig. 22) shows the quadrilateral lamina used in the exercises of this chapter. The boundary consists of four straight lines, and the angles are, for convenience, 60°, 75°, 135°, and 90°. At five or six points A B C D E, &c., taken at random on the surface, are drilled holes just large enough to permit a pencil-point to pass

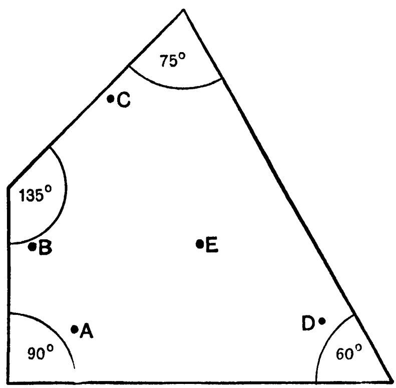


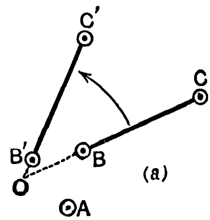
Fig 22

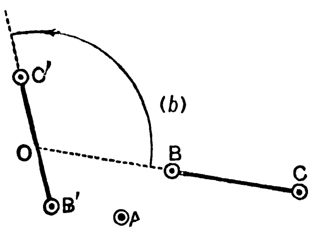
through and mark a sheet of paper underneath the lamina. Taking various groups of these points we can form various definite figures. Thus—

Triangles are formed by joining three points, as A B C, B D E, &c.—Quadrilaterals, by joining four points in succession, as A B C D, &c., and so on.

Any figure can be reproduced as often as necessary by marking the chosen points on the paper underneath and joining up by straight lines. With the lamina described we can realize the motion of a plane figure in its own plane and observe the geometrical results.

In order to measure the angle described by any chosen line in the figure during a rotation from the initial position B C to the final position B′C′ round the fixed point A, the lines B C, B′ C′ are drawn and produced to meet in O (if necessary). The required angle B O B′ or C O C′ is then measured by the protractor, the procedure being the same whether the angle described has been, as illustrated in fig. 23,





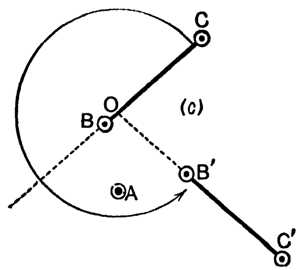


Fig 23

(*a*) acute, (*b*) obtuse, or (*c*) greater than two right angles.

If now we mark on the paper two edges of the mov-

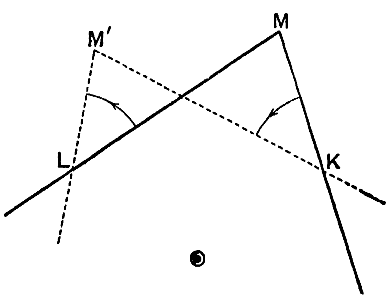


Fig 24

able plane and rotate the lamina round a point, as *e.g.* O in fig. 24, and again mark the positions of the two edges, it will be found on measurement with a protractor or compasses that the angle M K M′ through which an edge has been rotated is equal to the angle M L M′ through which the other edge has been rotated.

The pupil will find that if a triangle (or any other rectilineal figure) be rotated round a fixed point, each of the lines of the figure turns through the same angle.

Examine now the result of the movement of a figure in a plane without paying regard to its rotation round a point. Place the movable plane on the paper and mark a chosen figure in one position. Shift the movable plane. Mark the new position of the figure. Mark and measure the angles through which each side has to be rotated, so as to pass from the first to the second position. It will be found that "when a plane figure moves in its own plane every line in it turns through the same angle".

NOTE.—It is necessary to bear in mind that by the "amount of rotation" which a line experiences in moving from one position to another is meant either (1) the algebraic sum of the several amounts of rotation, having regard to whether these are positive or negative, by which the successive steps in the movement are made, or (2) the total amount of rotation in any direction by which the change in position might be effected.

SUMMARY OF RESULTS

1. A movable plane lamina with holes drilled at various points enables us to transfer from one position to another a variety of definite figures without changing their shapes.

2. The angle through which any chosen line rotates in a plane during a movement into a new position can be readily measured.

3. If a plane figure moves in its own plane every line in it rotates through the same angle.

4. **If the total rotation, therefore, of any line in that figure during a given movement is zero, so also is that of every other line in the figure.**

QUESTIONS

1. Describe a method of marking at various positions on a sheet of paper a figure of fixed size and shape.

2. If one point (say E) on this figure be fixed, and the figure rotated in its own plane round E, state some facts about the movement of—

(*a*) The points A, B, C, and D.
(*b*) The lines EA, EB, BA, BC, &c.

3. State the main theorem of this chapter.

EXERCISES

1. Draw a circle of 7 cm. radius. From A and B, the extremities of two radii, draw perpendiculars by means of a set-square. Measure the angle between these radii and the angle between the perpendiculars.

If O be the centre of the circle, draw figures for the following three cases:—

(*a*) When AOB is an acute angle.
(*b*) When AOB is greater than one but less than two right angles.
(*c*) When AOB is greater than two but less than three right angles.

Mark and measure the angles, and discuss the results from the point of view of rotation.

2. Draw AOB an angle of 50°. From a point P draw perpendiculars to OA and OB, and measure the angle between the perpendiculars. Repeat the exercise with P in various positions, and state carefully the general result.

3. AB is a straight line on a plane. P is a point not in this line. Join PA. With P as centre, and PA as radius describe an arc PA′. If A′ be taken as a new position of one end of a line AB rotating round P, find B′, the corresponding position of the other end.

4. Using the movable plane, draw a quadrilateral ABCD. Reproduce this figure on another portion of the paper. Mark and measure the angles through which the sides and the diagonals have turned.

CHAPTER VI

PARALLEL LINES

From experiments with the movable plane, the conclusion has been arrived at that "when a plane figure moves in its own plane every line on the figure turns through the same angle".

Accordingly, if the figure moves in such a way that one line in it is not subject to any rotation whatever, then, as a consequence of the foregoing theorem, no other line in the figure will experience rotation or turn through any angle.

In order to realize movement of a plane figure without rotation we have recourse to a straight-edge, and cause one straight line in the movable plane to slide along the fixed straight-edge.

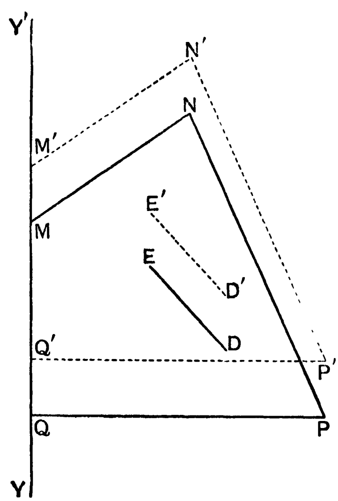


Fig 25

In the diagram (fig. 25) the movable plane is represented as sliding along the straight line Y Y′, and two positions M N P Q, M′ N′ P′ Q′ are shown.

Since there is no rotation in moving M Q into its new position M′ Q′, therefore there is no rotation in moving M N into its new position M′ N′, and so with all other lines in the figure.

Definition: **When a line can be brought from one position in a plane to another without rotation, the two positions of the line are said to be parallel.** Thus M N is parallel to M′ N′, N P to N′ P′, E D to E′ D′, and so on.

Euclid defines parallel lines to be "such as are in the same plane, and being produced ever so far both ways do not meet". Now, if one straight line A B can be brought into coincidence with another A′ B′ without rotation, then A B and A′ B′ cannot meet however far they are produced either way. For if they did meet, say at C, then rotation would be necessary to bring one line into coincidence with the other.

Theorems regarding Parallel Lines.

(i) If a straight line falling on two other straight lines in the same plane makes the exterior angle equal to the interior and opposite angle on the same side of the crossing line, the two straight lines are parallel.

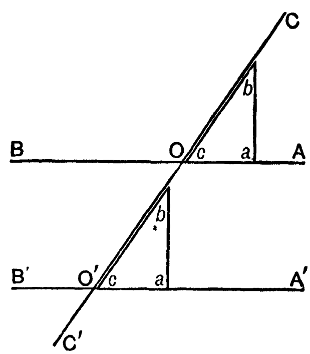


Fig. 26

In the diagram (fig. 26) the two lines B A and B′ A′ are crossed at O and O′ by a third line C C′, so that the angle A O C is equal to the angle A′ O′ C.

Consider a definite figure *c a b* fitted into the angle A O C, as in the diagram. Let *b c*, one of its edges, slide along C C′ until the angular point *c* which was at O reaches O′. The edge *c a* must lie along O′ A′, for the angle A O C is equal to the angle A′ O′ C. Therefore according to definition, O′ A′ is parallel to O A.

Corollaries: (1) If the alternate angles B O O′ and A′ O′ O are equal, the lines are parallel. For the angle B O O′ is equal to the vertically opposite angle A O C (see Chap. IV, Ex. 5), therefore A O C is equal to A′ O′ C, therefore A B is parallel to A′ B′.

NOTE.—The pupil should draw two lines crossing one another. He should mark each of a pair of vertically opposite angles, and, applying the principle of rotation, he should be satisfied as to the equality of the marked angles.

(2) If the two interior angles on the same side, A′ O′ O and O′ O A, are together equal to two right angles, the lines are parallel.

For $\angle$ O′ O A + $\angle$ A O C = two right angles,
∴ $\angle$ A′ O′ O + $\angle$ O′ O A = $\angle$ O′ O A + $\angle$ A O C,
∴ $\angle$ A′ O′ O = $\angle$ A O C,
∴ O A and O′ A′ are parallel lines by Theorem (i).

(ii) If a straight line falls on two parallel straight lines it makes the exterior angle equal to the interior and remote angle on the same side of the line.

Let B A and B′ A′ (fig. 26) be two parallel straight lines crossed at O and O′ by the straight line C C′. Consider a figure *c a b* fitted in the angle A O C, as in the diagram, and let it slide along C C′ until its angular point C which was at O reaches O′. The edge *c a* must lie along O′ A′, for as this edge has moved across the plane without rotation from the position O A it is now parallel to O A, (by Definition) and is therefore coincident with O′ A′, the given parallel to O A.

The angle *b c a* of the figure is equal to each of the angles A O C and A′ O′ C, therefore they are equal to one another.

Corollaries: (1) The alternate angles B O O′ and O O′ A′ are equal.

(2) The two interior angles A O O′ and O O′ A′ on the same side of C C′ are together equal to two right angles.

(3) If a straight line is perpendicular to one of two parallel lines it is also perpendicular to the other.

(iii) The three interior angles of a triangle are together equal to two right angles.

This result has been reached by the aid of a protractor in Chap. IV, Ex. 6, but a proof based on the definition of parallels should be considered.

Let A B C (fig. 27) be a triangle with the side A C produced to X. Let the angles of the figure A B C be marked 1, 2, 3, and let it slide along A X until A coincides with C. The line C B′ is by definition parallel to A B, therefore the alternate angles B′ C B and C B A are equal. The three angles 1, 2, 3 of the triangle arranged round C make up two right angles.

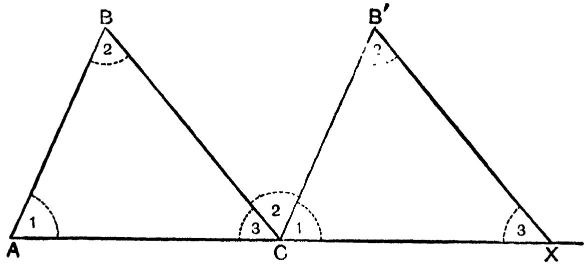


Fig. 27

Corollary: The exterior angle of a triangle is equal to the sum of the two interior and remote angles.

Thus, in the figure, the angle B C X is equal to the sum of the angles C A B and A B C.

SUMMARY OF RESULTS

1. When a plane figure moves in a plane so that one of its lines makes no rotation, every other line in it moves without rotation.

2. When a straight line moves without rotation in a plane from one position to another the two positions are parallel. Therefore parallel straight lines may be obtained by sliding one edge of a set-square along a straight-edge, and marking various positions of one of the other moving edges.

3. If a straight line crossing two other straight lines makes (*a*) the exterior angle equal to the interior and remote angle on the

same side of the line, or (*b*) the alternate angles equal, or (*c*) the two interior angles on the same side together equal to two right angles, the straight lines are parallel.

4. If two parallel lines are crossed by a third line, the relations (*a*), (*b*), and (*c*) stated in the previous paragraph will hold.

5. If a side of a triangle be produced, the exterior angle is equal to the sum of the two interior and remote angles.

6. The three angles of every triangle are together equal to two right angles.

QUESTIONS

1. Give a definition of parallel lines derived from a consideration of the movement of a rectilinear figure in its own plane.

2. How can movement without rotation of such a figure be secured?

3. How does Euclid define parallel lines? Show that your definition is equivalent to that of Euclid.

4. Show that two straight lines are parallel if, on being crossed by a third,

(*a*) the exterior angle is equal to the interior and remote on the same side; or

(*b*) the alternate angles are equal; or

(*c*) the two interior angles on the same side are together equal to two right angles.

5. Prove without referring to parallelism that if one of the relations (*a*), (*b*), (*c*) holds, so must the other two.

6. Prove that if two parallel straight lines are crossed by a third line, the relations (*a*), (*b*), and (*c*) hold.

7. Prove that if two straight lines are parallel to a third, they are parallel to one another.

8. If the lines 1 and 3 in the diagram (fig. 28) are respectively parallel to the lines 2 and 4, prove that the angle between 1 and 3 is equal to the angle between 2 and 4.

9. A parallelogram being defined as a four-sided rectilinear figure with its opposite sides parallel, show that the opposite angles of a parallelogram are equal.

10. What is the nature of the assumption on which this chapter rests?

EXERCISES

1. Across two parallel straight lines A B and C D draw a series of lines parallel to each other. The lines in the series may be

(*a*) perpendicular to A B and C D,
(*b*) oblique to A B and C D.

Prove by measurement that the intercepts made by A B and C D on each of a set of parallels are equal.

Hence note the following theorem (which appears also in Chap. VII):—"If a plane figure slide in its own plane so that a fixed line in it moves along a given straight line, every point on it describes a straight line parallel to the given line".

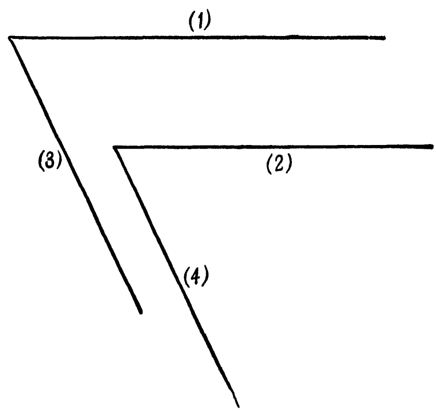


Fig. 28

Make a diagram in illustration of this theorem, and show several successive positions of the sliding figure.

2. Slide one edge of the quadrilateral lamina along a straight-edge. Mark the path of a pencil-point placed in one of the holes. Examine this path with regard to the line formed by running a pencil-point along the edge of the straight-edge.

3. Draw O A 10 cm. long and mark the cm. divisions. Draw O A′, say 8 cm. long, making an acute angle with O A. Using set-square and straight-edge, draw parallels to A A′ from the cm. marks to cut O A′. Note the relations of the divisions of O A′.

From each of the points of intersection on O A′ draw parallels to O A to cut A A′, and from these points of intersection on A A′ draw parallels to O A′. Study the results you obtain.

4. Mark on the page three points A, B, C not in the same straight line. Draw through each point a parallel to the join of the other two. Mark the intersection of these three lines, thus A′ where lines through B and C meet, B′ where lines through C and A meet, and C′ where lines through A and B meet. Study the drawing, and indicate any relations you find.

5. Draw two finite straight lines parallel to one another but unequal in length. Mark the four end-points. Two new points are determined by the crossing of two pairs of joins. Draw a straight line through these points. Examine the nature of the division of each of the original lines.

