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HIGHER ELEMENTARY
GEOMETRY

(AN INTRODUCTION)

For

PRE-UNIVERSITY STUDENTS

VENUGOPAL

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PREFACE.

This geometry text is written for the Pre-university students and also to serve as an introduction to Higher Elementary Geometry for the three year degree students offering Geometry either as a subsidiary or as a main subject. I believe that the elementary is the most vital and the object of writing this book is to give to the students certain Elementary ideas about the subject. Endeavours have been made to make everything as simple as possible without losing rigour.

A good number of worked examples are given to illustrate the theorems. As there are only a few theorems all exercises are given together at the end. Students can try these riders after they finish reading the entire text.

I hope that this text book will lay down a good foundation for students who intend to pursue their studies in geometry. It is my sincere wish that students must acquire that taste for Geometry without which the Greeks thought—and rightly, in my opinion,—that there is no real culture. If the book is found to be suitable for the class of students for whom it is intended the author will feel amply rewarded.

For any corrections and suggestions for the improvement of the book, I shall be thankfull.

VALSARAJ VILLA,
ILAKKOOL, TELLICHERRY.
August 15, 1958.

K. C. VENUGOPAL.

NOTE.

(As per Government Regulations)

This is neither an official nor an officially-sponsored Publication. The author is working in Government Brennen College, Tellicherry, as Tutor in Mathematics.

CONTENTS.

HIGHER ELEMENTARY GEOMETRY

(AN INTRODUCTION)

Chapter 1

PLANE FIGURES.

Plane figures are figures lying in a plane and not in a curved surface. Every plane figure with which we deal in Geometry is only an aggregate of points, though the point in itself is undefined. However a vague definition of a point can be given. A point is a geometrical entity which has position and nothing else. From points we come to curves. A curve is a collection of infinite (a very large) number of points arranged closely one after the other. We easily see that this definition of a curve also applies to circles and straight lines. So we conclude that circles and straight lines are particular cases of curves. In order to get a circle the points must be arranged in a particular order. If all the points taken are equidistant from a fixed point, we get a circle. Similarly to get a straight line the points must be arranged in another particular order. But this particular order cannot be easily described, though we are sure that a straight line can be obtained by arranging a number of points in a particular order. As a matter of fact straight line is only a special case of a circle. If we take a piece of wire of a definite length in the form of an arc of a circle and also a number of such pieces having the same shape and size it will be possible for us to get a complete circle of which each such piece is a part, by placing the pieces suitably on a table. This is the principle adopted by masons in shaping the stones for the purpose of erecting the round walls of a well. In fact, the circle mentioned above can be generated by one piece of such wire, shifting it from place to place in a particular order, till the complete circle is obtained. Now, by decreasing the curvature (amount of bending) of this piece (i. e., by increasing the length of the bounding chord of the arc. formed out of this piece of wire) we can generate a circle of radius greater than that of the previous one. Thus. as the curvature decreases, radius of the circle generated increases. Hence a circle of infinite radius can be generated with the help of the same piece of wire, by making its curvature zero i. e. by making it straight (a straight line). So this time the piece of the wire in the form of a straight line is a part of a very great circle. This shows that a straight line is a circle of infinite radius. If a mason uses ordinary stones which are used for ordinary buildings, he will be actually erecting the round wall of a well of infinite radius. In this case, if he starts construction from a particular point, he won't be able to come back to the original position, completing the construction.

It is because a straight line is a particular case of a circle, we find certain properties, common to circles and straight lines. For example, the student can compare Apollonius' theorem mentioned elsewhere in this text, and the elementary theorem that the locus of a point equidistant from two fixed points A, B is the perpendicular bisector of AB.

If we have a piece of wire in the form of any curve, at our disposal, we can make it either in the form of a circle or in the form of a straight line according as we wish. This itself is sufficient to show that circles and straight lines are both particular cases of curves. A curve may be a closed curve (like the circle) or may not be a closed one. If it is a closed one we can talk of its area. The area of a plane closed curve is the superficial space whose boundary is the closed curve. So we can talk of the area of a circle, for it is a closed curve.

In Elementary Geometry we deal only with the two particular cases of curves, viz., circle and straight line and the compound figures formed out of these two independently or together. Straight lines independently form what is known as Polygons. We shall discuss the different types of polygons in the following paragraphs.

Polygon:— A polygon is a geometrical figure formed by any number of straight lines. The point of intersection of two adjacent straight lines forming a polygon is called a vertex of the polygon. The segment, between two consecutive vertices, of a straight line forming a polygon is called a side. Line joining any two vertices other than consecutive vertices is called a diagonal of the polygon. The area af a polygon is the superficial space whose boundaries are the sides of the polygon.

A polygon in which each angle is less than two right angles is called a convex polygon. In a polygon if any angle is greater than two right angles it is called a re-entrant polygon.

A polygon in which all the sides are equal is called an equilateral polygon. If all the angles are equal in a polygon it is called an equiangular polygon. A regular polygon is one in which both these conditions are satisfied, viz, sides are equal and angles are equal.

Two or more polygons are said to be similar if (i) their angles are equal and (2) their corresponding sides are proportional. Two or more polygons are said to be similar and similarly situated or homothetic if (1) their angles are equal, (2) corresponding sides are proportional and (3) corresponding sides are parallel.

Polygon of sides 10, 9, 8, 7, 6, 5 are respectively called Decagon, Nonagon, Octagon, Heptagon, Hexagon and Pentagon. If these are also regular, they will be called Regular Decagon Regular Nonagon etc.

A polygon of four sides is known as a quadrilateral. If one pair of opposite sides of a quadrilateral are parallel it is called a Trapezium. If the two pairs of opposite sides are parallel, the quadrilateral is called a parallelogram. If all the sides of a parallelogram are equal it is called a Rhombus. If all the angles of a parallelogram are equal (right angles) it is called a rectangle. If all the sides of a rectangle are equal it is called a square. It is easy to see that Rhombus and rectangle are special cases of a parallelogram and that square is a special special case of a parallelogram.

A polygon of three sides is called a Triangle. From the definition of equilateral polygons, equiangular polygons etc. it follows that a triangle is equilateral if its sides are equal, equiangular if its angles are equal etc. (Note that equilateral triangles are equiangular and that equiangular triangles are equilateral). If two sides of a triangle are equal (or if two angles of a triangle are equal) it is known as an isosceles triangle. A triangle is said to be (1) acute angled if each angle is less than a right angle (2) right angled if one of the angles is a right angle (3) obtuse angled if one of the angles is greater than a right angle.

The side facing the right angle is called the hypotenuse of the right angled triangle. Hypotenuse is the unique feature of a right angled triangle and so if mention is made of a hypotenuse of a triangle it will follow that the triangle in question is a right angled triangle, the angle opposite to the hypotenuse being necessarily a right angle.

Two or more Geometrical figures are said to be congruent if they agree in shape as well as in size. Hence if there are two congruent figures one can be completely superposd on the other. Students will note that two similar figures agree in shape only. So if there are two similar figures one cannot be superposed on the other. But two similar figures can be placed such that their corresponding sides are parallel, if they are not already so, by rotating one of them. Then they become similar and similarly situated figures. Therefore if the positions of two similar figures are not given, they can always be made homothetic or similar and similarly situated. But the student must realise that in Geometry position is also very often important and that it is only the position that draws a line of demarcation between similar figures and homothetic figures.

As an exercise, students are advised to draw all Geometrical figures one after the other, strictly following the definition of each given above.

Chapter 2

RATIO AND PROPORTION.

Ratio:— Ratio is merely a relation between two quantities of the same kind, showing how many times one quantity is greater than the other, both of them being measured in the same unit of measurement.

If A B and C D are two lengths, A B being equal to 4 inches and C D equal to 2 inches the ratio of A B to C D (usually written as $\frac{AB}{CD}$) is $\frac{4}{2}$ or 4:2 i e the ratio is $\frac{2}{1}$ or 2:1. This shows that A B is two times greater than C D.

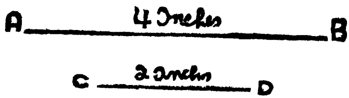


Fig. 1 (a)

In general if the length A B is a inches and the length C D is b inches the ratio of A B to C D is $\frac{a}{b}$ or a:b. This relation shows that A B is $\left(\frac{a}{b}\right)$ times greater than CD. for $a = \left(\frac{a}{b}\right) b$ and hence $AB = \left(\frac{a}{b}\right) CD$. Here a and b are called the terms of the ratio $\frac{a}{b}$.

If a and b are both multiples of the same quantity, say, k so that $a = pk$ and $b = qk$, the ratio of A B to C D $\left(\frac{AB}{CD}\right)$ becomes $\frac{a}{b} = \frac{pk}{qk} = \frac{p}{q}$. So the ratio of A B to C D is $\frac{p}{q}$. p and q are called the terms of the ratio $\frac{p}{q}$. From this it is clear that it is always customary to express a ratio in its simplest form.

If A B is a inches aud C D is b centimeters the ratio of A B to C D is not $\frac{a}{b}$ for by definition the lengths must be measured in the same unit of measurement.

Also, if A B is a length equal to a inches and P Q R is a triangle whose area is equal to b square inches we cannot talk of the ratio between A B and $\triangle$ P Q R, since by definition for the existence of a ratio the two quantities must be of the same kind.

Proportion:— If two ratios are equal the four terms taken in order are called proportionals and are said to be in proportion.

If $\frac{a}{b} = \frac{c}{d}$ a, b, c, d are proportionals. The proportion is written as a : b : : c : d and is read "a is to b as c is to d" Here b and c are called the means and a and d are called extremes of the proportion. d is called the fourth proportional to a, b and c.

If a, b, c are connected by the relation $\frac{a}{b} = \frac{b}{c}$ [$b^2 = ac$] b is called the mean proportional or geometric mean between a and c and c is called the third proportional to a and b. Also a, b, c in this case are said to be in continued proportion.

If a, b, c, d are connected by the relation $\frac{a}{b}=\frac{b}{c}=\frac{c}{d}$ then a, b, c, d are said to be in continued proportion and so on.

SIMPLE RESULTS IN RATIO AND PROPORTION.

(1) If $\frac{a}{b}=\frac{c}{d}$ each ratio is equal to $\frac{a+c}{b+d}$ Let $\frac{a}{b}=\frac{c}{d}=k$ so that $a=bk$ and $c=dk$ then to prove that $\frac{a+c}{b+d}$ is also equal to k.

$$a=bk$$
$$c=dk$$

adding $a+c=k(b+d)$

$$\therefore \quad \frac{a+c}{b+d}=k=\frac{a}{b}=\frac{c}{d}$$

In the following results also, the method of proof will be the same and therefore the proofs' are left to the students.

(2) If $\frac{a}{b}=\frac{c}{d}$ each ratio is equal to $\frac{a-c}{b-d}$

(3) If $\frac{a}{b} = \frac{c}{d}$ each ratio is equal to $\sqrt{\frac{ac}{bd}}$.

(4) If $\frac{a}{b} = \frac{c}{d}$ each ratio is equal to $\sqrt{\frac{a^2 + c^2}{b^2 + d^2}}$.

(5) If $\frac{a}{b} = \frac{c}{d}$, $\frac{a}{c} = \frac{b}{d}$ (Alternando)

for $\frac{a}{b} \times \frac{b}{c} = \frac{c}{d} \times \frac{b}{c}$ i. e. $\frac{a}{c} = \frac{b}{d}$.

(6) If $\frac{a}{b} = \frac{c}{d}$, $\frac{b}{a} = \frac{d}{c}$ (Invertendo)

for $\frac{1}{\frac{a}{b}} = \frac{1}{\frac{c}{d}}$ i. e. $\frac{b}{a} = \frac{d}{c}$

(7) If $\frac{a}{b} = \frac{c}{d}$, $\frac{a+b}{b} = \frac{c+d}{d}$ (Componendo)

for $\frac{a}{b} + 1 = \frac{c}{d} + 1$ i. e. $\frac{a+b}{b} = \frac{c+d}{d}$

(8) If $\frac{a}{b} = \frac{c}{d}$, $\frac{a-b}{b} = \frac{c-d}{d}$ (Dividendo)

for $\frac{a}{b} - 1 = \frac{c}{d} - 1$ i. e. $\frac{a-b}{b} = \frac{c-d}{d}$.

(9) If $\frac{a}{b} = \frac{c}{d}$, $\frac{a+b}{a-b} = \frac{c+d}{c-d}$ (Componendo et Dividendo)

for $\frac{\frac{a}{b}+1}{\frac{a}{b}-1} = \frac{\frac{c}{d}+1}{\frac{c}{d}-1}$ i. e. $\frac{a+b}{a-b} = \frac{c+d}{c-d}$

All these results though simple are very important. Hence the students must always bear these results in their mind.

Definition:— $\frac{a^2}{b^2}$ is called the duplicate ratio of $\frac{a}{b}$; $\frac{a^3}{b^3}$ is called the triplicate ratio of $\frac{a}{b}$; $\frac{\sqrt{a}}{\sqrt{b}}$ is called the sub-duplicate ratio of $\frac{a}{b}$.

POINTS OF DIVISION

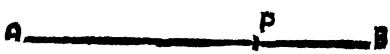


fig. 1 (b)

It AB is a straight line and P any point on it (whether between A and B as shown in the figure, on A B produced or on B A produced) then $\frac{AP}{PB}$ is the ratio in which the point P divides the straight line AB. Though the segments A B and B A are equal in magnitude they are opposite in sign For, the direction A B (i e. the direction from left to right) is taken as positive and the direction B A (consequently) as negative. Thus $AB \neq BA$, but $AB = -BA$ or $AB + BA = 0$ Again the segments A P and P B by our convention are positive and, B P and P A negative. If the point P lies within A and B on the straight line A B by our convention A P and P B are both positive and hence the ratio $\frac{AP}{PB}$ is positive. If P lies without A and B i. e. on A B or B A produced either A P or P B will be negative and hence the ratio $\frac{AP}{PB}$ will be negative. When the ratio $\frac{AP}{PB}$ is positive (i. e. when P lies within A and B) the point P is said to divide A B internally. In this case P is called the internal point of division. If the ratio $\frac{AP}{PB}$ is negative (i. e. when the point P lies without A and B) the point P is said to divide A B externally. Here P is called the external point of division. Hence the sign of a ratio will decide whether the point of division in question is an internal or external point of division.

Note 1:— The ratio in which the point P divides the straight line B A is $\frac{BP}{PA}$ wherever may be the position of the point P on the straight line (whether between A and B, on A B produced or on B A produced)

Note 2: — If there are two points P and P^1, P lying within A and B and P^1 lying without A and B (either on A B produced or on B A produced) $\frac{AP}{PB}$

is the ratio in which P divides A B (internally) and $\frac{AP^1}{P^1B}$ is the ratio in which P^1 divides A B (externally) It is evident that $\frac{AP}{PB}$ is positive and that $\frac{AP^1}{P^1B}$ as such is negative. Hence $\frac{AP}{PB}$ can never be equal to $\frac{AP^1}{P^1B}$ (for a positive quantity can never be equal to a negative quantity). But it may happen that $\frac{AP}{PB} = \left(-\frac{AP^1}{P^1B}\right)$ which is positive

$$\text{i. e} \quad \frac{AP}{PB} = \frac{AP^1}{-P^1B} = \frac{AP^1}{BP^1}$$

In this case i. e. when a point P divides a straight line A B internally and a point P^1 divides the same straight line externally in the same ratio, P and P^1 are said to divide A B harmonically. Also ($APBP^1$) is called a harmonic range. The student must note that if P and P^1 divide AB harmonically the internal ratio $\frac{AP}{PB}$ and the external ratio $\frac{AP^1}{P^1B}$ are only equal in magnitude and not in sign. In sign also if they are equal P and P^1 will coincide. This will be the truth of our first theorem.

Note 3:— Approximate position of the external point of division when the internal point of division is given:

If P, the internal point of division lies between O and B, where O is the middle point of line A B, P^1, the external point of division (or the harmonic conjugate of P, as it is usually called) will lie on A B produced. For, $\frac{AP}{PB}$ in this case is clearly greater than one and $\frac{AP^1}{BP^1}$ as long as P^1 lies on B A produced will be less than one; Hence there won't be any possibility of $\frac{AP}{PB}$ and $\frac{AP^1}{BP^1}$ becoming equal if P^1 lies on B A produced. So we conclude that P^1 must be on A B produced. Conversely if the external point P^1 lies on A B produced, P the internal point of division (or the harmonic conjugate of P^1 as it is usually called) must lie between O and B where O is the middle point of A B.

Similarly it can be shown that if P the internal point of division lies between A and O where O is the middle point of A B, P^1 its harmonic conjugate must lie on B A produced. Conversely if P^1, the external point of division lies on B A produced, P its harmonic conjugate will lie between A and O where O is the middle point of A B.

Note 4:— If the harmonic conjugate of O, the middle point of A B is represented by O^1, O^1 must be a point either on AB produced or on BA produced such that $\frac{AO^1}{BO^1} = 1$ i e. $AO^1 = BO^1$ Hence O^1 must be a point either on A B produced or on BA produced such that the distance AB becomes negligible when compared to the distance of O^1 from A and B i e. the distance of O^1 from A and B must be sufficiently great or in other words O^1 must be the point at infinity on the line A B.

