An Introduction to $\mathcal{D}$-Modules

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Abstract

The purpose of these notes is to introduce the reader to the algebraic theory of systems of partial differential equations on a complex analytic manifold. We start by explaining how to switch from the classical point of view to the point of view of algebraic analysis. Then, we perform a detailed study of the ring of differential operators and its modules. In particular, we prove its coherence and compute its homological dimension. After introducing the characteristic variety of a system, we prove Kashiwara's version of the Cauchy-Kowalewsky theorem. Next, we consider the behavior of systems under proper direct images. We conclude with an appendix giving the complements of algebra required to fully understand the exposition.

Contents

1991 *Mathematics Subject Classification.* 32C38, 35A27.
Key words and phrases. D-modules, algebraic analysis, partial differential equations on manifolds.

1 Introduction

The application of the powerful methods of homological algebra and sheaf theory to the study of analytic systems of partial differential equations started in the late '60s under the impulse of M. Sato and later M. Kashiwara. Many essential results were obtained both in the complex and real domain, and soon a whole theory was built. This theory, which is now often referred as "Algebraic Analysis", has grown steadily during the past years and important applications to other fields of mathematics and physics (e.g. singularities theory, group representations, Feynman integrals,...) were developed.

Many research areas are still open. However, the task facing a student who would like to work in these directions is quite impressive: the research literature is huge and few textbooks are available.

In these notes, we develop the elements of the algebraic theory of systems of partial differential equations in the complex domain. All the results are well-known and our contribution is only at the level of the presentation.

Of course, it is impossible to give a complete picture of the theory in these few pages, but we hope to give the reader the basic knowledge he needs to understand the main research papers on the subject.

To get a better perspective on algebraic analysis, the reader should have a look at the works [15, 2, 9, 16, 4, 12, 3] and their bibliographies. A natural extension of this course would be the study of the theory of holonomic systems as developed in [6, 7, 8, 11, 10].

Throughout the text, we assume the reader has a basic knowledge of linear algebra, homological algebra, sheaf theory and analytic geometry.

In the first section, as a motivation for the theory, we explain how to make the switch from the usual point of view of systems of partial differential equations to the point of view of $\mathcal{D}$-modules used in algebraic analysis.

In the second section, we study the ring $\mathcal{D}$ of differential operators on a complex manifold and its local finiteness properties (coherence, dimension, syzygies).

The third section is devoted to internal operations of the category of $\mathcal{D}$-modules on a complex manifold. After introducing the internal product, we explain how to switch between left and right $\mathcal{D}$-modules.

In the next section, as an example, we make a detailed study of the $\mathcal{D}$-module structure of the sheaf of holomorphic functions $\mathcal{O}$.

After a section on the characteristic variety, we study non-characteristic inverse images and proper direct images. In particular, we prove Kashiwara's version of the Cauchy-Kowalewsky theorem for systems.

In the section on non-singular systems, we identify the $\mathcal{D}$-modules corresponding to flat holomorphic connections and classify them.

The appendix is devoted to an informal study of homological dimension, Noether property and syzygies for rings, graded rings and filtered rings. Most of the results of filtered algebra needed for the study of $\mathcal{D}_X$ are proven there. Hence, the reader should have a quick look at the appendix before reading the main text and refer afterwards to the detailed proofs as the need arises.

These notes are based on lectures given at the "Journées d'Analyse Algébrique" held at the University of Liège in March 1993. Let me take this opportunity to thank the participants for their constructive comments.

Finally, I wish to thank M. Kashiwara and P. Schapira through the work of whom I learned to appreciate the power and beauty of algebraic analysis.

2 Motivations

Let U be an open subset of $\mathbb{C}^n$. Recall that a (complex analytic linear partial) differential operator on U is an operator of the form

$$P(z,\partial_z) = \sum_{|\alpha|\leq p} a_\alpha(z)\partial_z^\alpha$$

where $a_\alpha(z)$ is a holomorphic function on U, α being a multi-index $\alpha_1,\dots,\alpha_n$ of length $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and $\partial_z^\alpha = \partial_{z^1}^{\alpha_1}\dots\partial_{z^n}^{\alpha_n}$. Such an operator acts $\mathbb{C}$-linearly on holomorphic functions defined on U and is characterized by this action since

$$P((z-z_0)^\alpha)_{|z=z_0} = \alpha!\, a_\alpha(z_0).$$

One checks easily that differential operators form a subring of $\mathrm{End}_{\mathbb{C}}(\mathcal{O}(U))$. The restriction of a differential operator

$$P = \sum_{|\alpha|\leq p} a_\alpha \partial_z^\alpha$$

defined on U to an open subset V is the operator

$$P_{|V} = \sum_{|\alpha|\leq p} a_{\alpha|V}\partial_z^\alpha.$$

With these restriction morphisms, differential operators appear as a presheaf of rings on $\mathbb{C}^n$. We denote by $\mathcal{D}_{\mathbb{C}^n}$ the associated sheaf of rings. A section $P \in \Gamma(U;\mathcal{D}_{\mathbb{C}^n})$ may be written in a unique way as

$$\sum_\alpha a_\alpha(z)\partial_z^\alpha$$

where, $a_\alpha = 0$ in a neighborhood of z_0 for $|\alpha| \gg 0$. If there is an integer p such that $a_\alpha = 0$ for $|\alpha| > p$ and $a_\alpha \neq 0$ for at least an α of length p we say that P has order p. Any operator $P \in \Gamma(U,\mathcal{D}_{\mathbb{C}^n})$ has locally a finite order. Note that $\mathcal{O}_{\mathbb{C}^n}$ has a natural structure of $\mathcal{D}_{\mathbb{C}^n}$-module. For an open subset U of $\mathbb{C}^n$, we denote by $\mathcal{D}_U$ the restriction to U of the sheaf $\mathcal{D}_{\mathbb{C}^n}$.

Our basic problem in these lectures is to study systems of equations of the form

$$\sum_{j=1}^n P_{ij}(z,\partial_z)u_j = v_i \quad (i = 1,\dots,m) \tag{2.1}$$

where $P_{ij}(z,\partial_z)$ is a differential operator of some finite order p_{ij}, the $v_1,\dots,v_n$ being given data and $u_1,\dots,u_m$ being indeterminates in some function space $\mathcal{S}(U)$ on which the ring $\Gamma(U;\mathcal{D}_{\mathbb{C}^n})$ acts. In fact, since we want to be able to use the powerful methods of sheaf theory to glue together local results, we will assume that $\mathcal{S}(U)$ is the space of sections on U of a sheaf $\mathcal{S}$ which is a $\mathcal{D}_U$-module. For example, we may take $\mathcal{S} = \mathcal{O}_U$ the sheaf of holomorphic functions on U or $\mathcal{S} = \mathcal{A}_U$ the sheaf or real analytic functions or $\mathcal{S} = \mathcal{B}_U$ the sheaf of hyperfunctions. Other possible examples for $\mathcal{S}$ are $\mathcal{F}_U$ the sheaf of infinitely differentiable functions on U or $\mathcal{D}b_U$ the sheaf of distributions on U or even the sheaf $\mathcal{D}_U$ itself.

Our first step in studying systems like (2.1) will be to try to understand when two of them should be considered as equivalent.

Let us first consider the case of homogeneous equations

$$\sum_{j=1}^{n} P_{ij}(z,\partial_z)u_j = 0 \quad (i = 1,\dots,m).$$

Intuitively, equivalent systems are systems which have the same solutions in any solution space $\mathcal{S}$. To translate this intuition into a mathematical notion, we will proceed as follows. Denote by P the matrix (P_{ij}) and by $\mathcal{S}^P$ the solution sheaf of the system associated to P in the $\mathcal{D}_U$-module $\mathcal{S}$. We have

$$\Gamma(U;\mathcal{S}^P) = \{u \in \Gamma(U;\mathcal{S})^n : P \cdot u = 0\}.$$

The law $\mathcal{S} \mapsto \mathcal{S}^P$ defines a functor from the category of $\mathcal{D}_U$-modules to the category of sheaves of $\mathbb{C}$-vector spaces on U and two systems P, Q should be considered as equivalent when the associated functors $\mathcal{S} \mapsto \mathcal{S}^P$ and $\mathcal{S} \mapsto \mathcal{S}^Q$ are naturally equivalent. Now consider the $\mathcal{D}_U$-linear morphism

$$\begin{array}{ccc} \mathcal{D}_U^m & \xrightarrow{\cdot P} & \mathcal{D}_U^n \\ R & \mapsto & R \cdot P \end{array}$$

and denote by $\mathcal{M}_P$ its cokernel. From the exact sequence

$$\mathcal{D}_U^m \xrightarrow{\cdot P} \mathcal{D}_U^n \longrightarrow \mathcal{M}_P \longrightarrow 0$$

we get the exact sequence

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{H}om_{\mathcal{D}_U}(\mathcal{M}_P,\mathcal{S}) & \longrightarrow & \mathcal{S}^n & \xrightarrow{P\cdot} & \mathcal{S}^m \\ & & & & u & \mapsto & P \cdot u \end{array}$$

and hence a natural equivalence of functors

$$\mathcal{H}om_{\mathcal{D}_U}(\mathcal{M}_P,\mathcal{S}) \simeq \mathcal{S}^P.$$

This shows that the functor $\mathcal{S} \mapsto \mathcal{S}^P$ is representable by the $\mathcal{D}_U$-module $\mathcal{M}_P$. The homogeneous systems associated to P and Q are thus equivalent if and only if $\mathcal{M}_P \simeq \mathcal{M}_Q$ as $\mathcal{D}_U$-modules. So it appears that the natural object which represents a homogeneous system of P.D.E. is the $\mathcal{D}_U$-module $\mathcal{M}_P$ not the matrix of operators P.

What is the situation for inhomogeneous systems?

Assume that $u_1,\dots,u_m$; $v_1,\dots,v_m \in \Gamma(U,\mathcal{S})$ and that

$$\sum_{j=1}^{n} P_{ij}u_j = v_i \qquad (i = 1,\dots,m) \tag{2.2}$$

Then, for any differential operators $Q_1, \dots, Q_m$ on U, we have

$$\sum_{j=1}^{n}\left(\sum_{i=1}^{m} Q_i P_{ij}\right) u_j = \sum_{i=1}^{m} Q_i v_i,$$

and it follows that

$$\sum_{i=1}^{m} Q_i v_i = 0 \quad \text{if} \quad \sum_{i=1}^{m} Q_i P_{ij} = 0.$$

Such a vector of m differential operators defines what we will call an *algebraic compatibility condition* for the inhomogeneous system (2.2). By definition, it is clear that algebraic compatibility condition are the sections of the kernel $\mathcal{N}_P$ of the map

$$\mathcal{D}_U^m \xrightarrow{\cdot P} \mathcal{D}_U^n.$$

Denote by $\mathcal{I}_P$ the image of this same map. We have the exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{N}_P & \longrightarrow & \mathcal{D}_U^m & \xrightarrow{\alpha} & \mathcal{I}_P & \longrightarrow & 0 \\ 0 & \longrightarrow & \mathcal{I}_P & \xrightarrow{\beta} & \mathcal{D}_U^n & \longrightarrow & \mathcal{M}_P & \longrightarrow & 0 \end{array}$$

where $\beta \circ \alpha = \cdot P$. Hence, we get the exact sequences

$$0 \longrightarrow \mathcal{H}om_{\mathcal{D}_U}(\mathcal{M}_P, \mathcal{S}) \longrightarrow \mathcal{S}^n \longrightarrow \mathcal{H}om_{\mathcal{D}_U}(\mathcal{I}_P, \mathcal{S}) \longrightarrow \mathcal{E}xt^1_{\mathcal{D}_U}(\mathcal{M}_P, \mathcal{S}) \longrightarrow 0$$

$$0 \longrightarrow \mathcal{H}om_{\mathcal{D}_U}(\mathcal{I}_P, \mathcal{S}) \longrightarrow \mathcal{S}^m \longrightarrow \mathcal{H}om_{\mathcal{D}_U}(\mathcal{N}_P, \mathcal{S}) \longrightarrow \mathcal{E}xt^1_{\mathcal{D}_U}(\mathcal{I}_P, \mathcal{S}) \longrightarrow 0$$

The second one shows that

$$\mathcal{H}om_{\mathcal{D}_U}(\mathcal{I}_P, \mathcal{S}) = \{v \in \mathcal{S}^m : Q \cdot v = 0, \forall Q \in \mathcal{N}_P\}$$

so $\mathcal{I}_P$ is a $\mathcal{D}_U$-module which represents the system of algebraic compatibility conditions of the system P. Using the first exact sequence we see, at the level of germs, that the elements of $\mathcal{E}xt^1_{\mathcal{D}_U}(\mathcal{M}_P, \mathcal{S})_x$ are the classes of vectors v_x of $\mathcal{S}_x^m$ satisfying the algebraic compatibility conditions modulo those for which the system is truly compatible. Moreover, for $k \geq 1$,

$$\mathcal{E}xt^k_{\mathcal{D}_U}(\mathcal{I}_P, \mathcal{S}) \simeq \mathcal{E}xt^{k+1}_{\mathcal{D}_U}(\mathcal{M}_P, \mathcal{S}).$$

Thus, all the $\mathcal{E}xt^k_{\mathcal{D}_U}(\mathcal{M}_P, \mathcal{S})$ give meaningful information about the system P.

We hope that the preceding discussion has convinced the reader that the study of the system

$$\sum_{j=1}^{n} P_{ij} u_j = v_i \quad (i = 1, \dots, m)$$

is equivalent to the study of the $\mathcal{D}_U$-module $\mathcal{M}_P$ defined by the exact sequence

$$\mathcal{D}_U^m \xrightarrow{\cdot P} \mathcal{D}_U^n \longrightarrow \mathcal{M}_P \longrightarrow 0$$

and of its full solution complex

$$R\mathcal{H}om_{\mathcal{D}_U}(\mathcal{M}_P, \mathcal{S}).$$

In fact, restricting our interest to system which like $\mathcal{M}_P$ have a global finite free 1-presentation is not a very good point of view because the system $\mathcal{I}_P$ of algebraic compatibility conditions of P admits in general no finite free 1-presentation. However, we will show that it has locally such a presentation ($\mathcal{D}_U$ is a coherent sheaf of rings). So coherent $\mathcal{D}_U$-modules appear to be the natural algebraic objects representing systems of linear partial differential equations. We will see in the sequel that the study of coherent $\mathcal{D}$-modules also allows us to understand in a clear and intrinsic manner systems of linear partial differential equations on manifolds.

3 Differential operators on manifolds

Let X be a complex analytic manifold of dimension n. Let us recall that $\mathcal{O}_X$ denotes the sheaf of holomorphic functions on X and that Θ_X denotes the sheaf of holomorphic vector fields on X. Of course, $\mathcal{O}_X$ is a sheaf of $\mathbb{C}$-algebras. We know that Θ_X is the sheaf of holomorphic sections of the tangent bundle TX and that it is a locally free $\mathcal{O}_X$-module of rank n. Through Lie derivative, Θ_X acts $\mathbb{C}$-linearly on $\mathcal{O}_X$. In fact, Θ_X may be identified to the sheaf of $\mathbb{C}$-linear derivations of the $\mathbb{C}_X$-algebra $\mathcal{O}_X$. The Lie bracket turns Θ_X into a sheaf of Lie algebras over $\mathbb{C}$. As usual, one denotes by L_θ the Lie derivative along a vector field $\theta \in \Theta_X$.

3.1 The ring $\mathcal{D}_X$ and its modules

Definition 3.1.1 The *ring of linear partial differential operators with analytic coefficients on* X is the subsheaf of rings of $\mathcal{E}nd_{\mathbb{C}_X}(\mathcal{O}_X)$ generated by Lie derivatives along holomorphic vector fields and multiplications by holomorphic functions. We denote it by $\mathcal{D}_X$. Locally, a section $P \in \Gamma(U; \mathcal{D}_X)$ may be written in a unique way in the form

$$P = \sum_{|\alpha| \leq p} a_\alpha(z) \partial_z^\alpha.$$

where $(z : U \longrightarrow \mathbb{C}^n)$ is a local coordinate system on X.

Proposition 3.1.2 *The ring $\mathcal{D}_X$ is a simple sheaf of non-commutative $\mathbb{C}$-algebras with center $\mathbb{C}_X$.*

Proof: It is sufficient to work at the levels of germs and we may thus assume $X = \mathbb{C}^n$ and show that the center of $(\mathcal{D}_{\mathbb{C}^n})_0$ is $\mathbb{C}$.

For any central $P \in (\mathcal{D}_{\mathbb{C}^n})_0$ we have $[P, z^i] = 0$ and $[P, \partial_{z^i}] = 0$. Let $z' = (z^1, \dots, z^{n-1})$ and $z'' = z^n$. We may write P as

$$P(z, \partial_z) = \sum_{k=0}^{p} P_k(z', z'', \partial_{z'}) \partial_{z''}^k$$

where $P_p(z', z'', \partial_{z'}) \neq 0$ and p is the order of P in $\partial_{z''}$. We have

$$0 = [P, z''] = \sum_{k=1}^{p} P_k(z', z'', \partial_{z'}) k \partial_{z''}^{k-1}$$

hence $P = P_0(z', z'', \partial_{z'})$. Iterating this process, we see that P is a germ $a(z) \in (\mathcal{O}_{\mathbb{C}^n})_0$. Since

$$0 = [\partial_{z^i}, a] = \partial_{z^i} a$$

the germ a is constant and the conclusion follows.

Now, let I be a two sided ideal of $(\mathcal{D}_{\mathbb{C}^n})_0$. Assuming $I \neq 0$, we will show that $1 \in I$. With the same notations as above, we may find

$$P(z, \partial_z) = \sum_{k=1}^{p} P_k(z', z'', \partial_{z'}) \partial_{z''}^k \in I$$

where $P_p(z', z'', \partial_{z'}) \neq 0$. Since the order in $\partial_{z''}$ of $[P, \partial_{z''}]$ is $p-1$ we may even assume P is of order 0 in $\partial_{z''}$. Iterating this procedure, we find a non zero germ of holomorphic function $a(z) \in I$. Up to a change of coordinates, we may assume $a(z)$ is of Weierstrass type with respect to z'' at 0. This implies that

$$a(z) = \left(\sum_{k=0}^{p} b_k(z')(z'')^k \right) u(z)$$

where $u(z)$ is an invertible germ of holomorphic function on z and $b_0(z'), \dots, b_p(z')$ are germs of holomorphic function in z' with $b_0(0) = \dots = b_{p-1}(0) = 0$ and $b_p(z') = 1$. It follows that

$$b(z) = \sum_{k=0}^{p} b_k(z')(z'')^k \in I$$

and since

$$[\partial_{z''}, b(z)] = \sum_{k=0}^{p} b_k(z') k (z'')^{k-1}$$

we find by iteration that $p!\, b_p(z') \in I$. Hence $1 \in I$ and $I = (\mathcal{D}_{\mathbb{C}^n})_0$, and the proof is complete. $\square$

The category $\mathrm{Mod}(\mathcal{D}_X)$ (resp. $\mathrm{Mod}(\mathcal{D}_X^{\mathrm{op}})$) of left (resp. right) $\mathcal{D}_X$-modules is of course an Abelian category with enough injective objects as is any category of modules over a sheaf of rings.

The construction of left or right $\mathcal{D}_X$-modules is simplified by the following proposition.

Proposition 3.1.3 *Assume $\mathcal{M}$ is an $\mathcal{O}_X$-module and denote by*

$$\mu : \mathcal{O}_X \longrightarrow \mathcal{E}nd_{\mathbb{C}_X}(\mathcal{M})$$

the associated multiplication. Assume

$$\alpha : \Theta_X \longrightarrow \mathcal{E}nd_{\mathbb{C}_X}(\mathcal{M})$$

is a $\mathbb{C}_X$ linear action of holomorphic vector fields on $\mathcal{M}$ such that

(a) $[\alpha(\theta), \mu(h)] = \mu(L_\theta h)$ (resp. $-\mu(L_\theta h)$),

(b) $[\alpha(\theta), \alpha(\psi)] = \alpha([\theta, \psi])$ *(resp.* $-\alpha([\theta, \psi])$*),*

(c) $\mu(h) \circ \alpha(\theta) = \alpha(h\theta)$ *(resp.* $\alpha(\theta) \circ \mu(h) = \alpha(h\theta)$*)*

for any $\theta, \psi \in \Theta_X$ *and any* $h \in \mathcal{O}_X$*. Then there is a unique structure of left (resp. right)* $\mathcal{D}_X$*-module on* $\mathcal{M}$ *extending the given actions of* $\mathcal{O}_X$ *and* Θ_X*.*

Proof: The uniqueness of the extending structure is trivial since $\mathcal{D}_X$ is generated by $\mathcal{O}_X$ and Θ_X. The existence needs thus only to be proven locally. There, the structure of a differential operator dictates the definition of its action and one checks easily that this action is a left (resp. right) $\mathcal{D}_X$-module structure extending the given actions of $\mathcal{O}_X$ and Θ_X. □

Corollary 3.1.4 *(a) There is a unique left* $\mathcal{D}_X$*-module structure on* $\mathcal{O}_X$ *extending its structure of* $\mathcal{O}_X$*-module in such a way that*

$$\theta \cdot h = L_\theta h$$

for any $\theta \in \Theta_X$, $h \in \mathcal{O}_X$*.*

(b) There is a unique right $\mathcal{D}_X$*-module structure on* Ω_X *extending its structure of* $\mathcal{O}_X$*-module in such a way that*

$$\omega \cdot \theta = -L_\theta \omega$$

for any $\theta \in \Theta_X$, $\omega \in \Omega_X$*.*

Proof: (a) is obvious by the definition of $\mathcal{D}_X$.