6. Mark a point P′ in a given line P′A. Describe a positive angle A P′ P of say 40°, and mark P a point in the line P′ P. Make now a negative angle of the same size, P′ P A′, and note that a line can pass, by two equal and opposite rotations, from one position P′ A to another P A′ parallel to the former.

7. Draw any triangle A B C. Produce B C to X. Considering that an angle may be formed by the rotation of a line in a plane, show that the angle X C A is equal to the sum of the angles C B A and B A C. Mark the angles, and use the protractor to test the principle.

8. Draw a triangle A B C. By means of set-square and straight-edge draw through A a straight line parallel to B C. Measure the three angles at A; compare them with the three angles of the triangle, and hence show that the sum of the three angles of a triangle is equal to two right angles.

NOTE ON T-SQUARE.—This appliance, which is used with a drawing-board, is merely a set-square of a particular shape, the sliding edge being applied to the side of the board.

CHAPTER VII

PARALLELOGRAMS, RECTANGLES, SQUARES RIGHT-ANGLED TRIANGLES

It has been shown in the previous chapter that if a straight line moves without rotation from one position A B in a plane to another position A′ B′, then A B and A′ B′ are parallel.

Parallels are also obtained when a straight line moves in a plane so that the total rotation is two right angles or a multiple of two right angles. For (fig. 29) let the

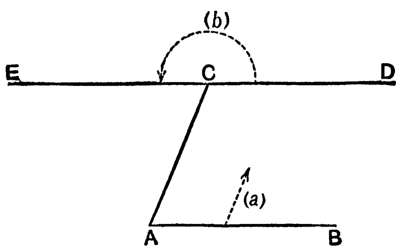


Fig. 29

straight line A B (*a*) move without rotation into the position C D, and then (*b*) rotate through an angle of 180° into the position C E. E C D is one right line, but C D is parallel to A B, therefore also E C is parallel to A B. A further rotation of 180° would bring the line back into the position C D, and so on.

It is to be noted that when the total rotation is zero, four right angles, eight right angles, &c., the initial and final positions of the end A are towards the same parts, but when the total rotation is two, six, ten, &c., right angles, the two positions of A are towards contrary parts.

Definition: A **parallelogram** is a four-sided rectilinear plane figure whose opposite sides are parallel.

Accordingly a parallelogram is easily constructed with the help of a straight-edge and a set-square. Draw two parallel lines A B and C D, and across them draw another pair of parallel lines. Mark the four points of crossing A B C D.

NOTE.—It will be seen that a parallelogram may be drawn so as to satisfy given conditions, such as that two adjacent sides and the included angle shall be respectively equal to two given straight lines and a given angle.

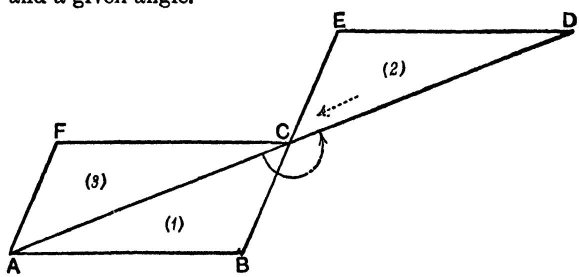


Fig. 30

The properties of a parallelogram may be arrived at in the following manner:—

Consider any triangle A B C (fig. 30), and let it rotate in its own plane round the point C from position (1) to position (2), so that the total rotation is two right angles. Then B C is brought into the position E C, and A C into the position D C, and B C E and A C D are straight lines. Now let the triangle D E C slide without rotation along the line D C A till C comes to A and consequently D to C. The line E C will take up the position A F and E D will come to F C. Thus on the whole a rotation of 180° has brought A B into the position C F and B C into the position F A. Consequently A B and C F are parallel, and B C and F A are parallel. So the quadrilateral formed in the manner described is a parallelogram.

Since the triangle C F A is just the triangle A B C in a new position, we have—

A B = F C
B C = A F
the angle A B C = the angle C F A
" F C A = " B A C
" F A C = " B C A
and the whole angle F A B = the whole angle B C F.

Summing up these results we have—

(i) The opposite sides and angles of a parallelogram

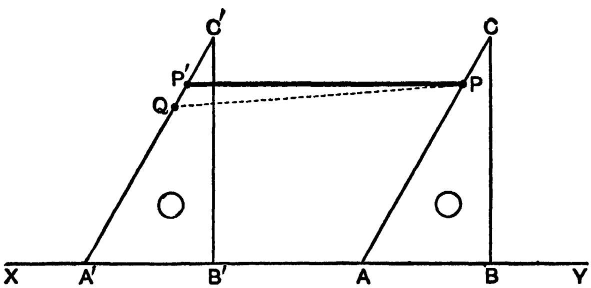


Fig. 31

are equal, and the diagonal divides the parallelogram into two identically equal triangles.

(ii) If a plane figure slide in its own plane so that any fixed line in it moves along a given straight line, then every point in it describes a straight line parallel to that given straight line.

For let the movable plane P A B (fig. 31) move in its own plane so that the line A B always remains in the fixed straight line X Y, and let P A B, P′ A′ B′ be two separate positions of the movable plane, P and P′ being separate positions of the same point. Join P P′. If P P′ is not parallel to X Y, draw P Q parallel to X Y so as to intersect P′ A′ in Q. Then P Q A′ A is a parallelogram, therefore Q A′ is equal to P A, which is impossible, since P′ A′ is equal to P A. Therefore P P′ is parallel to X Y. From this we derive—

(iii) The straight lines which join the extremities of two equal and parallel straight lines towards the same parts are themselves equal and parallel.

(iv) The diagonals of a parallelogram bisect one another.

Let A B C D (fig. 32) be the parallelogram, and D B one of its diagonals. Bisect D B in O and join O C and O A. Now let the triangle A B D rotate in its own plane round the point O, through an angle of 180°. The positions

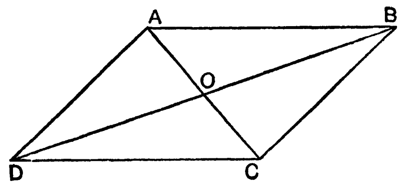


Fig 32

of the points D and B will be interchanged, the line B A will coincide with the line D C, the line D A with the line C B, and the line O A with the line O C. Thus a total rotation of 180° has brought O A to coincide exactly with O C, therefore C O A. is one straight line, viz. the second diagonal, and both diagonals are bisected at O.

Definition: A **rectangle** is a right-angled parallelogram.

If a parallelogram has one right angle, then clearly all its angles are right angles. For any two opposite angles are equal, and any two adjacent angles are together equal to two right angles. A rectangle is completely determined when the lengths of two adjacent sides are known.

(v) The diagonals of a rectangle are equal.

Let PQRS be a rectangle. Join QS, PR. It has to be shown that QS is equal to PR.

Let the triangle QPS (fig. 33) turn round QP as on a hinge until it is again in the plane of the paper and occupies the position QPS′: then SPS′ is one straight line. Now let QPS′ slide along S′PS till P comes to S,

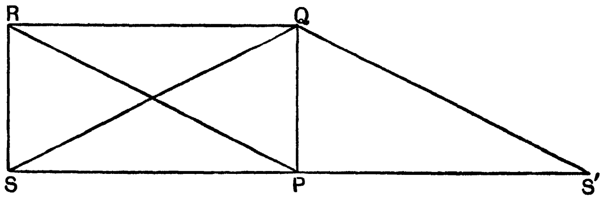


Fig. 33

and therefore also S′ to P. QP will coincide with SR and QS′ with RP. Thus by a combined movement of translation and a rotation we can make QS coincide exactly with PR, therefore PR and QS are equal.

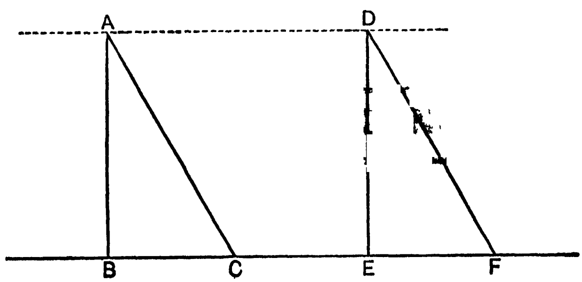


Fig. 34

Definition: A **right-angled triangle** is one which contains one right angle.

The other two angles must therefore be both acute, and from the foregoing treatment of a rectangle it is clear that every rectangle is divided by either diagonal into two exactly equal right-angled triangles.

(vi) If two right-angled triangles be such that the sides about the right angles are equal each to each, then the triangles are identically equal.

For, if ABC and DEF (fig. 34) be two triangles in which AB = DE and BC = EF, and the angles ABC, DEF both right angles, it is possible to place ABC and DEF so that BC and EF are in the same straight line, and then slide ABC along the line BCEF until B comes to E, and by consequence C to F; AB then coincides with DE and AC with DF.

Thus the triangles coincide completely, and are therefore equal in every respect.

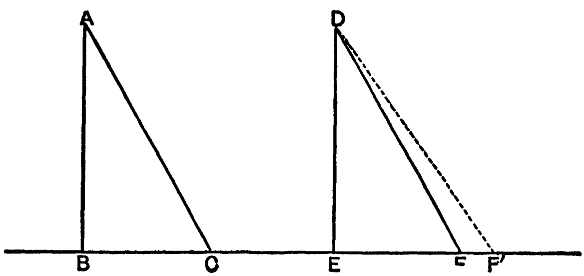


Fig 35

(vii) If two right-angled triangles have their hypotenuses equal, and one side of the one equal to one side of the other, they are identically equal.

For, if ABC and DEF (fig. 35) have the angles ABC, DEF right angles, AC = DF, and AB = DE, it is possible to place them so that BC, EF are in the same straight line, and then slide ABC over DEF until B comes to E, and by consequence AB coincides with DE; the point C will then fall on F, for if it did not, but took up another position such as DF′, there would be two equal obliques drawn from the same point to the same straight line, which is impossible, ∴ C falls on F and the two triangles exactly coincide, therefore they are identically equal.

Definition: A **square** is a rectangle whose sides are equal. Its angles are, of course, all right angles.

To construct a square on a given straight line A B.

From A (fig. 36) draw A D perpendicular to A B and make it equal to A B. Through B and D draw parallels to A D and A B meeting in G. A B C D is the required square.

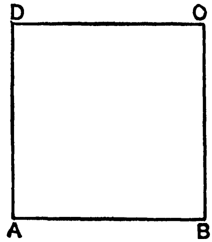


Fig. 36

(viii) From this we infer that squares described on equal straight lines are equal.

SUMMARY

1. A parallelogram is a quadrilateral with its opposite sides parallel.
2. A parallelogram is divided by a diagonal into two triangles, of which one is a replica of the other.
3. The opposite sides and angles of a parallelogram are equal, and either diagonal divides the figure into two exactly equal triangles.
4. The diagonals of a parallelogram bisect each other.
5. The diagonals of a rectangle are equal.
6. If two right-angled triangles have the sides about the right angle equal, each to each, the triangles are equal in every respect.
7. If two right-angled triangles have their hypotenuses equal, and a side of the one equal to a side of the other, the triangles are equal in every respect.
8. Squares described on equal straight lines are equal.

QUESTIONS

1. How do you show that the path traced out by a point in a figure, one line of which slides along a fixed straight line, is also a straight line parallel to that fixed line?
2. What assumption is made in the consideration of a parallelogram on page 54?

3. Define a parallelogram, a rectangle, a square, a right-angled triangle.

4. Give some facts about the diagonals of a parallelogram, a square, a rectangle.

5. What is meant by the hypotenuse of a right-angled triangle? Show that it is the greatest side.

6. Give the conditions which two right-angled triangles must satisfy before they can be pronounced equal.

EXERCISES

1. Using a set-square and straight-edge, draw a parallelogram whose adjacent sides are 7 cm. and 10 cm. respectively, and the angle contained by those sides 45°. Test the equality of the opposite sides and angles.

2. Draw A B and A C, two straight lines at right angles to one another. From a point D draw D B and D C at right angles to A B and A C, and prove that the resulting figure has (*a*) its opposite sides equal, (*b*) its diagonals equal.

3. Draw A C and A B, two straight lines at right angles. Join B C, bisect it in M; join A M, and produce it to D so that M D = A M; join B D, C D. Prove that A B C D is a rectangle.

4. Draw A C and A B, any two straight lines. Join B C, bisect it in M; join A M, and produce A M to D so that M D = A M; join B D, C D. Prove that A B C D is a parallelogram.

5. Show from the previous exercise how to bisect a line, using only ruler and set-square.

6. Draw a set of right-angled triangles on the same hypotenuse. Join the mid-point of this line with the vertex of each right angle. Prove that those joining lines are equal. What inference do you draw?

Exercises on Right-Angled Triangles

1. Prove that the perpendiculars at the mid-points of the sides of a triangle meet in a point which is equidistant from the three vertices. [Circum-centre.]

2. Show that the three perpendiculars drawn from the vertices of a triangle to the opposite sides meet in a point. [Ortho-centre.]

3. Show that the straight lines bisecting the angles of a triangle meet in a point which is equidistant from the three sides. [In-centre.]

4. Show that the lines that bisect two of the exterior angles of a triangle and the internal bisector of the third angle meet in a point which is equidistant from the three sides. [Ex-centre.]

5. Prove that in any circle the perpendicular from the centre on any chord bisects that chord, and conversely.

6. Prove that equal chords of a circle are equidistant from the centre, and conversely.

7. Find the locus of a point which moves so that it is always at equal distances from two fixed points.

8. Prove that the locus of a point which moves so that it is at a fixed distance from a given straight line and on one side of it is a straight line parallel to that given straight line.

9. Find the locus of a point which is always equidistant from two fixed straight lines.

10. Find the position of a point which is on a fixed line and is equidistant from two fixed points.

CHAPTER VIII

TRIANGLES

In the preceding chapters the pupil has become familiar with the **Triangle**, has found out for himself a few of its properties, and in particular certain relations that exist among its elements; that is, among its three sides and its three angles.

He has become convinced, for example, by experiment and reasoning founded thereon, or leading thereto, that the three angles of every triangle are together equal to two right angles, and that any two of the sides are together greater than the third.

In the present chapter he will be asked to extend his knowledge of the properties of the Triangle and to direct his attention specially to the consideration of these two questions—

(*a*) What elements must be specified in order that we shall be able to construct any required triangle?

(*b*) What relations must be known to hold among the elements of two given triangles before we are able to say definitely that they are equal in every respect?

It will be found convenient to adopt a short notation when speaking of the sides and angles of a triangle, and to indicate, when no confusion can arise thereby, (1) the three angles by the capital letters used to mark their vertices; (2) the sides opposite these angles by the corresponding small letters.

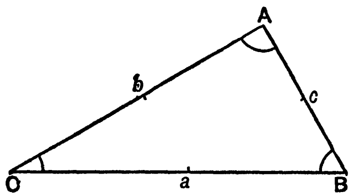


Fig. 37

Thus, in any triangle ABC (fig. 37) we shall sometimes indicate the angle BAC by the letter A

"	"	CBA	"	"	"	B
"	"	ACB	"	"	"	C
the	side	BC	"	"	"	*a*
"	"	CA	"	"	"	*b*
"	"	AB	"	"	"	*c*

Consider the phrase "the elements that completely determine a triangle". When is a triangle completely determined? The answer is—When so many of its sides and angles, with their relative position, are known that all triangles constructed so as to have these sides and angles as corresponding elements shall be exact replicas of each other.

How many elements must be specified before we are able to say that the triangle is quite definite?

Clearly *one* side is not sufficient, nor is *one* angle. Nor are *two* elements sufficient; for (*a*) the triangle formed so as to have two sides equal to two given lines as in fig. 38 (*a*) will vary with the size of the angle included by them, and (*b*) that formed so as to have two angles equal to two given angles as in fig. 38 (*b*) will vary with the length of the side to which both are adjacent.

A like remark applies when one side and one angle are specified.