In chapter 1 it has been already shown that a straight line is a circle of infinite radius. It is also easy to see that the tangent at any point to a straight line is itself. Hence the perpendicular erected at any point on a given straight line to itself is a radius of the straight line. Therefore the centre of the straight line (regarded as a circle of infinite radius) must be the point at infinity in a direction perpendicular to the straight line. Since the centre itself is at infinity, the other end of the diameter through the foot of any perpendicular to the given line, will also be at infinity in the same direction. So any straight line in a plane passes through the point at infinity in a direction perpendicular to the line or in other words a straight line is a circle passing through the point at infinity. Since this is so, we note that there is only one point at infinity on a straight line. Hence the question, whether O^1, the harmonic conjugate of O should be on A B produced or on B A produced does not arise. The two ends of the straight line A B when indefinitely produced will be coming to the point O^1. So we simply say that the harmonic conjugate of O, the middle point of A B is the point at infinity on the line A B.

Students will get some more ideas about the internal and external points of division when they study inverse points with respect to a circle under properties of circles.

THEOREM 1.

A straight line cannot be divided in the same ratio in more than one point (either internally or externally.)

Fig (2)

Let P be a point on A B dividing A B internally in the ratio $\frac{l}{k}$. Then it is evident that P must lic within A and B. Since P is a point on

A B that too within A and B, $\frac{AP}{PB}$ is the ratio in which the point P divides the straight line AB internally. But this ratio is given to be $\frac{l}{k}$

$$\therefore \quad \frac{AP}{PB} = \frac{l}{k}$$

If there is any other point dividing A B internally in the same ratio $\frac{l}{k}$, let it be P^1 Then

$$\frac{AP^1}{P^1B} = \frac{l}{k} \text{ but } \frac{AP}{PB} = \frac{l}{k}.$$

$$\therefore \quad \frac{AP}{PB} = \frac{AP^1}{P^1B}$$ Adding one to both sides and cancelling A B in the numerators of the two resulting ratios, we get,

$$\frac{1}{PB} = \frac{1}{P^1B} \text{ i. e. } PB = P^1B \text{ or } BP = BP^1.$$

$\therefore P^1$ coincides with P. ($\because$ Both of them being internal points of division lie within A and B) i. e. there is only one point dividing AB internally in the ratio $\frac{l}{k}$.

Similarly there is only one point dividing A B externally in the ratio $l:k$. Hence the theorem.

THEOREM 2.

A straight line drawn parallel to one side of a triangle cuts the other two sides or those sides produced proportionally. Let MN be parallel to the side BC of △ ABC cutting AB at M and AC at N. Divide AM into p equal parts and MB into q such equal parts. Draw parallels to BC through these points of division. Then AN will be divided into p equal parts and NC will be divided into q such equal parts (by a theorem)

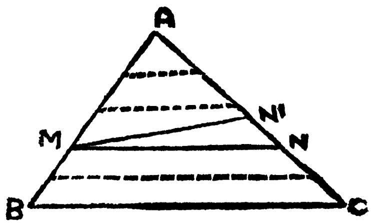


Fig 3.

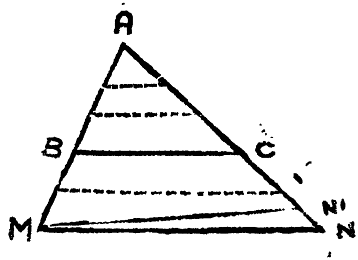


Fig. 4

$$\therefore \quad \frac{AN}{NC} = \frac{p}{q} \text{ But } \frac{AM}{MB} = \frac{p}{q} \text{ (construction)}$$

$$\therefore \quad \frac{AN}{NC} = \frac{AM}{MB} \text{ Hence the theorem.}$$

Conversely:— If a straight line cuts two sides of a triangle (both internally or both externally) in the same ratio it is parallel to the third side.

Let MN be a st line cutting the sides AB, AC of $\triangle$ ABC at M and N respectively such that $\frac{AM}{MB} = \frac{AN}{NC}$ Required to prove that MN || BC.

If MN is not parallel to BC let a parallel to BC be drawn through M cutting AC at N^1 then by the previous theorem, $\frac{AM}{MB} = \frac{AN^1}{N^1C}$ But by hypothesis $\frac{AM}{MB} = \frac{AN}{NC}$ $\therefore$ $\frac{AN}{NC} = \frac{AN^1}{N^1C}$ i. e. N and N^1 divide AC (both internally or both externally) in the same ratio which is impossible by theorem 1. $\therefore$ N^1 must coincide with N. i. e. MN is parallel to BC.

CONSTRUCTION.

1. Divide a straight line A B in the ratio $l : k$ internally. Take any line through A and mark A Q = l and QP = k along that line in the same direction. Join P, B. Then draw Q C parallel to P B to meet A B at C.

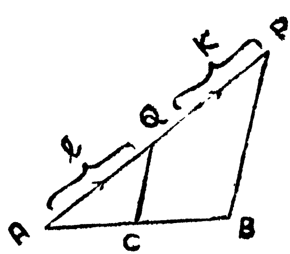


fig. 5.

Then C will be the required point. For, since QC || PB, $\frac{AC}{CB} = \frac{AQ}{QP}$ (By theorem 2), where AQ is exactly equal to l and QP exactly equal to k (by construction)

2. Divide a straight line AB externally in the ratio l : k.

A slight modification is necessary in this case.

Instead of taking QP in the same direction as AQ take QP in the opposite direction and proceed as before.

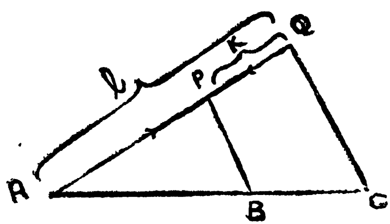


Fig. 6

Since QC || PB

$$\frac{AC}{CB} = \frac{AQ}{QP} = \frac{l}{-k} = -\frac{l}{k}$$

(By theorem 2 as before)

Thus C divides AB externally in the ratio $\frac{l}{k}$. When the word 'externally' is removed, we will have to say that C divides AB in the ratio $-\frac{l}{k}$. The negative sign indicates that the point of division is external.

3. To find a fourth proportional to three given lengths a, b, c.

Take two straight lines A B C and A P Q intersecting at A at any angle. on A B C, step off a length AB = a and BC = b. Along APQ measure a length AP = c. Join B, P. Draw a line through C parallel to BP to meet APQ at Q. Then P Q is the fourth proportional to a, b, c.

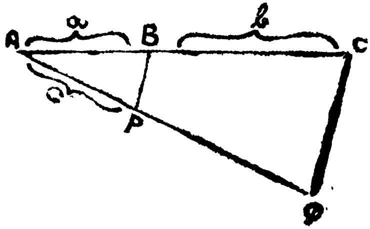


Fig. 7(a)

$$Proof: - \frac{AB}{BC} = \frac{AP}{PQ} \text{ i. e } \frac{a}{b} = \frac{c}{PQ}.$$

Hence the result.

4. To find a third proportional to two given lengths a, b.

As in the previous case take any two straight lines A B C and A P Q cutting each other at A at a convenient angle. On ABC mark off a length AB = a and a length BC = b. Also step off a length A P = b on A Q. Join B, P and draw a line through C parallel to

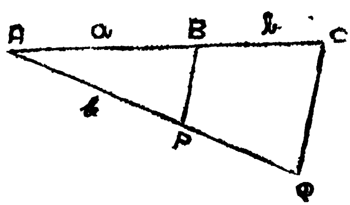


Fig. 7 (b)

BP to cut APQ at Q. Then PQ will be the third proportional.

$$Proof:- \frac{AB}{BC} = \frac{a}{b} \text{ (construction)}$$

$$\text{But } \frac{AP}{PQ} = \frac{AB}{BC}$$

$$\therefore \frac{AP}{PQ} = \frac{a}{b} \text{ i. e. } \frac{a}{b} = \frac{b}{PQ} \text{ Hence the result}$$

THEOREM 3.

Triangles and parallelograms of equal altitudes are to one another as their bases.

1. Triangles:—

Let △s ABC and PQR standing on bases BC and QR have equal altitudes h.

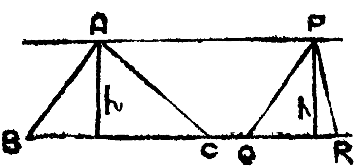


Fig. 7 (c)

Then $\triangle ABC = \frac{1}{2} h. BC$

" $\triangle PQR = \frac{1}{2} h. QR$

$$\therefore \frac{\triangle ABC}{\triangle PQR} = \frac{\frac{1}{2} h. BC}{\frac{1}{2} h. QR} = \frac{BC}{QR}$$

2. *Parallelograms:—*

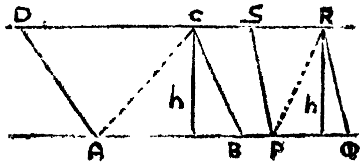


Fig 8

Let parallelograms ABCD and PQRS standing on bases AB and PQ have equal altitudes h.

Join A, C; P, R

Then $\square ABCD = 2\triangle ABC$ $(\because \triangle ABC \equiv \triangle CDA)$

$\square PQRS = 2\triangle PQR$ $(\because \triangle PQR \equiv \triangle RSP)$

$$\therefore \frac{\square ABCD}{\square PQRS} = \frac{2\triangle ABC}{2\triangle PQR} = \frac{\triangle ABC}{\triangle PQR} = \frac{AB}{PQ} \text{ by 1.}$$

WORKED EXAMPLE.

Three concurrent lines through the vertices A, B, C of a $\triangle$ ABC meet the opposite sides in D, E, F respectively. Prove that BD. CE. AF = DC. EA. FB (Ceva's Theorem)

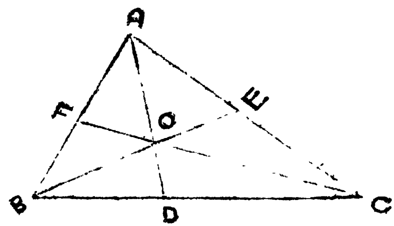


Fig. 9

Let the three lines concur at O. The altitudes from the vertex A for $\triangle$s BAD and DAC are the same.

Same

$\therefore \frac{BD}{DC} = \frac{\triangle BAD}{\triangle DAC}$. Also, $\triangle$s BOD and DOC have a common altitude, viz, the altitude from O.

$\therefore \frac{BD}{DC} = \frac{\triangle BOD}{\triangle DOC}$ Thus $\frac{BD}{DC} = \frac{\triangle BAD}{\triangle DAC} = \frac{\triangle BOD}{\triangle DOC}$

$= \frac{\triangle BAD - \triangle BOD}{\triangle DAC - \triangle DOC} = \frac{\triangle BAO}{\triangle OAC}$

(by ratio and proportion)

Similarly $\frac{CE}{EA} = \frac{\triangle CBO}{\triangle OBA}$ and $\frac{AF}{FB} = \frac{\triangle ACO}{\triangle OCB}$

Multiplying the three,

$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = \frac{\triangle BAO}{\triangle OAC} \cdot \frac{\triangle CBO}{\triangle OBA} \cdot \frac{\triangle ACO}{\triangle OCB} = 1$$

i. e. $\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = 1$ or BD. CE. AF=DC. EA. FB.

* THEOREM 4.

If an angle of a triangle is bisected internally (or externally), the bisector divides the opposite side internally (or externally) in the ratio of the other two sides of the triangle.

Let AD bisect $\angle$A of $\triangle$ ABC

Then to prove that $\frac{BD}{DC} = \frac{AB}{AC}$

* The method of proof adopted here for this theorem was first (i. e. in October 1956) given to Mr. Broadbent of Royal Naval college, Greenwich.

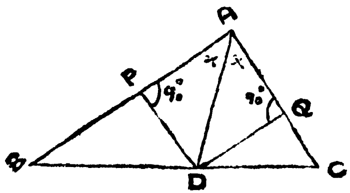


Fig. 10.

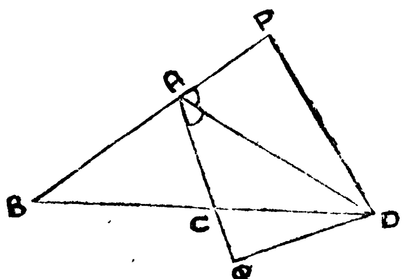


Fig. 11

Draw DP perpendicular to AB and DQ perpendicular to AC.

Then $\triangle APD \equiv \triangle AQD$

($\because P\hat{A}D = Q\hat{A}D$; $A\hat{P}D = A\hat{Q}D = 90^{\circ}$; AD is common)

$$\therefore DP = DQ$$

$$\therefore \frac{\triangle BAD}{\triangle DAC} = \frac{AB}{AC} \quad \text{(by theorem 3)}$$

But $\frac{\triangle BAD}{\triangle DAC} = \frac{BD}{DC}$ (by theorem 3 for the altitude from A for both $\triangle$s is common)

$$\therefore \frac{BD}{DC} = \frac{AB}{AC}$$

Conversely, if a side of a triangle is divided internally (or externally) in the ratio of the other two sides, then the line joining the point of division to the opposite vertex bisects the angle at that vertex internally (or externally.)

(Figure and construction the same as before) given that $\frac{BD}{DC} = \frac{AB}{AC}$ to prove that AD is a bisector of $\angle$A.

Now $\frac{BD}{DC} = \frac{\triangle BAD}{\triangle DAC}$ But $\frac{BD}{DC} = \frac{AB}{AC}$ (given)

$\therefore \frac{AB}{AC} = \frac{\triangle BAD}{\triangle DAC} = \frac{\frac{1}{2} AB.DP}{\frac{1}{2} AC.DQ} \therefore \frac{DP}{DQ} = 1$

or DP=DQ ; Further AD is the common hypotenuse for $\triangle$s APD and AQD $\therefore$ right angled $\triangle$s APD and AQD are congruent

$\therefore \angle PAD = \angle QAD$

i. e. AD is a bisector of $\angle$A (internal bisector in figure 10 and external bisector in fig. 11)

Note 1:— If AD, AE are respectively the internal and external bisectors of $\angle$A of $\triangle$ABC meeting the base BC at D and E then D and E divide BC in the

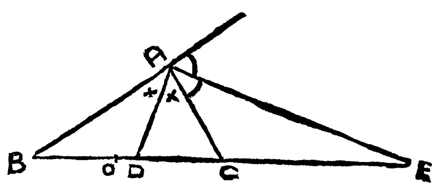


Fig. 12

same ratio $\frac{AB}{AC}$ one internally and the other externally. i. e. D and E divide BC harmonically. Hence (BDCE) is a harmonic range. D and E are called harmonic conjugates with respect to B and C. In the figure note that E, the external point of division lies on BC produced as D, the internal point of division lies between O and C where O is the middle point of BC.

Note 2:- If D and E divide BC harmonically,

(i) B and C divide DE harmonically (use Fig. 12 where line BC is divided harmonically at D and E)

$$\frac{BD}{DC} = \frac{BE}{CE} \quad \text{(by hypothesis)}$$

$$\therefore \frac{-BD}{DC} = \frac{-BE}{CE}$$

$$\frac{DB}{DC} = \frac{EB}{CE}$$

$$\therefore \frac{DB}{EB} = \frac{DC}{CE} \quad \text{(alternando)}$$

i. e. B and C divide DE harmonically.

(ii) BD, BC and BE are in harmonical progression.

$$\frac{BD}{DC} = \frac{BE}{CE}$$

$$\therefore \frac{BD}{(BD + DC) - BD} = \frac{BE}{(BC + CE) - BC}$$

$$\text{i. e. } \frac{BD}{BC - BD} = \frac{BE}{BE - BC}$$

$$\text{Inverting } \frac{BC - BD}{BD} = \frac{BE - BC}{BE}$$

$$\text{i. e. } \frac{BC}{BD} - 1 = 1 - \frac{BC}{BE}$$

$$BC\left(\frac{1}{BD} + \frac{1}{BE}\right) = 2$$

$$\therefore \frac{1}{BD} + \frac{1}{BE} = \frac{2}{BC} \text{ i. e. } \frac{1}{BD} + \frac{1}{BE} = \frac{1}{BC} + \frac{1}{BC}$$

or $\frac{1}{BD} - \frac{1}{BC} = \frac{1}{BC} - \frac{1}{BE}$ Hence by definition BD, BC, BE are in Harmonical progression.

$$\text{by (i) } \frac{DB}{EB} = \frac{DC}{CE}$$

$$\therefore \frac{DB}{EB} = \frac{-DC}{-CE} = \frac{CD}{EC}$$

$$\text{Inverting } \frac{EB}{DB} = \frac{EC}{CD} \text{ i. e. } \frac{EC}{CD} = \frac{EB}{DB}$$

∴ C and B divide ED harmonically.

Hence as before EC, ED, EB are also in harmonical progression.

So if (BDCE) is a harmonic range BD, BC, BE are in Harmonical progression and EC, ED, EB are also in Harmonical progression.

(iii) $OD.OE = OB^2 = OC^2$ where O is the midpoint of BC.

$$\frac{BD}{DC} = \frac{BE}{CE} \quad \text{(hypothesis)}$$

i. e. $\frac{BO+OD}{OC-OD} = \frac{BO+OE}{OE-OC}$ But $BO=OC$

$\therefore \frac{BO+OD}{BO-OD} = \frac{BO+OE}{OE-BO}$ Hence

$$\frac{BO+OD+BO-OD}{BO+OD-(BO-OD)} = \frac{BO+OE+OE-BO}{BO+OE-(OE-BO)}$$

(componendo et dividendo)

i. e. $\frac{2\,BO}{2\,OD} = \frac{2\,OE}{2\,BO}$ $\therefore BO^2 = OD.\,OE$

i. e. $OD.\,OE = BO^2 = (-OB)^2 = OB^2 = OC^2$

conversely if O is the midpoint of BC and D, E two points on it (on the same side of O) such that $OD.\,OE = OB^2 = OC^2$, then (BDCE) is a harmonic range.