(b) It is well known that Ω_X is an $\mathcal{O}_X$-module. We will show that the action of Θ_X on Ω_X defined by setting

$$\omega \cdot \theta = -L_\theta \omega$$

for $\omega \in (\Omega_X)_x, \theta \in (\Theta_X)_x$ satisfies the conditions of Proposition 3.1.3. The easiest way is to look what happens in a local coordinate system $z : U \longrightarrow \mathbb{C}^n$.

Recall that if

$$\theta = \sum_{i=1}^{n} \theta^i \partial_{z^i}$$

is a holomorphic vector field on U generating the complex flow φ_τ and if $\omega = a(z)dz^1 \wedge \ldots \wedge dz^n$ is a holomorphic n-form on U by definition

$$\begin{aligned}
L_\theta \omega &= \partial_\tau(\varphi_\tau^* \omega)|_{\tau=0} \\
&= \partial_\tau(a(\varphi_\tau)\, d\varphi_\tau^1 \wedge \ldots \wedge d\varphi_\tau^n)|_{\tau=0} \\
&= L_\theta a dz^1 \wedge \ldots \wedge dz^n + \sum_{i=1}^{n} a(z) dz^1 \wedge \ldots d\theta^i \wedge \ldots \wedge dz^n \\
&= \left[L_\theta a + \left(\sum_{i=1}^{n} \partial_{z^i} \theta^i \right) a(z) \right] dz^1 \wedge \ldots \wedge dz^n \\
&= \left[\sum_{i=1}^{n} \partial_{z^i}(\theta^i a) \right] dz^1 \wedge \ldots \wedge dz^n
\end{aligned}$$

Hence

$$\omega.\theta = \sum_{i=1}^{n} -\partial_{z^i}(\theta^i a) dz^1 \wedge \ldots \wedge dz^n$$

and the conditions of Proposition 3.1.3 are checked by easy direct computations. □

3.2 The order filtration of $\mathcal{D}_X$

Definition 3.2.1 We define inductively the subsheaves $\mathcal{F}_k\mathcal{D}_X$ ($k \in \mathbb{N}$) of $\mathcal{D}_X$ by setting:

$$\begin{aligned} \mathcal{F}_0\mathcal{D}_X &= \mathcal{O}_X, \\ \mathcal{F}_{k+1}\mathcal{D}_X &= \mathcal{F}_k\mathcal{D}_X + \Theta_X \cdot \mathcal{F}_k\mathcal{D}_X \quad (k \geq 0), \end{aligned}$$

and extend the preceding sequence to negative integers by setting:

$$\mathcal{F}_k\mathcal{D}_X = 0 \quad (k < 0).$$

By definition, $\mathcal{F}_k\mathcal{D}_X \subset \mathcal{F}_{k+1}\mathcal{D}_X$ for $k \in \mathbb{Z}$, and

$$\mathcal{D}_X = \bigcup_{k \in \mathbb{Z}} \mathcal{F}_k\mathcal{D}_X.$$

Hence, the sequence $(\mathcal{F}_k\mathcal{D}_X)_{k\in\mathbb{Z}}$ defines a filtration on the sheaf $\mathcal{D}_X$ which is easily seen to be compatible with its ring structure. The *order* of a section P of $\mathcal{D}_X$ at $x \in X$ is the smallest integer k such that $P_x \in (\mathcal{F}_k\mathcal{D}_X)_x$; we denote it by $\mathrm{ord}_x(P)$. We also set

$$\mathrm{ord}(P) = \sup_{x\in X} \mathrm{ord}_x(P).$$

Obviously, a section $P \in \Gamma(U, \mathcal{D}_X)$ is a section of $\mathcal{F}_k\mathcal{D}_X$ if and only if $\mathrm{ord}_x(P) \leq k$ for each $x \in U$. Therefore, we call the sequence $(\mathcal{F}_k\mathcal{D}_X)_{k\in\mathbb{Z}}$ the *order filtration* of $\mathcal{D}_X$. We denote by $\mathcal{F}\mathcal{D}_X$ the corresponding filtered ring and by $\mathcal{G}\mathcal{D}_X$ the associated graded ring.

Locally, a section $P \in \Gamma(U; \mathcal{F}_p\mathcal{D}_X)$ may be written in a unique way in the form

$$P = \sum_{|\alpha|\leq p} a_\alpha(z)\partial_z^\alpha.$$

where $(z : U \longrightarrow \mathbb{C}^n)$ is a local coordinate system on X.

3.3 Principal symbols and the symplectic structure of T^*X

Recall that to any local coordinate system $(z : U \longrightarrow \mathbb{C}^n)$ on X is associated a local coordinate system $((z;\xi) : T^*U \longrightarrow \mathbb{C}^n \times \mathbb{C}^n)$ of T^*X. The coordinates $\xi(\omega) \in \mathbb{C}^n$ of a form ω in T^*_uU are characterized by the formula

$$\omega = \sum_{j=1}^{n} \xi_j(\omega) dz^j.$$

Lemma 3.3.1 *Let $(z : U \longrightarrow \mathbb{C}^n)$ and $(z' : U \longrightarrow \mathbb{C}^n)$ be two coordinate systems on an open subset U of X. Assume $P \in \Gamma(U; \mathcal{F}_p\mathcal{D}_X)$ may be written as*

$$P = \sum_{|\alpha| \leq p} a_\alpha(z)\partial_z^\alpha; \quad P = \sum_{|\alpha| \leq p} a'_\alpha(z')\partial_{z'}^\alpha.$$

Then

$$\sum_{|\alpha| = p} a_\alpha(z)\xi^\alpha = \sum_{|\alpha| = p} a'_\alpha(z')(\xi')^\alpha.$$

Proof: Denoting by J the Jacobian matrix

$$\left(\frac{\partial z}{\partial z'}\right),$$

we have

$$\partial_{z'} = J^*\partial_z.$$

Therefore,

$$\sum_{|\alpha| \leq p} a'_\alpha(z')\partial_{z'}^\alpha = \sum_{|\alpha| \leq p} a'_\alpha(z')\left(J^*\partial_z\right)^\alpha.$$

and, retaining only the terms in ∂_z^α with $|\alpha| = p$, we get

$$\sum_{|\alpha| = p} a_\alpha(z)\xi^\alpha = \sum_{|\alpha| = p} a'_\alpha(z')(J^*\xi)^\alpha.$$

Since

$$\sum_{j=1}^n \xi_j dz^j = \sum_{j=1}^n \xi'_j dz'^j,$$

we also get

$$\xi' = J^*\xi,$$

and the conclusion follows easily. □

Definition 3.3.2 Let P be a section of $\mathcal{F}_p\mathcal{D}_X$ on X. In a local coordinate system $(z : U \longrightarrow \mathbb{C}^n)$ of X, P may be written in a unique way in the form

$$P = \sum_{|\alpha| \leq p} a_\alpha(z)\partial_z^\alpha.$$

The preceding lemma shows that

$$\sigma_U = \sum_{|\alpha| = p} a_\alpha(z)\xi^\alpha$$

is a holomorphic function on T^*U which depends only on P. Moreover, for another coordinate system $(z : V \longrightarrow \mathbb{C}^n)$ on X, we have

$$(\sigma_U)_{|U\cap V} = (\sigma_V)_{|U\cap V}.$$

Hence, there is a unique holomorphic function on T^*X gluing all the σ_U together; we call it the *symbol of order* p of P and denote it by $\sigma_p(P)$. The *principal symbol of* P is its symbol of order $\mathrm{ord}(P)$; we denote it by $\sigma(P)$.

Recall that a section of $\mathcal{O}_{T^*X}$ on an open subset W is homogeneous of degree $k \in \mathbb{Z}$ if for any $(x;\omega) \in U$ there is an open neighborhood D of 0 in $\mathbb{C}$ such that

$$h(x;t\omega) = t^k h(x;\omega)$$

for any $t \in D$. Obviously, these sections form a subsheaf of $\mathcal{O}_{T^*X}$. We denote it by $\mathcal{O}_{T^*X}(m)$. The reader will check easily that

$$\mathcal{O}_{T^*X}(m)\mathcal{O}_{T^*X}(n) \subset \mathcal{O}_{T^*X}(m+n).$$

Hence,

$$\bigoplus_{k\in\mathbb{N}} \pi_*\mathcal{O}_{T^*X}(k)$$

is canonically a sheaf of graded rings on X. In a local coordinate system $(z : U \longrightarrow \mathbb{C}^n)$, one checks easily that

$$\bigoplus_{k\in\mathbb{N}} \pi_*\mathcal{O}_{T^*U}(k) \simeq \mathcal{O}_U[\xi_1,\ldots,\xi_n].$$

Proposition 3.3.3 *The symbol maps induce a canonical isomorphism of graded rings*

$$\mathcal{G}\mathcal{D}_X \xrightarrow{\sim} \bigoplus_{k\in\mathbb{N}} \pi_*\mathcal{O}_{T^*X}(k).$$

Proof: By definition, the symbol $\sigma_k(P)$ of an operator $P \in \Gamma(X;\mathcal{F}_k\mathcal{D}_X)$ belongs to $\Gamma(X;\mathcal{O}_{T^*X}(k))$. Hence, for each $k \in \mathbb{Z}$, we get an intrinsic morphism

$$\sigma_k : \mathcal{F}_k\mathcal{D}_X \longrightarrow \pi_*\mathcal{O}_{T^*X}(k).$$

Since σ_k is zero on $\mathcal{F}_{k-1}\mathcal{D}_X$, it induces a morphism

$$\sigma_k : \mathcal{G}_k\mathcal{D}_X \longrightarrow \pi_*\mathcal{O}_{T^*X}(k).$$

Hence, we get a canonical morphism

$$\mathcal{G}\mathcal{D}_X \longrightarrow \bigoplus_{k\in\mathbb{N}} \pi_*\mathcal{O}_{T^*X}(k).$$

which is easily checked to be a morphism of graded rings. In a local coordinate system, this morphism is clearly bijective and the conclusion follows. $\square$

The manifold T^*X is endowed with a canonical 1-form α defined by

$$\alpha_{(x,\omega)}(\theta) = \langle\omega, \pi'\theta\rangle$$

for any $\theta \in T_{(x,\omega)}T^*X$, $\pi : T^*X \longrightarrow X$ being the canonical bundle map and π' denoting its tangent map. If $(z : U \longrightarrow V)$ is a local coordinate system on U and

$$(z,\xi) : T^*U \longrightarrow V \times \mathbb{C}^n$$

is the associated trivialization of T^*U, we have

$$\alpha_{|T^*U} = \sum_{k=1}^{n} \xi_k dz^k.$$

We turn T^*X into a symplectic manifold by endowing it with the non degenerate skew symmetric form $\sigma = d\alpha$. With the same notations as above, we have

$$\sigma_{|T^*U} = \sum_{k=1}^{n} d\xi_k \wedge dz^k.$$

As on any symplectic manifold, we define the Hamiltonian isomorphism

$$H : T^*T^*X \longrightarrow TT^*X$$

through the formula $\sigma_p(\theta, H(\omega)) = \langle \omega, \theta \rangle$ valid at any point $p \in T^*X$ and for any $\omega \in T_p^*T^*X$, $\theta \in T_pT^*X$.

Locally, we have

$$H(dz^k) = -\partial_{\xi_k}$$
$$H(d\xi_k) = \partial_{z^k}.$$

Hence, for a holomorphic function f defined on an open subset W of T^*U we have

$$H(df) = \sum_{k=1}^{n} \partial_{\xi_k} f \, \partial_{z^k} - \partial_{z^k} f \, \partial_{\xi_k}$$

on W which is the usual expression for the Hamiltonian field H_f of the function f in classical mechanics (see [1]).

Using Hamiltonian fields, we may introduce the classical notion of Poisson bracket of two differentiable functions f, g defined on an open subset of T^*X by the formula

$$\{f, g\} = H_f(g),$$

and one checks easily that $H_{\{f,g\}} = [H_f, H_g]$.

Proposition 3.3.4 *Let $P \in \Gamma(X; \mathcal{F}_k\mathcal{D}_X)$, $Q \in \Gamma(X; \mathcal{F}_\ell\mathcal{D}_X)$ be two differential operators on X. Then $[P, Q] \in \Gamma(X; \mathcal{F}_{k+\ell-1}\mathcal{D}_X)$ and*

$$\sigma_{k+\ell-1}([P, Q]) = \{\sigma_k(P), \sigma_\ell(Q)\}.$$

Proof: Since the problem is local, we may assume that X is an open subset of $\mathbb{C}^n$. In this case, the equality is easily checked by a direct computation. □

Remark 3.3.5 In the preceding proposition, note that the order of $[P, Q]$ may well be strictly lower than $k + \ell - 1$. Note also that the formula

$$\sigma([P, Q]) = \{\sigma(P), \sigma(Q)\}.$$

is false in general.

3.4 Local finiteness properties of $\mathcal{D}_X$

Proposition 3.4.1 *The ring $\mathcal{D}_X$ is Noetherian, syzygic and has finite homological dimension. In particular, there is an integer p such that any 1-presentation*

$$\mathcal{D}_X^{\ell_1} \longrightarrow \mathcal{D}_X^{\ell_0} \longrightarrow \mathcal{M} \longrightarrow 0$$

may be extended locally into a free resolution of length p

$$0 \longrightarrow \mathcal{D}_V^{\ell_p} \longrightarrow \mathcal{D}_V^{\ell_{p-1}} \longrightarrow \cdots \longrightarrow \mathcal{D}_V^{\ell_0} \longrightarrow \mathcal{M}_{|V} \longrightarrow 0.$$

Proof: Since the problem is local, we may assume that X is an open subset U of $\mathbb{C}^n$. We know from classical results of analytic geometry that $\mathcal{O}_U$ is a Noetherian ring on U. Moreover, for any $u \in U$, the fiber $(\mathcal{O}_U)_u$ is a regular Noetherian local ring of dimension n. Hence, by Corollary 10.1.7, $\mathcal{O}_U$ is syzygic and its homological dimension is lower than n. Therefore Proposition 10.1.1 combined with Corollary 10.2.5 and Proposition 10.4.10 shows that $\mathcal{O}_U[\xi_1, \ldots, \xi_n]$ is a syzygic Noetherian ring with homological dimension lower than $2n$. Hence, so is $\mathcal{G}\mathcal{D}_X$, and Proposition 10.4.8 shows that $\mathcal{F}\mathcal{D}_X$ is Noetherian and that $\mathcal{D}_X$ is Noetherian. Moreover, Propositions 10.4.5 and 10.4.7 show that $\mathcal{D}_X$ is syzygic with homological dimension $\operatorname{glhd}(\mathcal{D}_X) \leq 2n$. □

Remark 3.4.2 We will see in Corollary 6.3.2 that in fact $\operatorname{glhd}(\mathcal{D}_X)$ is the complex dimension of X.

4 Internal operations on $\mathcal{D}$-modules

In this section, we will define an internal tensor product and compute its right adjoint. We will also show that there is a canonical equivalence between the category of left $\mathcal{D}_X$-modules and that of right $\mathcal{D}_X$-modules.

4.1 Internal products

The proofs of the following results are easy verifications based on Proposition 3.1.3. We leave them to the reader.

Proposition 4.1.1 *Let $\mathcal{M}$, $\mathcal{N}$, $\mathcal{P}$ be three left $\mathcal{D}_X$-modules.*

(a) The action of Θ_X on the $\mathcal{O}_X$-module $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N}$ defined by setting

$$\theta(m \otimes n) = \theta m \otimes n + m \otimes \theta n$$

for any $\theta \in \Theta_X$, $m \in \mathcal{M}$, $n \in \mathcal{N}$ extends to a left $\mathcal{D}_X$-module structure on $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N}$.

(b) The action of Θ_X on the $\mathcal{O}_X$-module $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{N}, \mathcal{P})$ defined by setting

$$(\theta f)(n) = \theta f(n) - f(\theta n)$$

for any $\theta \in \Theta_X$, $f \in \mathcal{H}om_{\mathcal{O}_X}(\mathcal{N}, \mathcal{P})$, $n \in \mathcal{N}$, extends to a left $\mathcal{D}_X$-module structure on $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{N}, \mathcal{P})$.

(c) We have the adjunction isomorphism

$$\begin{array}{rcl} \operatorname{Hom}_{\mathcal{D}_X}(\mathcal{M}\otimes_{\mathcal{O}_X}\mathcal{N},\mathcal{P}) & \longrightarrow & \operatorname{Hom}_{\mathcal{D}_X}(\mathcal{M},\mathcal{H}om_{\mathcal{O}_X}(\mathcal{N},\mathcal{P})) \\ f & \mapsto & (m\mapsto(n\mapsto f(m\otimes n)). \end{array}$$

In the same way, we have

Proposition 4.1.2 *Let $\mathcal{M}$, $\mathcal{P}$ be two right $\mathcal{D}_X$-modules and let $\mathcal{N}$ be a left $\mathcal{D}_X$-module then*

(a) the action of Θ_X on $\mathcal{M}\otimes_{\mathcal{O}_X}\mathcal{N}$ defined by

$$(m\otimes n)\theta = m\theta\otimes n - m\otimes\theta n$$

extends to a right $\mathcal{D}_X$-module structure,

(b) the action of Θ_X on $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{N},\mathcal{P})$ defined by

$$(f\theta)(n) = f(\theta n) - f(n)\theta$$

extends to a right $\mathcal{D}_X$-module structure,

(c) the action of Θ_X on $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M},\mathcal{P})$ defined by

$$(\theta f)(n) = f(n\theta) - f(n)\theta$$

extends to a left $\mathcal{D}_X$-module structure,

(d) we have canonical adjunction isomorphisms

$$\begin{array}{rcl} \operatorname{Hom}_{\mathcal{D}_X}(\mathcal{M}\otimes_{\mathcal{O}_X}\mathcal{N},\mathcal{P}) & \simeq & \operatorname{Hom}_{\mathcal{D}_X}(\mathcal{M},\mathcal{H}om_{\mathcal{O}_X}(\mathcal{N},\mathcal{P})) \\ \operatorname{Hom}_{\mathcal{D}_X}(\mathcal{M}\otimes_{\mathcal{O}_X}\mathcal{N},\mathcal{P}) & \simeq & \operatorname{Hom}_{\mathcal{D}_X}(\mathcal{N},\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M},\mathcal{P})). \end{array}$$

Proposition 4.1.3 *Assume $\mathcal{M}$ is a right $\mathcal{D}_X$-module and $\mathcal{N},\mathcal{P}$ are left $\mathcal{D}_X$-modules then we have a natural isomorphism*

$$(\mathcal{M}\otimes_{\mathcal{O}_X}\mathcal{N})\otimes_{\mathcal{D}_X}\mathcal{P}\xrightarrow{\sim}\mathcal{M}\otimes_{\mathcal{D}_X}(\mathcal{N}\otimes_{\mathcal{O}_X}\mathcal{P}).$$

4.2 Side changing functors and adjoint operators

The preceding results give us a way to switch between left and right $\mathcal{D}_X$-modules.

Proposition 4.2.1 *The functor from $\operatorname{Mod}(\mathcal{D}_X)$ to $\operatorname{Mod}(\mathcal{D}_X^{op})$ defined by*

$$\mathcal{M}\mapsto\Omega_X\otimes_{\mathcal{O}_X}\mathcal{M}$$

is an equivalence of categories. Its quasi-inverse is given by the functor from $\operatorname{Mod}(\mathcal{D}_X^{op})$ to $\operatorname{Mod}(\mathcal{D}_X)$ defined by

$$\mathcal{N}\mapsto\mathcal{H}om_{\mathcal{O}_X}(\Omega_X,\mathcal{N}).$$

Proof: Note that, since Ω_X is locally free of rank 1 over $\mathcal{O}_X$,

$$\begin{array}{rcl}\mathcal{M} & \simeq & \mathcal{H}om_{\mathcal{O}_X}(\Omega_X, \Omega_X \otimes_{\mathcal{O}_X} \mathcal{M}) \\ \mathcal{N} & \simeq & \Omega_X \otimes_{\mathcal{O}_X} \mathcal{H}om_{\mathcal{O}_X}(\Omega_X, \mathcal{N})\end{array}$$

as $\mathcal{O}_X$-modules, and one checks easily that these isomorphisms are compatible with the canonical $\mathcal{D}_X$-module structure of their sources and targets. □

To better understand the preceding equivalence, let us introduce the notion of adjoint of an operator.

Proposition 4.2.2 *Let V be an open subset of X and assume to be given a generator ω of Ω_X on V. For any $P \in \Gamma(V; \mathcal{D}_X)$ there is a unique operator $P^{\sim\omega} \in \Gamma(V; \mathcal{D}_X)$ (the adjoint of P with respect to ω) such that*

$$(P^{\sim\omega} h)\omega = (h\omega).P$$

for any $h \in \Gamma(W; \mathcal{O}_X)$ (W open subset of V). Moreover, we have

$$(P^{\sim\omega})^{\sim\omega} = P,$$

and

$$(Q \circ P)^{\sim\omega} = P^{\sim\omega} \circ Q^{\sim\omega}$$

for any $Q \in \Gamma(V; \mathcal{D}_X)$.

Proof: The problem is clearly of local nature. In a local coordinate system $(z : U \longrightarrow \mathbb{C}^n)$ we have

$$\begin{array}{rcl}P & = & \sum\limits_{|\alpha| \leq p} a_\alpha(z)\partial_z^\alpha \\ \omega & = & a(z)dz^1 \wedge \ldots \wedge dz^n,\end{array}$$

where $a(z)$ is an invertible holomorphic function on U.