Also, when two angles of a triangle are given the

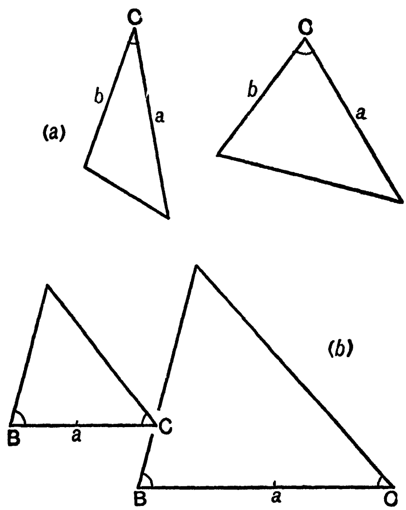


Fig. 38

third can be immediately found, since the sum of the three angles is always equal to two right angles; hence a knowledge of the third angle cannot make the triangle more definite—in other words, the *three angles* are not sufficient to determine the triangle.

This may be clearly seen otherwise (fig. 39). If from any point B′, in the side A B of a triangle A B C, B′ C′ be drawn parallel to B C, then the triangle A B′ C′ has its three angles respectively equal to those of the triangle A B C; yet, of course, the triangles are not equal.

Are we able to say that any three elements (other than the three angles) are sufficient to make the triangle perfectly definite? The pupil will best arrive at the answer

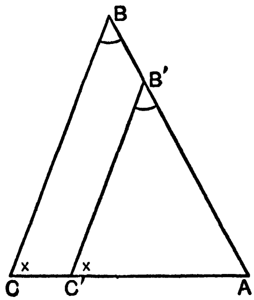


Fig. 39

to this question by attempting to construct triangles which shall have three given elements, the possible cases being—

(i) Given two sides and the included angle,
say, b, c, and A.

(ii) Given the three sides
a, b, and c.

(iii) Given two angles and the side adjacent to both,
say, B, C, and a.

(iv) Given two sides and the angle opposite to one of them, say, a, b, and A.

It will be seen from the method of construction that in each of these cases (with an exception to be presently noted) the triangle is quite determinate, and however often the same construction be repeated the resulting triangles will be exact replicas of each other.

The exception is to be found in case (iv) if it should happen that the side a opposite the given angle A is less than the side b, when it will be observed that we require to know also whether the angle B (opposite the longer side *b*) is acute or obtuse.

We shall here discuss the four cases in order—

CASE (i): **Given two sides and the included angle.**

Let *b*, *c* be the given sides and A the given angle (fig. 40). Draw any two lines A X, A Y containing an angle equal to the given angle A. From A X and A Y cut

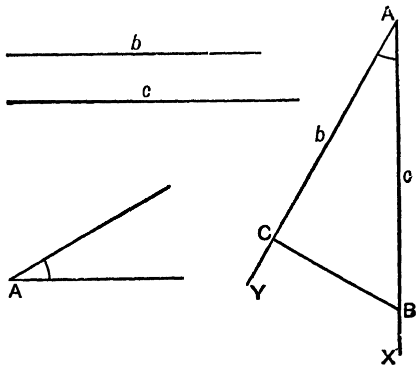


Fig 40

off parts A B and A C equal to the given sides *c* and *b* respectively. Join B C.

From the definite nature of this construction it is clear that the same triangle will result however often it be repeated, hence we conclude that—

" If two triangles have two sides of the one respectively equal to two sides of the other, and the contained angles equal, the triangles are equal in every respect "

EXERCISES

1. Draw any triangle ABC. Construct a new triangle XYZ such that the side XY=the side AB,
   ,, YZ= ,, BC,
   and the angle XYZ=the angle ABC.

Prove by using protractor and compasses that
the side XZ=the side AC,
the angle YXZ=the angle BAC,
and ,, YZX= ,, BCA.

2. Produce the sides BA, BC, YX, YZ through the ends of the oases to P, Q, R, S respectively, and prove that
   the angle PAC=the angle RXZ,
   and ,, QCA= ,, SZX.

3. Prove by cutting the triangles out in paper, and superposing one on the other, that they coincide perfectly, and hence are equal in every respect.

4. Draw an isosceles triangle of which the equal sides are each 10 cm. in length and the included angle 60°. Find the length of the third side and the value of the base angles.

5. Draw a triangle two of whose sides shall be respectively 5 cm. and 12 cm. in length, and shall include a right angle. Find the length of the third side and the value of the base angles.

6. Draw a right-angled triangle, the lengths of the sides containing the angle being 1·6″ and 3″ respectively. Measure the third side.

7. Draw a line AB, 3 inches long. Make a positive angle BAC of 45° and a negative angle BAC′ of 45°. Mark on AB and AC′ two points C and C′, each 2 inches from A. Join those points to B. Measure and compare the various elements in the triangles BAC and BAC′. Show that if triangle BAC be rotated round AB until it is again in the plane of the paper there will be coincidence with triangle BAC′.

8. Given

$$\left.\begin{array}{l} b=12 \text{ cm.} \\ c=\ 8 \text{ cm.} \\ A=60^\circ \end{array}\right\}$$ Construct the triangle (i) with the short side to the left, (ii) to the right.

Cut out the triangles and prove their congruency by superposition, and note that one has to be turned over to effect coincidence.

CASE (ii): **Given the three sides.**

Let a, b, c be the three sides (fig. 41). Draw a line B C equal to a. With B as centre and a line equal to c as radius describe a circle, and with C as centre and a line equal to b as radius describe a circle cutting the former in A. Join A B and A C.

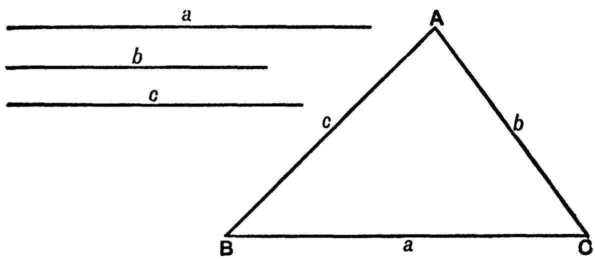


Fig. 41

However often the construction is repeated the same triangle will always result, hence we conclude that—

"If two triangles have the three sides of the one equal to the three sides of the other, each to each, the triangles are equal in all respects".

QUESTIONS AND EXERCISES

1. What condition must be fulfilled by the lengths of the given sides in order that the construction of the triangle may be possible. Attempt to construct triangles with the following data, in each case laying down the side a first.

(i)	a=10 cm.	b= 7 cm.	c=3 cm.
(ii)	a=10 cm.	b= 6 cm.	c=3 cm.
(iii)	a= 3 cm.	b=10 cm.	c=5 cm.

Observe and describe accurately the result.

2. Construct the triangle in which

a=4 cm.
b=8 cm.
c=6·93 cm.

Measure the angles, note their values, and prove that their sum is 180°.

3. Draw an isosceles triangle, having given

$$a = b = 10 \text{ cm.}$$
$$c = 6 \text{ cm.}$$

Find the angles A, B and C. Bisect C in M, join MA, and prove that the angles AMC, AMB are right angles. State the general principle in words.

4. You are required to construct a triangle whose sides are respectively 12 cm., 11 cm., and 10 cm. in length. Proceed as follows:—

Lay down the longest side first and approximately parallel to the upper edge of the page. Complete the construction (i) with the shortest side towards the left, (ii) with the shortest side towards the right. Then repeat (i) and (ii), using the second point of intersection of the circles. You have thus four triangles. Prove their equality in all respects by cutting out and superposing. Note that two of the triangles have to be turned over to effect complete superposition.

5. Construct a triangle whose sides are 1·6″, 3″, 3·4″. Measure the greatest angle. Confirm this result by calculation.

6. Upon a given base draw a set of triangles, each of which has the sum of its two sides equal to a given straight line.

CASE (iii): **Given one side and the angles adjacent to it.**

[NOTE.—If *any two* angles of a triangle are given the third is also given, since the sum of the three angles

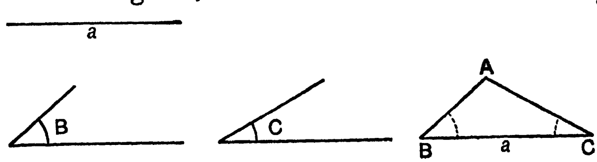


Fig. 42

is always equal to two right angles. Hence the present problem is equivalent to the more general one—Given one side and any two angles to construct the triangle.]

Let B, C be the given angles, *a* the given side (fig. 42).

Lay down a line BC equal to the given side *a*, and from its ends draw two lines B A, C A, making with B C angles which are respectively equal to the given angles B and C.

The construction being perfectly definite, the same triangle will always result, however often the operation is performed. Hence we conclude that—

"If two triangles have two angles of the one equal to two angles of the other, each to each, and a side of the one equal to a side of the other similarly situated with respect to those angles, the triangles are equal in every respect".

EXERCISES

1. In a triangle A B C—

$$\left.\begin{array}{l}C = 90^\circ \\ A = 45^\circ\end{array}\right. a = 6 \text{ cm.}$$

Draw the triangle and find the sides b and c.

2. In a right-angled triangle A B C one of the acute angles is 30°, and the side opposite it is 4 cm. in length. Draw the triangle and find the lengths of the remaining sides.

3. The distance between two trees, A and B, on one bank of a river is 120 yards. The line joining B to a third tree C on the opposite bank subtends at A an angle of 65°, while the line joining A to C subtends at B an angle of 45°.

Draw a plan of the triangle A B C, and find from it the distances B C and A C.

4. Draw two triangles A B C, A′ B′ C′, such that—

$$\left.\begin{array}{r}B = B' = 60^\circ \\ C = C' = 80^\circ \\ \text{and } a = 10 \text{ cm.} \\ a' = 15 \text{ cm.}\end{array}\right\}$$

Show that the angle A = the angle A′, and find from your drawing the ratios

$$\frac{a}{a'}, \frac{b}{b'}, \frac{c}{c'}$$

If a like result be assumed to hold for every pair of triangles whose angles are equal, each to each, state the general principle covering all cases.

5. Draw a triangle A B C, given $a = 5$ cms., A = 25°, and B = 40°,

Measure the other elements in the triangle. Obtain from your drawing the area.

6. If you are told that one side of a certain parallelogram is 10 cm. long, and that the two interior angles are 70° and 110°, construct the parallelogram if you can. You are further told that the longer diagonal is inclined at 30° to the given side. Construct the parallelogram.

CASE (iv *a*): **Given two unequal sides and the angle opposite the longer.**

Let *a*, *b* be the given sides (of which $a > b$) and A the given angle (fig. 43).

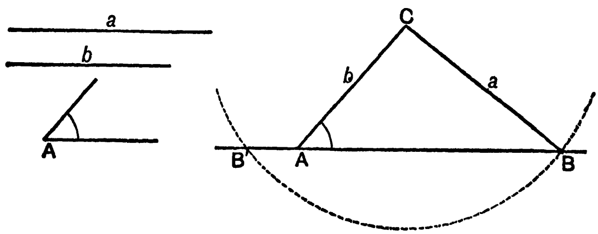


Fig. 43

Draw A P, A Q two lines containing an angle equal to A, and from A Q cut off A C equal to *b*.

With C as centre, and A as radius, describe a circle cutting A P in B. Join B C.

[NOTE.—The circle will cut P A produced in B′, but if B′ be joined to C the triangle so formed will not fulfil the given conditions.]

From the construction it is seen that a triangle is determined by two sides, and the angle opposite the longer. Hence—

"If two triangles have two sides of the one equal to two sides of the other, each to each, and the angle opposite the greater side in the one equal to the angle opposite the greater side in the other, the triangles are equal in every respect".

[NOTE.—If the two given sides are equal, the two angles opposite them are equal; hence if we have one of them given, so also is the other, and the problem belongs to Case iii.]

CASE (iv *b*): **Given two unequal sides and the angle opposite the less.**

Let *a*, *b* be the given sides (of which $a < b$) and A the given angle (fig. 44).

Draw A P, A Q two lines containing an angle equal to A, and from A Q cut off A C equal to *b*.

With centre C, radius *a*, describe a circle.

This circle will in general cut A P in two points B, B′. Join B C, B′ C.

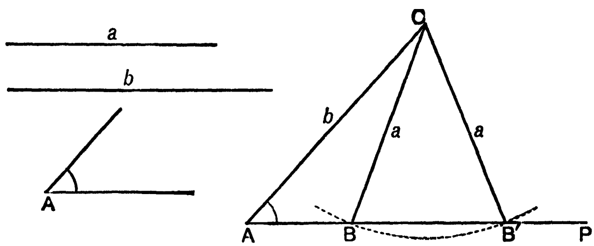


Fig. 44

Both the triangles A B C, A B′ C fulfil the given conditions. Thus—

When two sides and the angle opposite the less are given, two triangles can in general be constructed.

Let us consider the figure a little more closely.

The angles A B C and B′ B C are supplementary, and C B B′ is equal to C B′ B, hence also the angles A B C, A B′ C are supplementary.

Therefore if one is acute the other will be obtuse.

However often the construction is performed the same two triangles will always result, the triangles in which the angle B is obtuse being equal in all respects, and the triangles in which B is acute being equal in all respects. Hence—

"If two triangles have two sides of the one respectively equal to two sides of the other, and the angles opposite the two smaller of these sides equal, whilst the angles opposite the two greater are either both obtuse or both acute, then the triangles are equal in all respects".

EXERCISES

1. Construct a triangle with the following elements:—

$$A = 60^\circ.$$
$$b = 10 \text{ cm.}$$
$$a = 8{\cdot}66 \text{ cm.}$$

Is the triangle unique? What is the value of B?

2. Draw a triangle whose sides are respectively—

$$a = 14 \text{ cm.}$$
$$b = 8{\cdot}5 \text{ ,,}$$
$$c = 9{\cdot}0 \text{ ,,}$$

Find the angle B of this triangle.
Now construct two separate triangles having—

$$a' = 14 \text{ cm.}$$
$$b' = 8{\cdot}5 \text{ cm.}$$
$$B' = \text{the angle B of the former triangle.}$$

Show that these two triangles have the angles opposite the two longer of the given sides supplementary.

CHAPTER IX

AREAS

When a closed figure is drawn on a sheet of paper or on a black-board it occupies a certain portion of the surface of the paper or of the board. It is said to have a certain **area.** The unit of area used in expressing the size of such a figure is generally a square whose side is some unit of length, *e.g.* a yard, an inch, or a centimetre. The accompanying drawing (fig 45) shows two

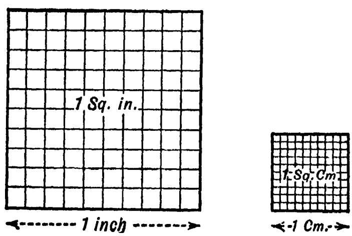


Fig. 45

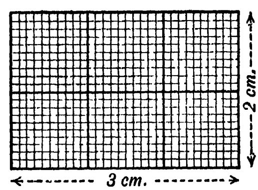


Fig. 46

of those units of area—(i) the square inch, (ii) the square centimetre. Each side of the square is divided into 10 equal parts, and consequently each area into 100 equal parts. In (i) we have 100 hundredths of a square inch, *i.e.* 100 × ·01 sq. in., and in (ii) we have 100 hundredths of a square centimetre, *i.e.* 100 × ·01 sq. cm.

Rectangle.—The area of a rectangle is known if we know the lengths of two adjacent sides.

Thus in fig. 46 we have a rectangle whose adjacent sides are 2 cm. and 3 cm. respectively.

Its area = 6 sq. cm. = (2 × 3) sq. cm.

In fig. 47 we have a rectangle whose adjacent sides are ·8 in. and 1·7 in. respectively.

Its area = 136 hundredths of a sq. in. = 136 × ·01 sq. in.
= 1·36 sq. in. = (·8 × 1·7) sq. in.

In fig. 48 we have a rectangle whose adjacent sides are 2·3 cm. and 2·4 cm. respectively. It will be convenient to divide up the area into four parts A, B, C, and D.