$OD.\,OE = BO^2$

i. e. $\frac{BO}{OD} = \frac{OE}{BO}$

$\therefore \frac{BO+OD}{BO-OD} = \frac{OE+BO}{OE-BO}$

i. ,e $\frac{BO+OD}{OC-OD} = \frac{OE+BO}{OE-OC}$ $(\because BO = OC)$

i. e. $\frac{BD}{DC} = \frac{BE}{CE}$

$\therefore$ D and E divide BC internally and externally in the same ratio. Hence by definition (BDCE) is a harmonic range.

Note 3:— In fig. 12, if ABC is an isosceles triangle (AB = AC) D will coincide with O and AD will be perpendicular to BC. ∠DAE = 90° always. Hence AE will be parallel to BC. i. e. E in this case is the point at infinity on the line BC. (or more generally the point at infinity in a direction parallel to BC). So we note that the harmonic conjugate of the middle point of BC is the point at infinity on the line BC.

WORKED EXAMPLE.

1. If A,B are fixed points and P a variable point such that the ratio of PA to PB is always constant prove that the locus of P is in general a circle.

[This is called Apollonius' Theorem. The student in future when he studies the Geometry of the conic, will note that the locus of a point which moves such that the ratio of its distance from a focus to its distance from the foot of the corresponding directrix is a constant equal to the eccentricity of the conic, is its auxiliary circle. Hence auxiliary circle of a conic may be appropriately called Apollonian circle of the conic.]

Let P be a point such that $\frac{PA}{PB} = \frac{l}{k}$ Divide AB internally and externally in the same fixed ratio $\frac{l}{k}$. Let C be the internal point of division and D the external point of division.

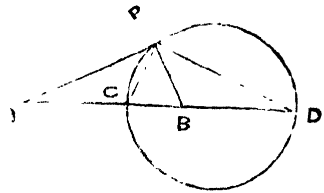


Fig. 13

$$\text{Then } \frac{PA}{PB} = \frac{l}{k} = \frac{AC}{CB} = \frac{AD}{BD}$$

i. e. $\frac{PA}{PB} = \frac{AC}{CB}$ $\therefore$ PC is the internal bisector of $\angle$APB.

and $\frac{PA}{PB} = \frac{AD}{BD}$ $\therefore$ PD is the external bisector of $\angle$APB

Hence $\angle CPD = 90^\circ$

Now if we describe a circle on CD as diameter this circle passes through the point P. ($\angle CPD = 90^\circ$). But P is any point satisfying the given condition. Thus any point satisfying the given condition (namely $\frac{PA}{PB} = \frac{l}{k}$) lies on this circle on CD as diameter and this circle is a fixed circle ($\because$ A and B are fixed points and the points C, D which divide the line joining these fixed points in the fixed ratio $\frac{l}{k}$ are also fixed. Consequently the circle on CD as diameter is also a fixed circle) Hence the locus of P is this circle on CD as diameter.

[Locus is by definition the aggregate of all points satisfying any geometrical condition. All loci however are found to be curves. Students must note that any point lying on a locus will satisfy the condition for the locus and that any point satisfying the condition for a locus will lie on the locus.]

Definition:— The circle on CD as diameter (fig. 13) is called the Apollonius' circle of the two fixed points A and B for the constant ratio $\frac{l}{k}$.

[Note that Apollonius' circle reduces to a straight line (a circle of infinite radius or a circle passing through the point at infinity) when the constant ratio is unity].

2. CA, CB are two tangents to a circle A and B being the points of contact. E is the foot of the perpendicular from B to AD the diameter through A. Prove that BA, BD bisect angle CBE. Deduce that CD bisects BE (March 1948)

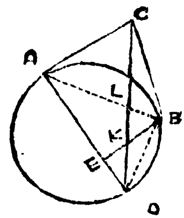


Fig. 14

Let BA and BE meet CD in L and K respectiveiy. CA, CB being tangents from A are equal

$\therefore \angle CAB = \angle CBA$

$\angle CAE + \angle BEA = 90^\circ + 90^\circ = 180^\circ$

$\therefore CA \parallel BE$

$\therefore \angle CAB = \angle ABE$

Hence $\angle CBA = ABE$

i. e. BA is the internal bisector of $\angle CBE$.

$\angle ABD = 90^\circ$ (since AD is a diameter)

Hence BD is the other bisector

$$\therefore \frac{KD}{CD} = \frac{LK}{CL}$$

Then to prove that EK = KB

Now $\triangle$ s DEK and DAC are ||| ($\because$ EK $\parallel$ AC)

$$\therefore \frac{EK}{AC} = \frac{KD}{CD}$$

|||ly $\triangle$s KBL and CAL are similar.

$$\therefore \frac{KB}{CA} = \frac{LK}{CL}$$

$$\frac{KD}{CD} = \frac{LK}{CL}$$

$$\therefore \frac{EK}{AC} = \frac{KB}{CA} \quad \text{or} \quad EK = KB$$

3, AE bisects the angle A of a $\triangle$ ABC and meets BC in E. If O and O^1 be the circumcentres of $\triangle$s ABE and ACE prove that $\frac{OE}{O^1E} = \frac{BE}{EC}$ (1939 M. U.)

Join O, B; O, E; O^1E; O^1C.

In $\triangle$s OBE and O^1EC

$B\hat{O}E = E\hat{O}^1C$ (Since $B\hat{A}E = E\hat{A}C$)

$= \theta$ (say)

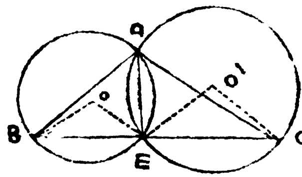


Fig. 15

also $O\hat{B}E = O\hat{E}B = \frac{O\hat{B}E + O\hat{E}B}{2} = \frac{180-\theta}{2}$

||| ly $O^1\hat{E}C = O^1\hat{C}E = \frac{180-\theta}{2}$

$\therefore$ $O\hat{B}E = O^1\hat{C}E$ and $O\hat{E}B = O^1\hat{E}C$

Hence $\triangle$s OBE and O^1CE are Equiangular

$\therefore$ $\frac{BE}{EC} = \frac{OE}{O^1E}$ ($\because$ If two $\triangle$s are Equiangular corresponding sides are proportional).

In the figure prove that $\triangle$s BEO^1 and CEO have the same area. (Proof is left to the student).

Aliter:—Let OP and O^1Q be $\perp$rs to BE and EC respectively from O and O^1.

Let $\angle BAE = \angle EAC = \theta$

Then $\angle BOE = \angle EO^1C = 2\theta$

and $\angle POE = \angle QO^1E = \theta$

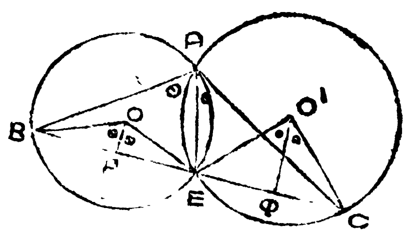


Fig 16

From rt. angled $\triangle$ POE

$PE = \frac{1}{2}BE = OE \sin\theta$ or $BE = 2\ OE \sin\theta$

|||ly $EC = 2\ O^1E \sin\theta$

$$\therefore \frac{BE}{EC} = \frac{OE}{O^1E}$$

CHAPTER 3.

SIMILAR TRIANGLES.

Definition:— Two $\triangle$s are said to be similar if (1) their angles are equal and (2) their corresponding sides (sides opposite to equal angles) are proportional. But it is found that if any one of the above conditions is satisfied the other will be automatically satisfied. This is indicated in the following two theorems.

THEOREM 1.

If two $\triangle$s are equiangular their corresponding sides are proportional.

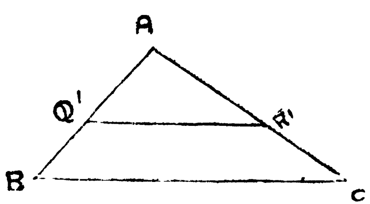


Fig. 17

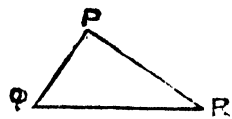


Fig. 18

In $\triangle$s ABC and PQR, let $\angle A = \angle P$, $\angle B = \angle Q$. (Then the third set of angles are obviously equal)

Place the $\triangle PQR$ so that $\angle P$ coincides with $\angle A$ and PQ coincides with AB. Then since $\angle P = \angle A$, PR coincides with AC.

Let AQ^1R^1 be the new position of $\triangle PQR$ Since $\angle B = \angle Q = \angle Q^1$ (or since $\angle C = \angle R = \angle R^1$)

$$Q^1R^1 \parallel BC \quad \therefore \quad \frac{AQ^1}{Q^1B} = \frac{AR^1}{R^1C}$$

i. e. $\frac{Q^1B}{AQ^1} = \frac{R^1C}{AR^1}$ adding 1 to both sides;

$$\frac{AB}{AQ^1} = \frac{AC}{AR^1} \quad \text{i. e.} \quad \frac{AB}{PQ} = \frac{CA}{RP}$$

($\triangle AQ^1R^1$ being the new position of $\triangle PQR$, $\triangle AQ^1R^1 \equiv \triangle PQR$)

Similarly by placing the $\triangle$ PQR so that $\angle R$ coincides with $\angle C$ and RP coincides with CA we can prove that $\frac{CA}{RP} = \frac{BC}{QR}$

But $\frac{AB}{PQ} = \frac{CA}{RP}$ (already proved)

$$\therefore \quad \frac{AB}{PQ} = \frac{BC}{QR} = \frac{CA}{RP}$$

Note:— The student should take care in writing down the corresponding sides in the two triangles. Sides opposite to equal angles in the two triangles are corresponding sides. Note also that the two triangles are now similar (by definition).

Conversely, if the sides of one triangle be proportional to the sides of another, the two triangles are equiangular.

Let ABC and PQR be two Δs in which

$$\frac{AB}{PQ} = \frac{BC}{QR} = \frac{CA}{RP}$$

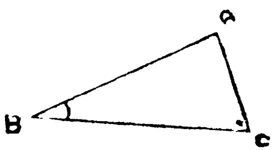


Fig. 19

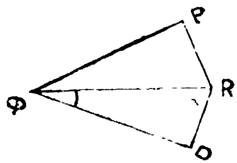


Fig. 20.

Required to prove that they are equiangular

Let D be a point on the side opposite to P, of QR such that $\angle DQR = \angle ABC$ and $\angle DRQ = \angle ACB$

Then Δs ABC and DQR are equiangular (by construction)

$\therefore \quad \frac{AB}{DQ} = \frac{BC}{QR} = \frac{CA}{RD}$ (by theorem 1, Chapter 3)

But $\frac{BC}{QR} = \frac{CA}{RP} = \frac{AB}{PQ}$ (Hypothesis)

$$\therefore \quad \frac{AB}{DQ} = \frac{AB}{PQ} \quad \text{and} \quad \frac{CA}{RD} = \frac{CA}{RP}$$

Hence PQ = DQ and RP = RD

QR is common to $\triangle$PQR and $\triangle$DQR

$\therefore$ $\triangle$ PQR $\equiv$ $\triangle$ DQR and hence they are equiangular.

But $\triangle$s ABC and DQR are equiangular.

$\therefore$ $\triangle$s ABC and PQR are equiangular.

Note that the two triangles are now similar (by definition).

CONSTRUCTION.

Divide a straight line AB internally and externally in the ratio l:k.

Erect a perpendicular AP, at A to line AB such that AP = l units. Also, erect a perpendicular BQ at B to line AB in the same side of AB as AP, such that BQ = k units. Produce QB to Q^1 such that $QB = BQ^1$. Join P, Q^1 to meet AB at C Then C will be the required internal point of division. Join P, Q and produce it to meet AB or BA produced as the case may be at D. Then D will be the required external point of of division.

(Draw the figure and supply proof)

The student will note that this method is very useful in dividing a straight line internally and externally in the same ratio.

WORKED EXAMPLE.

Two circles cut orthogonally at A and B. A diameter of one of the circles is drawn cutting the other in C and D. Show that BC. AD = AC. BD.

[Sept. 1950 M. U.]

Def:— Two circles are said to cut orthogonally if the angle between the tangents to the two circles at a common point is a right angle (or if the radius of one through a common point is the tangent to the other at that common point).

Let a diameter of circle I cut the circle II at C and D. Let O be the centre of circle I. Join O,A; O,B. Since OA by definition of orthogonal circles is a Tangent to circle II.

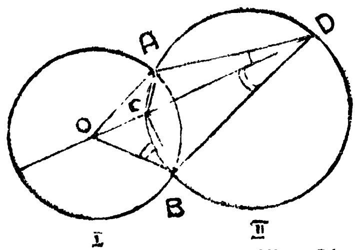


Fig. 21.

$\angle OAC = \angle ODA$.

Hence $\triangle$s OAC and ODA are |||

$$\therefore \quad \frac{CA}{AD} = \frac{OA}{OD}$$

Similarly since $\triangle$s OBC and ODB are |||

$$\frac{BC}{DB} = \frac{OB}{OD}$$

But OA = OB

$$\therefore \quad \frac{CA}{AD} = \frac{BC}{DB} \quad \text{i. e. } AC.\ BD = AD.\ BC.$$

THEOREM 2.

If two Δs have one angle of the one equal to one angle of the other and the sides about these equal angles proportional, the two Δs are similar.

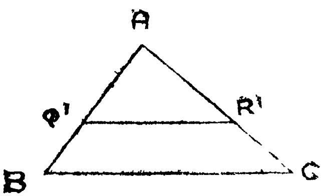


Fig. 22

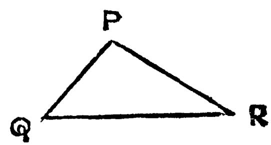


Fig. 23

In the two Δs ABC and PQR let

$$\angle A = \angle P \text{ and } \frac{AB}{PQ} = \frac{AC}{PR}.$$

Construction:— Place the ΔPQR on ΔABC such that $\angle$P coincides with $\angle$A and PQ falls along AB. Since $\angle A = \angle P$, PR will coincide with AC. Let Q^1, R^1, be the new positions of Q and R respectively.

$\therefore \quad AQ^1 = PQ; \ AR^1 = PR.$

$\frac{AB}{PQ} = \frac{AC}{PR}$ (given) i. e. $\frac{AB}{AQ^1} = \frac{AC}{AR^1}$

i. e. $\frac{AB}{AB-AQ^1} = \frac{AC}{AC-AR^1}$ i. e. $\frac{AB}{Q^1B} = \frac{AC}{R^1C}$ or

$\frac{AB}{BQ^1} = \frac{AC}{CR^1}$ $\therefore$ $Q^1R^1 \parallel BC$ (by Theorem 2. ch. 2.)

$\therefore \quad \angle Q^1 = \angle B; \ \angle R^1 = \angle C$ i. e. $\angle Q = \angle B$ and $\angle R = \angle C$.

Thus $\triangle$s ABC and PQR are equiangular and hence their corresponding sides are proportional (by Theorem 1, Ch. 3.)

i. e. The two $\triangle$s are similar.

WORKED EXAMPLE.

In a quadrilateral which is not cyclic prove that the rectangle contained by the diagonals is always less than the sum of the rectangles contained by pairs of opposite sides.

ABCD is a quadrilateral which is not cyclic. Let O be a point within the quadrilateral such that $\angle OAD = \angle CAB$ and $\angle ODA = \angle ACB$. By

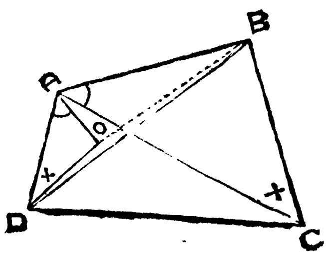


Fig. 24

construction $\triangle$s AOD and ABC are equiangular.

$$\therefore \quad \frac{AD}{AC} = \frac{OD}{BC} = \frac{AO}{AB} \quad (A) \text{ (Theorem 1. Ch. 3)}$$

Hence AD. BC = OD. AC. I

Now in $\triangle$s DAC and OAB

Evidently $\angle DAC = \angle OAB$

Further $\frac{AD}{AC} = \frac{AO}{AB}$ by (A)

$\therefore$ $\triangle$ DAC ||| $\triangle$OAB (by Theorem 2. Ch. 3)

$\therefore \frac{AC}{AB} = \frac{DC}{OB}$ i. e. AB. CD. = AC. OB. II.

Adding I and II,

AB. CD + AD. BC = AC (OD + OB)

But OD + OB > BD

$\therefore$ AB. CD + AD. BC. > AC. BD

When the quadrilateral is cyclic $\angle ACB = \angle ADB$ (Angles in the same segment)

But $\angle ACB = \angle ODA$ (by construction)

$\therefore$ $\angle ADB = \angle ADO$. Hence O lies on BD

$\therefore$ OD + OB = BD.

Thus when the quadrilateral is cyclic

AB. CD + AD. BC = AC. BD which is known as Ptolemy's Theorem.

Def:— If a, b, c are three quantities connected by the relation $\frac{a}{b}=\frac{b}{c}$ ($b^2=ac$) the three quantities, a, b, c are said to be in continued proportion. Also, b is called the mean proportional between a and c.

THEOREM 3.