Hence, we get

$$(ha\, dz^1 \wedge \ldots \wedge dz^n).P = \sum_{|\alpha| \leq p} \frac{(-\partial_z)^\alpha (a_\alpha(z) a h)}{a} a\, dz^1 \wedge \ldots \wedge dz^n$$

for any holomorphic function h on $W \subset U$; hence the conclusion. □

Proposition 4.2.3 *Assume $\mathcal{M}$ is a left $\mathcal{D}_X$-module. Then the right $\mathcal{D}_X$-module structure on $\Omega_X \otimes_{\mathcal{O}_X} \mathcal{M}$ is locally given by*

$$(\omega \otimes m).P = \omega \otimes P^{\sim\omega}.m$$

ω being a local generator of Ω_X.

Proof: It is sufficient to check the formula in a local coordinate system $(z : U \longrightarrow \mathbb{C}^n)$ when $P = \partial_{z^i}$ and $\omega = a(z)dz^1 \wedge \ldots \wedge dz^n$. In this case, we have

$$\begin{aligned}(\omega \otimes m).\partial_{z^i} &= \omega\partial_{z^i} \otimes m - \omega \otimes \partial_{z^i} m \\ &= (-\partial_{z^i} a)dz^1 \wedge \ldots \wedge dz^n \otimes m - adz^1 \wedge \ldots \wedge dz^n \otimes \partial_{z^i} m \\ &= dz^1 \wedge \ldots \wedge dz^n \otimes -\partial_{z^i}(a.m) \\ &= \omega \otimes \widetilde{\partial_{z^i}}^{\omega} m\end{aligned}$$

and the proof is complete. □

5 The de Rham system

Let X be a complex analytic manifold of complex dimension n. We will now study in details the $\mathcal{D}_X$-module $\mathcal{O}_X$. In a local coordinate system $(z : U \longrightarrow \mathbb{C}^n)$, we clearly have the exact sequence

$$\begin{array}{ccccccc} & & Q & \mapsto & Q \cdot 1 & & \\ \mathcal{D}_U^n & \longrightarrow & \mathcal{D}_U & \longrightarrow & \mathcal{O}_U & \longrightarrow & 0 \\ (Q_1, \ldots, Q_n) & \mapsto & \sum_{i=1}^n Q_i \partial_{z^i} & & & & \end{array}$$

which shows that $\mathcal{O}_U$ is the $\mathcal{D}_U$-module associated with the system

$$\begin{cases} \partial_{z^1} u = 0 \\ \quad\vdots \\ \partial_{z^n} u = 0 \end{cases}$$

In order to understand the full system of algebraic compatibility conditions of this system, we need an important algebraic tool: the Koszul complex.

5.1 Koszul complexes

Let A be any ring with unit and let M be an A-module. Denote by $e_1, \ldots, e_p$ the canonical basis of $\mathbb{Z}^p$ and set

$$M^{(k)} = M \otimes \Lambda^k \mathbb{Z}^p.$$

Note that any element of $M^{(k)}$ can be written in a unique way as

$$\sum_{1 \leq i_1 < \cdots < i_k \leq n} m^{i_1 \ldots i_k} \otimes e_{i_1 \ldots i_k}$$

where $m^{i_1 \ldots i_k} \in M$ and $e_{i_1 \ldots i_k} = e_{i_1} \wedge \ldots \wedge e_{i_k}$.

Definition 5.1.1 Assume $\varphi_1, \ldots, \varphi_p$ are p commuting endomorphisms of M. Let

$$d^k : M^{(k)} \longrightarrow M^{(k+1)}$$

be the A-linear morphism defined by setting

$$d^k(m \otimes e_{i_1 \ldots i_k}) = \sum_{j=1}^{p} \varphi_j(m) \otimes e_j \wedge e_{i_1 \ldots i_k}.$$

By the commutativity of the φ_j's, we have $d^{k+1} \circ d^k = 0$. We denote by $K^\cdot(\Phi, M)$ the complex

$$0 \longrightarrow M^{(0)} \xrightarrow{d^0} M^{(1)} \xrightarrow{d^1} \cdots \longrightarrow M^{(p)} \longrightarrow 0$$

where $M^{(0)}$ is in degree zero. This is the *positive Koszul complex* of $\Phi = (\varphi_1, \ldots, \varphi_p)$.

Proposition 5.1.2 *Assume φ_1 is surjective and φ_j induces a surjective endomorphism of $\ker \varphi_1 \cap \ldots \cap \ker \varphi_{j-1}$ for $j \in \{2, \ldots, p\}$. Then*

$$H^k(K^\cdot(\Phi, M)) = \begin{cases} \ker \varphi_1 \cap \ldots \cap \ker \varphi_p & \text{if } k = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Proof: Note that, for $p = 1$, the result is obvious. For $p > 1$, the complex

$$K^\cdot(\varphi_1, \ldots, \varphi_p; M)$$

is isomorphic to the mapping cone of the morphism

$$K^\cdot(\varphi_1, \ldots, \varphi_{p-1}; M) \xrightarrow{\varphi_p} K^\cdot(\varphi_1, \ldots, \varphi_{p-1}; M)$$

and the conclusion follows by induction on p. □

Definition 5.1.3 Consider the differential

$$\delta_k : M^{(k)} \longrightarrow M^{(k-1)}$$

defined by setting

$$\delta_k(m \otimes e_{i_1 \ldots i_k}) = \sum_{j=1}^{k} (-1)^{j-1} \varphi_{i_j}(m) \otimes e_{i_1 \ldots \widehat{i_j} \ldots i_k}.$$

The commutativity of the φ_j's assures that $\delta_{k-1} \circ \delta_k = 0$. We will denote by $K.(\Phi, M)$ the complex

$$0 \longrightarrow M^{(p)} \xrightarrow{\delta_p} M^{(p-1)} \xrightarrow{\delta_{p-1}} \cdots \longrightarrow M^{(0)} \longrightarrow 0$$

where $M^{(0)}$ is in degree 0. This is the *negative Koszul complex* of Φ.

Proposition 5.1.4 *Assume φ_1 is injective and φ_j induces an injective endomorphism of $M/(\operatorname{im} \varphi_1 + \cdots + \operatorname{im} \varphi_{j-1})$ for $j \in \{2, \ldots, p\}$. Then*

$$H_k(K.(\varphi, M)) = \begin{cases} M/(\operatorname{im} \varphi_1 + \cdots + \operatorname{im} \varphi_p) & \text{if } k = 0 \\ 0 & \text{otherwise} \end{cases}$$

Proof: Work as in the proof of 5.1.2. □

Note that, for any Abelian group G, $\mathrm{Hom}(M,G)$ is canonically an A^{op}-module and the endomorphisms $\varphi_1,\ldots,\varphi_p$ induce the A^{op}-linear endomorphisms

$$\psi_1 = \mathrm{Hom}(\varphi_1, \mathrm{id}_G), \ldots, \psi_p = \mathrm{Hom}(\varphi_p, \mathrm{id}_G)$$

of $\mathrm{Hom}(M,G)$ such that

$$\begin{aligned} \mathrm{Hom}^{\cdot}(K_{\cdot}(\varphi, M), G) &\simeq K^{\cdot}(\psi, \mathrm{Hom}(M,G)) \\ \mathrm{Hom}^{\cdot}(K^{\cdot}(\varphi, M), G) &\simeq K_{\cdot}(\psi, \mathrm{Hom}(M,G)). \end{aligned}$$

Let us denote

$$*^k : M^{(k)} \longrightarrow M^{(p-k)}$$

the only A-linear map such that

$$*^k(m \otimes e_{i_1\ldots,i_k}) = sm \otimes e_{j_1\ldots j_{p-k}}$$

where $\{i_1,\ldots,i_k\} \cup \{j_1,\ldots,j_{p-k}\} = \{1,\ldots,p\}$ and

$$s = \mathrm{sign}\begin{pmatrix} i_1 & \ldots i_k j_1 \ldots & j_{p-k} \\ 1 & \ldots & p \end{pmatrix}.$$

A direct computation shows that

$$d^{p-k} \circ *^k = (-1)^{k-1} *^{k-1} \circ \delta^k.$$

Hence, the maps

$$(-1)^{\frac{k(k-1)}{2}} *^k : M^{(k)} \longrightarrow M^{(p-k)}$$

define an isomorphism

$$K_{\cdot}(\Phi, M) \xrightarrow{\sim} K^{\cdot}(\Phi, M)[p]$$

which induces the isomorphism

$$H_k(\Phi; M) \simeq H^{p-k}(\Phi; M).$$

5.2 The universal Spencer and de Rham complexes

Definition 5.2.1 Let Θ_X^p denote the sheaf of holomorphic p-vector fields on X. The *universal Spencer complex of* X is the complex $\mathcal{SP}_{\cdot}^X$ defined by setting

$$\mathcal{SP}_p^X = \mathcal{D}_X \otimes_{\mathcal{O}_X} \Theta_X^p,$$

the differential

$$\delta_p : \mathcal{SP}_p^X \longrightarrow \mathcal{SP}_{p-1}^X$$

being defined by the formula

$$\begin{aligned} \delta_p(Q \otimes \theta_1 \wedge \ldots \wedge \theta_p) \;=\; & \sum_{i=1}^p (-1)^{i-1} Q\theta_i \otimes \theta_1 \wedge \ldots \wedge \widehat{\theta_i} \wedge \ldots \wedge \theta_p \\ & + \sum_{1 \le i < j \le p} (-1)^{i+j} Q \otimes [\theta_i, \theta_j] \wedge \theta_1 \wedge \ldots \widehat{\theta_i} \ldots \widehat{\theta_j} \ldots \wedge \theta_p. \end{aligned}$$

Note that δ_p is well defined since the right hand side of the preceding formula is an alternate $\mathcal{O}_X$-multilinear form in $\theta_1,\ldots,\theta_p$. A simple computation will confirm that $\delta_{p-1} \circ \delta_p = 0$.

Proposition 5.2.2 *The $\mathcal{D}_X$-linear morphism*

$$\begin{array}{ccc} \mathcal{D}_X & \longrightarrow & \mathcal{O}_X \\ Q & \mapsto & Q \cdot 1 \end{array}$$

induces a $\mathcal{D}_X$-linear quasi-isomorphism

$$\mathcal{SP}^X_\cdot \longrightarrow \mathcal{O}_X.$$

Proof: In a local coordinate system $(z : U \longrightarrow \mathbb{C}^n)$ we have

$$\delta_p(Q \otimes \partial_{z^{i_1}} \wedge \ldots \wedge \partial_{z^{i_p}}) = \sum_{j=1}^{p} (-1)^{j-1} Q\partial_{z^{i_j}} \otimes \partial_{z^{i_1}} \wedge \ldots \wedge \widehat{\partial_{z^{i_j}}} \wedge \ldots \wedge \partial_{z^{i_p}}.$$

Hence, the universal Spencer complex is isomorphic to the negative Koszul complex associated to the sequence $(\cdot\partial_{z^1}, \ldots, \cdot\partial_{z^n})$ of $\mathcal{D}_U$-linear endomorphisms of $\mathcal{D}_U$. Since $\cdot\partial_{z^1}$ is injective and $\cdot\partial_{z^k}$ induces an injective endomorphism of the quotient $\mathcal{D}_U/\mathcal{D}_U\partial_{z^1} + \cdots + \mathcal{D}_U\partial_{z^{k-1}}$, Proposition 5.1.4 allows us to conclude. □

Definition 5.2.3 Recall that Ω^k_X denotes the sheaf of holomorphic k-forms. Set

$$\mathcal{DR}^k_X = \Omega^k_X \otimes_{\mathcal{O}_X} \mathcal{D}_X$$

and define

$$d^k : \Omega^k_X \otimes_{\mathcal{O}_X} \mathcal{D}_X \longrightarrow \Omega^{k+1}_X \otimes_{\mathcal{O}_X} \mathcal{D}_X$$

in a local coordinate system $(z : U \longrightarrow \mathbb{C}^n)$ by the formula

$$d^k(\omega^k \otimes m) = d\omega^k \otimes m + \sum_{j=1}^{n} dz^j \wedge \omega^k \otimes \partial_{z^j} m.$$

One checks easily that this definition does not depend on the chosen coordinate system and that $d^{k+1} \circ d^k = 0$. The complex $\mathcal{DR}^\cdot_X$ is the *universal de Rham complex of X*.

Proposition 5.2.4 *The $\mathcal{D}^{\mathrm{op}}_X$-linear morphism*

$$\begin{array}{ccc} \Omega_X \otimes_{\mathcal{O}_X} \mathcal{D}_X & \longrightarrow & \Omega_X \\ \omega \otimes Q & \mapsto & \omega Q \end{array}$$

induces a $\mathcal{D}^{\mathrm{op}}_X$-linear quasi-isomorphism

$$\mathcal{DR}^\cdot_X[n] \longrightarrow \Omega_X.$$

Proof: Let us denote by ε the action morphism

$$\Omega_X \otimes_{\mathcal{O}_X} \mathcal{D}_X \longrightarrow \Omega_X.$$

A local computation shows that

$$\varepsilon \circ d^{n-1} = 0$$

hence ε induces a $\mathcal{D}_X^{\mathrm{op}}$-linear morphism of complexes

$$\mathcal{DR}^{\cdot}_X[n] \longrightarrow \Omega_X.$$

In a local coordinate system ($z : U \longrightarrow \mathbb{C}^n$), the complex $\mathcal{DR}^{\cdot}_U$ is isomorphic to the positive Koszul complex associated to the sequence $(\partial_{z^1 \cdot}, \ldots, \partial_{z^n \cdot})$ of right $\mathcal{D}_U$-linear endomorphisms of $\mathcal{D}_U$. Since $\partial_{z^1 \cdot}$ is injective and $\partial_{z^k \cdot}$ induces an injective endomorphism of $\mathcal{D}_U/\partial_{z^1}\mathcal{D}_U + \cdots + \partial_{z^{k-1}}\mathcal{D}_U$ we know that $H^j(\mathcal{DR}^{\cdot}_U) = 0$ for $j \neq n$, $H^n(\mathcal{DR}^{\cdot}_U)$ being isomorphic to $\mathcal{D}_U/\partial_{z^1}\mathcal{D}_U + \cdots + \partial_{z^n}\mathcal{D}_U$.

To conclude, we note that for $\omega = dz^1 \wedge \ldots \wedge dz^n$, the diagram of isomorphisms:

$$\begin{array}{ccccc}
 & P & \mapsto & P1 & \\
P & \mathcal{D}_U/\mathcal{D}_U\partial_{z^1} + \cdots + \mathcal{D}_U\partial_{z^n} & \longrightarrow & \mathcal{O}_U & h \\
\downarrow & \downarrow & & \downarrow & \downarrow \\
P^{\sim\omega} & \mathcal{D}_U/\partial_{z^1}\mathcal{D}_U + \cdots + \partial_{z^n}\mathcal{D}_U & \longrightarrow & \Omega_U & h\omega \\
 & P & \mapsto & \omega.P &
\end{array}$$

is commutative. □

Proposition 5.2.5 *There is a canonical isomorphism of complexes of $\mathcal{D}_X^{\mathrm{op}}$-modules*

$$\mathcal{DR}^{\cdot}_X \xrightarrow{\sim} \mathcal{H}om^{\cdot}_{\mathcal{D}_X}(\mathcal{SP}^X_{\cdot}, \mathcal{D}_X).$$

Proof: At the level of components, we have

$$\begin{aligned}
\mathcal{H}om_{\mathcal{D}_X}(\mathcal{SP}^X_k, \mathcal{D}_X) &= \mathcal{H}om_{\mathcal{D}_X}(\mathcal{D}_X \otimes_{\mathcal{O}_X} \Theta^k_X, \mathcal{D}_X) \\
&\simeq \Omega^k_X \otimes_{\mathcal{O}_X} \mathcal{D}_X.
\end{aligned}$$

The isomorphism

$$\mathcal{DR}^k_X \xrightarrow{\sim} \mathcal{H}om_{\mathcal{D}_X}(\mathcal{SP}^X_k, \mathcal{D}_X)$$

sends $\omega \otimes P$ to the $\mathcal{D}_X$-linear morphism

$$\begin{array}{rcl}
\mathcal{D}_X \otimes_{\mathcal{O}_X} \Theta^k_X & \longrightarrow & \mathcal{D}_X \\
(Q \otimes \theta) & \mapsto & Q(\langle \omega, \theta \rangle\, P).
\end{array}$$

In a local coordinate system, it is easy to check that this induces a morphism of complexes

$$\mathcal{DR}^{\cdot}_X \longrightarrow \mathcal{H}om^{\cdot}_{\mathcal{D}_X}(\mathcal{SP}^X_{\cdot}, \mathcal{D}_X)$$

hence the conclusion. □

5.3 The de Rham complex of a $\mathcal{D}_X$-module

Definition 5.3.1 The *de Rham complex of a left $\mathcal{D}_X$-module $\mathcal{M}$* is the complex

$$\Omega^{\cdot}_X(\mathcal{M}) = \mathcal{DR}^{\cdot}_X \otimes_{\mathcal{D}_X} \mathcal{M}.$$

Of course,

$$\Omega^k_X(\mathcal{M}) \simeq \Omega^k_X \otimes_{\mathcal{O}_X} \mathcal{M},$$

and, in a local coordinate system $(z : U \longrightarrow \mathbb{C}^n)$, we have

$$d^k(\omega^k \otimes m) = d\omega^k \otimes m + \sum_{j=1}^{n} dz^j \wedge \omega^k \otimes \partial_{z^j} m.$$

Proposition 5.3.2 *For any left $\mathcal{D}_X$-module $\mathcal{M}$ there is a canonical quasi-isomorphism*

$$R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{O}_X, \mathcal{M}) \simeq \Omega^{\cdot}_X(\mathcal{M}).$$

Proof: By Proposition 5.2.5, we have the isomorphism of complex

$$\mathcal{DR}^{\cdot}_X \xrightarrow{\sim} \mathcal{H}om^{\cdot}_{\mathcal{D}_X}(\mathcal{SP}^X_{\cdot}, \mathcal{D}_X).$$

Tensoring with $\mathcal{M}$, we get the isomorphism

$$\Omega^{\cdot}_X(\mathcal{M}) \xrightarrow{\sim} R\mathcal{H}om^{\cdot}_{\mathcal{D}_X}(\mathcal{SP}^X_{\cdot}, \mathcal{M}).$$

in $\mathbf{D}^{\mathrm{b}}(X)$. By Proposition 5.2.2, we have

$$\mathcal{SP}^X_{\cdot} \xrightarrow{\sim} \mathcal{O}_X,$$

and the proof is complete. □

Corollary 5.3.3 *In $\mathbf{D}^{\mathrm{b}}(X)$, we have the canonical isomorphisms*

$$R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{O}_X, \mathcal{O}_X) \simeq \Omega^{\cdot}_X \simeq \mathbb{C}_X.$$

Corollary 5.3.4 *In $\mathbf{D}^{\mathrm{b}}(\mathcal{D}^{\mathrm{op}}_X)$, we have the canonical isomorphisms*

$$R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{O}_X, \mathcal{D}_X) \simeq \mathcal{DR}^{\cdot}_X \simeq \Omega_X[-n].$$

In particular, $\mathrm{hd}_{\mathcal{D}_X}(\mathcal{O}_X) = n$.

6 The characteristic variety

6.1 Definition and basic properties

Definition 6.1.1 Through the isomorphism

$$\mathcal{GD}_X \xrightarrow{\sim} \bigoplus_{k \in \mathbb{N}} \pi_* \mathcal{O}_{T^*X}(k)$$

we may identify $\Sigma\mathcal{GD}_X$ to a subsheaf of $\pi_*\mathcal{O}_{T^*X}$. Using this identification, we define the *analytic localization* $\widetilde{\mathcal{GM}}$ of a $\mathcal{GD}_X$-module $\mathcal{GM}$ by the formula:

$$\widetilde{\mathcal{GM}} = \mathcal{O}_{T^*X} \otimes_{\pi^{-1}\Sigma\mathcal{GD}_X} \pi^{-1}\Sigma\mathcal{GM}.$$

Proposition 6.1.2 *The analytic localization functor*

$$\mathcal{GM} \mapsto \widetilde{\mathcal{GM}}$$

is faithful and exact. Moreover, a $\mathcal{GD}_X$-module is coherent if and only if its analytic localization is a coherent $\mathcal{O}_{T^*X}$-module.