Area of part A = (2 × 2) sq. cm. = 4·0 sq. cm.
B = (20 × 4) sq. mm. = 80 sq. mm. = ·8 sq. cm.
C = (20 × 3) sq. mm. = 60 sq. mm. = ·6 sq. cm.
D = (4 × 3) sq. mm. = 12 sq. mm. = ·12 sq. cm.
Total area = 5·52 sq. cm. = (2·3 × 2·4) sq. cm.

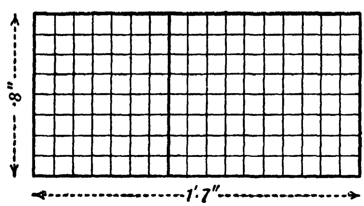


Fig. 47

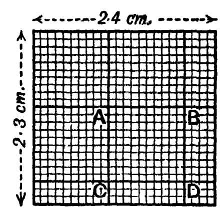


Fig. 48

Generally, if the sides of a rectangle are a cm. and b cm. the area is ab sq. cm., and the area of a square of a cm. side is $a \times a$ sq. cm., *i.e.* a^2 sq. cm.

It will be advisable to verify the principle here stated by using millimetre paper to test such cases as the following:—

(i) the area of a square of ·2 cm. side is ·04 sq. cm.
(ii) the area of a square of ·4 cm. side is ·16 sq. cm.
(iii) the area of a square of 2·3 cm. side is 5·29 sq. cm.
(iv) the area of a rectangle 2·3 cm. by 3·4 cm. is 7·82 sq. cm.

Should it be more convenient to use squared paper with inch and tenth-inch divisions, exercises similar to those just given should be worked with the inch as unit of length.

Triangle.—The area of a triangle is known if the lengths of one side and of the perpendicular to that side from the opposite vertex are known. In fig. 49 it

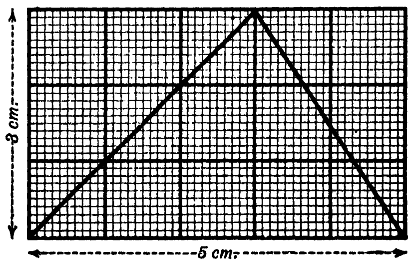


Fig. 49

will be found by counting the number of square millimetres within the triangle that its area is 750 sq. mm. or 7·5 sq. cm., *i.e.* $\frac{3 \times 5}{2}$ sq. cm. In all cases the area of a triangle is equal to one-half the area of a rectangle of the same base and altitude. Using symbols, we have

if a = the number of linear units in the side,
and p = „ „ „ perpendicular,
then $\frac{1}{2}\,ap$ = the number of square units in the area.

This result follows directly from the important theorem that a diagonal divides a rectangle into two identically equal triangles (*vide* ch. viii).

Thus in fig. 50 (i)

triangle ABC = triangle ABD + triangle ACD
„ = ½ rectangle EBDA + ½ rectangle ADCF
„ = ½ rectangle EBCF
„ = $\frac{1}{2}\,ap$ sq. cm.

In fig. 50 (ii)

triangle ABC = triangle ABD − triangle ACD
 „ $= \frac{1}{2}$ rectangle EBDA − $\frac{1}{2}$ rectangle ADCF
 „ $= \frac{1}{2}$ rectangle EBCF
 „ $= \frac{1}{2}\, ap$ sq. cm.

Parallelogram.—The area of a parallelogram is equal

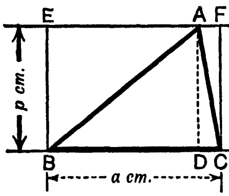


Fig. 50 (i)

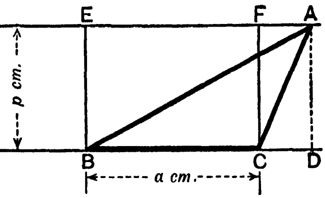


Fig. 50 (ii)

to that of a rectangle of the same base and same perpendicular height.

Let ABCD be a parallelogram (fig. 51). Join AC and draw AE perpendicular to DC.

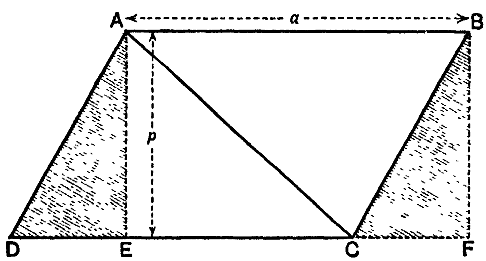


Fig. 51

Let the base DC be a cm. and the perpendicular height AE be p cm. From ch. vii, p. 52, we have

Area of the parallelogram ABCD = twice area of triangle ACD
$= 2 \times \frac{1}{2}\, ap$
$= ap$
= area of rectangle of base a and height p

NOTE.—Draw AE BF perpendicular to DC and DC produced.

The shaded triangle A D E may be imagined to slide along D C into the position B C F. Thus the parallelogram can be dissected and rearranged so as to exactly make up the rectangle A B F E.

Trapezium.—The area of a trapezium, which is a four-sided figure with two opposite sides parallel, is equal to that of a rectangle whose base is the mean of the two parallel sides and whose height is the perpendicular distance between them.

Let A B C D (fig. 52) be the trapezium, A B being parallel

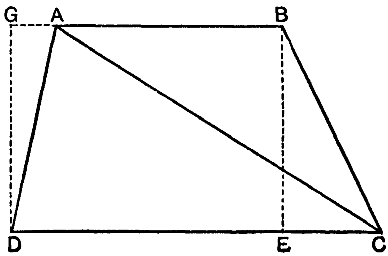


Fig. 52

to D C. Then if A B = a and C D = b, and B E or G D = p,

Area of trapezium = area of triangle A B C + area of triangle A C D

$$= \frac{1}{2}\,ap + \frac{1}{2}\,bp$$

$$= \frac{1}{2}\,(a + b)\,p$$

NOTE.—Producing A B (fig. 53), and drawing perpendiculars to C D through C, A, B, and D, we have

triangle A F D = triangle A D E, and
triangle B H C = triangle B C G;

therefore the rectangle F H C D is as much greater than the trapezium A B C D as the rectangle A B G E is less than it. The trapezium A B C D is thus the mean of the rectangles F H C D and A B G E, and is therefore equal to half their sum.

Trapezium A B C D = $\frac{1}{2}\,(ap + bp) = \frac{1}{2}\,(a + b)\,p$.

A great many theorems may be derived from the principles illustrated in the previous paragraphs.

(i) Parallelograms on the same base and between the same parallels, *i.e.* having the same altitude, are equal in area.

For if A B D C and E F D C (fig. 54) are parallelograms on the same base C D and between the same parallels A F

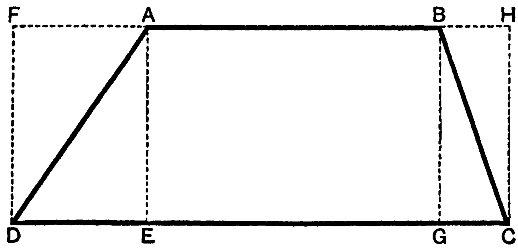


Fig. 53

and C D, the area of each is $a p$ where a is the length of the base and p that of the perpendicular distance between the parallels.

(ii) Parallelograms on equal bases and between the

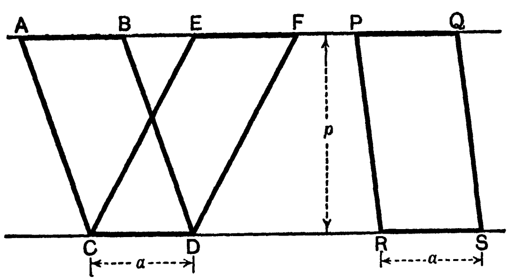


Fig. 54

same parallels, *i.e.* having the same altitude, are equal in area.

For if E F D C and P Q S R (fig. 54) be parallelograms on equal bases C D and R S and between the same parallels A Q and C S, the area of each is $a p$ as before.

It may be seen (fig. 55) in the same way that (iii) triangles on the same base, or on equal bases, and between the same parallels, *i.e.* having the same altitude, are equal in area.

For the area of each is $= \frac{1}{2}\, a\, p$.

(iv) If a parallelogram and a triangle have the same base and altitude the parallelogram is double the triangle.

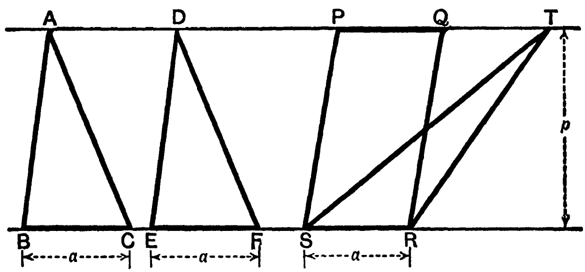


Fig. 55

For the area of the parallelogram is $a\, p$ and the area of the triangle is $\frac{1}{2}\, a\, p$.

(v) If any number of equal triangles such as A B C, D E F, T S R (fig. 55) stand on equal bases B C, E F, S R, in the same straight line B R, their vertices A, D, T lie on a straight line parallel to that straight line.

For since the triangles have equal areas and equal bases they must also have equal perpendiculars, that is, the vertices A, D, T must be equidistant from B R and are therefore (see chap. vii, ex. 8, p. 59) on a straight line parallel to B R.

The following problems are of considerable practical importance, and are easily solved by using the foregoing principles:—

(1) On a given base to construct a triangle which shall have the same area as a given triangle.

Let A B C (fig. 56) be the given triangle, and let B D the given base for the new triangle be marked off from C B produced. Join A D. Draw C E parallel to A D, cutting A B produced in E, and join E D.

Then since A C D and A E D are triangles on the same base and between the same parallels they are equal in area. Take away the common area A B D and the remaining triangle A B C is equal in area to the remaining triangle B E D.

If now we draw through the point E a straight line E X parallel to C D it is clear that *any* triangle having B D for base and its vertex in E X will be equal to A B C.

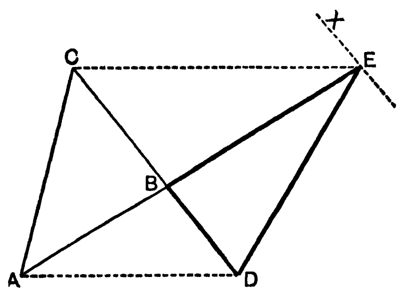


Fig 56

It is therefore possible to make the new triangle fulfil another given condition. For example, we may construct it so that one of the angles adjacent to B D shall be equal to a given angle. Thus, draw B F (fig. 57), making with B D an angle equal to the given angle and meeting E X in F, and join F D. The new triangle F B D has been constructed on the given base B D, having its area equal to A B C, and having the angle F B D equal to a given angle.

(2) To construct a triangle equal in area to a given quadrilateral A B C D (fig. 58).

This is merely an extension of the preceding. Join A C: draw D C′ parallel to A C: produce B C to meet D C′ in C′: join A C′.

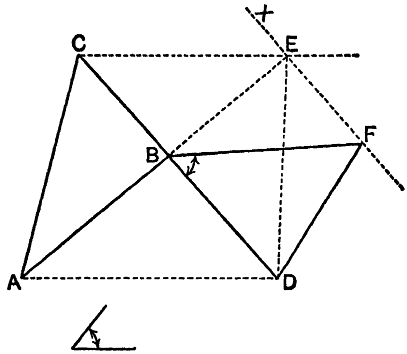


Fig. 57

The triangle A C C′ = the triangle A D C.
∴ the triangle A B C′ = the quadrilateral A B C D.

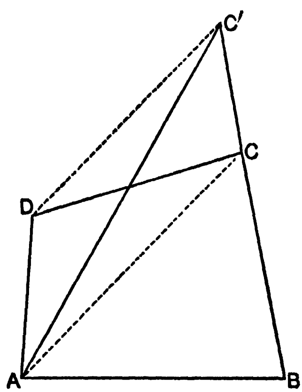


Fig. 58

Having obtained a triangle A B C′ equal in area to the quadrilateral, we can apply the methods of (1) to form a triangle of the same area in which other conditions will be satisfied.

(3) To construct a rectangle equal in area to a given polygon ABCDEF (fig. 59), and having one side equal to a given straight line.

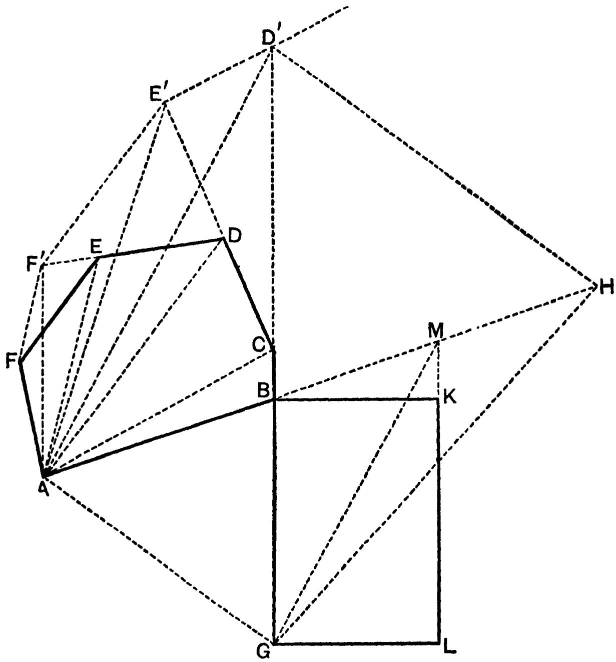


Fig. 59

Join AE, AD, AC.

Through	F	draw	FF′	parallel to	AE,	meeting	DE	produced	in	F′
„	F′	„	F′E′	„	AD	„	CD	„		E′
„	E′	„	E′D′	„	AC	„	BC	„		D

Join AF′, AE′, AD′.

Then the triangle ABD′
= the quadrilateral ABCE′ (∵ ACE′ = ACD′)
= the figure ABCDF′ (∵ ADF′ = ADE′)
= the figure ABCDEF (∵ AFE = AF′E)

Now produce CB to G, making BG equal to the given side of the rectangle: join AG: draw D′H parallel to AG meeting A B produced in H: bisect B H in M: through M draw M K L parallel to B G, and draw B K, G L perpendicular to B G.

The rectangle B K L G = twice the triangle B G M
= the triangle B G H
= " " A B D′
= the figure A B C D E F

and it has the given length B G for one of its sides.

(4) To bisect a triangle by a straight line through a point in one of its sides.

Let A B C be the given triangle (fig. 60), and P the

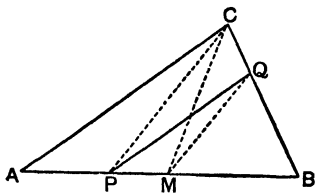


Fig. 60

given point in the side A B. Bisect A B in M: join C P: through M draw M Q parallel to C P and join C M, P Q.

Then the triangle P Q B = the triangle C M B
= half the triangle A B C.

SUMMARY

1. The unit of area being the area of a square whose side is the unit of length, we have the following expressions for the areas of certain common geometrical figures:—

square of side a	area $= a^2$
rectangle of sides a and b	area $= a\,b$
parallelogram of base a and perpendicular height p	area $= a\,p$
trapezium with two parallel sides a and b, and perpendicular breadth p	area $= \frac{1}{2}\,p\,(a + b)$
triangle of base a and perpendicular height p	area $= \frac{1}{2}\,a\,p$

2. Parallelograms on the same base, or on equal bases, and of the same perpendicular height are equal in area.

3. Triangles on the same base, or on equal bases, and of the same perpendicular height are equal in area.

4. If a parallelogram and a triangle are on the same base, or on equal bases, and have the same perpendicular height, the parallelogram is double of the triangle.

5. Equal triangles or parallelograms on the same base and on the same side of it are between the same parallels.

6. Equal triangles or parallelograms on equal bases in the same straight line, and on the same side of it, are between the same parallels.

7. It is possible by using the foregoing principles to draw a triangle or a rectangle on any given base equal in area to any rectangular figure.