If from the right angle A of a right angled triangle ABC, AD is drawn perpendicular to BC then (i) AD is the mean proportional between BD and DC (ii) BA is the mean proportional between BD and BC (iii) CA is the mean proportional between CD and CB.

Let ABC be a $\triangle$ right angled at A and AD $\perp$ to BC from A.

Proof:—

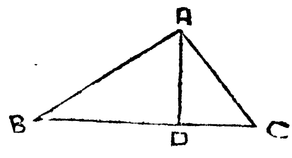


Fig. 25

$\angle ABD + \angle BAD = 90^\circ$

$\angle ABC + \angle ACB = 90^\circ$

$\therefore \quad \angle BAD = \angle ACB$

Also $\angle B$ is common to $\triangle$s ABD and CBA

Hence $\triangle$s ABD and CBA are similar.

|||ly $\triangle$s CAD and CBA are similar.

$\therefore$ $\triangle$s ABD and CAD are also similar.

(i) From similar $\triangle$s ABD and CAD

$$\frac{AD}{DC}=\frac{BD}{AD} \text{ or } AD^2 = BD.\ DC.$$

(ii) From similar $\triangle$s ABD and CBA

$$\frac{BA}{BC} = \frac{BD}{BA} \text{ or } BA^2 = BD.\ BC.$$

(iii) From similar $\triangle$s CAD and CBA

$$\frac{CA}{CB} = \frac{CD}{CA} \text{ or } CA^2 = CD.\ CB.$$

[An alternate method of proof is given below for these three results so that the student may recollect them easily to his memory without any confusion].

(i) Describe a circle on BC as diameter. Since $\angle BAC = 90°$ this passes through A. If AD meets this circle again at A^1, BC the diameter bisects this perpendicular chord ADA^1. Hence $AD = DA^1$.

By a property of the circle,

$AD.\ DA^1 = BD.\ DC$

i. e. $AD^2 = BD.\ DC.\ (AD = DA^1)$.

(ii) If we describe a circle on CA as diameter this passes through D ($\angle ADC = 90°$). Further since BA is perpendicular to CA, BA becomes the tangent to this circle on CA as diameter at A. Hence $BA^2 = BD.\ BC$.

(iii) Similarly by describing a circle on BA as diameter we get $CA^2 = CD.\ CB$.

[The student will note that Pythagoras' Theorem follows from (ii) and (iii), on addition].

Def:— If the values of two quantities vary, subject to the condition that their product is always constant those two quantities are said to be in inverse proportion.

WORKED EXAMPLE.

1. PA, PB are Tangents to a circle whose centre is O from any external point P. AB cuts OP in Q Prove that OP and OQ are in inverse proportion.

Join O to A. Then $\angle$ OAP $= 90°$.
Also AQ is $\perp$ to OP
($\because$ $\triangle$ OAP $\equiv$ $\triangle$ OBP and hence $\triangle$ QAP $\equiv$ $\triangle$ QBP.

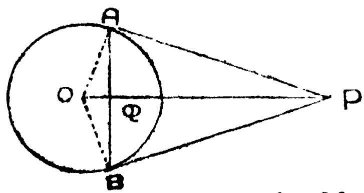


Fig. 26

Thus $\angle AQP = \angle BQP = \dfrac{A\hat{Q}P + B\hat{Q}P}{2} = \dfrac{180}{2} = 90°$)
Hence by theorem 3 chapter 3,

$OA^2 = OQ.\ OP.$

i. e. $OQ.\ OP = (\text{radius of the circle})^2$

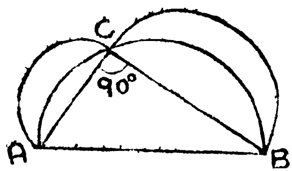
$= \text{a constant.}$

$\therefore$ OP and OQ are in inverse proportion.

2 C is a point on the semi-circle on the line AB as diameter. Semi-circles are outwardly described on AC and BC as diameters. Prove that the sum of the crescent shaped areas lying outside the semi-circle ACB is equal to the area of the △ ACB. (Inter March 1956)

The area of semi-circle on AB as diameter is $\frac{\pi}{2}\left(\frac{AB}{2}\right)^2 = \frac{\pi}{8} AB^2$

|||ly the Areas of semi-circles on AC and BC as diameters are respectively $\frac{\pi}{8} AC^2$ and $\frac{\pi}{8} BC^2$



Ffg. 27

Since AB is a diameter $\angle ACB = 90°$ and hence by Pythagoras' Theorem,

$AB^2 = AC^2 + BC^2$. Multiplying throughout by $\frac{\pi}{8}$

$$\frac{\pi}{8} AB^2 = \frac{\pi}{8} AC^2 + \frac{\pi}{8} BC^2$$

i. e. Area of semi-circle on AB as diameter is equal to the sum of the areas of semi-circles on AC and BC as diameters. Now let the area of the semi-circle ACB excluding △ ACB be S

Then, $\left(\frac{\pi}{8} AB^2 - S\right) = \left(\frac{\pi}{8} AC^2 + \frac{\pi}{8} BC^2\right) - S$

i. e. $\triangle$ ACB = Sum of the crescent shaped areas lying outside the semi-circle ACB.

3. Prove that the common tangent to two circles having external contact is a mean proportional between the diameters of the circles.

[Inter 1925 M. U.]

Let two circles, centres A and B touch externally at M. Also let PQ be one of their common tangents. (The other common tangent will also be equal in length by symmetry.)

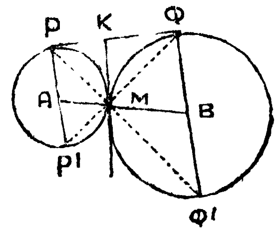


Fig. 28

Produce PA and QB to meet the circles again at P^1 and Q^1. Then PP^1 is a diameter of the circle A and QQ^1 is a diameter of the circle B.

Join P, M; Q, M; P^1, M and Q^1, M.

$\angle PMP^1 = 90°$. Also $\angle PMQ = 90°$ (For, if we draw the common Tangent at M to the two circles, cutting PQ at K, KP = KM = KQ)

$\therefore$ P^1MQ is a straight line. |||ly Q^1MP is a st. line.

In $\triangle$s PP^1Q and QPQ^1

$P^1PQ = PQQ^1 = 90°$

$\angle PP^1Q = \angle PP^1M = \angle QPM = \angle QPQ^1$

(Property of the circle)

$\therefore \quad \triangle PP^1Q \; ||| \; \triangle QPQ^1$

Hence $\dfrac{PQ}{QQ^1} = \dfrac{PP^1}{PQ}$ or $PQ^2 = PP^1 . QQ^1$

Aliter:—

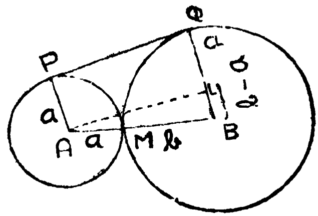


Fig. 29

Let the radii of the two circles centres A and B and touching externally at M, be a and b respectively;

Also let PQ be one of the two common tangents. Join P, A; Q, B. Draw a perpendicular from A to QB cutting it at L. Then evidently PQLA is a rectangle.

$\therefore \quad PQ = AL$ and $PA = QL = a$

$\therefore \quad BL = BQ - LQ = (b - a) \quad (\because BQ = b)$

$AB = (a + b)$

From right angled $\triangle$ ALB,

$AB^2 = AL^2 + BL^2$

i. e. $(a + b)^2 = PQ^2 + (b - a)^2$

$\therefore \quad PQ^2 = (a + b)^2 - (b - a)^2$

$= 4\,a\,b = (2\,a)\;(2\,b)$

Construction:— Find the mean proportional between two given lengths a and b.

Take any straight line AB. Let O be a point on it. Measure out OA = a and OB = b on opposite sides of O on the straight line AB.

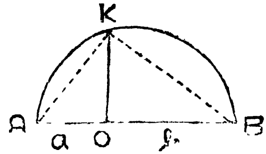


Fig. 30

Draw the circle on AB as diameter. Through O draw a perpendicular to cut the circle in K and K^1 (K^1 is not shown in the figure). Then OK or OK^1 will be the mean proportional between AO and OB. Join A, K; B, K.

Proof:— $\angle$AKB = 90° and KO is $\perp$r to AB. Hence applying Theorem **3**, Chapter 3.

$$AO.\ OB = OK^2 \quad \text{i. e.} \quad a\,b = OK^2$$

Hence OK is the mean proportional between a and b.

Note 1:— This construction geometrically illustrates the algebraic proposition that, if the sum of two positive quantities is given the product of them is greatest when the two quantities are equal. For, for all positions of the straight line OK (O lying between A and B) the relation AO. OB = OK^2 is true. The left hand side will be greatest when the right hand side is greatest. The

R. H. S evidently will be greatest when O coincides with the centre of the circle. Then $AO = OB$.

Note 2:— If OB and OA ($OA < OB$) are taken in the same side of O, the construction will be as follows:

Draw the circle on OB as diameter and erect a ⊥r to OB at A to meet the circle at one of the two points (say) at K. Join O. K and B, K

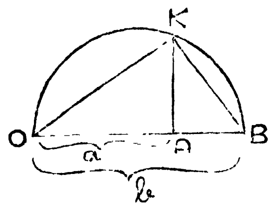


Fig. 31

Then OK will be the mean proportional between OA and OB.

Proof: – $\angle OKB = 90°$ and KA is ⊥r to OB.

∴ By applying Theorem 3, Chapter 3,

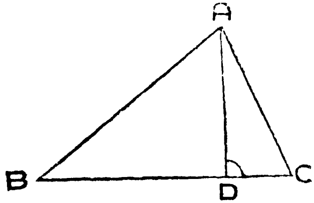
$$OK^2 = OA \; OB.$$

i. e. $OK^2 = a\ b$

∴ OK is the mean proportional between a and b.

TEOREM 4.

Similar △s are to one another as the squares on their corresponding sides. (Areas of two similar triangles are in the duplicate ratio of corresponding sides).



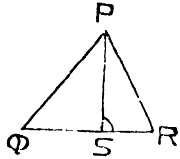


Fig. 32 Eig. 33

Let $\triangle$ ABC be similar to $\triangle$ PQR. Also let AD be $\perp$ to BC and PS be $\perp$ to QR. Then in the two right angled $\triangle$s ADC and PSR.

$$A\hat{D}C = P\hat{S}R = 90^\circ$$

$$A\hat{C}D = P\hat{R}S \quad (\angle C = \angle R)$$

$$\therefore \triangle ADC \ ||| \ \triangle PSR$$

$$\text{So} \quad \frac{AD}{PS} = \frac{AC}{PR} \quad \text{But} \quad \frac{BC}{QR} = \frac{CA}{RP} = \frac{AB}{PQ}$$

$$\therefore \quad \frac{AD}{PS} = \frac{CA}{RP} = \frac{AB}{PQ} = \frac{BC}{QR}$$ (Thus altitudes to two corresponding sides of two ||| $\triangle$s are in the ratio of corresponding sides).

$$\frac{\triangle ABC}{\triangle PQR} = \frac{\frac{1}{2} BC.AD}{\frac{1}{2} QR.PS} = \frac{BC}{QR} \cdot \frac{BC}{QR} \left(\because \frac{AD}{PS} = \frac{BC}{QR} \right)$$

$$= \frac{BC^2}{QR^2} = \frac{CA^2}{RP^2} = \frac{AB^2}{PQ^2}$$

WORKED EXAMPLE.

1. The tangent at A to the circumcircle of a triangle ABC meets BC produced at D. Show that $\frac{BD}{CD}=\frac{AB^2}{AC^2}$ (Inter 1934 M. U.)

In $\triangle$s ABD and CAD

$\angle$ D is common.

$\angle ABD = \angle CAD$ (angle between a tangent to a circle and any chord through the point of contact is equal to the angle subtended by that chord in the alternate segment),

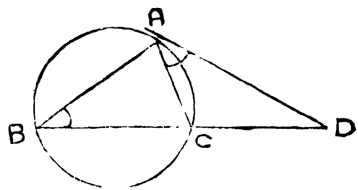


Fig. 34

Hence the two $\triangle$s are similar.

$\therefore \frac{\triangle ABD}{\triangle CAD}=\frac{AB^2}{AC^2}$. The altitude from A to the two $\triangle$s is Common.

$\therefore \frac{\triangle ABD}{\triangle CAD}=\frac{BD}{CD}$ Hence $\frac{BD}{CD}=\frac{AB^2}{AC^2}$

Aliter:—Since the two $\triangle$s ABD and CAD are equiangular, corresponding sides are proportional.

Thus, $\frac{AB}{AC}=\frac{BD}{AD}=\frac{AD}{CD}=\sqrt{\frac{BD.\ AD}{AD.\ CD}}=\sqrt{\frac{BD}{CD}}$

$\therefore \quad \frac{BD}{CD} = \frac{AB^2}{AC^2}$ or taking the two values of $\frac{AB}{AC}$ and multiplying together we get the result.

2. In two similar triangles corresponding lines such as (a) *medians (b) altitudes (c) circum-radii (d) in-radii etc. are in the ratio of corresponding sides.

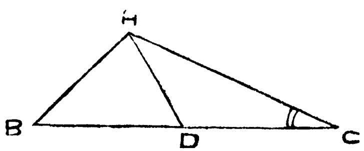


Fig. 35

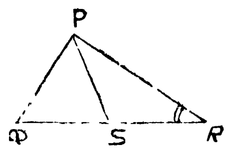


Fig 36

(a) Let ABC and PQR be two similar triangles and AD, PS medians to the sides BC and QR respectively.

Proof:— Since $\triangle$ ABC ||| $\triangle$ PQR

$$\frac{BC}{QR} = \frac{CA}{RP} \quad \therefore \quad \frac{\frac{1}{2}BC}{\frac{1}{2}QR} = \frac{CA}{RP} \quad \text{i. e.} \quad \frac{DC}{SR} = \frac{CA}{RP}$$

Further $\angle C = \angle R$

$\therefore$ $\triangle$ACD and $\triangle$PRS are similar by theorem 2, chapter 3.

---

* For definition of nedians of a plane triangle see appendix II.

Hence $\frac{AD}{PS} = \frac{CA}{RP} = \frac{AB}{PQ} = \frac{BC}{QR}$

Similarly the other medians are also in the ratio of corresponding sides.

(b) Already proved in theorem , chapter 3.

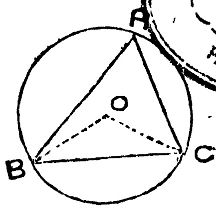


Fig. 37

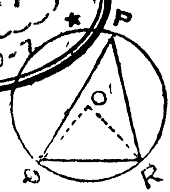


Fig. 38

(c) [Circumcentre of a triangle is the point of concurrence of the perpendicular bisectors of the sides of the △. Hence it is evident that the circumcentre of a traiangle is equidistant from the vertices of the triangle. i e. with the circumcentre as centre a circle can be drawn to pass through the vertices of the triangle This circle is called the circumcircle. The radius of this circle is called the circum-radius.]

Let O and O' be the circumcentres of two similar triangles ABC and PQR.

Join B, O; C, O; Q, O'; R, O'.

Proof:— Since $\angle A = \angle P$

$\angle BOC = \angle QO'R$

($\because \angle BOC = 2\angle A$ and $\angle QO'R = 2\angle P$)

$$\text{Further } \frac{BO}{OC} = \frac{QO'}{O'R} = 1$$

$$\text{or} \quad \frac{BO}{QO'} = \frac{OC}{O'R}$$

Hence $\triangle BOC \parallel\!| \triangle QO'R$ by Theorem 2, Chapter 3.

$$\therefore \quad \frac{OB}{O'Q} = \frac{BC}{QR} = \frac{CA}{RP} = \frac{AB}{PQ}$$

(d) [Incentre of a triangle is the point of concurrence of the internal bisectors of the angles

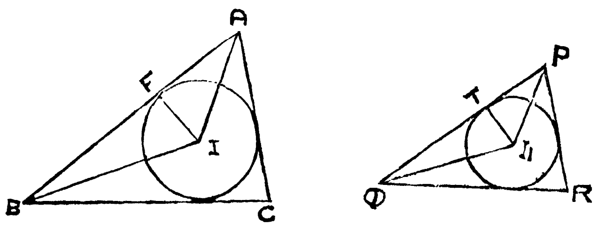


Fig. 39 Fig. 40

of the triangle. Hence it is evident that the perpendicular distances of the incentre of a triangle from the three sides are equal. i. e. We can draw a circle with the incentre as centre to touch the three sides of the triangle. This circle is called the incircle and the radius of this circle is called the in-radius.]

Let I and I_1 be the incentres of two similar triangles ABC and PQR. Let IF be perpendicular to AB and I_1T perpendicular to PQ. Then, evidently F is the point of contact of the side AB of △ ABC with the incircle of △ABC and T is the point of contact of the side PQ of △PQR with the incircle of △PQR. i. e. IF and I_1T are the in-radii of the two △s ABC and PQR.

Proof:— In △s ABC and PQR,

$\angle A = \angle P \quad \therefore \angle BAI = \angle QPI_1$ and

similarly $\angle ABI = \angle PQI_1$

$\therefore \quad \triangle ABI \;|||\; \triangle PQI_1$

IF and I_1T are altitudes to the corresponding sides AB and PQ of these similar △s.

Hence by (b) $\frac{IF}{I_1T} = \frac{AB}{PQ} = \frac{BC}{QR} = \frac{CA}{RP}$

3. In two △s if one angle of the one equals one angle of the other, their areas are in the ratio of the rectangles contained by sides about equal angles.