Proof: It is a direct consequence of well-known results of Serre [17]. □

Corollary 6.1.3 *The annihilating ideal $\mathcal{A}nn\,\widetilde{\mathcal{GM}}$ of a coherent $\mathcal{GD}_X$-module $\mathcal{GM}$ is generated by $\pi^{-1}\mathcal{A}nn\,\mathcal{GM}$. In particular, $\operatorname{supp}\widetilde{\mathcal{GM}}$ is a closed conic analytic subset of T^*X.*

Lemma 6.1.4 (Comparison of filtrations) *Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module. Assume that $\mathcal{FM}$ and $\mathcal{FM}'$ are two good filtrations on $\mathcal{M}$. Then for any $x \in X$, we may find a neighborhood U of x and integers d, d' such that*

$$\mathcal{F}_k\mathcal{M}_{|U} \subset \mathcal{F}_{k+d}\mathcal{M}'_{|U} \qquad \mathcal{F}_k\mathcal{M}'_{|U} \subset \mathcal{F}_{k+d'}\mathcal{M}_{|U}$$

Proof: Since $\mathcal{FM}$ is a coherent $\mathcal{FD}_X$-module, we may find a neighborhood U of x and generators $m_1, \dots, m_p$ of order $d_1, \dots, d_p$ of $\mathcal{FM}$ on U. After shrinking U, we may assume that $m_1, \dots, m_p$ have finite orders $d'_1, \dots, d'_p$ with respect to $\mathcal{FM}'$. Hence, for $k \in \mathbb{Z}$, we have

$$\mathcal{F}_k\mathcal{M}_{|U} \subset \mathcal{F}_{k+d}\mathcal{M}'_{|U}$$

if $d = \sup(d'_1 - d_1, \dots, d'_p - d_p)$. The other part of the result is obtained by reversing the roles of $\mathcal{FM}$ and $\mathcal{FM}'$. □

Proposition 6.1.5 *Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module. Assume that $\mathcal{FM}$ and $\mathcal{FM}'$ are two good filtrations on $\mathcal{M}$ and denote by $\mathcal{GM}$ and $\mathcal{GM}'$ the corresponding coherent $\mathcal{GD}_X$-modules. Then*

$$\operatorname{supp}\widetilde{\mathcal{GM}} = \operatorname{supp}\widetilde{\mathcal{GM}'}.$$

Proof: Since the problem is local on X and since the localization functor is shift invariant, the preceding lemma allows us to assume that there is an integer $d \in \mathbb{N}_0$ such that

$$\mathcal{F}_k\mathcal{M}' \subset \mathcal{F}_k\mathcal{M} \subset \mathcal{F}_{k+d}\mathcal{M}'$$

for every $k \in \mathbb{Z}$. We will proceed by increasing induction on d.

For $d = 1$, we define the auxiliary $\mathcal{GD}_X$-modules $\mathcal{GL}$ and $\mathcal{GN}$ by the formulas

$$\mathcal{GL} = \bigoplus_{k\in\mathbb{Z}} \mathcal{F}_k\mathcal{M}'/\mathcal{F}_{k-1}\mathcal{M}; \qquad \mathcal{GN} = \bigoplus_{k\in\mathbb{Z}} \mathcal{F}_k\mathcal{M}/\mathcal{F}_k\mathcal{M}'.$$

We get the exact sequences

$$0 \longrightarrow \mathcal{GL} \longrightarrow \mathcal{GM} \longrightarrow \mathcal{GN} \longrightarrow 0$$

$$0 \longrightarrow \mathcal{GN}(-1) \longrightarrow \mathcal{GM}' \longrightarrow \mathcal{GL} \longrightarrow 0.$$

Since $\mathcal{GM}$ and $\mathcal{GM}'$ are coherent, so are $\mathcal{GL}$ and $\mathcal{GN}$. Applying the localization functor, we get the exact sequences

$$0 \longrightarrow \widetilde{\mathcal{GL}} \longrightarrow \widetilde{\mathcal{GM}} \longrightarrow \widetilde{\mathcal{GN}} \longrightarrow 0$$

$$0 \longrightarrow \widetilde{\mathcal{GN}} \longrightarrow \widetilde{\mathcal{GM}'} \longrightarrow \widetilde{\mathcal{GL}} \longrightarrow 0.$$

The requested equality is then given by the formula

$$\operatorname{supp} \widetilde{\mathcal{GM}} = \operatorname{supp} \widetilde{\mathcal{GL}} \cup \operatorname{supp} \widetilde{\mathcal{GN}} = \operatorname{supp} \widetilde{\mathcal{GM}'}.$$

For $d > 1$, we define the auxiliary filtration $\mathcal{FM}''$ by the formula

$$\mathcal{F}_k\mathcal{M}'' = \mathcal{F}_k\mathcal{M} + \mathcal{F}_{k+1}\mathcal{M}'.$$

This is obviously a good filtration. Moreover, for any $k \in \mathbb{Z}$, we have

$$\mathcal{F}_k\mathcal{M} \subset \mathcal{F}_k\mathcal{M}'' \subset \mathcal{F}_{k+1}\mathcal{M}$$

and

$$\mathcal{F}_k\mathcal{M}' \subset \mathcal{F}_k\mathcal{M}'' \subset \mathcal{F}_{k+1+(d-1)}\mathcal{M}'.$$

By the induction hypothesis, we get

$$\operatorname{supp} \widetilde{\mathcal{GM}} = \operatorname{supp} \widetilde{\mathcal{GM}''} = \operatorname{supp} \widetilde{\mathcal{GM}'}.$$

and the conclusion follows. □

Definition 6.1.6 Since any coherent $\mathcal{D}_X$-module admits locally a good filtration, the preceding proposition shows that there is a unique closed conic analytic subvariety V of T^*X such that

$$V \cap T^*U = \operatorname{supp} \widetilde{\mathcal{GM}}$$

for any open subset U of X and any good filtration $\mathcal{FM}$ of $\mathcal{M}$ on U. We call V the *characteristic variety* of $\mathcal{M}$ and denote it by $\mathrm{char}\mathcal{M}$. For a complex $\mathcal{M}^\cdot \in \mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_X)$, we set

$$\mathrm{char}\mathcal{M}^\cdot = \bigcup_{k\in\mathbb{Z}} \mathrm{char}\mathcal{H}^k(\mathcal{M}^\cdot).$$

Proposition 6.1.7 *Let $\mathcal{FM}^\cdot$ be a complex of $\mathcal{FD}_X$-modules and denote by $\mathcal{M}^\cdot$ the underlying complex of $\mathcal{D}_X$-modules. Assume that the components of $\mathcal{FM}^\cdot$ are $\mathcal{FD}_X$-coherent. Then the components of $\mathcal{M}^\cdot$ are $\mathcal{D}_X$-coherent and*

$$\mathrm{char}\mathcal{H}^k(\mathcal{M}^\cdot) \subset \operatorname{supp} \mathcal{H}^k(\tilde{\mathcal{G}}r\mathcal{FM}^\cdot)$$

for $k \in \mathbb{Z}$. In particular,

$$\mathrm{char}\mathcal{M}^\cdot \subset \operatorname{supp} \tilde{\mathcal{G}}r\mathcal{FM}^\cdot.$$

Proof: Let us denote by $\mathcal{FH}^k$ the k-th cohomology group of $\mathcal{M}^\cdot$ endowed with the filtration induced by that of $\mathcal{FM}^k$. By construction, we have the exact sequence

$$0 \longrightarrow \operatorname{im} \mathcal{F}d^{k-1} \longrightarrow \ker \mathcal{F}d^k \longrightarrow \mathcal{FH}^k \longrightarrow 0.$$

Since both $\ker \mathcal{F}d^k$ and $\operatorname{im} \mathcal{F}d^{k-1}$ are $\mathcal{FD}_X$-coherent, so is $\mathcal{FH}^k$. By Proposition 10.3.2, we have the inclusions

$$\begin{aligned} \operatorname{im} \tilde{\mathcal{G}}r\mathcal{F}d^{k-1} &\subset \tilde{\mathcal{G}}r \operatorname{im} \mathcal{F}d^{k-1} \\ \tilde{\mathcal{G}}r \ker \mathcal{F}d^k &\subset \ker \tilde{\mathcal{G}}r\mathcal{F}d^k. \end{aligned}$$

Hence, in a neighborhood of a point of T^*X,

$$\operatorname{im} \tilde{\mathcal{G}}r\mathcal{F}d^{k-1} = \ker \tilde{\mathcal{G}}r\mathcal{F}d^k$$

implies

$$\tilde{\mathcal{G}}r \operatorname{im} \mathcal{F}d^{k-1} = \tilde{\mathcal{G}}r \ker \mathcal{F}d^k.$$

This shows that

$$\operatorname{supp} \tilde{\mathcal{G}}r\mathcal{FH}^k \subset \operatorname{supp} \mathcal{H}^k(\tilde{\mathcal{G}}r\mathcal{FM}^\cdot)$$

and the conclusion follows easily. □

Remark 6.1.8 In general, the inclusion of the preceding proposition is strict. For example, let $\mathcal{FM}^\cdot$ be the complex

$$\mathcal{FD}_X(-1) \xrightarrow{\mathcal{F}u} \mathcal{FD}_X$$

where $\mathcal{F}u$ is the morphism corresponding to the inclusions $\mathcal{F}_{k-1}\mathcal{D}_X \subset \mathcal{F}_k\mathcal{D}_X$. Then,

$$\operatorname{char}(\mathcal{M}^\cdot) = \emptyset,$$

but

$$\operatorname{supp}(\tilde{\mathcal{G}}r\mathcal{FM}^\cdot) = T^*X.$$

Proposition 6.1.9 *Assume $\mathcal{M}$ is a coherent $\mathcal{D}_X$-module. Then*

$$\operatorname{char}(R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X)) = \operatorname{char}(\mathcal{M}).$$

Moreover,

$$\operatorname{codim} \operatorname{char}\mathcal{E}xt^j_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X) \geq j$$

and

$$\mathcal{E}xt^j_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X) = 0$$

for $j < \operatorname{codim} \operatorname{char}\mathcal{M}$.

Proof: We only need to work locally. Hence, we may assume that $\mathcal{M}$ is the underlying $\mathcal{D}_X$-module of a coherent $\mathcal{FD}_X$-module $\mathcal{FM}$ and that we have a finite resolution

$$0 \longrightarrow \mathcal{FL}_p \longrightarrow \cdots \longrightarrow \mathcal{FL}_0 \longrightarrow \mathcal{FM} \longrightarrow 0$$

of $\mathcal{FM}$ by finite free $\mathcal{FD}_X$-modules. Forgetting the filtration, the complex

$$\mathcal{FH}om_{\mathcal{FD}_X}(\mathcal{FL}., \mathcal{FD}_X)$$

is quasi-isomorphic to $R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X)$. Moreover,

$$\mathcal{G}r\mathcal{FH}om_{\mathcal{FD}_X}(\mathcal{FL}., \mathcal{FD}_X) \simeq \mathcal{GH}om_{\mathcal{GD}_X}(\mathcal{G}r\mathcal{FL}., \mathcal{GD}_X).$$

Hence, by Proposition 6.1.7, we get

$$\mathrm{char}\mathcal{E}xt^j_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X) \subset \mathrm{supp}\,\mathcal{E}xt^j_{\mathcal{O}_{T^*X}}(\tilde{\mathcal{G}}r\mathcal{FM}, \mathcal{O}_{T^*X})$$

and the conclusion follows by a well-known result of analytic geometry. $\square$

6.2 Additivity

Proposition 6.2.1 *If*

$$0 \longrightarrow \mathcal{M}' \longrightarrow \mathcal{M} \longrightarrow \mathcal{M}'' \longrightarrow 0$$

is an exact sequence of $\mathcal{D}_X$-modules then

$$\mathrm{char}\mathcal{M} = \mathrm{char}\mathcal{M}' \cup \mathrm{char}\mathcal{M}''.$$

Proof: Since the problem is local on X, we may assume that $\mathcal{M}$ is endowed with a good filtration $\mathcal{FM}$. This filtration induces good filtrations $\mathcal{FM}'$ and $\mathcal{FM}''$ on $\mathcal{M}'$ and $\mathcal{M}''$ respectively and the sequence

$$0 \longrightarrow \mathcal{FM}' \longrightarrow \mathcal{FM} \longrightarrow \mathcal{FM}'' \longrightarrow 0$$

is strictly exact. Hence, the sequence

$$0 \longrightarrow \mathcal{GM}' \longrightarrow \mathcal{GM} \longrightarrow \mathcal{GM}'' \longrightarrow 0$$

is also exact, and, applying the localization functor, we get the exact sequence

$$0 \longrightarrow \widetilde{\mathcal{GM}}' \longrightarrow \widetilde{\mathcal{GM}} \longrightarrow \widetilde{\mathcal{GM}}'' \longrightarrow 0.$$

The conclusion follows by taking the supports. $\square$

Corollary 6.2.2 *Assume*

$$\mathcal{M}' \longrightarrow \mathcal{M} \longrightarrow \mathcal{M}'' \xrightarrow{+1}$$

is a distinguished triangle in $\mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_X)$. Then

$$\mathrm{char}\mathcal{M} \subset \mathrm{char}\mathcal{M}' \cup \mathrm{char}\mathcal{M}''.$$

Proof: This is a direct consequence of the snake's lemma and the preceding proposition. $\square$

6.3 Involutivity and consequences

For a linear subset $L \subset T_pT^*X$, we denote by $L^\perp$ the orthogonal of L with respect to the symplectic form of T^*X.

An analytic subvariety V of T^*X is *involutive* (resp. *Lagrangian*; *isotropic*) if for any smooth point $p \in V$, $T_pV^\perp \subset T_pV$ (resp. $T_pV^\perp = T_pV$; $T_pV^\perp \supset T_pV$). Note that for a non empty involutive (resp. Lagrangian, isotropic) analytic subvariety V of T^*X we have $\dim V \geq \dim X$ (resp. $\dim V = \dim X$, $\dim V \leq \dim X$).

Since $T_pV^\perp$ is generated by the values at p of the Hamiltonian fields of the holomorphic functions which are zero on V near p, we find that V is involutive if and only if the germ $\{f, g\}$ is zero on V near p for any germs of holomorphic functions f, g which are zero on V near p. Hence, a necessary and sufficient condition for an analytic subvariety of T^*X to be involutive is that

$$\{\mathcal{I}_V, \mathcal{I}_V\} \subset \mathcal{I}_V$$

where, as usual, $\mathcal{I}_V$ denotes the defining ideal of V in $\mathcal{O}_{T^*X}$.

This implies in particular that on the smooth part of an involutive analytic subvariety of T^*X, the sub-bundle $TV^\perp$ of TV satisfies the Frobenius integrability conditions. The *leaves* of V are the corresponding maximal integral immersed sub-manifolds. Their dimension is $\operatorname{codim}_{T^*X} V$.

Theorem 6.3.1 *The characteristic variety* $\operatorname{char}(\mathcal{M})$ *of a coherent* $\mathcal{D}_X$*-module* $\mathcal{M}$ *is a closed conic involutive analytic subset of* T^*X.

Proof: Locally, $\mathcal{M}$ is the underlying $\mathcal{D}_X$-module of a coherent $\mathcal{F}\mathcal{D}_X$-module $\mathcal{F}\mathcal{M}$. By Proposition 3.3.4, the Poisson bracket defined algebraically using the fact that $\mathcal{G}\mathcal{D}_X$ is commutative is the same as the usual Poisson bracket of classical mechanics. Hence, Theorem 10.3.7 shows that

$$\{\sqrt{\mathcal{A}nn\, \mathcal{G}r\mathcal{F}\mathcal{M}}, \sqrt{\mathcal{A}nn\, \mathcal{G}r\mathcal{F}\mathcal{M}}\} \subset \sqrt{\mathcal{A}nn\, \mathcal{G}r\mathcal{F}\mathcal{M}}.$$

Since $\sqrt{\mathcal{A}nn\, \mathcal{G}r\mathcal{F}\mathcal{M}}$ is the defining ideal of the characteristic variety of $\mathcal{M}$, the proof is complete. □

Corollary 6.3.2 *Assume* $\mathcal{M}$ *is a non zero coherent* $\mathcal{D}_X$*-module. Then,*

(a) $\dim \operatorname{char}(\mathcal{M}) \geq \dim X$,

(b) $\operatorname{hd}_{\mathcal{D}_X}(\mathcal{M}) \leq \dim X$,

(c) $\operatorname{glhd}(\mathcal{D}_X) = \dim X$.

Proof: Part (a) is an obvious consequence of the involutivity of $\operatorname{char}(\mathcal{M})$.

By Proposition 6.1.9, we know that

$$\operatorname{codim} \operatorname{char} \mathcal{E}xt^j_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X) \geq j$$

for any positive integer j. For $j > \dim X$, we see that $\mathcal{E}xt^j_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X) = 0$ since otherwise we would get a contradiction with the involutivity of $\mathrm{char}\mathcal{E}xt^j_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X)$. Part (b) is then a consequence of Proposition 10.1.2.

From part (b) we already know that

$$\mathrm{glhd}(\mathcal{D}_X) \leq \dim X.$$

To conclude, we note that the equality holds thanks to Corollary 5.3.4. □

Definition 6.3.3 We call the leaves of $\mathrm{char}(\mathcal{M})$ the *bicharacteristic leaves* of $\mathcal{M}$.

6.4 Definition of holonomic systems

Definition 6.4.1 A $\mathcal{D}_X$-module is *holonomic* if it is coherent and has a Lagrangian characteristic variety. Between non-zero $\mathcal{D}_X$-modules, the holonomic $\mathcal{D}_X$-modules are the ones with a characteristic variety of the smallest possible dimension (i.e. $\dim X$).

Proposition 6.4.2 *Assume $\mathcal{M}$ is a coherent $\mathcal{D}_X$-module. Then $\mathcal{M}$ is holonomic if and only if*

$$\mathcal{E}xt^j_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X) = 0$$

for $j \neq \dim X$.

Proof: Set $n = \dim X$.

Assume $\mathcal{M}$ is holonomic. Since

$$\mathcal{E}xt^j_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X) = 0$$

for $j < \mathrm{codim}\,\mathrm{char}(\mathcal{M})$ and for $j > n$ and $\mathrm{codim}\,\mathrm{char}(\mathcal{M}) = n$, the conclusion follows.

Assume now that

$$\mathcal{E}xt^j_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X) = 0$$

for $j \neq n$. Since

$$\mathrm{codim}\,\mathrm{char}(\mathcal{E}xt^n_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X)) \geq n,$$

the $\mathcal{D}_X^{\mathrm{op}}$-module $\mathcal{E}xt^n_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X)$ is a holonomic. Moreover, since

$$\mathcal{M} \simeq R\mathcal{H}om_{\mathcal{D}_X^{\mathrm{op}}}(\mathcal{E}xt^n_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X)[-n], \mathcal{D}_X),$$

we see that

$$\mathrm{char}\mathcal{M} \subset \mathrm{char}\mathcal{E}xt^n_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X),$$

and the proof is complete. □

7 Inverse Images of $\mathcal{D}$-modules

7.1 Inverse image functors and transfer modules

Definition 7.1.1 Let $f : X \longrightarrow Y$ be a complex analytic map. We set

$$\mathcal{D}_{X\to Y} = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{D}_Y.$$

Note that $\mathcal{D}_{X\to Y}$ is naturally both a left $\mathcal{O}_X$-module and a right $f^{-1}\mathcal{D}_Y$-module and these structure are clearly compatible. Let x_0 be a point in X and set $y_0 = f(x_0)$. For $\theta \in \Theta_{X,x_0}$, consider the map

$$\begin{array}{rcl} \theta\cdot : \Theta_{X,x_0} \otimes_{\mathcal{O}_{Y,y_0}} \mathcal{D}_{Y,y_0} & \longrightarrow & \Theta_{X,x_0} \otimes_{\mathcal{O}_{Y,y_0}} \mathcal{D}_{Y,y_0} \\ h \otimes m & \mapsto & \theta h \otimes m + \sum_{j=1}^{p} h\theta(y^j \circ f) \otimes \partial_{y^j} m \end{array}$$

where $(y^1, \dots, y^p)$ is a coordinate system in a neighborhood of y_0 in Y. Since this map is easily checked to be independent of the chosen local coordinate system, we get a canonical action morphism

$$\Theta_X \otimes \mathcal{D}_{X\to Y} \longrightarrow \mathcal{D}_{X\to Y}.$$

A direct computation shows that it satisfies the conditions of Proposition 3.1.3; hence $\mathcal{D}_{X\to Y}$ is canonically endowed with a structure of left $\mathcal{D}_X$-module compatible with its structure of right $f^{-1}\mathcal{D}_Y$-module. With this structure of bimodule, $\mathcal{D}_{X\to Y}$ is the *transfer module of f*. It has a canonical section

$$1_{X\to Y} = 1_X \otimes f^{-1}1_Y.$$

Definition 7.1.2 Let $f : X \longrightarrow Y$ be a morphism of complex analytic manifolds. We define the *inverse image functor for $\mathcal{D}$-modules*

$$\underline{f}^* : \mathbf{D}^-(\mathcal{D}_Y) \longrightarrow \mathbf{D}^-(\mathcal{D}_X)$$

by setting

$$\underline{f}^*(\mathcal{M}) = \mathcal{D}_{X\to Y} \otimes^L_{f^{-1}\mathcal{D}_Y} f^{-1}\mathcal{M}$$

for any object $\mathcal{M}$ in $\mathbf{D}^-(\mathcal{D}_Y)$. Note that, in $\mathbf{D}^-(\mathcal{O}_X)$, we have

$$\underline{f}^*(\mathcal{M}) \simeq \mathcal{O}_X \otimes^L_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{M}.$$

Proposition 7.1.3 *Assume $f : X \longrightarrow Y$, $g : Y \longrightarrow Z$ are two morphisms of complex analytic manifolds and set $h = g \circ f$. Then we have a canonical isomorphism of $(\mathcal{D}_X,\mathcal{D}_Z^{\mathrm{op}})$-bimodules*

$$\mathcal{D}_{X\xrightarrow{h}Z} \xrightarrow{\sim} \mathcal{D}_{X\xrightarrow{f}Y} \otimes^L_{f^{-1}\mathcal{D}_Y} f^{-1}\mathcal{D}_{Y\xrightarrow{g}Z}.$$

It induces a canonical isomorphism

$$\underline{h}^* \xrightarrow{\sim} \underline{f}^* \circ \underline{g}^*.$$

Proof: Forgetting the $\mathcal{D}_X$-module structure, the isomorphism is a direct consequence of the definition of the transfer module. To check that it is an isomorphism of $(\mathcal{D}_X, \mathcal{D}_Z^{\mathrm{op}})$-bimodules, we only need to prove its compatibility with the left action of Θ_X. This easily achieved in local coordinates. □

By the preceding proposition, we may use the graph embedding to reduce the study of inverse images to the case of a closed embedding or the case of the projection from a product to one of its factors. Since the problem is local, the following proposition will clarify the situation.