QUESTIONS

1. What is meant by saying that a rectangle is 2′ 6″ by 1′ 4″? Calculate its area.

2. If the area of a square is 200 square inches, what is the nearest whole number of inches in the side?

3. Compare the areas of two squares if a side of the larger is half as long again as that of the smaller. Give a sketch illustrating your answer.

4. A quadrilateral field can be divided into two triangles by a diagonal 1250 links long. The perpendicular distances of two corners from the diagonal are 400 links and 600 links respectively.

Express the area in (*a*) square chains
(*b*) „ yards
(*c*) „ miles
(*d*) „ inches

[100 links = 1 chain = 22 yards.]

5. The diagonals of a certain quadrilateral field cross one another at right angles and are in length 800 links and 750 links respectively. Find the area in acres.

6. ABCD is a quadrilateral figure, the angles ABC and CDA being right angles. AB = 25 cm., BC = 60 cm., CD = 52 cm., DA = 39 cm. Find the area in (*a*) sq. cm., (*b*) sq. metres.

7. How do you prove that if a line is perpendicular to one of a pair of parallel lines it is perpendicular to the other?

8. How do you prove that if points P and Q are on the same side of a straight line and equidistant from it, the straight line joining them is parallel to the given straight line?

9. Prove that the straight lines which join the mid-points of the sides of a triangle are parallel to the sides in order, and that they form a triangle whose area is equal to one quarter that of the original triangle.

10. The mid-points of the four sides of a quadrilateral figure are joined in order. Describe the figure enclosed by these joining lines. Compare its area with that of the given figure.

If through the four angular points of the quadrilateral, straight lines are drawn parallel respectively to these joining lines, compare the area of the figure thus formed with that of the quadrilateral.

EXERCISES IN DRAWING

1. Draw a square of 1·6″ side and note its area.

2. Draw a rectangle 2·3″ by 1·4″ and note its area.

3. Draw a parallelogram with its two adjoining sides 2·1″ and 1·8″ respectively and the included angle 30°. Find its area.

4. Draw a triangle with sides 1·5″, 2·0″, and 2·5″ respectively. Draw the perpendicular on the longest side from the opposite vertex, and measure its length. Calculate and note the area.

5. Draw a triangle with sides 2″, 2·5″, 3″ respectively, and without determining the area numerically draw another triangle equal in area to the first and having one side 3·5″ long. Find the perpendicular distance between this side and the opposite vertex.

6. Draw another triangle of the same area and with *two* of its sides 3·5″ and 4″ respectively.

7. Draw another equal triangle with one side 3·5″ long and the angle between it and an adjacent side 30°.

8. Draw an irregular six-sided rectilinear figure about 16 sq. cm. in area. Obtain a triangle of the same area as the figure. If one side of a rectangle of the same area as the triangle is 3 cm., what is the length of the adjoining side?

9. Two adjoining sides of a parallelogram are 10 cm. and 8 cm. long respectively. The angle between them is 50°. Draw the figure and find the area.

10. The angle between two adjoining sides of a parallelogram is 60°, one side is 10 cm. long. The area is 50 sq. cm. Draw the figure and find the length of the other side.

CHAPTER X

DISTRIBUTION OF RECTANGULAR AREAS

NOTE.—In working through this chapter the pupil should make free use of "squared" paper, and should mark plainly the lengths of the lines and the values of the areas. A convenient method of doing this is indicated in the text and accompanying figures.

It was established in Chapter IX that if the lengths of two adjacent sides of a rectangle be a and b units respectively, then the area of the figure is ab units of area. Here the expression ab which measures the area is a product, viz. the product of a and b. Accordingly, in dealing with areas we in reality deal with products. If, for example, we have three areas represented respectively by ab, ac, ad, the sum of these areas is $ab + ac + ad$. Now by the distributive law applied to numbers,

$$ab + ac + ad = a(b + c + d).$$

Here a is the length of one line, and $(b + c + d)$ the sum of the lengths of the three other lines.

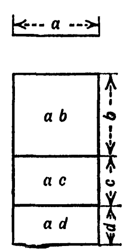


Fig. 61

The accompanying drawing (fig. 61) represents geometrically the fact stated in the equation $a(b + c + d) = ab + ac + ad$. From this we have—

(i) The rectangle contained by two given lines is equal to the sum of the rectangles contained by one of these, and the several parts into which the other may happen to be divided.

In (i) we dealt with an area expressed as a product, in which one of the factors was divided. If now the other is also divided we get an expression of the form $(a + b)(c + d)$.

$$\text{Now } (a + b)(c + d) = a(c + d) + b(c + d) \quad \text{(i)}$$
$$= ac + ad + bc + bd, \quad \text{(i)}$$

The geometrical representation of this theorem is given in fig. 62. So we have—

(ii) The rectangle contained by two lines, each divided into two parts, is equal to the sum of four rectangles, in each of which one side is a segment of one line and the adjoining side a segment of the other.

Consider now the case of (ii), where $c = a$ and $d = b$, *i.e.* where the two lines are equal and are similarly divided. The equation becomes

$$(a + b)(a + b) = aa + ab + ba + bb \quad \text{(ii)}.$$
$$\textit{i.e. } (a + b)^2 = a^2 + 2ab + b^2.$$

Hence the proposition illustrated in fig. 63, viz.:—

(iii) If a straight line be divided into two parts, the

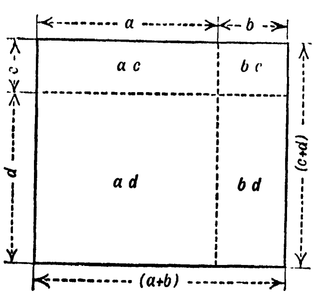


Fig. 62

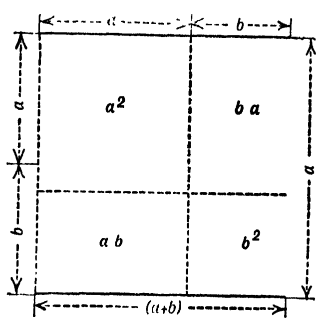


Fig. 63

square on the whole line is equal to the sum of the squares on the parts, and twice the rectangle contained by them.

In (iii) we have considered $(a + b)^2 = a^2 + 2ab + b^2$. Let us now see whether $(a - b)^2 = a^2 - 2ab + b^2$ has a similar geometrical interpretation.

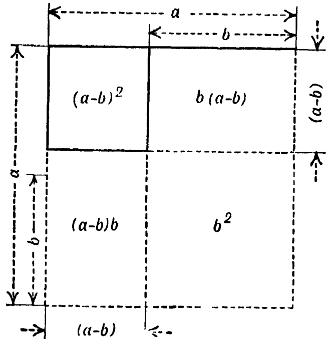


Fig. 64

Referring to fig. 64 we have the large square equal to a^2

and $(a - b)^2 = a^2 - 2b(a - b) - b^2$

$\therefore (a - b)^2 = a^2 - 2ab + b^2$. So we have—

(iv) The square on the difference of two lines is equal to the sum of the squares on these lines diminished by twice the rectangle contained by them.

The equation $(a + b)(a - b) = a^2 - b^2$, is treated in a similar way.

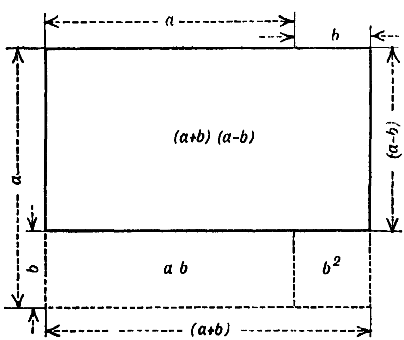


Fig 65

Referring to fig. 65, we have the whole rectangle equal to $a\,(a+b)$, and so we have

$$(a+b)\,(a-b) = a\,(a+b) - a\,b - b^2$$
$$\therefore\ (a+b)\,(a-b) = a^2 - b^2.$$ That is—

(v) The rectangle contained by the sum and the difference of two straight lines is equal to the difference of the squares on them.

A most important extension of (v) is shown in figs. 66 (*a*) and (*b*).

Here M is mid-point of A B

$AM = MB = a$

$MP = b$

$\therefore$ in fig. 66 (*a*) $PB = a - b$, and $AP = a + b$.

From (v) $(a+b)\,(a-b) = a^2 - b^2$

i.e. $AP \cdot PB = AM^2 - MP^2$.

Similarly in fig. 66 (*b*) $BP = b - a$ and $AP = b + a$.

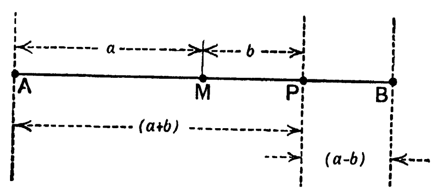


Fig 66 (*a*)

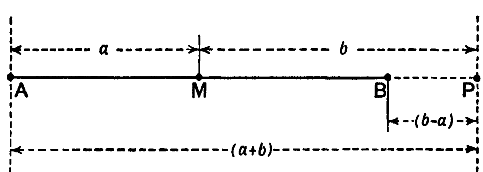


Fig 66 (*b*)

From (v) $(b+a)\,(b-a) = b^2 - a^2$

i.e. $AP \cdot BP = MP^2 - AM^2$. And thus—

(vi) If a straight line be bisected and be divided unequally, the rectangle contained by the unequal parts is equal to the difference of the squares on half the line, and on the line between the points of section.

Note.—If a straight line is bisected and divided internally,
the sum of the segments is equal to the original line; and the difference of the segments is equal to twice the line between the points of section.

But if the straight line is divided externally,
the difference of the segments is equal to the original line; and the sum of the segments is equal to twice the line between the points of section.

Again, we have (fig. 66) AP $= (a+b)$ and PB $= (a-b)$.
Now $(a+b)^2 + (a-b)^2 = 2a^2 + 2b^2$ (iii and iv),
i.e. $AP^2 + PB^2 = 2\,AM^2 + 2\,MP^2$. So we get—

(vii) If a straight line be bisected and be divided unequally, the sum of the squares on the two unequal parts is equal to twice the sum of the squares on half the line, and on the line between the points of section.

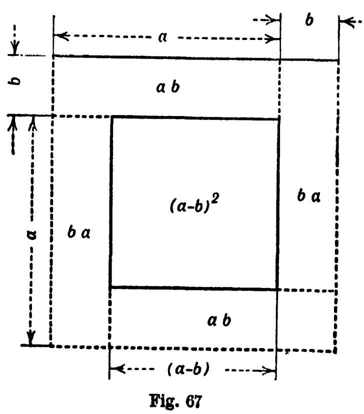


Fig. 67

Again, since $(a + b)^2 = a^2 + 2ab + b^2$ (iii),
and $(a - b)^2 = a^2 - 2ab + b^2$ (iv),
$\therefore\ (a + b)^2 - (a - b)^2 = 4ab.$

This equation, which is illustrated in fig. 67, gives—

(viii) The difference between the squares on the sum and on the difference of two lines is equal to four times the rectangle contained by these lines.

SUMMARY

If a, b, c, &c. be lengths of lines, ab, bc, &c. are areas;

and (1) $a(b + c + \&c.) = a(b + c + \&c.)$.
(2) $(a + b)(c + d) = ac + ad + bc + bd.$
(3) $(a + b)^2 = a^2 + b^2 + 2ab,$
(4) $(a - b)^2 = a^2 + b^2 - 2ab,$
(5) $(a + b)(a - b) = a^2 - b^2,$
(6) $(a + b)^2 + (a - b)^2 = 2(a^2 + b^2),$
(7) $(a + b)^2 - (a - b)^2 = 4ab.$

QUESTIONS

1. If a is a number, and if b represents a length, what does ab represent?
2. If a and b are lengths, what is ab?
3. If a is a number, and if b and c are lengths, what is abc?
4. If a and b are lengths, and c is a number, what are—

$$(1)\ \frac{ab}{c},\ (2)\ \frac{bc}{a},\ (3)\ \frac{a^2}{b},\ (4)\ \frac{b^2}{c}?$$

5. If a, b, c are lengths, what are—

$$(1)\ a(b + c),\ (2)\ (a + b)^2,\ (3)\ \frac{ab}{c},\ (4)\ \left(\frac{ab}{c}\right)^2,\ (5)\ abc?$$

6. Prove that the square on a line is four times the square on half the line.
7. What is the position of a point P between A and B in a line AB, when the rectangle AP . PB is the greatest possible?

8. One hundred yards of fencing are to be used to make the largest possible sheepfold. What are the dimensions of the enclosure?

9. Show how to divide a straight line into two parts so that the sum of the squares on these parts shall be the least possible.

10. If the lengths of two straight lines vary in such a way that the sum of the squares remains constant, when is the rectangle contained by the two lines of maximum area?

EXERCISES

(Squared paper may be used with advantage)

1. Show by a drawing that if a straight line be divided into any two parts, the square on the line is equal to the sum of the rectangles contained by the line and each of its parts. Give the equation.

2. Show by a drawing that if a straight line be divided into any two parts, the rectangle contained by the whole and one of the parts is equal to the sum of the square on that part and the rectangle contained by the two parts. Give the equation.

3. Show by a drawing that the square on a line is equal to 9 times the square on one-third of that line.

4. A, B, C, D are points in a straight line, and AB = BC = CD. Show by a drawing that the square on AD is equal to the sum of the squares on AC, BD, and BC. Give equation.

5. If A, B, C, D are points in order in a straight line, prove that the rectangle AC.BD = AB.CD + AD.BC.

6. Give a drawing to illustrate $(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$.

7. Prove and illustrate—

$$\begin{aligned}(a + b + c + d + \ldots\ldots)^2 = a^2 + b^2 + c^2 + d^2 + \ldots\ldots \\ + 2a(b + c + d + \ldots\ldots) \\ + 2b(c + d + \ldots\ldots) \\ + 2c(d + \ldots\ldots) \\ + \&c\ldots\ldots\end{aligned}$$

Give this theorem in words.

8. Give a drawing to illustrate $(a - b)(c - d) = ac - ad - bc + bd$.

9. Give a drawing to illustrate $(a + b)(c - d) = ac - ad + bc - bd$.

10. If a straight line AB is divided at P into two unequal parts AP and PB of lengths a and b respectively, show that the length

of half the line is $\frac{1}{2}(a + b)$, and that the length between the mid-point of the line and the point of section is $\frac{1}{2}(a - b)$.

Examine the equations—

$$(1) \quad ab = \left(\frac{a+b}{2}\right)^2 - \left(\frac{a-b}{2}\right)^2,$$

$$(2) \quad a^2 + b^2 = 2\left(\frac{a+b}{2}\right)^2 + 2\left(\frac{a-b}{2}\right)^2.$$

11. If a straight line A B is produced to a point P, and if the lengths of A P and P B, the segments into which the line is divided externally at P, be a and b respectively, show that the length of half the line is $\frac{1}{2}(a - b)$, and that the length between the mid-point and the point of section is $\frac{1}{2}(a + b)$.

Examine the equations—

$$(1) \quad ab = \left(\frac{a+b}{2}\right)^2 - \left(\frac{a-b}{2}\right)^2,$$

$$(2) \quad a^2 + b^2 = 2\left(\frac{a+b}{2}\right)^2 + 2\left(\frac{a-b}{2}\right)^2.$$

12. If in fig. 66 (a) the area of the squares described on A M and M P respectively be 36 sq. cm. and 16 sq. cm., find the length of P B. Make a full-size drawing.

13. Prove that the difference of the squares on A P and P B, fig. 66 (a), is equal to twice the rectangle contained by M P and A B.

14. If P lies in A B produced, fig. 66 (b), and if M is mid-point of A B, show that the difference of the squares on A P and P B is equal to twice the rectangle contained by M P and A B.

15. If a straight line be divided into two parts, the sum of the squares on the whole line and on one part is greater than twice the rectangle contained by the whole line and that part, by the square on the other part.

16. If a straight line A B be bisected at M and divided unequally at P, prove that $AB^2 = 4MP^2 + 4 . AP . PB$.

CHAPTER XI

SQUARES ON SIDES OF TRIANGLES—AREAS OF A TRIANGLE IN TERMS OF ITS SIDES

Let ACB be a right-angled triangle, ACB being the right angle. It is required to find the relation between the squares on AB, BC, and CA.