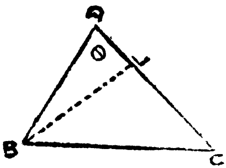


Fig. 41

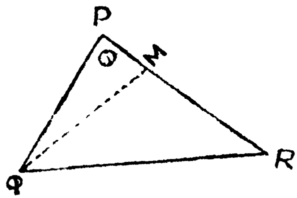


Fig. 42

Let ABC and PQR be two triangles such that $\angle BAC = \angle QPR = \theta$ (say)

Construction:— Draw BL, QM perpendiculars to AC and PR from B and Q respectively. Then $\triangle ABC = \frac{1}{2} AC.BL$; $\triangle PQR = \frac{1}{2} PR.QM$.

But $BL = AB \sin\theta$; $QM = PQ \sin\theta$.

$$\therefore \frac{\triangle ABC}{\triangle PQR} = \frac{\frac{1}{4} AC.AB \sin\theta}{\frac{1}{2} PR.PQ \sin\theta} = \frac{AB.AC}{PQ.PR} \quad (\sin\theta \neq 0)$$

Since, $\sin\theta = \sin(180 - \theta)$ it follows that in two $\triangle$s if one angle of the one is a supplement of one angle of the other their areas are in the ratio of the rectangles contained by sides about these supplementary angles.

From this problem it easily follows that the areas of two similar triangles are in the duplicate ratio of corresponding sides. For if ABC and PQR are two similar $\triangle$s.

$$\frac{\triangle ABC}{\triangle PQR} = \frac{AB \,.\, AC}{PQ.\, PR} \quad (\because \angle A = \angle P)$$

$$= \frac{AB}{PQ} \cdot \frac{AB}{PQ} \left(\because \frac{AB}{PQ} = \frac{AC}{PR}\right)$$

$$= \frac{AB^2}{PQ^2}$$

CONSTRUCTIONS.

I. Divide a triangle ABC into two parts whose areas are in the ratio l : m by drawing a straight line parallel to the side BC.

It is required to draw a line which is such that it divides the △ into two portions whose areas are in the ratio l:m and which must at the same time be parallel to BC.

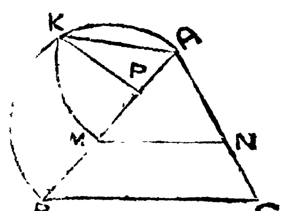


Fig. 43

To fix up a line either two points on it or any one point on it and its direction must be known. Here since the straight line must have to be parallel to BC its direction can be taken as given. Hence it is sufficient if we find a point on it.

Analysis:— Suppose MN is the required parallel line cutting AB and AC at M and N respectively.

$$\text{Then } \frac{\triangle AMN}{\square MNCB} = \frac{l}{m}$$

$$\therefore \frac{\triangle AMN}{\triangle AMN + \square MNCB} = \frac{l}{l+m}$$

$$\text{i. e. } \frac{\triangle AMN}{\triangle ABC} = \frac{l}{l+m}$$

But $\triangle$s AMN and ABC are also similar (MN $\parallel$ BC)

$$\therefore \frac{\triangle AMN}{\triangle ABC} = \frac{AM^2}{AB^2}$$

$$\text{Hence } \frac{AM^2}{AB^2} = \frac{l}{l+m}$$

i. e. M is a point on AB such that $\frac{AM^2}{AB^2} = \frac{l}{l+m}$

(|||ly N will be a point on AC such that $\frac{AN^2}{AC^2} = \frac{l}{l+m}$)

Now if $\frac{l}{l+m}$ is a perfact square the ratio $\frac{AM}{AB}$ can be easily found out and hence the point M can be easily fixed up on AB. If $\frac{l}{l+m}$ is not a perfect square M will have to be found out, employing some other geometrical result. In this case if P is a point on AB such that $AM^2 = AP. AB$. (i. e. AM becomes the mean proportional between AP and AB).

$$\frac{AM^2}{AB^2} = \frac{AP.\ AB.}{AB.\ AB.} = \frac{AP.}{AB.} = \frac{l}{l+m}$$

So if we find a point P such that $\frac{AP}{AB} = \frac{l}{l+m}$ the point M can be found out by finding, AM the mean proportional between AP and AB.

Construction:— Let P be a point on AB such that $\frac{AP}{AB} = \frac{l}{l+m}$ $\left(\text{i. e. } \frac{AP}{PB} = \frac{l}{m}\right)$. Find the mean proportional between AP and AB. (use construction under Theorem 3, Chapter 3). Let it be AM. Then M is the point through which the required parallel line passes. Hence draw a line through M parallel to BC. This will be the required line.

Proof: — $\frac{\triangle AMN}{\triangle ABC} = \frac{AM^2}{AB^2}$ (Since the two $\triangle$s are similar)

But $AM^2 = AP.\ AB$ (construction)

$$\therefore \frac{AM^2}{AB^2} = \frac{AP\ AB}{AB.\ AB} = \frac{AP}{AB} \therefore \frac{\triangle AMN}{\triangle ABC} = \frac{AP}{AB} = \frac{l}{l+m}$$

Hence $\frac{\triangle AMN}{\triangle ABC - \triangle AMN} = \frac{l}{l+m-l}$

i. e. $\frac{\triangle AMN}{\square MNCB} = \frac{l}{m}$.

2. Divide a triangle ABC into two parts whose areas are in the ratio $l:m$ by drawing a perpendicular to BC.

Analysis:— Let MN be the required perpendicular to BC. Draw AK perpendicular to BC.

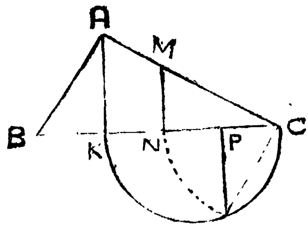


Fig. 44

$$\frac{\triangle CMN}{\square MNBA} = \frac{l}{m}$$

(Hypothesis)

$$\therefore \quad \frac{\triangle CMN}{\triangle CAB} = \frac{l}{l+m}$$

Now $\dfrac{\triangle CMN}{\triangle CAK} = \dfrac{CN^2}{CK^2}$ (1) (MN $\parallel$ AK)

$\dfrac{\triangle CAK}{\triangle CAB} = \dfrac{CK}{CB}$ (2) (Altitude AK is common)

(1) $\times$ (2) gives $\dfrac{\triangle CMN}{\triangle CAB} = \dfrac{CN^2}{CK.\,CB}$

$$\therefore \quad \frac{CN^2}{CK.\,CB} = \frac{l}{l+m}$$

$$\frac{CN^2}{CK} = \frac{l}{l+m}\, CB.$$

i. e CN is the mean proportional between CK and CP where CP is equal to $\frac{l}{l+m}$ CB.

$$\frac{CP}{CB}=\frac{l}{l+m} \quad \therefore \frac{CP}{PB}=\frac{l}{m}$$

So P divides CB in the ratio $l:m$.

Construction:— Divide CB in the ratio $l:m$ at P (this is the same thing as dividing BC in the ratio $m:l$ at P). Find the mean proportional between CP and CK, where K is the foot of the perpendicular from A on BC. Let it be CN. Then N is the point at which the required perpendicular will have to be erected to BC. Hence draw a line through N perpendicular to BC, cutting AC at M. Then MN will be the required line.

Proof:— $\frac{CP}{PB}=\frac{l}{m} \quad \therefore \frac{CP}{CB}=\frac{l}{l+m}$

(by ratio and proportion)

$$\frac{\triangle CMN}{\triangle CAB}=\frac{\triangle CMN}{\triangle CAK}\cdot\frac{\triangle CAK}{\triangle CAB}$$

But $\frac{\triangle CMN}{\triangle CAK}=\frac{CN^2}{CK^2}$ ($\because$ they are similar)

and $\frac{\triangle CAK}{\triangle CAB}=\frac{CK}{CB}$ (they have the same altitude)

$$\therefore \frac{\triangle CMN}{\triangle CAB}=\frac{CN^2}{CK^2}\cdot\frac{CK}{CB}=\frac{CN^2}{CK\cdot CB}=\frac{CP.\,CK}{CK.\,CB}$$

(CN is the mean proportional between CP and CK.)

$$\therefore \frac{\triangle CMN}{\triangle CAB}=\frac{CP}{CB}. \quad \text{Hence} \quad \frac{\triangle CMN}{\square MNBA}=\frac{l}{m}$$

3. Draw a triangle equal in area to a given triangle and similar to another given △.

Let PQR and ABC be two given triangles. Then it is required to construct another triangle which is equal in area to △ ABC and similar to △ PQR.

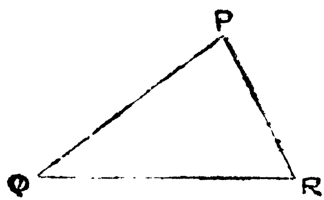


Fig. 45

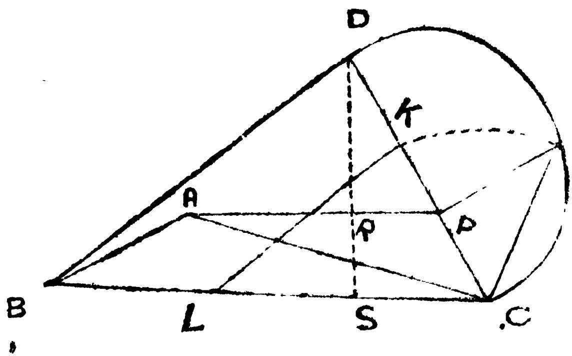


Fig. 46

Construction:— On BC construct a triangle DBC equiangular to △ PQR. Through A draw a parallel to BC, cutting DC at P. Find the mean proportional between CP and CD. Let it be CK.

Through K draw a parallel to DB, cutting BC at L. Then △ KLC will be the required triangle.

Proof:— △PQR and △DBC by construction are equiangular and hence similar.

Since LK ∥ BD, △KLC and △DBC are equiangular and hence similar.

∴ △PQR and △KLC are also similar... (1)

If we draw DRS ⊥ to parallel lines AP and BC cutting them at R and S respectively

$$\frac{CP}{CD}=\frac{SR}{SD} \text{ (by parallels)}$$

But $\frac{\triangle KLC}{\triangle DBC}=\frac{CK^2}{CD^2}=\frac{CP.CD}{CD.CD}=\frac{CP}{CD}$ (CK is the mean proportional between CP and CD)

$\therefore \quad \frac{\triangle KLC}{\triangle DBC}=\frac{SR}{SD}=\frac{\triangle ABC}{\triangle DBC}$ (△ABC and △DBC have got the same base BC)

$\therefore \quad \triangle KLC=\triangle ABC$......... (2)

From (1) and (2) it follows that △KLC is the required triangle.

EXCERCISES.

1. If a straight line m inches in length is divided internally and externally in the ratio $l : k$ find the lengths of the segments in each case.

2 A, B, C, D are four collinear points (points lying in a line) such that AC is divided at B and D in the ratio k : l. Show that DB is divided at C and A in the ratio $(k+l) : (k-l)$

3. AB is divided internally and externally in the same ratio at C and D: If O is the middle point of AB, prove that $OC.\ OD = OA^2 = OB^2$.

4. Two diameters AB and CD of a circle are divided in the same ratio internally and externally at P, Q; R, S respectively, the ratio for AB and CD being the same or different. Prove that P, Q, R, S, are concyclic.

5. Show that three or more parallel lines cut any two transversals proportionally.

6. ABC is a triangle inscribed in a circle. Show that the perpendicular from A on BC is a mean proportional between the perpendiculars from B and C on the tangent at A.

[Hint.— If AP is the $\perp$r from A to BC and BQ, CR $\perp$rs to the tangent at A from B and C, then figure AP BQ is cyclic $\therefore \frac{AP}{BQ}=\frac{OP}{OQ}$ where O is the point of intersection of the tangent at A and BC. |||ly since figure ARCP is cyclic $\frac{CR}{AP}=\frac{OC}{OA}$. But since OA is a tangent at A, AC||QP $\therefore \frac{OP}{OQ}=\frac{OC}{OA}$. Hence $\frac{AP}{BQ}=\frac{CR}{AP}$ or $AP^2=BQ.CR$].

7. ABCD is a quadrilateral; show that if the bisectors of the angles A and B meet in the diagonal BD, the bisectors of the angles B and D will meet on AC.

8. The bisector of angle A of $\triangle$ABC meets BC in D and DE is drawn parallel to AC to cut AB at E and DF parallel to AB to cut AC at F. Prove that $\frac{BE}{CF}=\frac{AB^2}{AC^2}$

9. The median AD of a $\triangle$ABC meets BC in D. The internal bisectors of angles ADB and ADC meet AB and AC in P and Q respectively. Prove that PQ is parallel to BC.

10. CA, CB are two tangents to a circle, A and B being the points of contact, E is the foot of the

perpendicular from B to AD, the diameter through A. Prove that BA, BD bisect angle CBE. Deduce that CD bisects BE.

11. The straight line BC is divided in the same ratio at D and E. DE subtends a right angle at P. Show that PD and PE bisect $\angle$BPC.

12. Show how to construct a triangle on a a given base so as to have its vertical angle bisected by a given straight line.

[Hint:— Let the given straight line cut the given base, say BC or the base BC produced at the point D Then D divides BC in the ratio $\frac{BD}{DC}$. Find the point E on BC or BC produced as the case may be, such that $\frac{BE}{CE} = \frac{BD}{DC}$. Draw the circle on DE as diameter to cut the straight line again at A. Then ABC will be the required triangle. The teacher is expected to explain the property of Apollonius' circle.]

13. Construct a parallelogram whose sides are 3·5″, 2 7″ and whose diagonals are in the ratio of 2 : 1.

14. I is the in-centre of the $\triangle ABC$. A straight line drawn through I perpendicular to AI meets AB, AC in D, E respectively. Show that BD. CE $= ID^2$.

15. Two circles of radii a and b touch each other externally at the point k. AB is one of their common tangents, A and B being the points of contact. Show that AB subtends a right angle at k and that $AB^2 = 4\ ab$.

16. PM and QN are the perpendiculars from two given points P and Q to a given straight line AB. PN and QM intersect in R. If RS. be drawn perpendicular to AB, show that PS and QS make equal angles with AB

17 ABC is an isosceles triangle right angled at A. Any point P is taken on AB and BD is drawn perpendicular to BC on the side of BC opposite to A, of such length that $\frac{BD}{BC} = \frac{AP}{AC}$.
Prove that CPD is a right angle and CP=DP.

18. AB is a diameter of a circle. PQ is a parallel chord. The tangent at A meets BP, BQ in R and S respectively. Prove that BP. BR=BQ. BS$=AB^2$.

19. AB is a diameter of a circle and C is any point on the circumference. From a point on AB the perpendicular to AB is drawn cutting CA, CB and the circumference at D, E, F respectively. Prove that $PF^2 = PD. PE$.

20. ABC is a triangle right angled at A. The bisector of the angle A meets the circumcircle in D. Show that $2\ AD^2 = (AB+AC)^2$.

21. PA, PB are tangents from any point P on a circle to a concentric circle lying within it. AB and PB are produced to cut the outer circle in C and D. Show that $CB : CA = CD^2 : CP^2$.

[Hint:— Use that the length of a tangent drawn from any point on the onter circle to the inner concentric circle is constant]

22. Two circles intersect at A and B. The tangents at A meet the circles again in X and Y. Prove that $\triangle ABX : \triangle ABY = (AB^2 + BX^2) : (AB^2 + BY^2)$

23. If from the vertex of a $\triangle$, a perpendicular be drawn to the base, prove that the rectangle contained by the two sides is equal to the rectangle contained by the altitude of the base and the circumdiameter.

Hence deduce that in any triangle, $abc = 4R\Delta$

24. A transversal PQR cuts the sides BC, CA, AB of a $\triangle ABC$ in P, Q, R respectively. Prove that

$$\frac{BP}{PC} \times \frac{CQ}{QA} \times \frac{AR}{RB} = -1$$ (Menelaus' Theorem)

[Hint·— Draw perpendiculars from the vertices of the $\triangle ABC$ to the transversal and consider a set of similar triangles]

25. If one diagonal of a quadrilateral bisects the angle between two of the sides and be a mean propoıtional between them prove that the segments of the other diagonal are in the duplicate ratio of the other sides.

[Hint:— If ABCD is a quadrilateral in which AC bisects angle A and $AC^2 = AB.\ AD$, ΔsDAC and CAB are similar.

Hence $\frac{AD}{AC} = \frac{DC}{BC} = \frac{CA}{AB} \therefore \frac{DC^2}{BC^2} = \frac{AD}{AB} = \frac{DO}{OB}$

where O is the intersection of AC and BD]

26. ABCD is a cyclic quadrilateral. If $AB.\ BC = CD.\ DA$ prove that AC bisects BD.

[Hint:— Draw $\perp$rs BP, DQ on AC from B and D respectively. Then AB. BC = BP. 2R and CD. DA = DQ. 2R where R is the circum-radius. Since AB. BC = CD. DA, BP = DQ. Hence $\triangle$s ODQ and OBP are congruent, where O is the point of intersection of AC and BD.

$\therefore$ OB = OD]

27. The tangent at A to the circumcircle of a triangle ABC meets BC produced at D. Show that $\frac{BD}{DC} = \frac{AB^2}{AC^2}$

Hence deduce that, if the tangents at A, B, C of a $\triangle$ABC to the circumcircle meet the opposite sides in D, E, F respectively, then $\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = -1.$

28. AB is a diameter, of a circle and AM, BN are drawn perpendicular to the tangent at any point C on the circle. Show that the area ABC is the sum of the areas of ACM and CBN.