Proposition 7.1.4 *Let U, V be two open neighborhoods of 0 in $\mathbb{C}^n$ and $\mathbb{C}^m$ respectively $(m, n \neq 0)$. Then :*

(a) *The transfer module of the second projection*

$$\begin{array}{rcl} p : U \times V & \longrightarrow & V \\ (u, v) & \mapsto & v \end{array}$$

is given by the formula

$$\mathcal{D}_{U\times V\to V} \simeq \{\sum_\alpha a_\alpha(u,v)\partial_v^\alpha : a_\alpha(u,v) \in \mathcal{O}_{U\times V}\}.$$

As a left $\mathcal{D}_{U\times V}$-module it is isomorphic to

$$\mathcal{D}_{U\times V}/\mathcal{D}_{U\times V}\partial_{u^1} + \cdots + \mathcal{D}_{U\times V}\partial_{u^n}.$$

It is a coherent module generated by $1_{U\times V\to V}$. The Koszul complex

$$K_\cdot(\mathcal{D}_{U\times V}; \cdot\partial_{u^1}, \cdots, \cdot\partial_{u^n})$$

is a free resolution of length n of this module.

As a right $\mathcal{D}_V$-module it is flat but not finitely generated.

b) *The transfer module of the closed embedding*

$$\begin{array}{rcl} i : U & \longrightarrow & U \times V \\ u & \mapsto & (u, 0) \end{array}$$

is given by the formula

$$\mathcal{D}_{U\to U\times V} = \{\sum_{\alpha,\beta} a_{\alpha\beta}(u)\partial_u^\alpha\partial_v^\beta : a_{\alpha\beta}(u) \in \mathcal{O}_U\}$$

As a right $\mathcal{D}_{U\times V}$-module, it is isomorphic to

$$\mathcal{D}_{U\times V}/v^1\mathcal{D}_{U\times V} + \cdots + v^m\mathcal{D}_{U\times V}.$$

It is a coherent module generated by $1_{U\to U\times V}$. *The Koszul complex*

$$K.(\mathcal{D}_{U\times V}; v^1\cdot, \cdots, v^m\cdot)$$

is a free resolution of length m *of this module.*

As a left $\mathcal{D}_U$*-module, it is flat but not finitely generated (it is a countable direct sum of copies of* $\mathcal{D}_U$*).*

Corollary 7.1.5 *For any morphism* $f : X \longrightarrow Y$ *of complex analytic manifolds, the functor* $\underline{f}^*$ *has bounded amplitude and induces functors*

$$\begin{array}{rcl} \underline{f}^* : \mathbf{D}^{\mathrm{b}}(\mathcal{D}_Y) & \longrightarrow & \mathbf{D}^{\mathrm{b}}(\mathcal{D}_X) \\ \underline{f}^* : \mathbf{D}(\mathcal{D}_Y) & \longrightarrow & \mathbf{D}(\mathcal{D}_X). \end{array}$$

In general, the inverse image of a coherent $\mathcal{D}$-module is not coherent. It is however the case for non-characteristic inverse images which we shall now investigate.

7.2 Non-characteristic maps

Consider a map $f : X \longrightarrow Y$ between complex analytic manifolds. From a microlocal point of view (i.e. at the level of cotangent bundles), we get the diagram

$$\begin{tikzcd} T^*X \arrow[d, "\pi_X"'] & X \times_Y T^*Y \arrow[l, "\rho"'] \arrow[r, "\varpi"] \arrow[d, "\pi"'] & T^*Y \arrow[d, "\pi_Y"'] \\ X & X \arrow[l, "\sim"'] \arrow[r, "f"'] & Y \end{tikzcd}$$

where vertical arrows are the canonical projections of the various bundles and

$$\begin{array}{rcl} \rho(x; (f(x); \xi)) & = & (x; f^*\xi) \\ \varpi(x; (f(x); \xi)) & = & (f(x); \xi). \end{array}$$

The kernel of ρ will be denoted by $T^*_X Y$. In general, it is a conic complex analytic subset of $X \times_Y T^*Y$. If f has locally constant rank, it becomes a holomorphic bundle.

Proposition 7.2.1 *Assume that* V *is a closed conic complex analytic subset of* T^*Y. *Then the following conditions are equivalent :*

(i) ρ *is proper on* $\varpi^{-1}(V)$,

(ii) ρ *is finite on* $\varpi^{-1}(V)$,

(iii) $T^*_X Y \cap \varpi^{-1}(V)$ *is in the zero section of* $X \times_Y T^*Y$.

Proof: The equivalence of (i) and (ii) comes from the fact that a compact analytic subset of $\mathbb{C}^n$ has a finite number of points. The equivalence of (i) and (iii) is a consequence of Lemma 7.2.2 below. □

Lemma 7.2.2 *Let Γ_1, Γ_2 be two closed cones in a real vector bundle $p : E \longrightarrow X$. Then the map*

$$+ : \Gamma_1 \times_X \Gamma_2 \longrightarrow \Gamma_1 + \Gamma_2$$

is proper if and only if

$$\Gamma_1 \cap \Gamma_2^a \subset E_0$$

where E_0 is the zero section of E and $(\Gamma_2)^a$ is the antipodal of Γ_2.

Definition 7.2.3 Let $f : X \longrightarrow Y$ be a morphism of complex analytic manifolds and let V be a conic complex analytic subset of T^*Y. We say that f *is non-characteristic for V* if the equivalent conditions of Proposition 7.2.1 hold for f and V. If $\mathcal{M}$ is an object of $\mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_Y)$ and f is non-characteristic for $\mathrm{char}(\mathcal{M})$ then we say, for short, that *f is non characteristic for $\mathcal{M}$.*

The meaning of the non-characteristicity condition may be clarified by the following simple remark.

Remark 7.2.4 (i) An analytic submersion is non-characteristic for any coherent $\mathcal{D}_Y$-module.

(ii) Assume Z is a closed analytic hypersurface of X. Then, the embedding $i : Z \longrightarrow X$ is non-characteristic for the $\mathcal{D}_X$-module $\mathcal{D}_X/\mathcal{D}_X P$ associated to an operator P of $\mathcal{D}_X$ if and only if $\sigma(P)(z;\xi) \neq 0$ for any non zero conormal covector $\xi \in T^*_Z X$.

7.3 Non-characteristic inverse images

Let us first investigate a very special but important case.

Proposition 7.3.1 *Let U, V be open neighborhoods of 0 in $\mathbb{C}^n$ and $\mathbb{C}$ respectively and let P be an operator in $\Gamma(U \times V; \mathcal{D}_{U\times V})$. Consider the closed embedding*

$$\begin{array}{rcl} i : U & \longrightarrow & U \times V \\ u & \mapsto & (u, 0) \end{array}$$

and assume i is non-characteristic for $\mathcal{D}_{U\times V}/\mathcal{D}_{U\times V}P$. Then up to shrinking U and V,

$$P = A_0(u,v)\partial_v^d + \sum_{k=1}^{d} A_k(u,v,\partial_u)\partial_v^{d-k}$$

where A_0 is an invertible holomorphic function and A_k is a differential operator of degree at most k in ∂_u. For such a P, the map

$$\begin{array}{rcl} \mathcal{D}_U^d & \longrightarrow & \mathcal{D}_{U\to U\times V} \\ \bigoplus\limits_{\ell=0}^{d-1} Q_k(u,\partial_u) & \mapsto & \sum\limits_{\ell=0}^{d-1} Q_k(u,\partial_u)\partial_v^k \end{array}$$

is left $\mathcal{D}_U$-linear and induces a quasi-isomorphism between $\mathcal{D}_U^d$ and the complex

$$\mathcal{D}_{U\to U\times V} \xrightarrow{\cdot P} \mathcal{D}_{U\to U\times V}.$$

In particular, we have an isomorphism

$$\mathcal{D}_U^d \xrightarrow{\sim} \underline{f}^*(\mathcal{D}_{U\times V}/\mathcal{D}_{U\times V}P).$$

Proof: Since i is non-characteristic for $\mathcal{D}_{U\times V}/\mathcal{D}_{U\times V}P$,

$$\sigma(P)((u,0);(0,\tau)) \neq 0$$

for any $u \in U$ and any $\tau \in \mathbb{C}^\times$. Hence, up to shrinking U and V, Weierstrass lemmas show that

$$\sigma(P)((u,v);(\eta,\tau)) = a_0(u,v)\tau^d + \sum_{k=1}^{d} a_k(u,v,\eta)\tau^{d-k}$$

where a_0 is an invertible holomorphic function and a_k is a holomorphic function which is homogeneous of degree k in η. The first part of the proposition follows easily. The second part is obvious and so is the third one since it is clear that any operator S of order s in $\Gamma(U\times V;\mathcal{D}_{U\times V})$ may be written as

$$S = Q\cdot P + R$$

where R is of degree strictly lower than d in ∂_v. $\square$

In the general case, we have the following result.

Theorem 7.3.2 *Let $f : X \longrightarrow Y$ be a morphism of complex analytic manifolds and let $\mathcal{N}$ be a coherent $\mathcal{D}_Y$-module. Assume f is non-characteristic for $\mathcal{N}$. Then*

(i) $\underline{f}^*\mathcal{N}$ *is a coherent $\mathcal{D}_X$-module,*

(ii) $\mathrm{char}(\underline{f}^*\mathcal{N}) \subset \rho\varpi^{-1}(\mathrm{char}(\mathcal{N}))$.

Proof: Using the factorization of f through its graph embedding, we may restrict ourselves to the special cases where f is a closed embeddings or a projection from a product to one of its factors. Since the problem is local on X, we may even assume f is the embedding (case a)

$$\begin{array}{rcl} i : U & \longrightarrow & U\times V \\ u & \mapsto & (u,0) \end{array}$$

or the projection (case b)

$$\begin{array}{rcl} p : U\times V & \longrightarrow & U \\ (u,v) & \mapsto & u \end{array}$$

where U, V are open neighborhoods of 0 in $\mathbb{C}^n$ and $\mathbb{C}^m$ respectively. Finally, by factorizing i and p we may assume $m = 1$.

Let us prove part (i) of the theorem. In case (b), it is a direct consequence of Proposition 7.1.4 so we only have to consider case (a).

Let s be a section of $\mathcal{N}$. Since $\mathcal{D}_{U\times V}s \subset \mathcal{N}$ it is clear from our hypothesis that

$$T^*_U U \times V \cap \operatorname{char}(\mathcal{D}_{U\times V}s) \subset T^*_{U\times V} U \times V$$

Let us denote $\mathcal{FI}$ the annihilating ideal of s in $\mathcal{D}_{U\times V}$ endowed with the order filtration. Since $\mathcal{N}$ is coherent, we know that $\operatorname{char}(\mathcal{D}_{U\times V}s)$ is the zero variety of the $\mathcal{O}_{T^*U\times V}$-ideal generated by $\mathcal{G}r\mathcal{FI}$. Hence, up to shrinking U and V, we may assume that there is an operator P in $\Gamma(U \times V; \mathcal{D}_X)$ such that

$$\sigma(P)((u,0);(0,\tau)) \neq 0$$

for some $\tau \in \mathbb{C}^\times$. For such a P, i is non-characteristic for $\mathcal{D}_{U\times V}/\mathcal{D}_{U\times V}P$.

We will now prove that $\underline{i}^*\mathcal{N}$ is concentrated in degree zero. By definition, as an $\mathcal{O}_U$-modules

$$\underline{i}^*\mathcal{N} = \mathcal{O}_U \otimes^L_{i^{-1}\mathcal{O}_{U\times V}} i^{-1}\mathcal{N}.$$

since the sequence

$$0 \longrightarrow i^{-1}\mathcal{O}_{U\times V} \xrightarrow{\cdot v} i^{-1}\mathcal{O}_{U\times V} \longrightarrow \mathcal{O}_U \longrightarrow 0$$

is exact, all we have to prove is that the kernel of the morphism

$$\mathcal{N}_{|U} \xrightarrow{v\cdot} \mathcal{N}_{|U}$$

is zero. If it is not the case, up to shrinking U and V, we would have a non zero section s of $\mathcal{N}$ such that $v \cdot s = 0$. Let P be the operator associated to s by the construction of the preceding paragraph. Since $v \cdot s = 0$ and $P \cdot s = 0$ we get by induction

$$\operatorname{ad}^n_v(P)s = 0$$

for any $n \in \mathbb{N}_0$. From Proposition 7.3.1 it follows that $\operatorname{ad}^d_v(P) = d!A_0(u,v)$. Hence s is zero and we get a contradiction.

To prove that $\underline{i}^*\mathcal{N}$ is $\mathcal{D}_{U\times V}$-coherent, we will proceed as follows. Up to shrinking U and V, we may assume $\mathcal{N}$ is generated by a finite number of sections $s_1, \ldots, s_N$. Denote by $P_1, \ldots, P_N$ the non-characteristic operators associated to these sections by the above procedure. By construction, we have an epimorphism

$$\bigoplus_{k=1}^N \mathcal{D}_{U\times V}/\mathcal{D}_{U\times V}P_k \longrightarrow \mathcal{N} \longrightarrow 0.$$

Let us denote by $\mathcal{N}'$ its kernel. Applying the inverse image functor, we get the exact sequence

$$i^*\mathcal{N}' \longrightarrow \bigoplus_{k=1}^N i^*\mathcal{D}_{U\times V}/\mathcal{D}_{U\times V}P_k \longrightarrow i^*\mathcal{N} \longrightarrow 0.$$

By construction i is non-characteristic for each P_k. Hence, by Proposition 7.3.1, the middle term is a finite free $\mathcal{D}_U$-module and $i^*\mathcal{N}$ is of finite type. Since i is non-characteristic for $\mathcal{N}'$, $i^*\mathcal{N}'$ is also of finite type and the conclusion follows.

Let us now prove part (ii) of the theorem.

Up to shrinking U nd V, we may assume $\mathcal{N}$ is endowed with a good filtration $\mathcal{FN}$. Let us denote by $\underline{f}^*\mathcal{FN}$ the $\mathcal{FD}_X$-module obtained by filtering $\mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{FN}$ by the images of the sheaves $\mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{F}_k\mathcal{N}$. From the exact sequence

$$0 \longrightarrow \mathcal{F}_{k-1}\mathcal{N} \longrightarrow \mathcal{F}_k\mathcal{N} \longrightarrow \mathcal{G}r_k\mathcal{FN} \longrightarrow 0$$

we get the exact sequence

$$\mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{F}_{k-1}\mathcal{N} \longrightarrow \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{F}_k\mathcal{N} \longrightarrow \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{G}r_k\mathcal{FN} \longrightarrow 0$$

Hence we have a canonical epimorphism of $\mathcal{G}r\mathcal{FD}_X$-module

$$\mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{G}r\mathcal{FN} \longrightarrow \mathcal{G}r f^*\mathcal{FN} \longrightarrow 0.$$

Since ρ is finite on $\operatorname{supp}\tilde{\mathcal{G}}r\mathcal{FN}$, we get a canonical epimorphism

$$\rho_*\varpi^*\tilde{\mathcal{G}}r\mathcal{FN} \longrightarrow \tilde{\mathcal{G}}r f^*\mathcal{FN} \longrightarrow 0$$

and the conclusion follows. $\square$

Remark 7.3.3 In fact, the equality holds in part (ii) of the preceding proposition. To prove this fact, we would need the theory of characteristic cycles which is not developed here.

7.4 Holomorphic solutions of non-characteristic inverse images

In this section, following Kashiwara, we will obtain a wide generalization of the Cauchy-Kowalewsky theorem in the form of a formula for the holomorphic solutions of non characteristic inverse images of $\mathcal{D}$-modules.

First, we recall the classical Cauchy-Kowalewsky theorem for one operator.

Proposition 7.4.1 *Let U, V be two open neighborhoods of 0 in $\mathbb{C}^n$ and $\mathbb{C}$ respectively and let P be an operator of degree d in $\Gamma(U \times V; \mathcal{D}_{U\times V})$. Consider the embedding*

$$\begin{array}{rcl} i : U & \longrightarrow & U \times V \\ u & \mapsto & (u, 0) \end{array}$$

and assume that

$$\sigma(P)((u,0);(0,\tau)) \neq 0$$

for $\tau \in \mathbb{C}^\times$. Then for any $g \in \Gamma(U \times V; \mathcal{O}_{U\times V})$ and any $g_0, \ldots, g_n \in \Gamma(U; \mathcal{O}_U)$, there is an open neighborhood W of U in $U \times V$ such that the Cauchy problem

$$\begin{cases} Pf = g \\ (\partial_v^k f) \circ i = g_k \quad ; \quad k = 0, \ldots, d-1 \end{cases}$$

has a unique holomorphic solution in W.

Next, we consider the general case.

Definition 7.4.2 Let $f : X \longrightarrow Y$ be a morphism of complex analytic manifolds. For any pair of $\mathcal{D}_Y$-modules $\mathcal{N}$, $\mathcal{N}'$, we have an obvious canonical morphism

$$f^{-1}R\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{N},\mathcal{N}') \longrightarrow R\mathcal{H}om_{\mathcal{D}_X}(\underline{f}^*\mathcal{N},\underline{f}^*\mathcal{N}').$$

Since we have

$$\underline{f}^*\mathcal{O}_Y \simeq \mathcal{O}_X,$$

we get a canonical *restriction morphism*

$$f^{-1}R\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{N},\mathcal{O}_Y) \longrightarrow R\mathcal{H}om_{\mathcal{D}_X}(\underline{f}^*\mathcal{N},\mathcal{O}_X).$$

Theorem 7.4.3 *Let $f : X \longrightarrow Y$ be a morphism of complex analytic manifolds and let $\mathcal{N}$ be a coherent $\mathcal{D}_Y$-module. Assume f is non-characteristic for $\mathcal{N}$. Then the canonical restriction morphism*

$$f^{-1}R\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{N},\mathcal{O}_Y) \longrightarrow R\mathcal{H}om_{\mathcal{D}_X}(\underline{f}^*\mathcal{N},\mathcal{O}_X)$$

is an isomorphism in $\mathbf{D}^{\mathrm{b}}(X)$.

Proof: As in the proof of Theorem 7.3.2, we may assume f is (case a) the embedding

$$\begin{array}{rcl} i : U & \longrightarrow & U \times V \\ u & \mapsto & (u,0) \end{array}$$

or (case b) the projection

$$\begin{array}{rcl} p : U \times V & \longrightarrow & U \\ (u,v) & \mapsto & u \end{array}$$

where U, V are open neighborhoods of 0 in $\mathbb{C}^n$ and $\mathbb{C}$ respectively.

For case (a), an iteration of the procedure used in the proof of Theorem 7.3.2 shows that $\mathcal{N}$ has a projective resolution $\mathcal{R}^{\cdot}$ by finite sums of $\mathcal{D}_{U\times V}$-module of the type $\mathcal{D}_{U\times V}/\mathcal{D}_{U\times V}P$ where i is non-characteristic for the operator P. Hence, we may assume from the beginning that $\mathcal{N}$ is of this special kind. On one hand, Proposition 7.3.1 shows that

$$\mathcal{D}_U^d \xrightarrow{\sim} \underline{i}^*\mathcal{D}_{U\times V}/\mathcal{D}_{U\times V}P,$$

and we get

$$R\mathcal{H}om_{\mathcal{D}_U}(\underline{i}^*\mathcal{D}_{U\times V}/\mathcal{D}_{U\times V}P,\mathcal{O}_U) \xrightarrow{\sim} \mathcal{O}_U^d.$$

On the other hand,

$$i^{-1}R\mathcal{H}om_{\mathcal{D}_{U\times V}}(\mathcal{D}_{U\times V}/\mathcal{D}_{U\times V}P,\mathcal{O}_{U\times V})$$

is represented by the complex

$$\mathcal{O}_{U\times V|U} \xrightarrow{P_{|U}\cdot} \mathcal{O}_{U\times V|U}$$

which, by Proposition 7.4.1, is quasi-isomorphic to $\ker(P_{|U}\cdot)$. With the preceding explicit representations, the restriction map becomes

$$\begin{array}{rcl} \ker(P_{|U}\cdot) & \longrightarrow & \mathcal{O}_U^d \\ h & \mapsto & \bigoplus\limits_{l=0}^{d-1} (\partial_v^l h) \circ i \end{array}$$

and the conclusion follows from Proposition 7.4.1.

In case (b), we may assume $\mathcal{N} = \mathcal{D}_U$. Then

$$p^{-1} R\mathcal{H}om_{\mathcal{D}_U}(\mathcal{N}, \mathcal{O}_U) \simeq p^{-1}\mathcal{O}_U.$$

Moreover, since

$$\underline{p}^*\mathcal{D}_U \simeq \mathcal{D}_{U\times V\to V},$$

Proposition 7.1.4 shows that

$$R\mathcal{H}om_{\mathcal{D}_{U\times V}}(\underline{p}^*\mathcal{D}_U, \mathcal{O}_{U\times V})$$

is represented by the complex

$$\mathcal{O}_{U\times V} \xrightarrow{\partial_v} \mathcal{O}_{U\times V}$$

which is canonically quasi-isomorphic to $p^{-1}\mathcal{O}_U$. Using these explicit representations, the restriction map becomes the identity on $p^{-1}\mathcal{O}_U$ and the proof is complete. □

8 Direct images of $\mathcal{D}$-modules

In this section, it is convenient to work with right $\mathcal{D}$-modules. The reader will easily obtain the corresponding results for left $\mathcal{D}$-modules by applying the side changing functors.