Let CBDE (fig. 68) be the square described on CB. From C and B draw perpendiculars CP and DL meeting

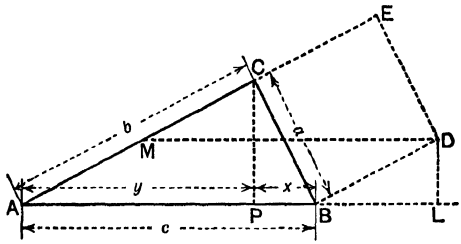


Fig 68

the line AB in P and L respectively. Through D draw DM parallel to AB meeting AE in M. Now ACE is a straight line, and the figure ABDM is a parallelogram.

On examination, the two triangles BPC and DLB are equal in every respect, and thus PB = DL.

Represent the three sides AB, BC, CA by c, a, b respectively, and BP and PA the parts of BA by x and y respectively. Since the parallelograms ABDM and CBDE are on the same base BD, and between the same parallels, they are of equal area (page 76).

The area of the parallelogram CBDE = a^2, and the area of the parallelogram ABDM = AB . DL = cx,

$$\therefore\ a^2 = cx$$

$$\text{similarly } b^2 = cy$$

$$\therefore\ a^2 + b^2 = c(x + y) = c^2.$$

And so we have (i)—The square on the hypotenuse of a right-angled triangle is equal to the sum of the squares on the other two sides.

The converse of (i) is true, viz.—If the square on one side of a triangle is equal to the sum of the squares on the other two sides, the angle contained by these two sides is a right angle.

If we have in a certain triangle $c^2 = a^2 + b^2$, where c, a, b are lines of fixed length, the triangle must be right-angled, for a right-angled triangle with its two sides equal to a and b has by theorem (i) its hypotenuse

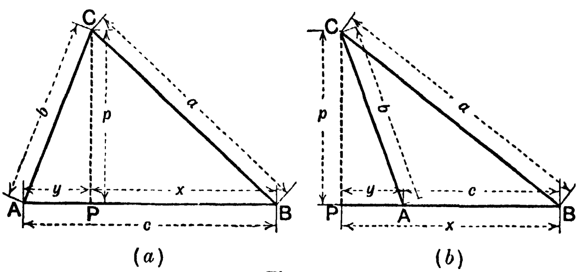


Fig. 69

equal to c, and three lines determine a triangle completely (chap. viii).

Many theorems can be deduced directly from the above proposition. The following are important examples:—

A B C is a triangle. From C a perpendicular C P is drawn to A B, meeting A B in P. The point P may lie between A and B, or may lie in A B produced (fig. 69, b).

Represent sides A B, B C, C A by c, a, b respectively, the perpendicular C P by p, and the segments B P and P A by x and y respectively.

$$\text{We have } a^2 = p^2 + x^2 \quad \text{(i)}$$
$$\text{and } b^2 = p^2 + y^2$$
$$\therefore\ a^2 - b^2 = x^2 - y^2. \text{ So we have—}$$

(ii) The difference between the square on two sides of a triangle is equal to the difference of the squares on the segments into which the third side is divided by a perpendicular from the vertex opposite.

Take now result (ii): $a^2 - b^2 = x^2 - y^2$.

$$\begin{aligned} \text{Note that } x^2 - y^2 &= (x+y)^2 - 2xy - 2y^2 \\ &= (x+y)^2 - 2y(x+y), \\ \text{and that } x + y &= c. \end{aligned}$$

$$\begin{aligned} \text{Thus we obtain } a^2 - b^2 &= c^2 - 2cy, \\ \text{and therefore } a^2 &= b^2 + c^2 - 2cy. \end{aligned}$$

Referring to fig. 69 (a), $BC^2 = CA^2 + AB^2 - 2AB.AP$. As the angle CAB is an acute angle, we have—

(iii) The square on the side opposite an acute angle is less than the sum of the squares on the other two sides by twice the rectangle contained by one of these two, and the projection of the other on it.

Note on Projections.—If from the ends A and B of a given line AB perpendiculars AA′, BB′ are drawn to a given straight line XY produced if necessary, the line A′B′ measured on XY is said to be the projection of AB on XY.

Consider fig. 69 (b)—

The equation $a^2 - b^2 = x^2 - y^2$, applies here as in fig. 69 (a). But as BP and PA, the segments of the line, are represented by x and y respectively, the side $c = x - y$.

So with this in view, we write—

$$\begin{aligned} a^2 - b^2 &= x^2 - y^2 = (x-y)^2 + 2xy - 2y^2 \\ \text{or } a^2 - b^2 &= (x-y)^2 + 2y(x-y) = c^2 + 2yc \\ \therefore a^2 &= c^2 + b^2 + 2cy \\ BC^2 &= AB^2 + CA^2 + 2AB.AP \end{aligned}$$

and accordingly as CAB is an obtuse angle, we have—

(iv) The square on the side opposite an obtuse angle is greater than the sum of the squares on the other two sides by twice the rectangle contained by one of these two, and the projection of the other on it.

Let a line C M (a median) be drawn from C to mid-

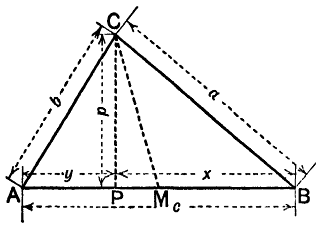


Fig 70 (*a*)

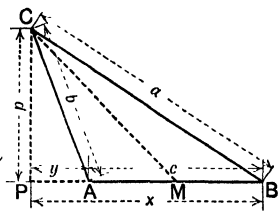


Fig 70 (*b*)

point M of A B (fig. 70). From C draw C P perpendicular to A P.

Unless C P and C M (the median) coincide, one of the two angles C M A and C M B must be acute and the other obtuse.

In both figures angle C M A is acute and angle C M B is obtuse.

$$\therefore\ b^2 = \text{CM}^2 + \text{AM}^2 - 2\,\text{AM}\,.\,\text{MP}, \qquad \text{(iii)}$$

$$\text{and } a^2 = \text{CM}^2 + \text{MB}^2 + 2\,\text{BM}\,.\,\text{MP}. \qquad \text{(iv)}$$

$$\text{Now } 2\,\text{AM}\,.\,\text{MP} = 2\,\text{BM}\,.\,\text{MP},$$

$$\therefore\ a^2 + b^2 = 2\,\text{CM}^2 + 2\,\text{MB}^2.$$

This gives us—

(v) The sum of the squares on the two sides of a triangle is equal to twice the sum of the squares on the median, and on half the third side.

In the following method of obtaining an expression for the area of a triangle in terms of its sides, the same symbols for the sides, the segments of the base, and the perpendicular from vertex to base are used.

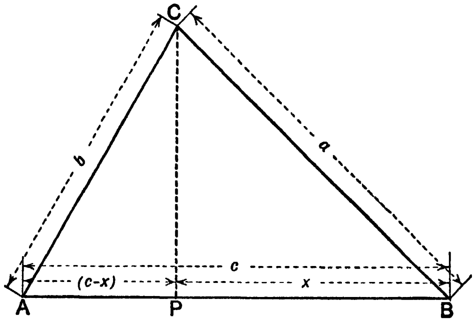


Fig. 71

$$\text{We have } a^2 - b^2 = x^2 - y^2,$$

$$\text{but } y = c - x,$$

$$\therefore\ a^2 - b^2 = x^2 - (c - x)^2 = 2\,cx - c^2,$$

$$\therefore\ x = \frac{c^2 + a^2 - b^2}{2\,c};$$

$$\text{and } p^2 = a^2 - x^2 = a^2 - \left(\frac{c^2 + a^2 - b^2}{2\,c}\right)^2,$$

$$\textit{i.e. } p^2 = \left(a + \frac{c^2 + a^2 - b^2}{2\,c}\right)\left(a - \frac{c^2 + a^2 - b^2}{2\,c}\right),$$

$$\therefore\ p^2 = \frac{(c + a)^2 - b^2}{2\,c} \times \frac{b^2 - (c - a)^2}{2\,c},$$

$$\therefore\ p^2 = \frac{(c + a + b)\ (c + a - b)\ (b + c - a)\ (b - c + a)}{4\,c^2}$$

Let $a + b + c = 2\,s$ (here s = semiperimeter),

$$\therefore\ -a + b + c = 2\,(s - a)$$

$$a - b + c = 2\,(s - b)$$

$$a + b - c = 2\,(s - c).$$

$$\text{So we have } p^2 = 4\,.\,\frac{s\,(s - a)\,(s - b)\,(s - c)}{c^2},$$

$$\textit{i.e. } \frac{p^2\,c^2}{4} = s\,.\,(s - a)\,(s - b)\,(s - c),$$

$$\therefore\ \frac{p\,c}{2} = \sqrt{s\,(s - a)\,(s - b)\,(s - c)}.$$

But area of triangle ABC $= \frac{pc}{2}$,

$\therefore$ area of triangle $= \sqrt{s(s-a)(s-b)(s-c)}$.

SUMMARY

1. The square on the hypotenuse of a right-angled triangle is equal to the sum of the squares on the other two sides.

2. If the square on one side of a triangle is equal to the sum of the squares on the other two sides, the angle contained by these two sides is a right angle.

3. The difference of the squares on two sides of a triangle is equal to the difference of the squares on the segments into which the third side is divided by a perpendicular from the vertex opposite.

4. In any triangle the square on the side opposite an acute angle is less than the sum of the squares on the other two sides by twice the rectangle contained by one of these sides and the projection of the other on it.

5. In an obtuse-angled triangle, the square on the side opposite the obtuse angle is greater than the sum of the squares on the other two sides by twice the rectangle contained by one of these sides and the projection of the other on it.

6. The sum of the squares on two sides of a triangle is equal to twice the sum of the squares on half the third side and on the median drawn to it from the opposite vertex.

7. The area of a triangle of semiperimeter s, and sides a, b, and c is $\sqrt{s(s-a)(s-b)(s-c)}$.

EXERCISES

1. In an equilateral triangle ABC, if AM is drawn perpendicular to BC, prove that—

   (i) BM = MC
   (ii) $AM^2 = 3BM^2$ and $AM = \sqrt{3}.BM$
   (iii) the angle ABM = 60° and the angle BAM = 30°.

2. In an isosceles triangle ABC, right angled at C, prove that—

   (i) $AB^2 = 2BC^2$ and $AB = \sqrt{2}.BC$
   (ii) the angle BAC = the angle ABC = 45°.

3. A square is equal to half the square on its diagonal.

4. If a point is taken within a square the sum of the squares on its joins with the corners is equal to twice the sum of the squares on the perpendiculars from it on the sides.

5. Squares BCYX, CNMA, APQB are described on the three sides BC, CA, AB of a triangle ABC, all externally, and AX CQ are joined. Prove—

(i) the triangle ABX is equal in all respects to the triangle QBC,

(ii) Hence show by the method of rotation that AX is perpendicular to CQ.

6. In any right-angled triangle the sum of the hypotenuse and the perpendicular on it from the opposite vertex is greater than the sum of the two sides containing the right angle.

7. If from the vertices ABC of a triangle perpendiculars AX, BY, CZ are drawn to the opposite sides, then

$$AZ^2 + BX^2 + CY^2 = BZ^2 + CX^2 + AY^2.$$

8. If from any point perpendiculars are drawn on all the sides of any rectilineal figure, the sum of the squares on the alternate segments are equal.

9. In a triangle ABC, in which BN is drawn perpendicular to AC,

the angle A = 30°
side c = 12″
side b = 15″,
find (i) BN (ii) AN (iii) BC.

10. With a similar notation,

if the angle A = 45°
side c = 12″
side b = 15″,
find (i) BN (ii) AN (iii) BC.

11. With a similar notation,

if the angle A = 60°
side c = 12″
side b = 15″,
find (i) BN (ii) AN (iii) BC.

12. With a similar notation,

if the angle A = 135°
side c = 12″
side b = 15″,
find (i) BN (ii) AN (iii) BC,

13. With a similar notation,

if the angle A = 120°
side c = 12″
side b = 15″,
find (i) B N (ii) A N (iii) B C.

[*Note.*—The last five questions are to be worked by calculation and from drawings.]

14. Prove theorem (2) in the summary by using theorems (4) and (5) of the same.

15. By using theorem (6), summary, prove that the median drawn to the mid-point of the hypotenuse of a right-angled triangle is equal to half the hypotenuse.

16. If any point P is drawn to the corners of a rectangle A B C D, then $PA^2 + PC^2 = PB^2 + PD^2$.

17. The sum of the squares on the sides of a quadrilateral is equal to the sum of the squares on the diagonals, together with four times the squares of the lines joining the mid-points of the diagonals.

18. If A B is a finite straight line bisected in M, and X Y is an indefinite straight line, and if A′ B′ M′ are the projections of A B and M on the line X Y, then

(i) A′ M′ = M′ B′
(ii) M M′ = $\frac{1}{2}$ (A A′ + B B′) if X Y intersects A B
M M′ = $\frac{1}{2}$ (A A′ ~ B B′) if X Y does not intersect A B.

19. If the sides of a triangle A B C are projected on any straight line X Y in the same plane, then the projection of any side is equal to the sum of the projections of the other two.

20. If any closed figure be projected on any straight line, the projection of any part of the boundary is equal in length to the projection of the remainder.

21. In a triangle whose sides are
a = 25″
b = 52″
c = 63″, find s, $s - a$, $s - b$, $s - c$, and $\sqrt{s(s-a)(s-b)(s-c)}$.

22. In a triangle whose sides are
a = 50
b = 120
c = 130, find (i) s, $s - a$, $s - b$, $s - c$, and $\sqrt{s(s-a)(s-b)(s-c)}$.
(ii) The perpendicular on c from the opposite vertex.

[*Note.*—The last two questions are to be worked (i) by calculations, (ii) by drawings.]

CHAPTER XII

CHORDS—SEGMENTS INTO WHICH A CHORD IN A CIRCLE IS DIVIDED BY A POINT THROUGH WHICH IT PASSES —RECTANGLE CONTAINED BY THE SEGMENTS OF A CHORD—TANGENTS

As preliminary to the study of the main propositions of this chapter, let us take in brief review some of the more obvious properties of the circle.

(i) A circle is the figure described in a plane when a point moves at a constant distance round a fixed point.

(ii) If a straight line cuts the circumference of a circle there are two points of intersection. The part of the straight line intercepted by the circumference is a chord of the circle.

Make a drawing like fig. 72. Here we have a set of lines drawn through a point P, and cutting the circle. The distance between the points of intersection is different for each of the lines, being greatest in the case of the line passing through the centre. So the diameter is the greatest chord in a circle. As we move out from the centre the chords diminish in length. The points of intersection get nearer and nearer, until finally we have, as at T, coincidence of the two points of intersection. The line P T, which cuts the circumference in two coincident points, is said to be a **tangent** to the circle.

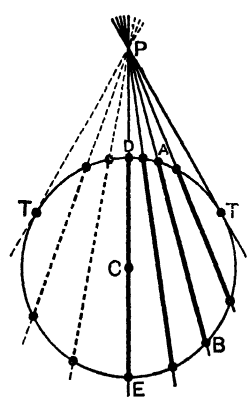


Fig. 72

(iii) The centre of any circle which passes through the end-points of a finite straight line must lie on the right bisector of that line.

Let A B (fig. 73) be a given line, and let its right bisector pass through M.

Consider what would follow if the centre C′ of a certain circle which passed through A and B did not lie on the right bisector. On joining C′A we cross the right bisector at C, and when C B is joined we have an isosceles triangle C A B. But, as C′ is assumed to be the centre of a circle passing through A and B, C′A B will also be an isosceles triangle. This obviously is impossible, and therefore C′ the centre of a circle which passes through A and B must lie on the right bisector of A B.