29. Prove that the opposite angles of a quadrilateral inscribed in a circle are together equal to two right angles. Deduce the following:— The sides AB, DC of a quadrilateral ABCD meet at E and the sides AD, BC meet at F. Prove that the circles circumscribed to the four triangles

ABF, BCE, CDF and DAE have a common point, say, K. Prove further that if ABCD is cyclic, K lies on EF.

30 Draw a $\triangle ABC$, having $AB = 4\cdot2$ cm; $AC = 5\cdot3$ cm; and $\angle ABC = 62°$. Construct a $\triangle$ similar to $\triangle ABC$ so as to have its greatest side less than the sum of the remaining two sides by $1\cdot2$ cm. Explain your construction.

31. Construct an equilateral triangle equal in area to a $\triangle PQR$ having $PQ = 2\cdot5''$, $QR = 3''$ and the $\angle Q = 40°$.

32. Given that $AB = 3\cdot2''$, $C = 75°$ and $\frac{AC}{BC} = 5:3$ show how to construct the triangle.

[Hint:— Construct a segment containing the angle 75° on AB. Divide AB internally and externally in the ratio 5 : 3 (say) at D and E and describe the circle on DE as diameter to cut the segment at C. Then ABC is the required triangle- The circle on DE as diameter becomes the Apollonius' circle of the two fixed points A and B for the ratio $\frac{5}{3}$. In numerical problems like this the alternate method is advisable. Draw a $\triangle AC_1B_1$ such that $C_1A = 5''$, $C_1B_1 = 3''$ and $\angle AC_1B_1 = 75°$. Then take a point B on AB_1 or AB_1 produced if necessary such that $AB = 3\cdot2''$

Through B draw a line parallel to B_1C_1 meeting AC_1 at C. Then ABC is the required triangle]

33. Draw a $\triangle ABC$ given $a : b : c = 3 : 4 : 6$ and $R = 2''$. Measure the sides and angles of the $\triangle$.

[Hint:— Draw a $\triangle PQR$ such that $p = 3''$, $q = 4''$ and $r = 6''$. Find the circumcentre T of $\triangle PQR$. With T as centre and $2''$ as radius draw a circle meeting TP, TQ, TR in A, B, C respectively. Then ABC is the required triangle]

34. ABC is a $\triangle$ in which $a = 1{\cdot}2''$, $b = 1{\cdot}6''$ and $c = {\cdot}9''$.

Draw an isosceles $\triangle$ of equal area having the vertical angle equal to A. Measure its sides.

35. Draw a $\triangle$ ABC having AB = 9 cm; BC = 8 cm; $\angle ABC = 60^\circ$. Construct a triangle equal in area to $\triangle ABC$ and having its sides in the ratio 4 : 5 : 7. State the construction and prove it.

[Hint:— Draw a $\triangle$ whose sides are 4 cm, 5 cm and 7 cm and call it $\triangle PQR$. Then construct a $\triangle$ similar to $\triangle PQR$ and equal in area to $\triangle ABC$]

36. The sides of a $\triangle$ are 3, 5 and 7 cm, Bisect the area of the $\triangle$ by a line drawn (1) perpendicular to (2) parallel to the longest side.

37. Construct an equilateral $\triangle$ which is equal in area to a triangle ABC in which $a = 2{\cdot}6''$, $b = 3''$, $B = 48°$. Measure its sides.

38. Construct a parallelogram ABCD in which $AB = 2{\cdot}5''$, $AD = 2''$, $AC = 3{\cdot}4''$. Divide the parallelogram into three equal parts by straight lines parallel to the diagonal AC.

39. Construct a $\triangle$ whose sides have the ratios 5 : 7 : 9 and whose area is 10 sq. Inches.

[Hint:— Apply $abc = 4R\triangle$, find R and proceed as in exercise No. 33].

40. Draw a $\triangle ABC$ having the sides $a = 2{\cdot}5''$ $b = 2''$ and $c = 1{\cdot}5''$. Construct a similar triangle having two-thirds of the area of the $\triangle ABC$.

41. Construct a parallelogram of area equal to 6 square inches and having its sides in the ratio 3 : 2 and having an angle 70°.

42. The sides of a $\triangle$ are 5, 12 and 13 cm. respectively. Show how to trisect the area of the $\triangle$ by a line drawn parallel to the longest side.

[Hint:— Divide the area of the $\triangle$ into two portions whose areas are in the ratio 1 : 2 (this is trisection) by a straight line parallel to the longest side, as in construction 1.]

43. In a triangle ABC, AP and AQ are drawn perpendicular to the internal bisectors of the angles B and C. Prove that PQ is parallel to BC.

[Hint:— Apply angle chasing method]

44. One circle touches another internally at P. A straight line touches the inner circle at A and meets the outer circle at B and C. Prove that PB : PC = AB : AC

[*Hint:*— Apply, angle between the tangent at a point on a circle and the chord through the point of contact is equal to the angle subtended by that chord in the alternate segment of the circle, repeatedly and prove that PA is a bisector of $\angle$CPB].

45. The tangent to a circle at a point A on it, meets two parallel tangents at B and C. If O is the centre of the circle prove that $OA^2 = AB.\ AC$.

[The points of contact of parallel tangents and the centre O are collinear. Hence OB, OC become bisectors of supplementary angles and are therefore at right angles. i. e. $\angle BOC = 90°$. Further, OA is perpendicular to BC, the hypotenuse. $\therefore\ OA^2 = AB.\ AC$.]

* APPENDIX I.

THEOREM 1.

If two triangles have one angle of the one equal to one angle of the other and the sides about one other angle in each proportional, then the third angles of the two triangles are either equal or supplementary and in the former case the two triangles are similar.

In triangles ABC and PQR let $\angle C = \angle R$ and $\frac{AB}{PQ} = \frac{CA}{RP}$. Then to prove that $\angle B$ and $\angle Q$ are either equai (in which case the two triangles will be similar) or supplementary.

One of the two things may happen for the two $\triangle$s ABC and PQR. $\angle A$ will either be equal to $\angle P$ or will not be equal to $\angle P$. (These are the only possibilities).

If $\angle A$ and $\angle P$ are equal (as in fig. 47 and fig. 48(a)) since $\angle C$ and $\angle R$ are already equal $\angle B$ will be equal to $\angle Q$ i. e. the two $\triangle$s become equiangular and hence similar.

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* This may be omitted by the Pre-university students.

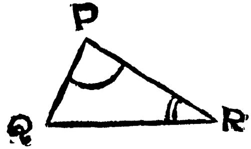


Fig. 48 (a)

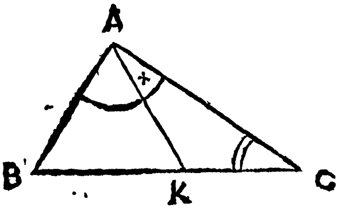


Fig. 47

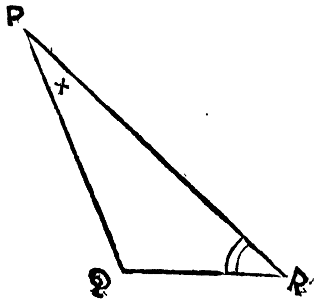


Fig. 48 (b)

If $\angle A \neq \angle P$ (as in fig. 47 and fig. 48(b)) draw a line AK through A meeting BC at K such that $\angle CAK = \angle RPQ$.

[Construction must be so effected that $\triangle$CAK formed includes $\angle C$]

$\angle CAK = \angle RPQ$ (construction).

$\angle KCA = \angle QRP$ (given).

$\therefore$ $\triangle CAK \;|||\; \triangle RPQ$

$$\therefore \quad \frac{CA}{RP} = \frac{AK}{PQ}$$

But $\frac{AB}{PQ} = \frac{CA}{RP}$ (given)

$$\therefore \quad \frac{AK}{PQ} = \frac{AB}{PQ}$$

i. e. $AK = AB$

$\therefore \quad \angle ABK = \angle AKB$

i. e. $\angle ABC = \angle AKB$

$\angle PQR = \angle AKC$ ($\because$ $\triangle RPQ \parallel\!\mid \triangle CAK$

Now $\angle AKB$ and $\angle AKC$ are evidently supplementary angles.

$\therefore$ $\angle ABC$ and $\angle PQR$ are also supplementary angles.

Note:— In fig. 47, AB<CA. Therefore in figures 48(a) and 48(b) PQ is less than RP in order that $\frac{AB}{PQ} = \frac{CA}{RP}$ i. e in order that $\frac{AB}{CA} = \frac{PQ}{RP}$.
This point must be borne in mind while drawing the three figures.

SECOND METHOD.

(figure is not necessary)

Given that in triangles ABC and PQR $\angle C = \angle R$ and $\frac{AB}{PQ} = \frac{CA}{RP}$ to prove that $\angle B$ and $\angle Q$ are either equal (in which case the two triangles will be equiangular and hence similar) or supplementary.

Proof:— Since $\angle C = \angle R$

$\frac{\triangle ABC}{\triangle PQR} = \frac{BC.\ CA}{QR.\ RP}$ ($\because$ In two $\triangle$s if an angle of the one is equal to an angle of the other, their areas are proportional to the rectangles contained by the sides about those equal angles.)

$$\text{But } \frac{CA}{RP} = \frac{AB}{PQ} \text{ (given)}$$

$$\therefore \quad \frac{\triangle ABC}{\triangle PQR} = \frac{BC.\ CA}{QR.\ RP} = \left(\frac{BC}{QR}\right)\left(\frac{CA}{RP}\right)$$

$$= \left(\frac{BC}{QR}\right)\left(\frac{AB}{PQ}\right) = \frac{AB.\ BC}{PQ.\ QR}$$

$$\frac{\triangle ABC}{\triangle PQR} = \frac{\frac{1}{2}\ AB.\ BC.\ \text{Sin } \angle ABC}{\frac{1}{2}\ PQ.\ QR.\ \text{Sin } \angle PQR}$$

(From Trigonometry)

$$\therefore \frac{AB.\ BC}{PQ.\ QR} = \frac{AB.\ BC.\ \text{Sin}\ \angle ABC}{PQ.\ QR.\ \text{Sin}\ \angle PQR}$$

Hence $\frac{\text{Sin}\ \angle ABC}{\text{Sin}\ \angle PQR} = 1$ or $\text{Sin}\ \angle ABC = \text{Sin}\ \angle PQR$

$\therefore$ either $\angle ABC = \angle PQR$ or $\angle ABC = 180° - \angle PQR$
(each angle of a $\triangle$ is less than 180°)

i. e either $\angle B = \angle Q$ or $\angle B + \angle Q = 180°$

If $\angle B = \angle Q$, as $\angle C = \angle R$ already, the two $\triangle$s ABC and PQR will be equiangular and hence similar.

EXERCISES.

1. A and B are the centres of two circles whose radii are respectively r_1 and r_2. If S is a point dividing AB internally in the ratio $\frac{r_1}{r_2}$ prove that any secant through S cuts the circles at the extremities of parallel radii one in each circle.

[Hint:— If a secant cuts the circle A at P, Q and circle B at P^1, Q^1 (draw a figure and in it take Q, Q^1 within P, P^1 or P, P^1 within Q, Q^1) consider $\triangle$s SAP and SBP^1. To rule out the possibility of the third set of angles in the two triangles becoming supplementary use that the base angles of any isosceles triangle are acute angles and that two angles which are both acute or both obtuse cannot be supplementary. $\triangle$ SAP ||| $\triangle$ SBP^1. $\therefore$ AP $\parallel$ BP^1. Similarly consider $\triangle$s SAQ and SBQ^1. They are similar. $\therefore$ AQ $\parallel$ BQ^1. Hence the result. The point S is usually called the internal centre of similitude of the two circles A and B.]

2. Prove the above result for a point S^1 dividing AB externally in the ratio $\frac{r_1}{r_2}$.

[This point S^1 is usually called the external centre of similitude of the two circles A and B.

3. A and B are the centres of two circles whose radii are respectively r_1 and r_2. S and S^1 are points dividing AB internally and externally in the ratio $\frac{r^1}{r_2}$. Prove that the lengths of the tangents drawn from any point on the circle on SS^1 as diameter to the two circles A and B are in the ratio $\frac{r_1}{r_2}$.

[Hint:— Use that the circle on SS^1 as diameter is the Apollonius' circle of the two fixed points

A, B for the constant ratio $\frac{r_1}{r_2}$. Now, if P is any point on this circle, $\frac{PA}{PB} = \frac{r_1}{r_2}$. Let PL, PM be the tangents from P to the two circles A and B.

Then $\frac{PA}{PB} = \frac{r_1}{r_2} = \frac{AL}{BM}$

Further $\angle PLA = \angle PMB = 90°$.

$\therefore$ $\angle APL$ and $\angle BPM$ are either equal or Supplementary. But these two angles being both acute, cannot be supplementary. Hence they are equal and therefore $\triangle ALP \ ||| \ \triangle BMP$

$\therefore \frac{PL}{PM} = \frac{AL}{BM} = \frac{r_1}{r_2}$. The circle on SS^1 as diameter is usually called the circle of similitude of the two circles A and B]

The application of theorem 1 given in appendix I will also be found in the Geometry of the conic while proving the theorem that the tangent at any point on a central conic is equally inclined to the focal distances of the point.

THEOREM 2.

In equal circles, angles whether at the centres or at the circumferences, are in the ratio of the arcs on which they stand.

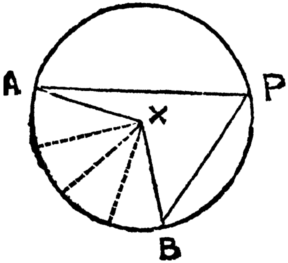


Fig. 49

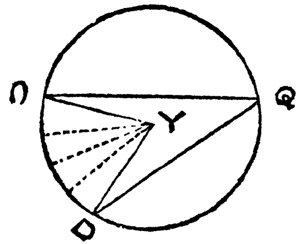


Fig. 50

Let APB and CQD be two equal circles whose centres are X and Y respectively; Also let AXB, CYD be the angles at the centres and APB, CQD those at the circumferences, standing on the arcs AB and CD respectively.

Then to prove that

(i) $\frac{\angle AXB}{\angle CYD} = \frac{\text{Arc } AB}{\text{Arc } CD}$

(ii) $\frac{\angle APB}{\angle CQD} = \frac{\text{Arc } AB}{\text{Arc } CD}$

Proof:— Suppose there is a common measure, say, k for the lengths of the two arcs AB and CD. [Making the two arcs straight, suppose we measure them and find AB to be 5 inches and CD to be 3 inches. This implies that Arc AB can be divided into 5 equal parts each equal to 1 inch.

and that arc CD can be divided into 3 equal parts each equal to 1 inch. If this is so, the common measure of the two arcs AB and CD is an inch and the ratio of AB to CD is $\frac{5}{3}$. Now, if AB=4·5 inches and CD=3 inches the common measure for the two is ·5 of an inch, for, AB in this case can be divided into 9 equal parts each equal to ·5 of an inch and CD can be divided into 6 equal parts each equal to ·5 of an inch. The ratio of AB to CD in this case is $\frac{9}{6}$ $\left(=\frac{4\cdot5}{3}\right)$. Hence whatever may be the lengths of the two arcs AB and CD it will be quite possible to find out a common measure for the two and so our supposition is justified]

Let the arc AB be divided into p equal arcs each equal to k and arc CD be divided into q such equal arcs (each equal to k).

In each circle, let radii be drawn through the points of division of the arcs AB and CD.

Then, $\angle$AXB is divided into p equal angles each equal to θ (say) and $\angle$CYD is divided into q such equal angles (each equal to θ). [$\because$ In equal circles, equal arcs subtend equal angles at the centres. This theorem is also true in the same circle.]

$$\therefore \quad \angle AXB = p\theta; \ \angle CYD = q\theta$$

So $\dfrac{\angle AXB}{\angle CYD} = \dfrac{p\theta}{q\theta} = \dfrac{p}{q}$

But Arc AB $=$ pk and Arc CD $=$ qk and hence

$$\frac{\text{Arc AB}}{\text{Arc CD}} = \frac{pk}{qk} = \frac{p}{q}$$

$$\therefore \quad \frac{\angle AXB}{\angle CYD} = \frac{\text{Arc AB}}{\text{Arc CD}}$$

Now $\angle APB = \frac{1}{2}\angle AXB$ and $\angle CQD = \frac{1}{2}\angle CYD$.
[$\because$ The angle subtended by an arc of a circle at the centre is equal to double the angle subtended by that arc at any point on the circumference]

$$\therefore \quad \frac{\angle APB}{\angle CQD} = \frac{\frac{1}{2}\angle AXB}{\frac{1}{2}\angle CYD} = \frac{\angle AXB}{\angle CYD} = \frac{\text{Arc AB}}{\text{Arc CD}}.$$

Note:— This theorem is true in the same circle also.

WORKED EXAMPLE.

Prove that a radian is a constant angle.

Radian by definition is an angle subtended at the centre of any circle by an arc equal in length to the radius of the circle.

This definition does not explicitly tell us that rdian is a constant angle. If radian is not a constant angle we won't be able to accept radian as a unit of measurement for angles. So it is imperative to prove that radian is a constant angle. It is because right angle is a constant angle (all right angles are equal) that we accept right angle as a unit of measurement for angles. Any angle can be measured in terms of right angles and their parts.