8.1 Direct image functors

Definition 8.1.1 Let $f : X \longrightarrow Y$ be a morphism of complex analytic manifolds. Since the functors $(\cdot) \otimes^L_{\mathcal{D}_X} \mathcal{D}_{X\to Y}$ and $Rf_!(\cdot)$ have bounded amplitude they define by composition a functor

$$\underline{f}_! : \mathbf{D}(\mathcal{D}_X^{\mathrm{op}}) \longrightarrow \mathbf{D}(\mathcal{D}_Y^{\mathrm{op}})$$

such that

$$\underline{f}_!(\mathcal{M}) = Rf_!(\mathcal{M} \otimes^L_{\mathcal{D}_X} \mathcal{D}_{X\to Y}).$$

for $\mathcal{M}$ in $\mathbf{D}(\mathcal{D}_X^{\mathrm{op}})$. This is the *direct image functor* (with proper support) for right $\mathcal{D}_X$-modules. It has bounded amplitude, and induces a functor

$$\underline{f}_! : \mathbf{D}^{\mathrm{b}}(\mathcal{D}_X^{\mathrm{op}}) \longrightarrow \mathbf{D}^{\mathrm{b}}(\mathcal{D}_Y^{\mathrm{op}}).$$

The direct image functor is compatible with the composition of morphisms.

Proposition 8.1.2 *Let $f : X \longrightarrow Y$, $g : Y \longrightarrow Z$ be two morphisms of complex analytic manifolds and set $h = g \circ f$. Then we have the canonical isomorphism*

$$\underline{h}_! \xrightarrow{\sim} \underline{g}_! \circ \underline{f}_!.$$

Proof: One has successively :

$$\begin{aligned} \underline{g}_! \circ \underline{f}_!(\mathcal{M}) &= Rg_!(Rf_!(\mathcal{M} \otimes^L_{\mathcal{D}_X} \mathcal{D}_{X\to Y}) \otimes^L_{\mathcal{D}_Y} \mathcal{D}_{Y\to Z}) \\ &\xrightarrow{\sim} Rg_! Rf_!(\mathcal{M} \otimes^L_{\mathcal{D}_X} \mathcal{D}_{X\to Y} \otimes^L_{f^{-1}\mathcal{D}_Y} f^{-1}\mathcal{D}_{Y\to Z}) \\ &\xrightarrow{\sim} Rh_!(\mathcal{M} \otimes^L_{\mathcal{D}_X} \mathcal{D}_{X\to Z}), \end{aligned}$$

where the last isomorphism comes from the first part of Proposition 7.1.3. □

In general, the direct image of a coherent $\mathcal{D}$-module is not coherent. This is however the case for proper direct images provided that $\mathcal{M}$ is good as will be shown in the next section.

8.2 Proper direct images

Definition 8.2.1 Let X be a complex analytic manifold.

A right coherent $\mathcal{D}_X$-module is *generated by a coherent $\mathcal{O}$-module* if $\mathcal{M}$ has a coherent $\mathcal{O}_X$-submodule $\mathcal{G}$ for which the natural morphism

$$\mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{D}_X \longrightarrow \mathcal{M}$$

is an epimorphism.

A right coherent $\mathcal{D}_X$-module is *good* if, in a neighborhood of any compact subset K of X, it has a finite filtration $(\mathcal{M}^k)_{k=1,\dots,N}$ by right coherent $\mathcal{D}_X$-submodules such that each quotient $\mathcal{M}^k/\mathcal{M}^{k-1}$ is generated by a coherent $\mathcal{O}_X$-module.

Good $\mathcal{D}_X$-modules form a thick subcategory of $\mathrm{Coh}(\mathcal{D}_X)$ denoted by $\mathrm{Good}(\mathcal{D}_X)$. The associated full triangulated subcategory of $\mathbf{D}^{\mathrm{b}}(\mathcal{D}_X)$ is denoted by $\mathbf{D}^{\mathrm{b}}_{\mathrm{good}}(\mathcal{D}_X)$.

Theorem 8.2.2 *Let $f : X \longrightarrow Y$ be a morphism of complex analytic manifolds. Assume $\mathcal{M} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{good}}(\mathcal{D}_X^{\mathrm{op}})$ and f is proper on* $\mathrm{supp}(\mathcal{M})$. *Then*

(i) $\underline{f}_!\mathcal{M}$ *is in* $\mathbf{D}^{\mathrm{b}}_{\mathrm{good}}(\mathcal{D}_Y^{\mathrm{op}})$,

(ii) $\mathrm{char}(\underline{f}_!\mathcal{M}) \subset \varpi\rho^{-1}(\mathrm{char}\mathcal{M})$.

Proof: Since the problem is local on Y and f is proper on $\mathrm{supp}\,\mathcal{M}$, we only need to work in a neighborhood of a compact subset of X. Hence, we may assume that $\mathcal{M}$ has a finite filtration $(\mathcal{M}_k)_{k=1,\dots,N}$ by coherent $\mathcal{D}_X$-modules such that each quotient $\mathcal{M}_k/\mathcal{M}_{k-1}$ is generated by a coherent $\mathcal{O}_X$-module. It is clear that the theorem will be true for $\mathcal{M}$ if it is true for each quotient $\mathcal{M}_k/\mathcal{M}_{k-1}$. Hence, we may assume from the beginning that $\mathcal{M}$ is generated by a coherent $\mathcal{O}_X$-module $\mathcal{G}$. We endow $\mathcal{M}$ with the filtration induced by the order filtration of $\mathcal{D}_X$ through the epimorphism

$$\mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{D}_X \longrightarrow \mathcal{M}$$

and denote by $\mathcal{F}\mathcal{M}$ the resulting $\mathcal{F}\mathcal{D}_X$-module.

For any coherent $\mathcal{F}\mathcal{D}_X$-module $\mathcal{F}\mathcal{N}$ and any compact subset K of X, there is an integer k such that, in a neighborhood of K, the natural morphism

$$\bigoplus_{d\leq k} \mathcal{F}_d\mathcal{N} \otimes^L_{\mathcal{O}_X} \mathcal{F}\mathcal{D}_X(-d) \longrightarrow \mathcal{N}$$

becomes a strict epimorphism. Thus, we may assume that $\mathcal{F}\mathcal{M}$ has a filtered projective resolution $\mathcal{F}\mathcal{P}^{\cdot}$ such that

$$\mathcal{F}\mathcal{P}^k = \bigoplus_{d\in I_k} \mathcal{G}^k_d \otimes_{\mathcal{O}_X} \mathcal{F}\mathcal{D}_X(-d)$$

where I_k is a finite subset of $\mathbb{Z}$ and $\mathcal{G}^k_d$ is a coherent $\mathcal{O}_X$-module with f-proper support. We denote by $\mathcal{P}^{\cdot}$ the underlying complex of $\mathcal{D}_X$-module.

Let $R^{\cdot}$ be a bounded f-soft resolution of $\mathbb{C}_X$ by sheaves of $\mathbb{C}$-vector spaces. Since $R^j \otimes \mathcal{G}^k_d$ is f-soft, the simple complex associated to

$$\mathcal{S}^{\cdot,\cdot} = f_!(R^{\cdot} \otimes \mathcal{P}^{\cdot} \otimes_{\mathcal{D}_X} \mathcal{D}_{X\to Y})$$

is quasi-isomorphic to $\underline{f}_!(R^{\cdot} \otimes \mathcal{P}^{\cdot}) \simeq \underline{f}_!\mathcal{M}$. Let us endow the $\mathcal{D}_Y$-module

$$\mathcal{S}^{j,k} = \bigoplus_{d\in I_k} f_!(R^j \otimes \mathcal{G}^k_d) \otimes_{\mathcal{O}_Y} \mathcal{D}_Y$$

with the structure of right $\mathcal{F}\mathcal{D}_Y$-module defined by setting

$$\mathcal{F}\mathcal{S}^{j,k} = \bigoplus_{d\in I_k} f_!(R^j \otimes \mathcal{G}^k_d) \otimes_{\mathcal{O}_Y} \mathcal{F}\mathcal{D}_Y(-d).$$

A simple computation shows that the differentials of $\mathcal{S}^{\cdot,\cdot}$ are then filtered morphisms. We denote by $\mathcal{F}\mathcal{S}^{\cdot,\cdot}$ the corresponding double complex of $\mathcal{F}\mathcal{D}_Y$-modules and by $\mathcal{F}\mathcal{S}^{\cdot}$ its associated simple complex.

By Grauert's coherence theorem for proper direct images, $Rf_!(\mathcal{G}^k_d)$ is quasi-isomorphic to a bounded complex of coherent $\mathcal{O}_Y$-modules. Hence, $\mathcal{F}\mathcal{S}^{\cdot}$ is quasi-isomorphic to a bounded complex of coherent $\mathcal{F}\mathcal{D}_Y$-modules. Moreover, one checks easily that

$$\tilde{\mathcal{G}}r\mathcal{F}\mathcal{S}^{\cdot} \simeq \varpi_*\rho^{-1}\tilde{\mathcal{G}}r\mathcal{F}\mathcal{R}^{\cdot}.$$

From this formula, it follows that

$$\mathrm{char}(\mathcal{S}^{\cdot}) \subset \mathrm{supp}\,\tilde{\mathcal{G}}r\mathcal{F}\mathcal{R}^{\cdot}$$

and Proposition 6.1.7 allows us to conclude. □

8.3 The $\mathcal{D}$-linear integration morphism

Let X and Y be complex analytic manifolds of complex dimension d_X and d_Y respectively. In this section, we will associate to any morphism $f : X \longrightarrow Y$ of complex analytic manifolds a canonical integration morphism

$$\underline{f}_!\Omega_X[d_X] \longrightarrow \Omega_Y[d_Y]$$

in $\mathbf{D}^{\mathrm{b}}(\mathcal{D}_Y^{\mathrm{op}})$. This map will play for direct images a role which is similar to the role of the restriction morphism

$$\underline{f}^*\mathcal{O}_Y \longrightarrow \mathcal{O}_X$$

in the study of inverse images.

Let X be a complex manifold. In the sequel, we denote by $\mathcal{D}b_X^{p,q}$ (resp. $\mathcal{F}_X^{p,q}$) the sheaf of forms of type (p,q) with distributions (resp. differentiable functions) as coefficients. We also denote by $\mathcal{D}b_X^{\cdot}$ (resp. $\mathcal{F}_X^{\cdot}$) the associated Dolbeault complex with differential $d = \partial + \overline{\partial}$.

Let $f : X \longrightarrow Y$ be a morphism of complex analytic manifolds. Recall that we have a pull-back morphism

$$f^* : f^{-1}\mathcal{F}_Y^{p,q} \longrightarrow \mathcal{F}_X^{p,q}$$

and a push-forward morphism

$$f_! : f_!\mathcal{D}b_X^{p+d_X,q+d_X} \longrightarrow \mathcal{D}b_Y^{p+d_Y,q+d_Y}.$$

Both are compatible with the differentials. Moreover, if $g : Y \longrightarrow Z$ is another morphism of complex analytic manifolds, we have $(g\circ f)^* = f^*\circ g^*$ and $(g\circ f)_! = g_!\circ f_!$. The two morphisms are linked by the formula

$$(f_!u)\wedge\omega = f_!(u\wedge f^*\omega).$$

Definition 8.3.1 Let $\mathcal{M}$ be a left $\mathcal{D}_X$-module. We set

$$\mathcal{D}b_X^{p,q}(\mathcal{M}) = \mathcal{D}b_X^{p,q}\otimes_{\mathcal{O}_X}\mathcal{M}.$$

The differentials

$$\begin{array}{rcl}\partial^{p,q} : \mathcal{D}b_X^{p,q}(\mathcal{M}) & \longrightarrow & \mathcal{D}b_X^{p+1,q}(\mathcal{M}) \\ u\otimes P & \mapsto & \partial^{p,q}u\otimes P + \sum_{i=1}^{x} dz_i\wedge u\otimes D_{z_i}P \\ \overline{\partial}^{p,q} : \mathcal{D}b_X^{p,q}(\mathcal{M}) & \longrightarrow & \mathcal{D}b_X^{p,q+1}(\mathcal{M}) \\ u\otimes P & \longrightarrow & \overline{\partial}^{p,q}u\otimes P\end{array}$$

do not depend on the local coordinate system $(z : U \longrightarrow \mathbb{C}^n)$. We denote $\mathcal{D}b_X^{\cdot}(\mathcal{M})$ the simple complex associated to the bi-graded sheaf $\mathcal{D}b_X^{\cdot\cdot}(\mathcal{M})$ and the differential $d = \partial + \overline{\partial}$. We call it the *distributional de Rham complex of* $\mathcal{M}$.

Lemma 8.3.2 *The differential of $\mathcal{D}b_X^{\cdot}(\mathcal{D}_X)$ is compatible with the right $\mathcal{D}_X$-module structure of its components, and, in $\mathbf{D}^{\mathrm{b}}(\mathcal{D}_X^{\mathrm{op}})$, one has a canonical isomorphism :*

$$\mathcal{D}b_X^{\cdot}(\mathcal{D}_X)[d_X] \xrightarrow{\sim} \Omega_X.$$

Proof: The compatibility of the differential of $\mathcal{D}b_X^{\cdot}(\mathcal{D}_X)$ with the right $\mathcal{D}_X$-module structure of its components is a direct consequence of the local forms of ∂ and $\overline{\partial}$.

Since $\mathcal{D}_X$ is flat over $\mathcal{O}_X$, the Dolbeault quasi-isomorphism

$$\Omega_X^p \xrightarrow{\sim} \mathcal{D}b_X^{p,\cdot}$$

induces the quasi-isomorphisms

$$\Omega_X^p \otimes_{\mathcal{O}_X} \mathcal{D}_X \xrightarrow{\sim} \mathcal{D}b_X^{p,\cdot} \otimes_{\mathcal{O}_X} \mathcal{D}_X \xrightarrow{\sim} \mathcal{D}b_X^{p,\cdot}(\mathcal{D}_X).$$

Hence, Weil's lemma shows that the natural morphism

$$DR_X^{\cdot}(\mathcal{D}_X) \xrightarrow{\sim} \mathcal{D}b_X^{\cdot}(\mathcal{D}_X)$$

from the holomorphic to the distributional de Rham complex of $\mathcal{D}_X$ is a quasi-isomorphism of complexes of right $\mathcal{D}_X$-modules and the conclusion follows from the quasi-isomorphism

$$DR_X^{\cdot}(\mathcal{D}_X) \simeq \Omega_X[-d_X].$$

of Proposition 5.2.4. □

Lemma 8.3.3 *To any morphism $f : X \longrightarrow Y$ of analytic manifolds is associated a canonical integration morphism*

$$f_! : f_!\mathcal{D}b_X^{\cdot}(\mathcal{D}_{X\to Y})[2d_X] \longrightarrow \mathcal{D}b_Y^{\cdot}(\mathcal{D}_Y)[2d_Y].$$

in the category of bounded complexes of right $\mathcal{D}_Y$-modules.

Proof: At the level of components the integration morphism is obtained as the following chain of morphisms :

$$\begin{aligned} f_!\mathcal{D}b_X^{p+d_X,q+d_X}(\mathcal{D}_{X\to Y}) &\xrightarrow{\sim} f_!(\mathcal{D}b_X^{p+d_X,q+d_X} \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{D}_Y) \\ &\xrightarrow{\sim} f_!\mathcal{D}b_X^{p+d_X,q+d_X} \otimes_{\mathcal{O}_Y} \mathcal{D}_Y \\ &\longrightarrow \mathcal{D}b_Y^{p+d_Y,q+d_Y} \otimes_{\mathcal{O}_Y} \mathcal{D}_Y \\ &\xrightarrow{\sim} \mathcal{D}b_Y^{p+d_Y,q+d_Y}(\mathcal{D}_Y). \end{aligned}$$

To get the second morphism one has used the projection formula, the fact that $\mathcal{D}b_X$ is a soft sheaf and the fact that $\mathcal{D}_Y$ is locally free over $\mathcal{O}_Y$. The third arrow is deduced from the push-forward of distributions along f.

To conclude, we need to show that the integration morphism is compatible with the differentials of the complexes involved. Thanks to the local forms of the differentials, this is an easy computational verification and we leave it to the reader. □

Lemma 8.3.4 *Let $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ be morphisms of complex analytic manifolds. One has the following commutative diagram:*

$$\begin{array}{ccc} g_!(f_!\mathcal{D}b_X^{\cdot}(\mathcal{D}_{X\to Y})[2d_X] \otimes_{\mathcal{D}_Y} \mathcal{D}_{Y\to Z}) & \xrightarrow{1} & g_!(\mathcal{D}b_Y^{\cdot}(\mathcal{D}_Y)[2d_Y] \otimes_{\mathcal{D}_Y} \mathcal{D}_{Y\to Z}) \\ \downarrow{\scriptstyle 2} & & \downarrow{\scriptstyle 3} \\ g_!f_!\mathcal{D}b_X^{\cdot}(\mathcal{D}_{X\to Z})[2d_X] & \xrightarrow{4} & \mathcal{D}b_Z^{\cdot}(\mathcal{D}_Z)[2d_Z]. \end{array}$$

In this diagram, arrow (1) is deduced from $f_!$ by tensor product and proper direct image, arrow (2) is an isomorphism deduced from the projection formula, arrow (3) is $g_!$ and arrow (4) is equal to $(g \circ f)_!$.

Proof: Going back to the definition of the various morphisms, one sees easily that the commutativity of the preceding diagram is a consequence of the Fubini theorem for distributions, that is, the formula

$$(g \circ f)_!(u) = g_!(f_!(u))$$

where g_* and f_* denotes the push-forward of distributions along g and f respectively, u being a distribution with $g \circ f$ proper support. $\square$

Proposition 8.3.5 *To any morphism $f : X \longrightarrow Y$ of complex analytic manifolds is associated a canonical integration arrow*

$$\int_f : \underline{f}_!(\Omega_X[d_X]) \longrightarrow \Omega_Y[d_Y]$$

in $\mathbf{D}^{\mathrm{b}}(\mathcal{D}_Y^{\mathrm{op}})$. Moreover, if $g : Y \longrightarrow Z$ is a second morphism of complex analytic manifolds then

$$\int_{g\circ f} = \int_g \circ \underline{g}_!(\int_f)$$

Proof: One gets the arrow $\int_f$ by composing the morphisms :

$$\begin{aligned} \underline{f}_!(\Omega_X[d_X]) &\xrightarrow{\sim} Rf_!(\Omega_X[d_X] \otimes^L_{\mathcal{D}_X} \mathcal{D}_{X\to Y}) \\ &\xrightarrow{\sim} Rf_!(\mathcal{D}b^{\cdot}_X(\mathcal{D}_{X\to Y})[2d_X]) \\ &\xrightarrow{\sim} f_!(\mathcal{D}b^{\cdot}_X(\mathcal{D}_{X\to Y})[2d_X]) \\ &\longrightarrow \mathcal{D}b^{\cdot}_Y(\mathcal{D}_Y)[2d_Y] \\ &\xrightarrow{\sim} \Omega_Y[d_Y] \end{aligned}$$

Let us point out that the second and last isomorphisms come from Lemma 8.3.2, that the third one is deduced from the fact that $\mathcal{D}b^{p,q}_X(\mathcal{D}_{X\to Y})$ is c-soft and that the fourth arrow is given by Lemma 8.3.3.

The compatibility of integration with composition is then a direct consequence of Lemma 8.3.4. $\square$

Corollary 8.3.6 *To any morphism $f : X \longrightarrow Y$ of complex analytic manifolds and any $\mathcal{M} \in \mathbf{D}^{\mathrm{b}}(\mathcal{D}_X^{\mathrm{op}})$ is associated a morphism*

$$Rf_* R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \Omega_X[d_X]) \longrightarrow R\mathcal{H}om_{\mathcal{D}_Y}(\underline{f}_!\mathcal{M}, \Omega_Y[d_Y])$$

which is functorial in $\mathcal{M}$ and compatible with composition in f.