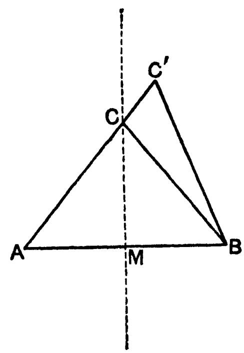


Fig. 73

What is true of C′ is true of the centres of all circles passing through A and B. This property may be stated thus—the locus of the centres of circles passing through two fixed points is the right bisector of the line joining these points. It follows that (*a*) the line joining the mid-point of a chord to the centre of the circle is the right bisector of that chord, and (*b*) the perpendicular on a chord from the centre of the circle is the right bisector of the chord.

(iv) The centre of a given circle can be found by applying (iii).

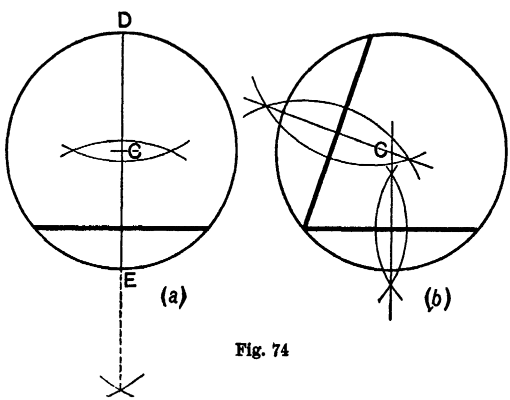


Fig. 74

In fig. 74 (*a*) you are given a circle. A chord is drawn. The right bisector of this chord is constructed. As D E the part of the right bisector intercepted by the circumference is the diameter (iii), C the mid-point of D E is the centre of the circle.

In fig. 74 (*b*) two chords are drawn from a point on the circumference. The crossing-point of the right bisectors of these chords is the centre (iii).

In fig. 74(*c*) the right bisectors of two chords which meet outside the circle intersect at C the centre.

(v) A circle can be drawn to pass through any three fixed points not in line.

Thus in fig. 75 P, Q, R are three points not in line. C is the point common to the right bisectors of two joins, PQ and QR. So CP = CQ = CR. The point C is thus equidistant from P and R, and is therefore on the right bisector of the third join, PR.

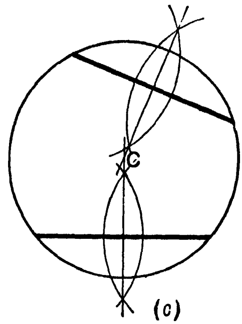


Fig 74

NOTE.—Here it is established that the three right bisectors of the sides of a triangle meet in a point equidistant from the three angular points of the triangle. A circle which will circumscribe the triangle can be described with this point as centre (the circum-centre).

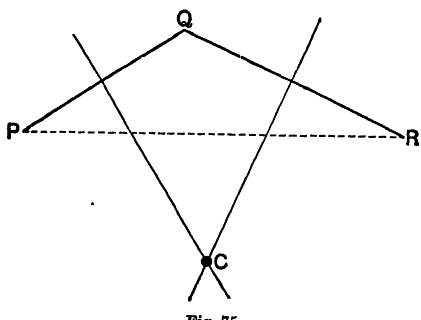


Fig. 75

(vi) Chords equidistant from the centre of a circle are equal, and equal chords in a circle are equidistant from the centre.

In the circle fig. 76 are two chords AB and A′B′. From C the centre perpendiculars CM and CM′ are drawn to the chords. M and M′ are therefore the mid-points of AB and A′B′. Join CA and CA′.

If CM and CM′ are equal, the right-angled triangles CMA and CM′A′ are equal in every respect (*vide* page 22). and therefore AB = A′B′.

Again, if the chords AB and A′B′ are equal the halves of the chords AM and A′M′ are equal, and therefore the right-angled triangles AMC and A′M′C are equal in every respect. So CM = CM′.

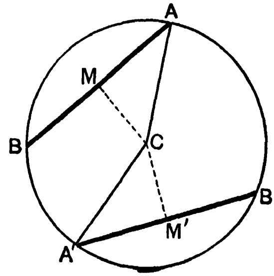


Fig. 76

NOTE.—The pupil should test the equality of the triangles (1) by the method of superposition, (2) by applying the relation between the hypotenuse and the sides of a right-angled triangle (page 92), by referring to Chap. VIII.

The rectangle contained by the segments of a chord through a fixed point within a circle.

In fig. 77 we have a circle of definite radius OA, and in that circle a fixed point P through which an unlimited number of chords may be drawn.

Consider AB one of the chords through P. Its segments are AP and PB. From O the centre of the circle a perpendicular OM is drawn to AB. From (iii) we learn that M is the mid-point of AB.

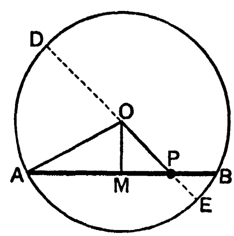


Fig. 77

Join OA and OP and draw the diameter DOPE.

Represent OA by r (radius of circle).

„ OP by d (distance of point from centre).

„ OM by p (distance of chord from centre).

With a fixed point in a given circle the values of r and d are fixed, but the value of p changes with the position of the chord.

Since AMO is a right-angled triangle,

$$\therefore AM^2 + p^2 = r^2. \quad \text{(p. 92.)}$$
$$\text{Similarly } MP^2 + p^2 = d^2.$$
$$\therefore AM^2 - MP^2 = r^2 - d^2 = \text{a constant.}$$
$$\therefore (AM + MP)(AM - MP) = \text{a constant.}$$
$$\therefore AP . PB = \text{a constant.}$$

Hence we have—

"If a chord in a circle passes through a fixed point in that circle, the rectangle contained by the segments into which the chord is divided at that point has a fixed value".

As this magnitude is, as we have just seen, quite independent of the direction in which the chord is drawn through the fixed point, it must be equal to the rectangle contained by the segments into which the diameter is divided at that point. The rectangle contained by the segments of the diameter $= DP . PE = (DO + OP)(OE - OP) = (r + d)(r - d) = r^2 - d^2$.

When the chord through P is in such a position that the diameter is its right bisector, as in fig. 78, M and P coincide, and so $DP . PE = AP^2$. Or thus—

$$AP^2 = AO^2 - OP^2 = r^2 - d^2 = (r + d)(r - d) = DP . PE.$$

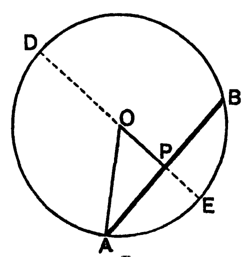


Fig. 78

The rectangle contained by the segments of a chord through a fixed point without the circle.

Take in a given circle (fig. 79) a chord AB passing through a fixed external point P. The segments into which AB is divided at P are AP and PB.

From centre O draw OM perpendicular to chord AB.

Represent OA by r (radius of circle).
" OP by d (distance of point from centre).
" OM by p (distance of chord from centre).

$$\text{We have } MP^2 + p^2 = d^2.$$
$$AM^2 + p^2 = r^2.$$
$$\therefore MP^2 - AM^2 = d^2 - r^2 = \text{a constant.}$$
$$\therefore (MP + AM)(MP - AM) = \text{a constant.}$$
$$\therefore AP \cdot PB = \text{a constant.}$$

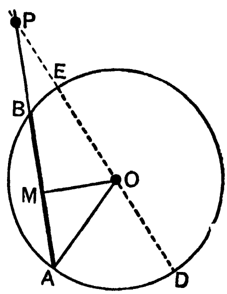


Fig. 79

Hence we have—

"The rectangle contained by the segments into which a chord in a given circle is divided externally by a fixed point is of fixed value".

Accordingly all chords which pass through P are divided so that the rectangles contained by their segments are of the same magnitude. One of these chords is the diameter DE, and so $DP \cdot PE = AP \cdot PB$.

Note that—

$$DP \cdot PE = (OP + DO)(OP - OE) = (d + r)(d - r)$$
$= d^2 - r^2$ (the constant to which $AP \cdot PB$ is equal).

Tangent to a circle.

If we suppose the line PBA (fig. 80) to rotate round P, passing in its motion through such positions as PB′A′ and PB″A″, we shall see that as it moves from the centre of the circle the points in which it cuts the circumference get nearer and nearer to one another until finally they coincide at T, when the line from P becomes a tangent to the circle.

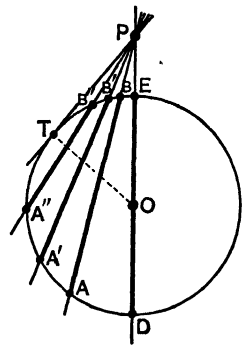


Fig. 80

Since for all positions of this line the rectangles contained by the segments are of the same magnitude, we have—

$$DP.PE = AP.PB = A'P.PB' = \ldots = PT.PT = PT^2.$$

$$\text{Now } DP.PE = d^2 - r^2,$$

$$\therefore PT^2 = d^2 - r^2 = OP^2 - OT^2,$$

and so the angle OTP is a right angle (p. 92). Hence we have—

"A straight line drawn at right angles to a diameter of a circle from its extremity is a tangent to the circle".

NOTE.—As only one perpendicular can be drawn to a given straight line from a given point, it follows that—

(i) The perpendicular to a tangent from the point of contact passes through the centre.

(ii) The perpendicular to a tangent from the centre passes through the point of contact.

(iii) If on one of two lines which issue from a point P a fixed length PT be taken, and on the other two segments PA and PB, such that $PT^2 = PA.PB$, then PT will be a tangent to the circle which passes through T, A, and B.

Concyclic Points.—We saw in (v) that a circle could be described so as to pass through any three points not in line. The two theorems just established give conditions under which a circle which passes through three points A, B, and C will also pass through a fourth point D.

Fig. 81 shows the circle which passes through A, B, and C. If that circle passes also through D, then the following relations hold—$AP.PC = DP.PB$, $AP'.P'B = DP'.P'C$, and $AP''.P''D = BP''.P''C$—where P, P′, and P″ are crossing points of the various joins of the four points A, B, C, and D.

It may easily be shown that if one or other of these

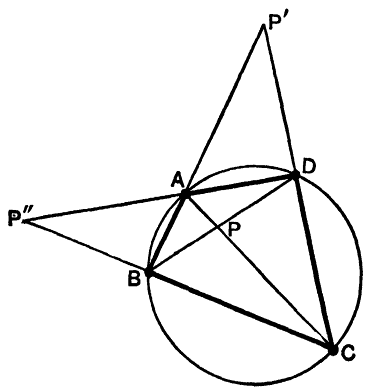


Fig. 81

relations holds, the points A, B, C, and D are concyclic. One proof has regard to what would follow if the circle which passes through A, B, and C did not also pass through D.

SUMMARY OF RESULTS

1. The right bisector of a chord of a circle passes through the centre.
2. A circle can be drawn through any three points not in line.
3. If through a fixed point within a given circle a chord is drawn, the rectangle contained by its segments is constant.
4. When the fixed point is without the circle, the rectangle con-

tained by the segments into which the chord is externally divided is constant.

5. The rectangle contained by the segments into which a chord is divided by a point without the circle is equal to the square on the tangent drawn from that point to the circle.

6. When the two points of intersection of a line and a circle coincide, the line is a tangent to the circle.

7. A straight line drawn at right angles to a diameter of a circle from its extremity is a tangent to the circle.

QUESTIONS

1. How would you prove that a straight line cannot cut a circle in more than two points?

2. How would you prove that a straight line drawn across a closed curve must cut it in an even number of points?

3. How would you apply the obvious theorem, "the two sides of a triangle are together greater than the third side", to prove that the diameter is the greatest chord in a circle?

4. What is meant by distance of chord from centre?

5. What is the locus of the foot of the perpendicular from the centre of a given circle to a chord of given length?

6. What is the locus of the mid-point of a chord of given length in a given circle?

7. How would you use the relation between the sides and the hypotenuse of a right-angled triangle to prove that of two unequal chords in a circle the longer is nearer the centre?

8. If p and p' are the respective distances of two chords from the centre of a circle of radius r, what is the ratio of the lengths of these chords?

9. How would you show that the shortest chord that can be drawn through a point in a circle is perpendicular to the diameter through that point?

10. If a chord in a circle of radius r passes through a given point at distance d from the centre of that circle, what is the magnitude of the rectangle contained by the segments into which the chord is divided by the given point? Consider various positions of this point. State the theorems arising from your results.

11. Of the straight lines that may be drawn from a point without a circle to its circumference, two, and only two, are tangents to the circle. What is meant by this?

12. How do you show that a tangent is at right angles to a radius?

13. How could you find the centre of a given circle by means of a straight-edge, a set-square, and a pencil? Give a full proof.

14. Two circles are concentric. How would you show that all tangents drawn to the inner circle from points on the outer circumference are of the same length?

EXERCISES

1. Draw straight lines across an irregular closed figure, and examine the number of points in the line that cut the boundary line of the figure.

2. Draw a circle of 5 cm. radius. Mark on circumference two points A and B 6 cm. apart. Draw a straight line through these points. Show (by measurements) that the distance from the centre of the circle to any point on that part of the line intercepted by A and B is less than the radius, and that the distance of any other point on the line, but not on the part between A and B, is greater than the radius.

3. In a circle of 5 cm. radius place a chord 6 cm. long. Join mid-point of this chord with centre. Show, using protractor, that this line is at right angles to the chord. Test also by measurement and calculation.

4. From centre of a circle of radius 2·6 cm. draw a perpendicular on a chord 2 cm. long. Measure the segments of the chord. What property of the circle does this illustrate? What result does a calculation give?

5. Draw several parallel chords in a circle. Bisect each independently of the others. Join the mid-points of each pair of chords. Derive a theorem.

6. Draw several parallel chords in a circle. Draw the right bisector of one of the chords. As this right bisector passes through the centre, and is perpendicular to each of the other chords, it bisects each of them. Test this by careful drawing. Show that a diameter of a circle is the locus of the mid-points of parallel chords in the circle.

7. Draw a circle of 5 cm. radius. Take a point P 4 cm. from the centre. Draw chords through P. Measure the lengths of the greatest and least chords, and also of one or two chords of intermediate length. Measure also the segments in which each chord

is divided at P. Calculate the area of the rectangle contained by the segments of each chord. Tabulate your results thus—

Chords.	Lengths in cm.	Segments in cm.	Rectangles in sq. cm.
A B	....	A P = ... P B = ...	....
A′B′	....	A′P = ... P B′ = ...	....
A″B″	....	A″P = ... P B″ = ...	....
⋮	⋮	⋮	⋮
D E	....	D P = ... P E = ...	....

8. In a circle of 5 cm. radius place a set of chords (say 20 or 30) each 8 cm. long. Note the result.

9. Draw two concentric circles of radius 5 cm. and 3 cm. respectively. Draw several (say 3 or 4) chords in the larger circle, each to be a tangent to the smaller. Measure the length of each chord. What theorem is suggested by the results?

10. Practise drawing tangents to a circle from points in the circumference. Test with protractor.

11. Draw a circle of 3·7 cm. radius. Place in that circle several chords 2·4 cm. long. Find the distance of the middle point of each from the centre. Obtain a theorem.

12. Practise drawing tangents to a circle from points without the circle. Test with protractor.

13. Take two points P and O 3·4″ apart. With O as centre, draw a circle of radius 1·6″. Draw and measure the length of a tangent from point P to the circle. Measure the lengths of the segments into which a chord passing through P is divided at P. Calculate the rectangle contained by these segments. Tabulate your results.

14. Draw two concentric circles, and measure the lengths of tangents to the inner circle from points on the circumference of the outer circle. Give the proof of the theorem relative to this result.

15. Take O the centre of two concentric circles. Join P, a point on the outer circle, to O, cutting inner circle at T. Through T draw a straight line at right angles to P O, cutting outer circle in P′ and P″. Join P′O and P″O, cutting inner circle in T′ and T″

respectively. Join PT′ and PT″. Prove that PT′ and PT″ are tangents to the circle.

16. Practise the method of 11 for drawing tangents to a circle from a given point without it. Test with protractor.

17. Mark a point P. From P draw a straight line PAB, making PA and PB 1″ and 2¼″ long respectively. Draw the right bisector of AB. Describe circles passing through A and B. Draw and measure the tangents from P to the several circles.