(1 degree $=\frac{1}{90}$th of a right angle, 1 minute $=\frac{1}{60}$th of a degree and 1 second $=\frac{1}{60}$th of a minute). When we say that an angle is equal to (say) 240 degrees, this statement actually means that the angle measured is equal to 2 right angles plus 60 degrees or is equal to $2\frac{2}{3}$ right angles. We will have to note that radians are not divided and subdivided like right angles.

All right angles are equal (this is axiomatic). Hence we can say that right angle is a constant angle. In the case of radians, if it is possible to prove that all radians are equal, it will follow that radian is a constant angle. Suppose OA and OB are two radii of a circle whose centre is O, such that Arc AB $=$ OA $=$ OB $=$ r where r is the radius of the circle. Producing OA and OB to meet a concentric circle of radius r_1 at A_1 and B_1 if we are able to prove that arc $A_1 B_1 = OA_1 = OB_1 = r_1$, then it will follow that radian is a constant angle. But it is not very easy to establish this result. Hence we shall proceed with a circle and find out the value of a radian defined in terms of the radius and arc, equal to the radius, of this circle, in degrees and thus see that the value of a radian has nothing to do with the radius of the circle taken to define it.

Let OA and OB be two radii of a circle, centre O and radius r, such that arc AB = r. (Draw a figure). Then by definition $\angle AOB$ is a radian. Produce AO to meet the circle again at the point **K**.

Then, $\text{arc } AK = \frac{2\pi r}{2} = \pi r.$

Hence $\frac{\angle AOK}{\angle AOB} = \frac{\text{Arc } AK}{\text{Arc } AB}$ (Vide note given under theorem 2, appendix I)

i. e. $\frac{\angle AOK}{\text{a radian}} = \frac{\pi r}{r} = \frac{\pi}{1}$

$\therefore \quad \angle AOK = \pi$ of a radian or π radians.

But $\angle AOK = 180°$.

Hence $180° = \pi$ radians.

(π represents only a number whose approximate value is $\frac{22}{7}$ and hence the student must not try to replace π by 180°. The equation 180 degrees = π radians implies that an angle measured in degrees and found to be 180° when measured in radians will be equal to π radians. It is important to remember that the value or magnitude of an angle does not change whether it is measured in degrees or in radians. So there must have a relationship between degrees and radians and that relationship is given above.)

$$\pi \text{ radians} = 180^\circ$$

$$\therefore \quad \text{a radian} = \frac{180^\circ}{\pi}$$

(||| ly starting with another circle we can prove that, a radian $= \dfrac{180^\circ}{\pi}$ i. e. this result is independent of the radius of the circle, taken)

Hence a radian is a constant angle.

The value of a radian is approximately equal to 57° 17′ 45″ (taking the approximate value of π = 3 14159) — [The ratio of the circumference to the diameter is found to be the same for all circles i. e. this ratio is a constant. This constant ratio (a number) though its value cannot be actually found out is represented by π. Since its value cannot be found out, this number π (this number represented by π) is called a transcendental number]

Thus we have proved that a radian is a constant angle and henceforth we can talk of "the radian"

$$\pi \text{ radian} = 180^\circ$$

$$\therefore \quad 2\pi \text{ radians} = 360^\circ$$

So, if equal arcs each equal to the radius are cut off from the circumference of a circle we will be getting six such equal arcs each subtending a radian at the centre of the circle. The remaining portion r $(2\pi-6)$ of the circumference will be subtending an angle equal to $(2\pi-6)$ of a radian at the centre.

Now if OA, OB are two radii of a circle, centre O and radius r such that arc AB = r and OC is any other radius, as before,

$$\frac{\angle AOC}{\angle AOB} = \frac{\text{Arc } AC}{\text{Arc } AB}$$

i. e. $$\frac{\angle AOC}{\text{a radian}} = \frac{\text{Arc } AC}{r}$$

$$\therefore \quad \angle AOC = \left(\frac{\text{Arc } AC}{r}\right) \text{ of a radian}$$

Result 1:— From this it follows that the length of an arc of a circle divided by the radius of the circle gives the magnitude of the angle which the arc subtends at the centre in radians.

Result 2:— When two straight lines meet they are said to contain an angle. Suppose, two straight lines OX and OY meet at O and that we want to measure the angle which they contain, viz. $\angle$XOY in radians. For this, draw a circle with O as centre and any length as radius to cut OX at A and OY at B. Measure the length of the arc AB and also the radius of the circle drawn, both in the same units of measurement.

Then $\dfrac{\text{the length of the arc AB}}{\text{radius of the circle drawn}}$ will give the magnitude of the angle XOY in radians.

Result 3:— The length of an arc of a circle is equal to the product of its radius and the rodian measure of the angle the arc subtends at the centre of the circle.

APPENDIX II

NEDIANS OF A PLANE TRIANGLE.

Definition of medians has been recently extended by Mr. John Satterly, thus: "If the sides of a triangle are, in order, divided such that the short section of each side is $\frac{1}{n}$th the length of the side, and the points of subdivision are joined to the opposite angular points, the joining lines may be called the Nedians of the triangle (the name recalls the n) and the triangle formed by the nedians may be called the nedian triangle". We get the medians from this definition of nedians by putting $n=2$.

Again according to Satterly if A_1, B_1, C_1 are points on the sides BC, CA, AB respectively of a $\triangle ABC$ such that $\frac{BA_1}{BC}=\frac{CB_1}{CA}=\frac{AC_1}{AB}=\frac{1}{n}$ (where BA_1, CB_1, AC_1 are short sections of the sides BC, CA, AB respectively) then AA_1, BB_1, CC_1 are the forward nedians and the triangle formedby them is the forward nedian triangle; If A_2, B_2, $C_2$are points on the sides BC, CA, AB respectively such that

* This may be omitted by the Pre-university students.

$\frac{CA_2}{CB} = \frac{BC_2}{BA} = \frac{AB_2}{AC} = \frac{1}{n}$ (where CA_2, BC_2, AB_2, are short sections of the sides CB, BA, AC respectively) then AA_2, BB_2, CC_2 are the backward nedians and the triangle formed by them is the backward nedian triangle. When $n=2$, A_1 and A_2 both coincide with the middle point of BC. Hence AA_1, AA_2 both coincide with the median through the vertex A. ||| ly BB_1, BB_2, both coincide with the median through the vertex B and CC_1, CC_2 both coincide with the median through the vertex C. So the medians can be regarded both as forward nedians and as backward nedians. It is interesting to note that in two similar triangles corresponding nedians are also in the ratio of corresponding sides.

THEOREM 1

(Satterly's Theorem)

If AA_1, BB_1, CC_1 are Nedians (either forward nedians or backward nedians) to the sides BC, CA, AB respectively, of $\triangle ABC$, then

$$AA_1^2 + BB_1^2 + CC_1^2 = \left(\frac{n^2 - n + 1}{n^2}\right)(AB^2 + BC^2 + CA^2)$$

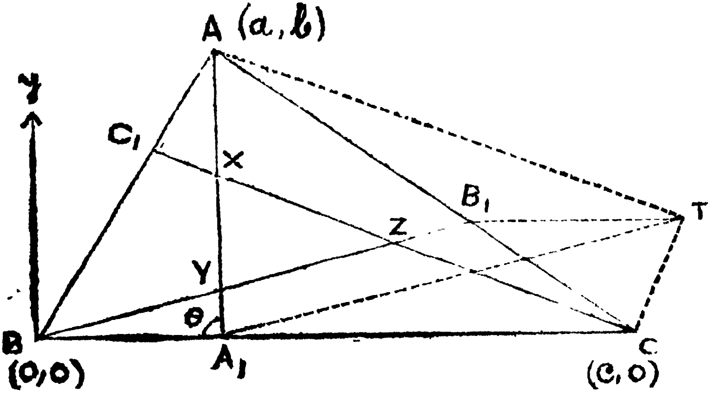


Fig. 51

Let us suppose that AA_1, BB_1, CC_1 are the forward nedians so that

$$\frac{BA_1}{BC} = \frac{CB_1}{CA} = \frac{AC_1}{AB} = \frac{1}{n}$$

from this, $\dfrac{BA_1}{A_1C} = \dfrac{CB_1}{B_1A} = \dfrac{AC_1}{C_1B} = \dfrac{1}{n-1}$

i. e. $(n-1)\ BA_1 = A_1C$; $(n-1)\ CB_1 = B_1A$; $)n-1)\ AC_1 = C_1B$.

If $\angle AA_1B = \theta$, $\angle AA_1C = (180 - \theta)$

$\therefore AB^2 = AA_1^2 + BA_1^2 - 2AA_1 . BA_1 . \cos\theta$ (1) and

$CA^2 = AA_1^2 + A_1C^2 + 2AA_1 . A_1C . \cos\theta$ (2)

Multiplying (1) by $(n-1)$ and then adding to (2), we get, $(n-1)\ AB^2 + CA^2 = nAA_1^2 + (n-1)\ BA_1^2 + A_1C^2$

But $BA_1 = \dfrac{BC}{n}$ and $A_1C = \dfrac{n-1}{n} BC$.

$[\ (n-1)\ BA_1 = A_1C\ ]$

$$\therefore\ (n-1)\ AB^2 + CA^2 = nAA_1^2 + (n-1)\ \frac{BC^2}{n^2} + \frac{(n-1)^2}{n^2} BC^2$$

$$= nAA_1^2 + \frac{n-1}{n^2} BC^2\ [\ 1 + n - 1\ ]$$

$$= nAA_1^2 + \frac{n-1}{n} BC^2$$

i. e. $(n-1)AB^2 + CA^2 = nAA_1^2 + \left(1 - \frac{1}{n}\right)BC^2$ ...I

|||ly $(n-1)\ BC^2 + AB^2 = nBB_1^2 + \left(1 - \frac{1}{n}\right)CA^2$ ...II

and $(n-1)\ CA^2 + BC^2 = nCC_1^2 + \left(1 - \frac{1}{n}\right)AB^2$ ...III

I + II + III gives,

$$n\,[\,AB^2 + BC^2 + CA^2] = n\,[AA_1^2 + BB_1^2 + CC_1^2] + \left(1 - \frac{1}{n}\right)(AB^2 + BC^2 + CA^2)$$

i. e. $(AB^2+BC^2+CA^2)\left(n+\frac{1}{n}-1\right)=n\,(AA_1{}^2+BB_1{}^2+CC_1{}^2)$

$\therefore\ AB^2+BC^2+CA^2=\left(\frac{n^2}{n^2-n+1}\right)(AA_1{}^2+BB_1{}^2+CC_1{}^2)$

i. e. $AA_1{}^2+BB_1{}^2+CC_1{}^2=\left(\frac{n^2-n+1}{n^2}\right)(AB^2+BC^2+CA^2)$.

If the lengths of the nedians AA_1, BB_1, CC_1 are respectively represented by n_1, n_2, n_3,

$$n_1{}^2+n_2{}^2+n_3{}^2=\left(\frac{n^2-n+1}{n^2}\right)(a^2+b^2+c^2)$$

where a, b, c as usual denote the lengths of the sides BC, CA and AB, respectively, of $\triangle ABC$

$$\text{or}\quad \Sigma n_1{}^2=\left(\frac{n^2-n+1}{n^2}\right)\Sigma a^2$$

As a particular case, if AA_1, BB_1, CC_1 are the medians $(n=2)$

$$AA_1{}^2+BB_1{}^2+CC_1{}^2=\tfrac{3}{4}\,(AB^2+BC^2+CA^2)$$

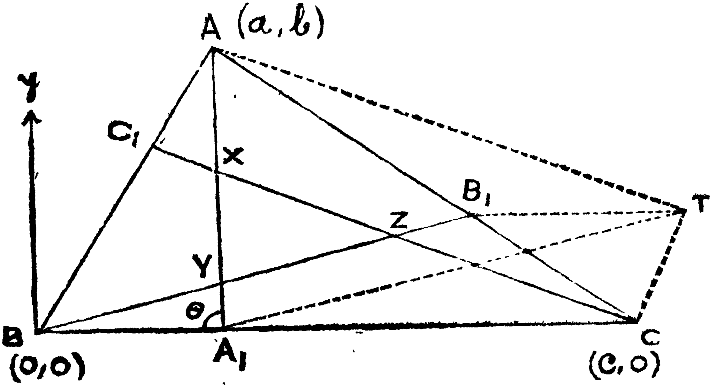
(This result can also be proved separately as in the case of nedians).

Note:— The result is true in the case of backward nedians also.

THEOREM 2.

(Satterly's Theorem)

The area of a triangle whose sides are equal to the lengths of the nedians of a triangle is $\left(\frac{n^2-n+1}{n^2}\right)$ times the area of the original triangle.



ABC is the original triangle and AA_1. BB_1, CC^1 are its nedians to the sides BC, CA, AB respectively.

Draw a line through B_1, parallel to BC and another line through C parallel to AB. Let these two lines meet at the point T. Join A, T: A_1,T. Then we shall first prove that $\triangle AA_1T$ is a triangle whose sides are equal to the lengths of the nedians and further that the area of $\triangle AA_1T$ is equal to $\left(\frac{n^2-n+1}{n^2}\right)$ times the area of the triangle ABC.

Proof (Analytical proof):— Let line BC be taken as the x-axis and the perpendicular at B to BC be taken as the y-axis. Also let the x and y co-ordinates of the point A be a and b respectively and the x co-ordinate of the point C be c. (The y co-ordinate of the point C is evidently zero).

The point A_1 divides BC in the radio $1:n-1$ $\left(\frac{BA_1}{BC}=\frac{1}{n}\right)$ $\therefore$ The point A_1 is $\left(\frac{c}{n}, 0\right)$ |||ly the point B_1 is $\left[\frac{a+c(n-1)}{n}, \frac{b}{n}\right]$ and the point C_1 is $\left[\frac{a(n-1)}{n}, \frac{b(n-1)}{n}\right]$.

Equation of the line B_1T which is passing through the point B_1 $\left[\frac{a+c(n-1)}{n}, \frac{b}{n}\right]$ and is parallel to the x-axis is $y=\frac{b}{n}$ I

Slope of line CT is the same as that of AB (CT || AB) and is equal to $\frac{b}{a}$

Hence equation of the line CT is $\frac{y-0}{x-c} = \frac{b}{a}$ or

$$ay = bx - bc$$

$$\therefore bx = ay + bc \quad ......II$$

Solving I and II we get the co-ordinates of the point T

$$bx = \frac{ab}{n} + bc$$

$$\therefore \quad x = \frac{a+cn}{n}$$

$$y = \frac{b}{n}$$

$\therefore$ The point T is $\left(\frac{a+cn}{n}, \frac{b}{n}\right)$

The slope of TA is equal to $\dfrac{\frac{b}{n} - b}{\frac{a+cn}{n} - a} = \dfrac{b - nb}{a + n(c-a)}$

The slope of $CC_1 = \dfrac{\left[\frac{b\,(n-1)}{n} - 0\right]}{\left[\frac{a\,(n-1)}{n} - c\right]} = \dfrac{nb - b}{n\,(a-c) - a}$

$$= \frac{b - nb}{a + n\,(c-a)}$$

$\therefore$ TA $\parallel$ CC_1. But CT $\parallel$ AB (Construction)

Hence TA C_1C is a ▱.

$\therefore$ $TA = CC_1$

The slope of $A_1T = \dfrac{\left(\frac{b}{n} - 0\right)}{\left(\frac{a+cn}{n} - \frac{c}{n}\right)} = \dfrac{b}{a+c(n-1)}$

The slope of $BB_1 = \dfrac{\left(\frac{b}{n} - 0\right)}{\left\{\frac{a+c\,(n-1)}{n} - 0\right\}}$

$$= \frac{b}{a+c\,(n-1)}$$

$\therefore A_1T \parallel BB_1$. But $B_1T \parallel BC$ (construction)
Hence $TB_1\ BA_1$ is a ▱ $\therefore\quad A_1T = BB_1$

The nedian AA_1 is already a side of the $\triangle AA_1T$. Hence the sides of the $\triangle\ AA_1T$ are equal to the lengths of the nedians of the $\triangle ABC$.

By the formula, $\triangle = \frac{1}{2} \Sigma x_1 (y_2 - y_3)$,
we get, area of the $\triangle AA_1T =$

$$\tfrac{1}{2}\left[a\left((0 - \frac{b}{n}\right) + \frac{c}{n}\left(\frac{b}{n} - b\right) + \left(\frac{a+cn}{n}\right)(b-0) \right]$$

$$= \tfrac{1}{2}\left[-\frac{ab}{n} + \frac{bc}{n^2} - \frac{bc}{n} + \frac{ab}{n} + bc \right]$$

$$= \frac{bc}{2}\left[\frac{n^2 - n + 1}{n^2}\right]$$

But area of $\triangle ABC = \frac{1}{2} \times BC \times$ (altitude to the base BC)

$$= \tfrac{1}{2} \times c \times b = \frac{bc}{2}.$$

Hence, area of $\triangle AA_1T = \left(\frac{n^2-n+1}{n^2}\right)$ (area of the $\triangle ABC$)

Note the common foctor $\frac{n^2-n+1}{n^2}$ in the two theorems

As a particular case, the area of a $\triangle$ whose sides are equal to the lengths of the medians of a triangle is $\frac{3}{4}$ times the area of the original triangle, for the nedians AA_1, BB_1, CC_1 become medians when $n=2$.

(This result can also be separately proved as in the previous case).