Proof: For any $\mathcal{M}, \mathcal{N} \in \mathbf{D}^{\mathrm{b}}(\mathcal{D}_X^{\mathrm{op}})$, we have a canonical

$$Rf_* R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{N}) \longrightarrow R\mathcal{H}om_{\mathcal{D}_Y}(\underline{f}_!\mathcal{M}, \underline{f}_!\mathcal{N}).$$

which is functorial both in $\mathcal{M}$ and in $\mathcal{N}$ and which is compatible with composition in f. To end the construction, we compose the preceding morphism for $\mathcal{N} = \Omega_X[d_X]$ with the morphism

$$R\mathcal{H}om_{\mathcal{D}_Y}(\underline{f}_!\mathcal{M}, \underline{f}_!\Omega_X[d_X]) \longrightarrow R\mathcal{H}om_{\mathcal{D}_Y}(\underline{f}_!\mathcal{M}, \Omega_Y[d_Y])$$

deduced from the integration morphism. $\square$

Proposition 8.3.7 *Let $f : X \longrightarrow Y$ be a morphism of complex analytic manifolds. The canonical section $1_{X\to Y}$ of $\mathcal{D}_{X\to Y}$ induces a morphism*

$$Rf_!\Omega_X[d_X] \longrightarrow \underline{f}_!\Omega_X[d_X]$$

which, by composition with the $\mathcal{D}$-linear integration morphism

$$\underline{f}_!\Omega_X[d_X] \longrightarrow \Omega_Y[d_Y],$$

gives the $\mathcal{O}$-linear integration morphism

$$Rf_!\Omega_X[d_X] \longrightarrow \Omega_Y[d_Y].$$

Proof: Since the $\mathcal{O}$-linear integration morphism is deduced from the push-forward

$$f_! : \underline{f}_!\mathcal{D}b^{d_X,\cdot}[d_X] \longrightarrow \mathcal{D}b^{d_Y,\cdot}[d_Y]$$

through the Dolbeault quasi-isomorphisms

$$\Omega_X \xrightarrow{\sim} \mathcal{D}b^{d_X,\cdot} \quad ; \quad \Omega_Y \xrightarrow{\sim} \mathcal{D}b^{d_Y,\cdot},$$

the conclusion follows easily from the explicit construction of the $\mathcal{D}$-linear integration morphism. □

Corollary 8.3.8 *Let $f : X \longrightarrow Y$ be a morphism of complex analytic manifolds. Assume $\mathcal{G}$ is an object of $\mathbf{D}^{\mathrm{b}}(\mathcal{O}_X)$. Then, we have the commutative diagram*

$$\begin{array}{ccc}
Rf_*R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{G}\otimes_{\mathcal{O}_X}\mathcal{D}_X, \Omega_X[d_X]) & \longrightarrow & R\mathcal{H}om_{\mathcal{D}_Y}(\underline{f}_!(\mathcal{G}\otimes_{\mathcal{O}_X}\mathcal{D}_X), \Omega_Y[d_Y]) \\
\downarrow & & \downarrow \\
Rf_*R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{G}, \Omega_X[d_X]) & \longrightarrow & R\mathcal{H}om_{\mathcal{O}_Y}(Rf_!(\mathcal{G}), \Omega_Y[d_Y])
\end{array}$$

where the horizontal arrows are deduced from the $\mathcal{D}$-linear and $\mathcal{O}$-linear integration morphisms, the vertical ones being canonical isomorphisms.

8.4 Holomorphic solutions of proper direct images

Proposition 8.4.1 *Let $f : X \longrightarrow Y$ be a morphism of complex analytic manifolds. Assume $\mathcal{M}$ is an object of $\mathbf{D}^{\mathrm{b}}_{\mathrm{good}}(\mathcal{D}_X^{\mathrm{op}})$ with f-proper support. Then the canonical integration morphism*

$$Rf_*R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \Omega_X[d_X]) \longrightarrow R\mathcal{H}om_{\mathcal{D}_Y}(\underline{f}_!\mathcal{M}, \Omega_Y[d_Y])$$

is an isomorphism in $\mathbf{D}^{\mathrm{b}}(Y)$.

Proof: Working as in Theorem 8.2.2, we may assume that

$$\mathcal{M} = \mathcal{G}\otimes_{\mathcal{O}_X}\mathcal{D}_X$$

where $\mathcal{G}$ is a coherent $\mathcal{O}_X$-module with f-proper support. Hence, we are reduced to prove that

$$Rf_*R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{G}\otimes_{\mathcal{O}_X}\mathcal{D}_X, \Omega_X[d_X]) \longrightarrow R\mathcal{H}om_{\mathcal{D}_Y}(\underline{f}_!(\mathcal{G}\otimes_{\mathcal{O}_X}\mathcal{D}_X), \Omega_Y[d_Y])$$

is an isomorphism. Thanks to Corollary 8.3.8, this is nothing but the well-known duality theorem of analytic geometry. □

9 Non-singular $\mathcal{D}$-modules

9.1 Definition

Set
$$\dot{T}^*X = T^*X \setminus T^*_X X$$
and denote by
$$p : \dot{T}^*X \longrightarrow P^*X$$
the canonical projection to the cotangent projective bundle. Denote by
$$q : P^*X \longrightarrow X$$
the projection to the base manifold. For any coherent $\mathcal{D}_X$-module $\mathcal{M}$, $\operatorname{char}(\mathcal{M})$ is a conic analytic subvariety of T^*X. Therefore, it is possible to find an analytic subvariety $V \subset P^*X$ such that $p^{-1}(V) = \operatorname{char}(\mathcal{M}) \cap \dot{T}^*X$. Since q is proper, $q(V)$ is an analytic subvariety of X. This is the *singular support* of $\mathcal{M}$; we denote it by $\operatorname{sing}(\mathcal{M})$. This is obviously a subset of $\operatorname{supp}(\mathcal{M})$.

A *non singular $\mathcal{D}_X$-module* is a coherent $\mathcal{D}_X$-module $\mathcal{M}$ such that $\operatorname{sing}(\mathcal{M}) = \emptyset$. In other words, a coherent $\mathcal{D}_X$-module $\mathcal{M}$ is non-singular if and only if $\operatorname{char}(\mathcal{M}) \subset T^*_X X$.

9.2 Linear holomorphic connections

To refresh the reader's memory, we start with an informal survey of the basic definitions in the theory of linear connections.

Let $p : E \longrightarrow X$ be a holomorphic vector bundle on the complex analytic manifold X. Here, we will identify a *(linear) holomorphic connection on E* with its *covariant differential*
$$\nabla : E \longrightarrow T^*X \otimes E$$
which is a first order differential operator such that
$$\nabla(he) = dh \otimes e + h\nabla e$$
for any holomorphic function h and any holomorphic section e of E. A holomorphic section e of E is *horizontal* if $\nabla e = 0$. For any holomorphic vector field θ, we define the *covariant derivative* of a holomorphic section e of E along θ by the formula
$$\nabla_\theta e = \langle \theta, \nabla e \rangle .$$
Note that
$$\nabla_\theta(he) = L_\theta h \otimes e + h\nabla_\theta e$$
for any holomorphic function h and any holomorphic section e of E. Hence, for any $\theta, \psi \in \Theta_X$, the operator
$$[\nabla_\theta, \nabla_\psi] - \nabla_{[\theta,\psi]} : E \longrightarrow E$$

has order 0. Since it is also $\mathcal{O}_X$-linear and antisymmetric in θ and ψ, it defines a vector bundle homomorphism

$$R_\nabla : E \longrightarrow \bigwedge^2 T^*X \otimes E.$$

This is the *curvature of* ∇. A *flat* holomorphic connection is a connection with 0 curvature (i.e. $R_\nabla = 0$). The covariant differential has a unique extension

$$\nabla^p : \bigwedge^p T^*X \otimes E \longrightarrow \bigwedge^{p+1} T^*X \otimes E$$

such that

$$\nabla^p(\omega \otimes e) = d\omega \otimes e + (-1)^p \omega \wedge \nabla e$$

for any holomorphic p-form ω and any holomorphic section e of E. A simple computation shows that

$$\nabla^{p+1} \circ \nabla^p(\omega \otimes e) = (-1)^p \omega \wedge R_\nabla(e).$$

The *dual* of a holomorphic connection ∇ on E is the only holomorphic connection ∇^* on E^* such that

$$d\langle e^*, e\rangle = \langle \nabla^* e^*, e\rangle + \langle e^*, \nabla e\rangle$$

for any holomorphic section e^*, e of E^* and E respectively. A simple computation shows that

$$R_{\nabla^*} = -R^*_\nabla.$$

9.3 Non-singular $\mathcal{D}$-modules and flat holomorphic connections

Definition 9.3.1 Let E be a holomorphic vector bundle on X endowed with a flat holomorphic connection ∇. Since $R_\nabla = 0$,

$$\nabla_{[\theta,\psi]} = [\nabla_\theta, \nabla_\psi]$$

for any holomorphic vector fields θ, ψ. Therefore, by Proposition 3.1.3 there is a unique structure of left $\mathcal{D}_X$-module on the sheaf $\mathcal{E}$ of holomorphic sections of E which extends its structure of $\mathcal{O}_X$-module in such a way that

$$\theta \cdot e = \nabla_\theta e$$

for any $\theta \in \Theta_X$ and $e \in \mathcal{E}$. We denote $\mathcal{E}_\nabla$ the corresponding $\mathcal{D}_X$-module.

Proposition 9.3.2 *Let E be a holomorphic vector bundle endowed with a flat holomorphic connection ∇. Then*

(i) $\mathcal{E}_\nabla$ *is a coherent* $\mathcal{D}_X$*-module,*

(ii) $\mathrm{char}\mathcal{E}_\nabla \subset T^*_X X$,

(iii) $\Omega^\cdot(\mathcal{E}_\nabla)$ *is the complex*

$$0 \longrightarrow \mathcal{E} \xrightarrow{\nabla} \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{E} \xrightarrow{\nabla^1} \Omega^2_X \otimes_{\mathcal{O}_X} \mathcal{E} \xrightarrow{\nabla^2} \cdots \longrightarrow \Omega_X \otimes_{\mathcal{O}_X} \mathcal{E} \longrightarrow 0.$$

Proof: We know by Proposition 4.1.3 that

$$(\mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{E}_\nabla) \otimes_{\mathcal{D}_X} \mathcal{O}_X \simeq \mathcal{D}_X \otimes_{\mathcal{D}_X} (\mathcal{E}_\nabla \otimes_{\mathcal{O}_X} \mathcal{O}_X).$$

Therefore,

$$(\mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{E}_\nabla) \otimes_{\mathcal{D}_X} \mathcal{SP}^X_\cdot$$

is a resolution of $\mathcal{E}_\nabla$ by left $\mathcal{D}_X$-modules. Since $\mathcal{E}$ is a locally free $\mathcal{O}_X$-module, the components of this resolution are locally free $\mathcal{D}_X$-modules. Hence, $\mathcal{E}_\nabla$ is coherent and the filtration

$$\mathcal{F}_k\mathcal{E}_\nabla = \begin{cases} \mathcal{E}_\nabla & k \geq 0 \\ 0 & k < 0 \end{cases}$$

on $\mathcal{E}_\nabla$ is good. Since $\tilde{\mathcal{G}r}\mathcal{F}\mathcal{E}_\nabla$ is supported by T^*_XX, it follows that $\mathrm{char}(\mathcal{E}_\nabla) \subset T^*_XX$. Part (iii) being a direct consequence of the definitions, the proof is complete. □

Proposition 9.3.3 *Let E be a holomorphic vector bundle on X endowed with a flat holomorphic connection ∇. Then, as $\mathcal{D}_X$-modules, we have*

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}_\nabla, \mathcal{O}_X) \simeq \mathcal{E}^*_{\nabla^*}.$$

Hence, in $\mathbf{D}^{\mathrm{b}}(X)$,

$$R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{E}_\nabla, \mathcal{O}_X) \simeq \Omega^\cdot(\mathcal{E}^*_{\nabla^*}).$$

Proof: The first isomorphism is clear. To get the second one, we note that

$$R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{E}_\nabla, \mathcal{O}_X) \simeq R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{O}_X, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}_\nabla, \mathcal{O}_X)).$$

□

Corollary 9.3.4 *Let E be a holomorphic vector bundle on X endowed with a flat holomorphic connection ∇. Then the $\mathcal{D}_X$-module $\mathcal{E}^*_{\nabla^*}$ represents the homogeneous system*

$$\nabla u = 0$$

defining the horizontal sections of E.

Proposition 9.3.5 *For a coherent $\mathcal{D}_X$-module $\mathcal{M}$, the following conditions are equivalent:*

(a) $\mathcal{M}$ *is non-singular,*

(b) $\mathrm{char}\mathcal{M} \subset T^*_XX$,

(c) $\mathcal{M}$ *is $\mathcal{O}_X$-coherent,*

(d) $\mathcal{M}$ *is the $\mathcal{D}_X$-module associated to a flat holomorphic connection.*

Proof: By definition, (a) and (b) are equivalent.

(b) $\Rightarrow$ (c). The problem being local on X, we may also assume that $\mathcal{M}$ is endowed with a good filtration $\mathcal{FM}$ and that X is an open subset of $\mathbb{C}^n$. Since

$$\operatorname{char}(\mathcal{M}) \subset T^*_X X,$$

we may find integers $\alpha_1, \ldots, \alpha_n$ such that

$$\xi_k^{\alpha_k} \mathcal{GM} = 0$$

for $k = 1, \ldots, n$. Let x be a point in X. Since $\mathcal{GM}$ is a coherent $\mathcal{GD}_X$-module, we may find a neighborhood V of x and generators $m_1, \ldots, m_p$ of order $d_1, \ldots, d_p$ of $\mathcal{GM}_{|V}$. Choose

$$k \geq \sup(d_1, \ldots, d_p) + \sum_{k=1}^{n} \alpha_k$$

and let m be a germ of order ℓ of $\mathcal{GM}$ at $y \in V$. By construction, we may find germs $s_1, \ldots, s_p$ of order $\ell - d_1, \ldots, \ell - d_p$ of $\mathcal{GD}_X$ such that

$$m = s_1 m_1 + \cdots + s_p m_p.$$

The s_ℓ's may be written as

$$s_\ell = \sum_{|\beta| = \ell - d_i} a_\ell^\beta \xi^\beta$$

with $a_\ell^\beta \in \mathcal{O}_X$. In each term of this sum we have $\beta_j \geq \alpha_j$ for at least a $j \in \{1, \ldots, n\}$. From this fact we deduce that the s_ℓ's annihilate $\mathcal{GM}$; hence $m = 0$. By the preceding discussion we know that locally $\mathcal{G}_k\mathcal{M} = 0$ for $k \gg 0$. This means that the filtration $\mathcal{FM}$ is locally stationary and thus $\mathcal{M}$ is $\mathcal{O}_X$ coherent.

(c) $\Rightarrow$ (d). Note that it is sufficient to prove that $\mathcal{M}$ is a locally free $\mathcal{O}_X$-module. As a matter of fact, $\mathcal{M}$ is then the sheaf of holomorphic section of a holomorphic vector bundle E on X and the action

$$\theta \cdot : \mathcal{M} \longrightarrow \mathcal{M}$$

of a holomorphic vector field $\theta \in \Theta_X$ may clearly be interpreted as the covariant derivative along θ associated to a flat holomorphic connection ∇ on E.

Since the problem is local on X, we may assume X is an open neighborhood of 0 in $\mathbb{C}^n$ and prove only that $\mathcal{M}$ is a locally free $\mathcal{O}_X$-module near 0.

Let us choose $m_1, \ldots, m_p \in \mathcal{M}_0$ such that the associated classes $[m_1], \ldots, [m_p]$ form a basis of the finite dimensional vector space

$$\mathcal{M}_0/(z^1, \ldots, z^n)\mathcal{M}_0.$$

By Nakayama's lemma, $m_1, \ldots, m_p$ generate $\mathcal{M}_0$ on $(\mathcal{O}_X)_0$.

Let us define the order of a relation

$$\sum_{k=1}^{p} b_k m_k = 0 \qquad (b_1, \ldots, b_p \in (\mathcal{O}_X)_0)$$

as the minimum of the vanishing orders of the b_i's at 0.

Suppose we have a non trivial relation

$$\sum_{k=1}^{p} b_k m_k = 0$$

of order δ. By construction, we have also

$$\partial_{z^\ell} m_k = \sum_{r=1}^{p} a^r_{\ell k} m_r$$

for some $a^r_{\ell k} \in (\mathcal{O}_X)_0$. Applying ∂_{z^ℓ} to the relation we started with gives us the relation

$$\sum_{r=1}^{p} (\partial_{z^\ell} b_r + \sum_{k=1}^{p} a^r_{\ell k} b_k) m_r = 0$$

which is clearly of order $\delta - 1$ for a suitable ℓ.

Hence, if we can find a non trivial relation of order δ we can also find one which has order 0. Such a non trivial relation is impossible since by construction $[m_1]$, ..., $[m_p]$ form a basis. □

9.4 Local systems

Here, by a *local system* on X we mean a locally constant sheaf of $\mathbb{C}$-vector spaces with finite dimensional fibers. Recall that a locally constant sheaf on $[0,1]^n$ is constant. In particular, if $\gamma : [0,1] \longrightarrow X$ is a path in X from x_0 to x_1, the sheaf $\gamma^{-1}\mathcal{L}$ is constant on $[0,1]$. Therefore, we have a canonical *monodromy* isomorphism

$$m_\gamma : \mathcal{L}_{x_0} \xrightarrow{\sim} \mathcal{L}_{x_1}.$$

Moreover, for any path γ' joining x_1 to x_2, we get

$$m_{\gamma'\gamma} = m_{\gamma'} \circ m_\gamma.$$

Let

$$h : [0,1]^2 \longrightarrow X$$

be a homotopy between two paths γ_0, γ_1 joining x_0 to x_1. Since the sheaf $h^{-1}\mathcal{L}$ is constant on $[0,1]^2$, we get

$$m_{\gamma_0} = m_{\gamma_1}.$$

Recall that the Poincaré groupoid of X is the category π_X which has the points of X as objects; a morphism from x_0 to x_1 being a homotopy class of paths from x_0 to x_1. The preceding discussion shows that a local system $\mathcal{L}$ on X induces a functor from the Poincaré groupoid to the category of finite dimensional $\mathbb{C}$-vector spaces. It is easily checked that this process induces an equivalence between the corresponding categories. Let us fix a point x_0 in X. Since

$$\operatorname{Hom}_{\pi_X}(x_0, x_0) = \pi_1(X, x_0),$$

we also get an equivalence between the category of local systems on X and the category of finite dimensional $\mathbb{C}$-linear representations of the Poincaré group $\pi_1(X, x_0)$ of X.

9.5 Non-singular $\mathcal{D}$-modules and local systems

We will now establish a link between local systems and non-singular $\mathcal{D}$-modules.

Proposition 9.5.1 *Let X be a complex analytic manifold and let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module. Assume $\mathcal{M}$ is non-singular. Then, locally,*

$$\mathcal{M} \simeq \mathcal{O}_X^p \quad (p \in \mathbb{N})$$

as a left $\mathcal{D}_X$-module.

Proof: Let x be a point of X and denote by $i : \{x\} \longrightarrow X$ the canonical embedding. Since $\mathcal{M}$ is non-singular, $\mathrm{char}(\mathcal{M}) \subset T_X^*X$. Therefore, i is non-characteristic for $\mathcal{M}$ and, by Theorem 7.3.2, $\underline{i}^*\mathcal{M}$ is a coherent $\mathcal{D}_{\{x\}}$-module. Since $\mathcal{D}_{\{x\}}$ is nothing but the field of complex numbers, there is an integer $p \in \mathbb{N}$ and an isomorphism

$$\alpha : \underline{i}^*\mathcal{M} \xrightarrow{\sim} \mathbb{C}^p.$$

By the Cauchy-Kowalewsky-Kashiwara theorem,

$$R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X^p)_x \simeq \mathcal{H}om(\underline{i}^*\mathcal{M}, \mathbb{C}^p).$$

Therefore, in a neighborhood of x, we can find a morphism of $\mathcal{D}_X$-modules

$$\beta : \mathcal{M} \longrightarrow \mathcal{O}_X^p$$

such that $\alpha = \underline{i}^*\beta$. Denote $\mathcal{N}_0$ is kernel and $\mathcal{N}_1$ its cokernel. Both are non-singular $\mathcal{D}_X$-modules and coherent $\mathcal{O}_X$-modules. Since $\underline{i}^*\mathcal{N}_1$ is the cokernel of α, we have

$$\underline{i}^*\mathcal{N}_1 = 0$$

and Nakayama's lemma shows that $\mathcal{N}_1 = 0$ in a neighborhood of x. Since the sequence

$$0 \longrightarrow \underline{i}^*\mathcal{N}_0 \longrightarrow \underline{i}^*\mathcal{M} \xrightarrow{\alpha} \mathbb{C}^p \longrightarrow 0$$

is exact, $\underline{i}^*\mathcal{N}_0 = 0$ and $\mathcal{N}_0 = 0$ in a neighborhood of x. Combining the preceding results, we see that β is an isomorphism in some neighborhood of x. $\square$

Remark 9.5.2 In the language of connections, the preceding result means that any holomorphic vector bundle E endowed with a flat holomorphic connection has horizontal local frames.

Proposition 9.5.3 *Let X be a complex analytic manifold. Assume $\mathcal{M}$ is a non-singular coherent $\mathcal{D}_X$-module. Then,*

$$R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X) \simeq \mathcal{L}$$

where $\mathcal{L}$ is a local system on X. Moreover, the functor

$$\mathcal{M} \mapsto \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)$$

induces an equivalence between the category of non-singular coherent $\mathcal{D}_X$-modules and the category of local systems on X with a quasi-inverse given by the functor

$$\mathcal{L} \mapsto \mathcal{H}om(\mathcal{L}, \mathcal{O}_X).$$

Proof: We know that, locally,

$$\mathcal{M} \simeq \mathcal{O}_X^p.$$

Therefore, locally,

$$R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X) \simeq R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{O}_X^p, \mathcal{O}_X) \simeq \mathbb{C}^p$$

and the first part of the proposition is clear. The canonical map

$$\mathcal{M} \longrightarrow \mathcal{H}om(\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X), \mathcal{O}_X)$$

is obviously an isomorphism for $\mathcal{M} = \mathcal{O}_X$; therefore it is also an isomorphism for any non-singular $\mathcal{D}_X$-module $\mathcal{M}$. In the same way, we see that the canonical map

$$\mathcal{L} \longrightarrow \mathcal{H}om_{\mathcal{D}_X}(\mathcal{H}om(\mathcal{L}, \mathcal{O}_X), \mathcal{O}_X),$$

is an isomorphism for local systems on X since it is so for $\mathcal{L} = \mathbb{C}_X$. □

10 Appendix

To split the difficulties, we first recall some basic results of linear algebra. Then, we develop filtered algebra, and finally we switch to its sheaf theoretical counterpart.