18. Draw two straight lines OA and OB, inclined at an angle of say 50°. Draw OC, the bisector of this angle. From any point in OC draw perpendiculars on OA and OB. Measure these perpendiculars. Prove that they are equal. Describe a set of circles which will touch both OA and OB.

19. Draw OA and OB, as in 18, and describe a circle of radius 1″ which will touch both lines.

20. Draw OA and OB, as in 18. Mark on OA a point P, 3 cm. distant from O. Draw a circle to touch both lines and pass through P.

21. Mark on OA, one of the two lines in 18, two points P and Q so that OP = 3 cm. and OQ = 5 cm. Draw a circle which will pass through P and Q and touch OB. [Note that the square on the tangent from O to any circle passing through P and Q is equal to the rectangle OP.OQ. Therefore cut off from OB a part OR equal to the tangent from O to any circle through P and Q, and describe a circle through P, Q, R (see page 105).]

22. In a given straight line find a point such that the straight line drawn from it to a given circle shall be of a given length. [Note that if the straight line and circle are given in position, the length of the line to which the tangent is to be equal may be as large as you please, but cannot be less than a certain length.]

23. Through a given point draw a straight line which shall cut a given circle so that the chord intercepted shall be a given length.

24. Describe a circle which shall pass through a given point and touch a given straight line at a given point.

25. Describe a circle which shall have its centre in a given straight line, pass through a given point in that line, and touch another straight line given in position.

26. Describe the circles which pass through a given point and touch two given straight lines.

27. Describe a circle touching a given straight line in a given point, and cutting off from a second line a chord of given length.

28. Draw two circles intersecting in A and B. Prove that the common chord is at right angles to the line joining the centres.

29. Draw several figures, showing the two circles in A and B, with centres farther and farther apart. [Note that the common chord becomes less and less, but is always at right angles to the line of centres.]

30. Draw a figure, showing the coincidence of A and B. [Here the two circles touch, and the common tangent is at right angles to the line of centres.]

31. Describe a circle passing through two given points and touching a given circle.

CHAPTER XIII

ANGLES IN SEGMENTS OF CIRCLES

Definition: A **segment** of a circle is the figure bounded by an arc and the chord joining its ends.

Every diameter divides the circle into two exactly equal portions, or semicircles, while every chord not a diameter divides the circle into two unequal segments and the circumference into two unequal arcs.

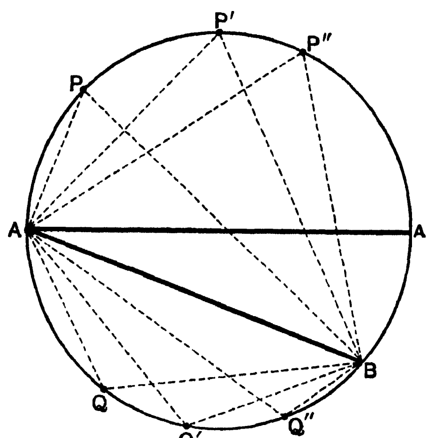


Fig. 82

Thus in fig. 82 the chord A B divides the circle into two segments A P B and A Q B, the former of which is greater than a semicircle and the latter less than a semicircle.

The chord A B subtends the angles A P B, A P′ B, A P″ B, &c. at the points P, P′, P″, &c. on the major arc, and subtends the angles A Q B, A Q′ B, A Q″ B, &c. at the points Q, Q′, Q″, &c. on the minor arc.

The angles A P B, A P′ B, A P″ B, &c. are said to be angles in the major segment A P B and to stand on the arc A Q B, whilst a like remark applies to the angles A Q B, A Q′ B, A Q″ B, &c. The protractor may here be used (as in Chapter III) to show by actual measurement that all angles (such as A P B, A P′ B, A P″ B, &c., fig. 82) in the same segment of a circle, or standing on the same arc of a circle, are equal.

This important result may easily be reached from the following elementary considerations.

We have seen (Chapter VI) that if in an isosceles triangle (fig. 83) either of the equal sides be produced through the vertex, the exterior angle so formed is equal to the sum of the base angles, and is therefore twice as large as either of these angles.

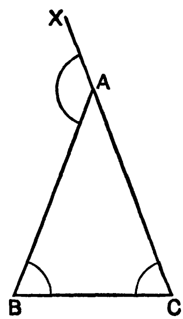


Fig. 83

Take now as in fig. 84 three segments of a circle—that in (1) being less than a semicircle, that in (2) being equal to a semicircle, and those in (3*a*) and (3*b*) being greater than a semicircle.

In each case a point P is taken on the arc of the segment and joined to the ends A and B of the chord, thus forming the angle A P B. It is desired to show that in each segment the angle A P B has a constant value.

Join the three points A, B, and P to O, the centre of the circle, and produce P O to D.

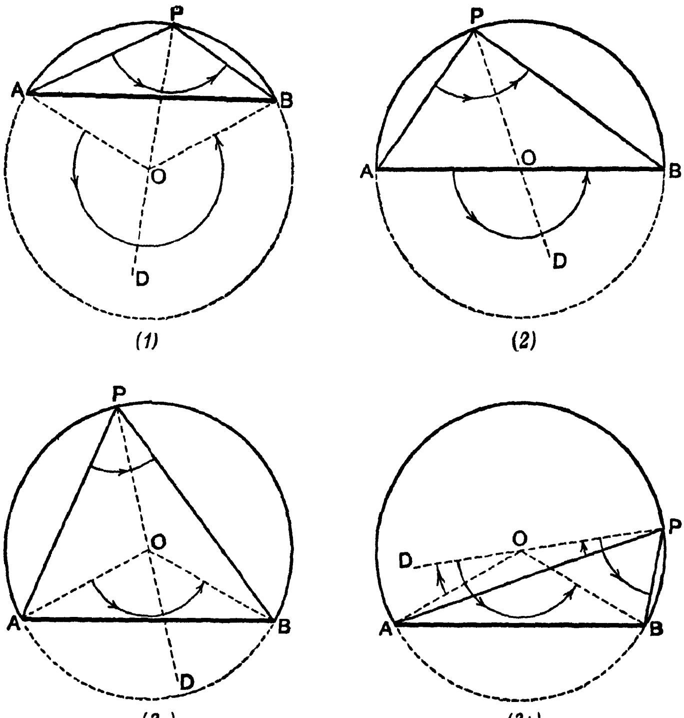


Fig. 84

The point O will lie

outside the segment in (1),
on the diameter AB in (2),
within the segment and within the angle in (3*a*),
within the segment and outside the angle in (3*b*);

but the reasoning is the same in all cases. For

the angle APO is equal to half the angle AOD,
and the angle OPB is equal to half the angle DOB.
Therefore the angle APB is equal to half the sum of the angles AOD and DOB.

Now the sum of these angles is in each case independent of the position of P on the arc A P B. Hence the magnitude of the angle A P B is independent of the position of P on the arc; that is, the angle A P B is constant.

[It is to be noted that in figure (3*b*), as in the other figures, if we look at the angles from the point of view of rotation, the angle A P B is the algebraic *sum* of the angles D P B and A P D, whilst in like manner the angle A O B (as marked) is the algebraic *sum* of the angles D O B and A O D.]

Hence in all cases—

(i) "The angles in the same segment of **a circle are** equal to one another".

Incidentally we see also that—

(ii) "The angle at the centre is double of the angle at the circumference standing on the same arc of the circumference".

Thus in all the cases of fig. 84 the angle A O B (as marked) is equal to twice the angle A P B.

Referring again to the same figure, it is clear that in—

(1) the sum of the angles A O D and D O B is greater than two right angles;
(2) the sum of the angles A O D and D O B is equal to two right angles;
(3*a*) and (3*b*) the sum of the angles A O D and D O B is less than two right angles.

Therefore in—

(1) the angle A P B is greater than one right angle;
(2) the angle A P B is equal to one right angle;
(3*a*) and (3*b*) the angle A P B is less than one right angle.

Hence we conclude that—

(iii) "The angle in a semicircle is a right angle; that in a segment greater than a semicircle is acute; that in a segment less than a semicircle is obtuse".

Now let two chords of a circle A B and C D intersect in P (fig. 85) inside the circle, as in (*a*), or outside the circle, as in (*b*).

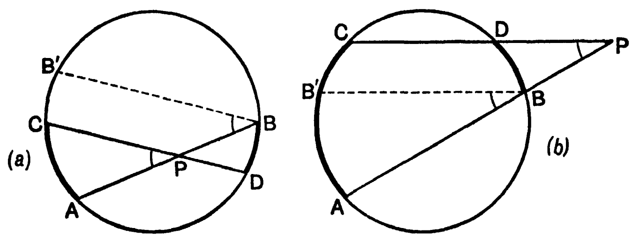


Fig. 85

In each case draw the chord B B′ parallel to C D. It follows that the arc B D is equal to the arc B′ C.

Then the angle A P C = the angle A B B′
= the circumferential angle standing on the arc A B′
= the circumferential angle standing on an arc which is equal to the *sum* of the arcs intercepted by the chords in (*a*); and to the difference of the arcs intercepted by the chords in (*b*).

Hence we have the following useful principle—

(iv) "If two chords of a circle intersect one another, the angle between them is equal to the angle at the circumference standing on an arc equal to the sum of the two arcs subtended by the angle considered when the chords meet within the circle, and equal to the difference of those arcs when the chords meet outside the circle".

Cyclic Quadrilateral.—If a circle can be drawn through the four angular points of a quadrilateral, these four points are said to be concyclic and the rectilineal figure is said to be a cyclic quadrilateral.

Let A B C D (fig. 86) be a quadrilateral inscribed in a circle whose centre is O. Join O to two opposite vertices A and C of the quadrilateral.

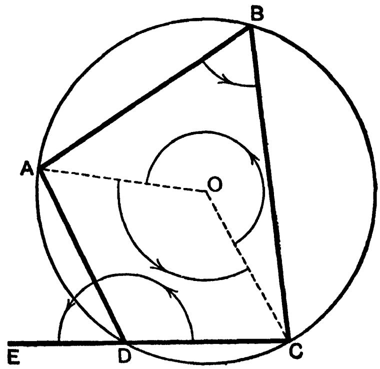


Fig. 86

Then, considering the angles from the point of view of rotation as shown by the arrow-heads,

the angle ABC = half the angle AOC, and the angle CDA = half the angle COA.

Therefore the sum of the angles A B C and C D A is equal to half the sum of the angles A O C and C O A; that is, is equal to half a revolution.

Thus—

(v) "The sum of two opposite angles of a quadrilateral inscribed in a circle is equal to two right angles".

Now produce the side C D to E (fig. 86).

The sum of the angles A D E and C D A is equal to two right angles, and the sum of the angles A B C and C D A is equal to two right angles; therefore the angle A D E is equal to the angle A B C.

That is—

(vi) "If a side of a cyclic quadrilateral be produced, the exterior angle is equal to the interior and opposite angle".

If now we consider the points A, B, and C as fixed, and let D move along the arc until finally it coincides with C,

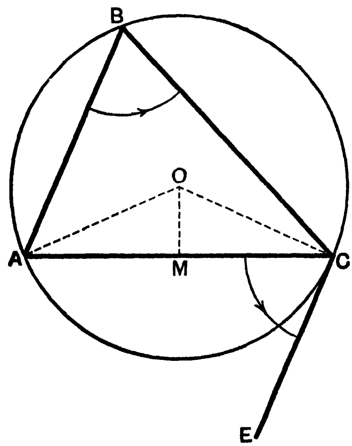


Fig 87

we have in this position the line E C a tangent to the circle at C (fig. 87).

But as the angle A B C has remained unaltered during this movement of D towards C, so also the angle A D E has remained unaltered, and in the final position when A D coincides with A C this angle becomes the angle between a chord A C and the tangent at its end, while the angle A B C becomes the angle subtended by the chord in the alternate segment.

Thus—

(vii) "If a straight line touch a circle and a chord be drawn from the point of contact, the angle which this chord makes with the tangent is equal to the angle in the alternate segment".

As an application of this theorem let us construct a triangle on a given base A C (fig. 88), and having a vertical angle equal to a given angle.

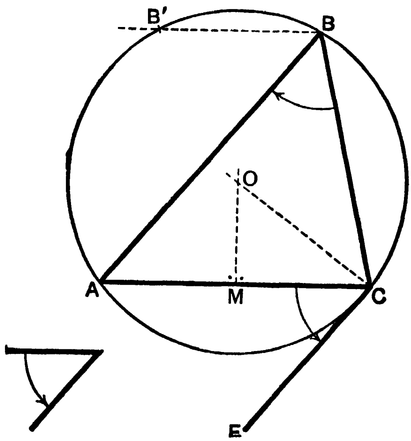


Fig. 88

From C draw C E so that A C E shall be equal to the given angle.

Let the right bisector of A C meet at O a perpendicular through C to C E. A circle described with O as centre and O C as radius will pass through A and C and will touch the line C E at C.

The triangle formed by joining any point B on the arc of the alternate segment A B C will have its vertical angle A B C equal to the given angle.

This triangle may be made to satisfy a further condition. Thus, if it is to have a definite area, since its base is fixed so also is its perpendicular height, and thus the points where a parallel to A C at a distance from it equal to this height cuts the circumference will when joined to A and C form a pair of triangles satisfying the additional condition.

It is evident that the perpendicular height given may be such that there is only one solution (or even none) possible.

SUMMARY

(i) The angles in the same segment of a circle are equal.

(ii) The angle at the centre of a circle is double of the angle at the circumference standing on the same arc of the circumference.

(iii) The angle in a semicircle is a right angle; that in a segment greater than a semicircle is acute; that in a segment less than a semicircle is obtuse.

(iv) If two chords of a circle intersect one another, the angle between them is equal to the angle at the circumference standing on an arc equal to the sum of the two arcs subtended by the angle considered when the chords meet within the circle, and equal to the difference of those arcs when the chords meet without the circle.

(v) The sum of the opposite angles of a cyclic quadrilateral is equal to two right angles.

(vi) If a side of a cyclic quadrilateral be produced, the exterior angle is equal to the interior and remote angle.

(vii) If a straight line touch a circle and if a chord be drawn from the point of contact, the angles made by this chord with the tangent are equal respectively to the angles in the alternate segments.

EXERCISES

1. Two circles intersect in X and Y, and a line P X Q is drawn through X to meet the circles in P and Q. Prove that the angle P Y Q is constant.

2. Show also that if a second line P′ X Q′ be drawn, as in Ex. 1, then P P′ and Q Q′ on being drawn and produced to meet will cut at a constant angle.

3. If two circles touch externally in X and two straight lines PXQ and P′XQ′ be drawn to meet the circles again, then, if PQ and P′Q′ be drawn, these two straight lines are parallel.

4. Write down the values of the angles at the circumference standing on an arc equal to (*a*) half the circumference, (*b*) one-third the circumference, (*c*) a quadrant, (*d*) one-fifth, (*e*) 1-*n*th the circumference.

5. A triangle moves in its plane so that two of its sides always pass through two fixed points. Find the locus of the vertex (the intersection of these two sides).

6. Through a given point P chords are drawn to a fixed circle whose centre is O. Find the locus of the mid-points of the chords.

7. The radius of one circle is a diameter of another. Prove that any chord of the greater drawn through the point of contact is bisected by the less.

8. If the four angles of any quadrilateral be bisected internally, prove that the four bisectors form on production a cyclic quadrilateral.

9. Two equal circles cut in X and Y, and a straight line PXQ is drawn through X and terminated by the circumferences; show that PY and QY are equal.

10. AB and CD are parallel chords of a circle; the tangent at B meets CD produced in T. Prove that the triangles ABD and BDT are equiangular.

11. If two equal circles cut in X and Y, and if a circle with X as centre and XY as radius cut the circles again in P and Q, then each of the lines PY and QY is a tangent to one of the circles.

12. If a straight line be drawn to cut two circles which touch externally and if the four points of crossing be joined to the point of contact, the angle between the two middle joins is supplementary to that between the extremes.

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