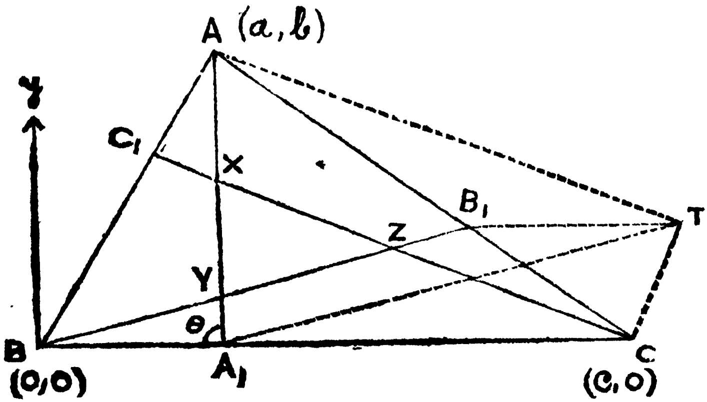
Def:— The triangle formed by the nedians of a triangle is called the nedian triangle of that triangle.

In the figure, XYZ is the triangle formed by the nedians AA_1, BB_1, CC_1 of $\triangle ABC$ and hence is the nedian triangle (Strictly speaking it is the forward nedian triangle, as AA_1, BB_1, CC_1 are forward nedians.) of $\triangle ABC$

THEOREM 3

(Satterly's Theorem)

The sum of the squares of the sides of the nedian triangle of a triangle is equal to $\frac{(n-2)^2}{n^2-n+1}$ times the sum of the squares of the sides of the original triangle.



Equation of line AA_1 is, $nbx-(na-c)\,y=bc$ ...I

Equation of line BB_1 is, $y=\left(\frac{b}{a+cn-c}\right)x$ II

Equation of line CC_1 is, $x\,(nb-b)+y\,(a+cn-an)$
$=nbc-bc$ ...III

Solving I and III we get the co-ordinates of the point X.

$\therefore$ The point X is $\left[\frac{a(n-1)^2+c}{n^2-n+1}, \frac{b(n-1)^2}{n^2-n+1}\right]$

Solving I and II we get the co-ordinates of the point Y

$\therefore$ The point Y is $\left[\frac{a+c(n-1)}{n^2-n+1}, \frac{b}{n^2-n+1}\right]$

Now AX, AY and AA_1, are in the ratio of the perpendicular distances of the points X, Y and A_1 from the line through A perpendicular to BC. ($\because$ Corresponding sides in similar triangles are proportional)

i. e. $AX:AY:AA_1 :: \left(\frac{a(n-1)^2+c}{n^2-n+1} - a\right)$

$$: \left(\frac{a+c(n-1)}{n^2-n+1} - a\right) : \left(\frac{c}{n} - a\right)$$

i. e. $AX:AY:AA_1 :: \frac{1}{n^2-n+1} : \frac{n-1}{n^2-n+1} : \frac{1}{n}$

$\therefore$ $\frac{AY}{AX} = \frac{n-1}{1}$ Henee $\frac{XY}{AX} = \frac{n-2}{1}$ ...... (1)

$\frac{AX}{AA_1} = \frac{n}{n^2-n+1}$ .... (2)

(1) $\times$ (2) gives $\frac{XY}{AA_1} = \frac{n(n-2)}{n^2-n+1}$

If AA_1, BB_1, CC_1 are represented by n_1, n_2, n_3, and YZ, ZX, XY by x, y, z,
$A_1T = BB_1 = n_2$; $TA = CC_1 = n_3$

Now $\triangle$s XYZ and AA_1T are similar.

$$\therefore \frac{XY}{AA_1} = \frac{YZ}{A_1T} = \frac{ZX}{TA}$$

$$\text{i. e.} \quad \frac{z}{n_1} = \frac{x}{n_2} = \frac{y}{n_3} = \frac{n(n-2)}{n^2-n+1}$$

$$\therefore \quad x^2+y^2+z^2 = \left\{ \frac{n(n-2)}{n^2-n+1} \right\}^2 \cdot (n_1^2+n_2^2+n_3^2)$$

$$\text{But} \quad n_1^2+n_2^2+n_3^2 = \left(\frac{n^2-n+1}{n^2} \right) \times (a^2+b^2+c^2)$$

where a, b, c represent the sides BC, CA, AB of $\triangle$ABC. *(by theorem 1 Appendix II)*

$$\therefore \quad x^2+y^2+z^2 = \left\{ \frac{n(n-2)}{n^2-n+1} \right\}^2 \times \frac{(n^2-n+1)}{n^2} \times (a^2+b^2+c^2)$$

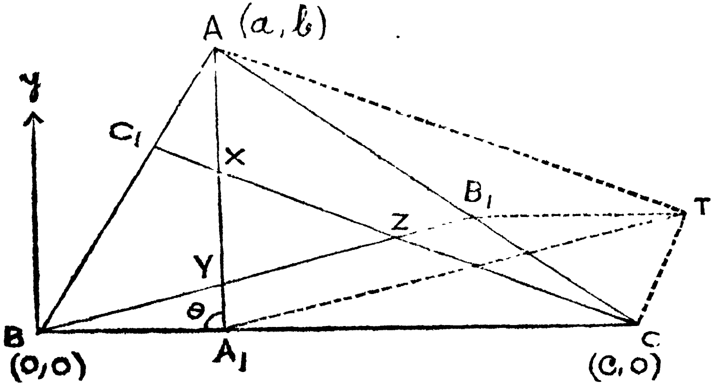
$$= \left\{ \frac{(n-2)^2}{n^2-n+1} \right\} (a^2+b^2+c^2)$$

Note:— When AA_1, BB_1, CC_1 become the medians (n = 2). $x^2+y^2+z^2 = 0$ $\therefore$ $x = 0$, $y = 0$, $z = 0$. Hence the medians form a 'point triangle' or in other words, the medians of a triangle are concurrent.

THEOREM 4

(Satterly's Theorem)

The area of the nedian triangle of a triangle is equal to $\frac{(n-2)^2}{n^2-n+1}$ times the area of the original triangle.



$$\frac{\Delta XYZ}{\Delta AA_1T} = \frac{z^2}{n_1^2} = \frac{x^2}{n_2^2} = \frac{y^2}{n_3^2} = \frac{x^2+y^2+z^2}{n_1^2+n_2^2+n_3^2}$$

$$= \left\{\frac{n(n-2)}{n^2-n+1}\right\}^2$$

$\therefore$ Area of $\Delta XYZ = \left\{\frac{n(n-2)}{n^2-n+1}\right\}^2$ (area of $\Delta\ AA_1T$)

But area of $\Delta AA_1T = \left(\frac{n^2-n+1}{n^2}\right)$ (area of triangle ABC)

Hence, area of $\triangle XYZ$

$$= \left\{ \frac{n(n-2)}{n^2-n+1} \right\}^2 \left(\frac{n^2-n+1}{n^2} \right) \cdot \text{(area of triangle ABC)}$$

$$= \left\{ \frac{(n-2)^2}{n^2-n+1} \right\} \text{(Area of triangle ABC)}$$

Note:— When AA_1, BB_1, CC_1, become the medians ($n=2$), area of the triangle formed by these three lines i. e. area of $\triangle XYZ=0$ i. e. the medians of a $\triangle$ are concurrent.

Note the common factor $\frac{(n-2)^2}{n^2-n+1}$ in the two theorems.

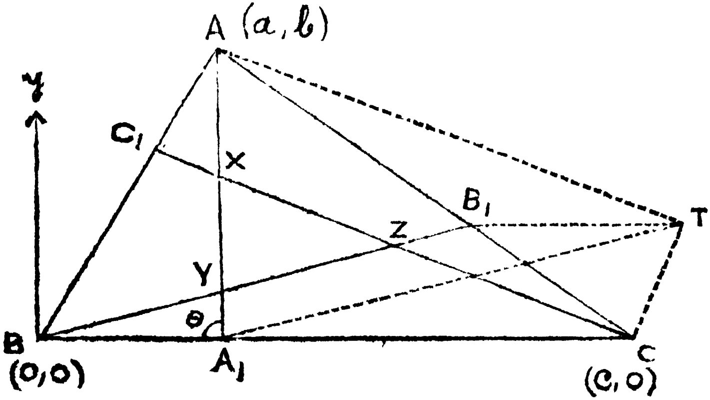
Definition:— If A_1, B_1, C_1 are points on the sides BC, CA, AB respectively of a $\triangle$ ABC such that $\frac{BA_1}{BC} = \frac{CB_1}{CA} = \frac{AC_1}{AB} = \frac{1}{n}$ the triangle formed by the lines joining the points A_1, B_1 and C_1 is called the Aliquot triangle or the $\frac{1}{n}$ th point-division triangle of $\triangle$ABC.

THEOREM 5.

(Satterly's Theorem)

The sum of the squares of the sides of the aliquot triangle of a triangle is equal to

$\left(\frac{n^2-3n+3}{n^2}\right)$ times the sum of the squares of the sides of the original triangle.



Join A_1, B_1; B_1, C_1 and C_1, A_1.

From $\Delta\ AB_1\ C_1$ we get

$B_1C_1{}^2 = AB_1{}^2 + AC_1{}^2 - 2AB_1 .\ AC_1 .\ \cos A.$

If a_1, b_1, c_1 denote the lengths of the sides B_1C_1, C_1A_1 and A_1B_1 respectively of the Aliquot triangle $A_1B_1C_1$,

$$a_1{}^2 = \frac{(n-1)^2}{n^2}\ b^2 + \frac{c^2}{n^2} - \frac{2(n-1)}{n^2}\ bc \cos A \ldots\ldots\ldots(1)$$

$$\left(\because\ \frac{AB_1}{AC} = \frac{n-1}{n};\ \frac{AC_1}{AB} = \frac{1}{n}\right)$$

$$|||\text{ly}\ \ b_1{}^2 = \frac{(n-1)^2}{n^2} c^2 \quad \frac{a^2}{n^2} - \frac{2(n-1)}{n^2}\ ca \cos\ B \ldots(2)$$

and $c_1^2 = \frac{(n-1)^2}{n^2}a^2 + \frac{b^2}{n^2} - \frac{2(n-1)}{n^2}$ ab cos C...(3)

(1) + (2) + (3) gives,

$$a_1^2+b_1^2+c_1^2=(a^2+b^2+c^2)\left\{\frac{(n-1)^2}{n^2}+\frac{1}{n^2}\right\} - \frac{(n-1)}{n^2}(2\,bc\cos A+2\,ca\cos B+2\,ab\cos C)$$

$$=(a^2+b^2+c^2)\left\{\frac{(n-1)^2}{n^2}+\frac{1}{n^2}\right\} - \frac{(n-1)}{n^2}(a^2+b^2+c^2)$$

$(\because\ a^2=b^2+c^2-2\,bc\cos A$ etc.)

i. e. $a_1^2+b_1^2+c_1^2$

$$=(a^2+b^2+c^2)\left\{\frac{(n-1)^2}{n^2}+\frac{1}{n^2}-\frac{(n-1)}{n^2}\right\}$$

$$=\left(\frac{n^2-3n+3}{n^2}\right)(a^2+b^2+c^2)$$

As a particular case, AA_1, BB_1, CC_1 become medians ($n=2$),

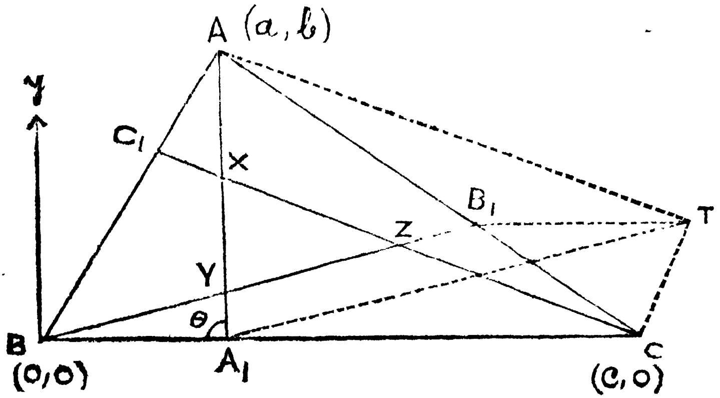
$$a_1^2+b_1^2+c_1^2=\tfrac{1}{4}(a^2+b^2+c^2)$$

i. e. the sum of the squares of the sides of the medial triangle of a triangle is $\frac{1}{4}$th of the sum of the squares of the sides of the original triangle. **This result is even otherwise evident. (The △ formed by the lines joining the mid-points of the sides of a △ is called the medial triangle of that triangle.)**

THEOREM 6

(Satterly's Theorem)

The area of the aliquot triangle of a triangle is equal to $\left(\frac{n^2-3n+3}{n^2}\right)$ times the area of the original triangle.



Area of $\triangle C_1B_1A = \frac{1}{2}AB_1 . AC_1 . \text{Sin } A$

$$= \frac{1}{2}\frac{(n-1)}{n} . b . \frac{c}{n} \text{Sin } A$$

$$= \frac{(n-1)}{n^2} . \frac{1}{2} bc \text{ Sin } A \quad ...(1)$$

|||ly Area of $\triangle A_1C_1B = \frac{(n-1)}{n^2} . \frac{1}{2} ca \text{ Sin } B \quad ...(2)$

and Area of $\triangle B_1A_1C = \frac{(n-1)}{n^2} . \frac{1}{2} ab \text{ Sin } C \quad (3)$

(1)+(2)+(3) gives

Area of $\triangle$ ABC − Area of $\triangle A_1B_1C_1$

$$=\frac{(n-1)}{n^2}\{\tfrac{1}{2}\,bc\ \text{Sin}\ A+\tfrac{1}{2}\,ca\ \text{Sin}\ B+\tfrac{1}{2}ab\ \text{Sin}\ C\}$$

But $\frac{1}{2}$ bc Sin A $=\frac{1}{2}$ca Sin B $=\frac{1}{2}$ ab Sin C = area of $\triangle$ABC

$\therefore$ Area of $\triangle$ABC — Area of $\triangle A_1B_1C_1$

$$=\frac{3(n-1)}{n^2}\ (\text{area of}\ \triangle\ ABC)$$

$\therefore$ Area of $\triangle A_1B_1C_1=\left\{1-\frac{3(n-1)}{n^2}\right\}$ (area of $\triangle$ABC)

$$=\left(\frac{n^2-3n+3}{n^2}\right)\ (\text{area of}\ \triangle ABC)$$

As a particular case, the area of the medial triangle of a triangle is $\frac{1}{4}$ th of the area of the original $\triangle$ (Put n = 2). This result is even otherwise evident.

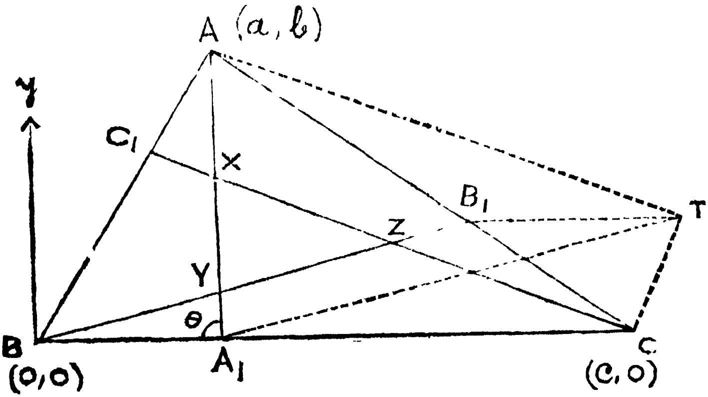
Note the common factor $\frac{n^2-3n+3}{n^2}$ in the above two theorems.

THEOREM 7.

(Pappus' Theorem)

If A_1, B_1, C_1 are points on the sides BC, CA, AB respectively of $\triangle$ ABC such that $\frac{BA_1}{BC}=\frac{CB_1}{CA}=\frac{AC_1}{AB}=\frac{1}{n}$, the centroids of the two triangles $A_1B_1C_1$ and ABC are one and the same point.

(Note that $A_1B_1C_1$ by definition, becomes the aliquot triangle of $\triangle ABC$)



As the centroid of a triangle is a point of trisection of each median, if the vertices of a triangle are given to be the points (x_1, y_1), (x_2, y_2), (x_3, y_3), its centroid will be the point

$$\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right).$$

Applying this, we easily see that the centroid of $\triangle$ ABC is the point $\left[\frac{a+c}{3}, \frac{b}{3}\right]$

ly the centroid of the $\triangle A_1B_1C_1$ is the point

$$\frac{\frac{c}{n}+\frac{a+c(n-1)}{n}+\frac{a(n-1)}{n}}{3}, \frac{o+\frac{b}{n}+\frac{b(n-1)}{n}}{3}$$

i. e. it is the point $\left[\frac{a+c}{3}, \frac{b}{3}\right]$
which is the same as the centroid of $\triangle$ABC

THEOREM 8

(Satterly's Theorem)

The centroids of the nedian triangle and the aliquot triangle of a triangle and that of the original triangle are one and the same point.

By Pappus' theorem, the centroids of the aliquot triangle and the original triangle are one and the same point. We have found that this common centroid for the $\triangle$ABC and its aliquot triangle $A_1B_1C_1$ is $\left[\frac{a+c}{3}, \frac{b}{3}\right]$

So prove that the same point $\left[\frac{a+c}{3}, \frac{b}{3}\right]$ is also the centroid of $\triangle$XYZ.

(*Proof is left to the student*)

For more details about nedians, nedian triangle, aliquot triangle etc., vide Mathematical Gazette Vol. XXXVIII, No. 324 (May 1954) and Vol. XL No. 332 (May 1956).