10.1 Linear algebra

Let A be a ring with unit. Without explicit mention of the contrary, an A-module is a *left* A-module. With this convention, we identify A^{op}-modules and right A-modules.

10.1.1 Homological dimension

The *homological dimension* of an A-module M is the smallest integer $n \in \mathbb{N}$ such that one of the following equivalent conditions is satisfied.

(a) There is a resolution

$$0 \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

of M by projective A-modules.

(b) For any A-module N

$$\mathrm{Ext}^j_A(M, N) = 0$$

for $j > n$.

We denote it by $\mathrm{hd}_A(M)$.

The *global homological dimension* of the ring A to the smallest integer $n \in \mathbb{N}$ such that one of the following equivalent conditions is satisfied

(a) Any A-module M has a projective resolution

$$0 \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

of length n.

(b) Any A-module N has an injective resolution

$$0 \longrightarrow N \longrightarrow I_0 \longrightarrow I_1 \longrightarrow \cdots \longrightarrow I_n \longrightarrow 0$$

of length n.

(c) For any A-modules M, N we have

$$\mathrm{Ext}^j_A(M, N) = 0$$

for $j > n$.

We will denote it by $\mathrm{glhd}(A)$. Note that, since an A-module N is injective if and only if

$$\mathrm{Ext}^j_A(A/I, N) = 0$$

for any $j > 0$ and any left ideal I of A, we have

$$\mathrm{glhd}(A) = \sup\{\mathrm{hd}_A(A/I) : I \text{ left ideal of } A\}.$$

So the global homological dimension depends only on cyclic A-modules.

Proposition 10.1.1 *Let A be a ring and denote by $A[x]$ the ring of polynomials in one unknown with coefficients in A. Then*

$$\mathrm{glhd}(A[x]) \leq \mathrm{glhd}(A) + 1.$$

Proof: Thanks to the exact sequence

$$0 \longrightarrow A[x] \xrightarrow{\cdot x} A[x] \longrightarrow A \longrightarrow 0,$$

the result is a direct consequence of the formula

$$\mathrm{RHom}_A(A \otimes^L_{A[x]} M, N) \simeq \mathrm{RHom}_{A[x]}(M, N)$$

holding for every $A[x]$-module M and N. □

10.1.2 Noether property

We say that an A-module M is of *finite type* if it is generated by a finite number of elements. It is *finite free* if it is isomorphic to A^p for some $p \in \mathbb{N}$.

For a ring A the following conditions are equivalent

(a) Any submodule of an A-module of finite type is of finite type.

(b) Any ideal of A is finitely generated.

(c) Any increasing sequence of submodules of an A-module of finite type is stationary.

(d) The subcategory of A-modules of finite type is stable by kernels, cokernels and extensions.

A ring satisfying one of these conditions is said to be *Noetherian*. In some sense, Noetherian rings are the only rings for which finite type modules are well-behaved.

If A is Noetherian, any A-module of finite type M has an infinite resolution

$$\cdots \longrightarrow L_n \xrightarrow{d_n} L_{n-1} \xrightarrow{d_{n-1}} \cdots \longrightarrow L_0 \longrightarrow M \longrightarrow 0$$

by finite free A-modules. If $\mathrm{hd}_A(M) \leq n$, then $P_n = \ker d_{n-1}$ is a projective A-module of finite type and we have the finite projective resolution

$$0 \longrightarrow P_n \longrightarrow L_{n-1} \longrightarrow \cdots \longrightarrow L_0 \longrightarrow M \longrightarrow 0.$$

To compute the homological dimension of an A-module of finite type, we have the following proposition.

Proposition 10.1.2 *Assume A is Noetherian and $\mathrm{glhd}(A) < +\infty$. Then, for any A-module of finite type M, the homological dimension $\mathrm{hd}_A(M)$ is the smallest integer $n \in \mathbb{N}$ such that*

$$\mathrm{Ext}^i_A(M, A) = 0$$

for $i > n$.

Proof: Let N be any A-module. We have an exact sequence

$$0 \longrightarrow R \longrightarrow A^{(I)} \longrightarrow N \longrightarrow 0$$

where I is some set of generators of N. Since M has a resolution by finite free A-modules

$$\mathrm{Ext}^i_A(M, A^{(I)}) = \mathrm{Ext}^i_A(M, A)^{(I)} = 0$$

for $i > n$. Then, the exact sequence

$$\mathrm{Ext}^i_A(M, A^{(I)}) \longrightarrow \mathrm{Ext}^i_A(M, N) \longrightarrow \mathrm{Ext}^{i+1}_A(M, R) \longrightarrow \mathrm{Ext}^{i+1}_A(M, A^{(I)})$$

shows that

$$\mathrm{Ext}^i_A(M, N) \simeq \mathrm{Ext}^{i+1}_A(M, R)$$

for $i > n$. By induction, we find for any $p \in \mathbb{N}$ an A-module R_p such that

$$\mathrm{Ext}^i_A(M, N) = \mathrm{Ext}^{i+p}_A(M, R_p)$$

for $i > n$. As soon as $i > n$ and $i + p > \mathrm{glhd}(A)$, we get

$$\mathrm{Ext}^i_A(M, N) = 0,$$

and the conclusion follows. $\square$

10.1.3 Syzygies

Let us now investigate when any A-module of finite type has a finite resolution by finite *free* A-modules.

Proposition 10.1.3 *Assume the ring A is Noetherian. Then, the following conditions are equivalent:*

(a) any A-module of finite type M has a finite resolution

$$0 \longrightarrow L_n \longrightarrow L_{n-1} \longrightarrow \cdots \longrightarrow L_0 \longrightarrow M \longrightarrow 0$$

by finite free A-modules,

(b) any A-module of finite type M has finite homological dimension and any finite type projective A-module P is stably finite free (i.e. there is a finite free A-module L such that $P \oplus L$ is finite free).

Proof: (a) $\Rightarrow$ (b). Obviously, M has a finite projective resolution and $\mathrm{hd}_A(M) < +\infty$. We will prove by induction on n that a finite type projective A-module P which has a finite free resolution of length n is stably finite free. For $n = 0$, it is obvious. For $n = 1$, the exact sequence

$$0 \longrightarrow L_1 \longrightarrow L_0 \longrightarrow P \longrightarrow 0$$

splits since P is projective; hence $L_0 \simeq L_1 \oplus P$ and the conclusion follows. For $n > 1$, we have the resolution

$$0 \longrightarrow L_n \xrightarrow{d_n} L_{n-1} \longrightarrow \cdots \longrightarrow L_1 \xrightarrow{d_1} L_0 \xrightarrow{d_0} P \longrightarrow 0$$

denote P_1 the kernel of d_0. Since the sequence

$$0 \longrightarrow P_1 \longrightarrow L_0 \longrightarrow P \longrightarrow 0$$

is exact, the module P_1 is projective. Moreover, it has a finite free resolution of length $n - 1$

$$0 \longrightarrow L_n \longrightarrow L_{n-1} \longrightarrow \cdots \longrightarrow L_1 \longrightarrow P_1 \longrightarrow 0.$$

The induction hypothesis gives us a finite free A-module L'_1 such that $L'_1 \oplus P_1$ is finite free. Since the sequence

$$0 \longrightarrow L'_1 \oplus P_1 \longrightarrow L'_1 \oplus L_0 \longrightarrow P \longrightarrow 0$$

is exact, the conclusion follows from the case $n = 1$.

(b) $\Rightarrow$ (a). Let M be an A-module of finite type. Let $n = \sup(1, \mathrm{hd}_A(M))$. We have a resolution

$$0 \longrightarrow P_n \xrightarrow{i_n} L_{n-1} \longrightarrow \cdots \longrightarrow L_0 \longrightarrow M \longrightarrow 0$$

where L_k is finite free and P_n is a projective module of finite type. Let L'_n be a finite free A-module such that $L'_n \oplus P_n$ is finite free. Since the sequence

$$0 \longrightarrow L'_n \oplus P_n \xrightarrow{id_{L'_n} \oplus i_n} L'_n \oplus L_{n-1} \longrightarrow \cdots \longrightarrow L_0 \longrightarrow M \longrightarrow 0$$

is still exact, the proof is complete. □

A ring A is *syzygic* when it satisfies the equivalent conditions of the preceding proposition and has finite global homological dimension.

Corollary 10.1.4 *Assume the ring A is syzygic. Then, any finite type A-module M has a finite free resolution of length* $\sup(1, \mathrm{hd}_A(M))$.

10.1.4 Local rings

Recall that a ring A is *local* if it has a unique maximal ideal.

Lemma 10.1.5 (Nakayama's lemma) *Assume A is a local ring with maximal ideal* m *and let M be an A-module of finite type. Then, $M = 0$ if and only if $M/\mathrm{m}M = 0$.*

Proof: The proof will work by induction on the number of generators of M. Denote this number by p and assume $g_1, \ldots, g_p$ are these generators. If $p = 1$ then $M = \mathrm{m}M$ implies that there is $m_1 \in \mathrm{m}$ such that $g_1 = m_1 g_1$. Hence $(1 - m_1)g_1 = 0$ in M and since $1 - m_1$ is invertible in A, $g_1 = 0$ and $M = 0$. If $p > 1$ then there are $m_{ij} \in \mathrm{m}$ for $i = 1, \ldots, p$; $j = 1, \ldots, p$ such that

$$\begin{array}{rcl} g_1 & = & m_{11}g_1 + \cdots + m_{1p}g_p \\ \vdots & & \qquad \vdots \\ g_p & = & m_{p1}g_1 + \cdots + m_{pp}g_p \end{array}$$

From the first line we see that g_1 is a linear combination of $g_2, \ldots, g_p$; hence the conclusion by the induction hypothesis. □

Proposition 10.1.6 *Assume A is a Noetherian local ring with maximal ideal* m *and M is an A-module of finite type. Then, a map*

$$A^p \xrightarrow{u} M$$

is surjective if and only if so is the associated map

$$(A/\mathrm{m})^p \xrightarrow{v} M/\mathrm{m}M.$$

If, moreover, $\mathrm{Tor}_1^A(A/\mathrm{m}, M) = 0$, then u is bijective if and only if so is v.

Proof: Denote by N_0 and N_1 the kernel and the cokernel of u. Clearly, v is surjective if and only if $N_1/\mathrm{m}N_1 = 0$ since the sequence

$$A/\mathrm{m} \otimes_A A^p \longrightarrow A/\mathrm{m} \otimes_A M \longrightarrow A/\mathrm{m} \otimes_A N_1 \longrightarrow 0$$

is exact. If v is bijective and $\mathrm{Tor}_1^A(A/\mathrm{m}, M) = 0$, the sequence

$$0 \longrightarrow A/\mathrm{m} \otimes_A N_0 \longrightarrow A/\mathrm{m} \otimes_A M \longrightarrow A/\mathrm{m} \otimes_A M \longrightarrow 0$$

is exact and $N_0/\mathrm{m}N_0 = 0$; hence $N_0 = 0$. □

Corollary 10.1.7 *Assume A is a Noetherian local ring and M is a finite type A-module. Then*

(a) $\mathrm{Tor}_1^A(A/\mathrm{m}, M) = 0$ *implies M finite free,*

(b) $\mathrm{Tor}_j^A(A/\mathrm{m}, M) = 0$ *for $j > n$ implies* $\mathrm{hd}_A(M) \leq n$.

In particular, if $\mathrm{glhd}(A) < +\infty$, *$A$ is syzygic.*

Proof: (a) is obvious from the preceding proposition since A/m is a field.

(b) Let

$$\cdots \longrightarrow L_n \xrightarrow{d_n} L_{n-1} \longrightarrow \cdots \longrightarrow L_0 \longrightarrow M \longrightarrow 0$$

be a finite free resolution of M. Let $F_n = \ker d_n$. By construction, $\mathrm{Tor}_1^A(A/\mathrm{m}, F_n) = 0$ and F_n is of finite type hence F_n is finite free by (a) and $\mathrm{hd}_A(M) \leq n$. □

10.2 Graded linear algebra

A graded ring GA is the data of a ring with unit A and a family of subgroups $(G_kA)_{k\in\mathbb{Z}}$ of A satisfying:

a. $A = \bigoplus_{k\in\mathbb{Z}} G_kA$,

b. $1 \in G_0A$,

c. $G_kA.G_\ell A \subset G_{k+\ell}A$.

Here we always also assume that $G_kA = 0$ for $k < 0$ (i.e. GA is positively graded).

A graded module GM over the graded ring GA is the data of an A-module M and a family of subgroups $(G_kM)_{k\in\mathbb{Z}}$ of M such that

a. $M = \bigoplus_{k\in\mathbb{Z}} G_kM$

b. $G_kA.G_\ell M \subset G_{k+\ell}M$

Here we will always also assume that $G_kM = 0$ for $k \ll 0$.

A morphism of a graded GA-module GM to a graded GA-module GN is a morphism of A-modules

$$f : M \longrightarrow N$$

such that $f(G_kM) \subset G_kN$.

These morphisms form a group denoted by $\mathrm{Hom}_{GA}(GM, GN)$.

With this notion of morphisms the category of graded GA-modules is Abelian. The reader will easily find the form of the kernel, cokernel, image, and coimage of a morphism in this category.

For $r \in \mathbb{Z}$ and any GA-module GM we define the shifted GA-module $GM(r)$ to be the A-module M endowed with the graduation defined by

$$G_kM(r) = G_{k+r}M.$$

Let us denote by $\mathrm{GHom}_{GA}(GM, GN)$ the graded group defined by setting

$$\mathrm{G}_k\mathrm{Hom}_{GA}(GM, GN) = \mathrm{Hom}_{GA}(GM, GN(k)).$$

10.2.1 Noether property

A GA-module GL is finite free if there are integers $d_1, \ldots, d_n$ such that

$$GL \simeq \bigoplus_{i=1}^{n} GA(-d_i).$$

A finite free presentation of length s of a GA-module GM is an exact sequence

$$GL_s \longrightarrow GL_{s-1} \longrightarrow \cdots \longrightarrow GL_0 \longrightarrow GM \longrightarrow 0$$

where GL_i is finite free.

A GA-module GM is of finite type if it has a finite free presentation of length 0. This means that there are homogeneous elements $m_1 \in G_{d_1}M, \ldots, m_n \in G_{d_n}M$ such that any $m \in G_dM$ may be written as

$$m = \sum_{i=1}^{n} a_{d-d_i} m_{d_i}$$

where $a_{d-d_i} \in G_{d-d_i}A$.

A graded ring GA is *Noetherian* if any GA-submodule of a GA-module of finite type is of finite type. An equivalent condition is that the subcategory of GA-modules of finite type is stable by kernels, cokernels and extensions.

If GA is Noetherian then any GA-module of finite type has finite free presentations of arbitrary length.

10.2.2 Homological dimension

The homological dimension of a GA-module GM is the smallest integer n such that $\operatorname{Ext}^j_{GA}(GM, GN) = 0$ for $j > n$ and any GA-module GN. It is also the smallest n such that GM has a projective resolution of length n. We denote it by $\operatorname{hd}_{GA}(GM)$.

The global homological dimension of GA is the supremum of the homological dimensions of all GA-modules. It is denoted by $\operatorname{glhd}(GA)$. It is also the supremum of the homological dimensions of finite type GA-modules.

We denote by Σ the functor which associates to any GA-module its underlying A-module. We have

$$\Sigma GM = \bigoplus_{k \in \mathbb{Z}} G_k M$$

and Σ is an exact functor.

Proposition 10.2.1 *If GA is Noetherian and GM is a GA-module of finite type then*

$$\Sigma \operatorname{GExt}^i_{GA}(GM, GN) = \operatorname{Ext}^i_{\Sigma GA}(\Sigma GM, \Sigma GN)$$

for $i \in \mathbb{N}$. In particular $\operatorname{hd}_{GA}(GM) = \operatorname{hd}_{\Sigma GA}(\Sigma GM)$ and $\operatorname{glhd}(GA) = \operatorname{glhd}(\Sigma GA)$.

Proof: Since GA is Noetherian, we have a resolution

$$\cdots \longrightarrow GL_n \longrightarrow GL_{n-1} \longrightarrow \cdots \longrightarrow GL_0 \longrightarrow GM \longrightarrow 0$$

of GM by finite free GA-modules. If $d_1, \ldots, d_p$ are integers, we have

$$\operatorname{GHom}_{GA}\left(\bigoplus_{i=1}^{p} GA(-d_i), GN\right) = \bigoplus_{i=1}^{p} \operatorname{GHom}_{GA}(GA, GN)(d_i) = \bigoplus_{i=1}^{p} GN(d_i).$$

Thus,

$$\Sigma \operatorname{GHom}_{GA}(GL, GN) \simeq \operatorname{Hom}_{\Sigma GA}(\Sigma GL, \Sigma GN)$$

when GL is a finite free GA-module. Since ΣGL is a finite free ΣGA-module, the isomorphism

$$\Sigma \operatorname{GHom}^{\cdot}_{GA}(GL_{\cdot}, GN) \simeq \operatorname{Hom}^{\cdot}_{\Sigma GA}(\Sigma GL_{\cdot}, \Sigma GN)$$

gives the requested result in cohomology. □

Corollary 10.2.2 *Assume GA is Noetherian and $\operatorname{glhd}(GA)$ is finite. Then, for any finite type GA-module GM, $\operatorname{hd}_{GA}(GM)$ is the smallest integer n such that*

$$\operatorname{GExt}^i_{GA}(GM, GA) = 0$$

for $i > n$.

10.2.3 Syzygies

The study of finite free resolutions of finite length in the preceding section may be reproduced in the graded case and we get

Proposition 10.2.3 *Assume GA is Noetherian. Then, the following conditions are equivalent :*

(a) Any finite type GA-module GM has a finite free resolution of finite length.

(b) Any finite type GA-module GM has finite homological dimension and finite type projective GA-modules are stably finite free.

A ring GA with finite global homological dimension satisfying the equivalent conditions of the preceding proposition is called *syzygic*.

Proposition 10.2.4 *Let GA be a graded ring. Assume that projective G_0A-modules of finite type are finite free (resp. stably finite free). Then, so are the projective GA-modules of finite type.*

Proof: Consider the ring morphism

$$p_0 : GA \longrightarrow G_0A.$$

If p_0 is an isomorphism, we have nothing to prove. Otherwise, denote by GI its kernel. We have

$$G_0A \otimes_{GA} GM = GM/GI.GM$$

for any GA-module GM. If GM is of finite type then

$$GM = GI.GM$$

if and only if $GM = 0$. As a matter of fact, assuming the contrary there is a lowest integer $k_0 \in \mathbb{Z}$ such that $G_{k_0}M \neq 0$. A non zero element of $G_{k_0}M$ is thus in $GI.GM$ and should be of degree greater than $k_0 + 1$; hence a contradiction.

Now, let GP be a finitely generated projective GA-module. It is clear that $G_0A \otimes_{GA} GP$ is a finitely generated projective G_0A-module. Hence it is finite free (resp. stably finite free) and (up to adding to GP a finite free GA-module) there is an isomorphism

$$G_0A^p \xrightarrow{\sim} G_0A \otimes_{GA} GP.$$

This isomorphism comes by, scalar extension, from a morphism

$$GL \longrightarrow GP$$

where GL is a finite free GA-module. Denote by GN_0 the kernel and by GN_1 the cokernel of this morphism. We have the exact sequence

$$G_0A \otimes_{GA} GL \longrightarrow G_0A \otimes_{GA} GP \longrightarrow G_0A \otimes_{GA} GN_1 \longrightarrow 0$$

hence $G_0A \otimes_{GA} GN_1 = 0$ and $GN_1 = 0$. The exact sequence

$$0 \longrightarrow GN_0 \longrightarrow GL \longrightarrow GP \longrightarrow 0$$

splits since GP is projective. Hence we have the exact sequence

$$0 \longrightarrow G_0A \otimes_{GA} GN_0 \longrightarrow G_0A \otimes_{GA} GL \longrightarrow G_0A \otimes_{GA} GP \longrightarrow 0$$

and $GN_0 = 0$. The conclusion follows. □

Corollary 10.2.5 *Assume A is syzygic then the graded ring $A[x]$ is syzygic.*

10.3 Filtered linear algebra

A filtered ring FA is the data of a ring with unit A and a family of subgroups $(F_kA)_{k\in\mathbb{Z}}$ of A satisfying

a. $\bigcup_{k\in\mathbb{Z}} F_kA = A$,

b. $F_kA \subset F_{k+1}A \quad (k \in Z)$,

c. $F_kA.F_\ell A \subset F_{k+\ell}A \quad (k, \ell \in \mathbb{Z})$.

d. $1 \in F_0A$.

Here, we always also assume that $F_kA = 0$ for $k < 0$ (i.e. FA is positively filtered).

The order of a non zero element $a \in A$ is the smallest integer $k \in \mathbb{Z}$ such that $a \in F_kA$. We denote it by $\mathrm{ord}(a)$.

A (filtered) module over the filtered ring FA is the data of an A-module M and a family $(F_kM)_{k\in\mathbb{Z}}$ of subgroups of M such that

a. $\bigcup_{k\in\mathbb{Z}} F_kM = M$

b. $F_kM \subset F_{k+1}M \quad ; \quad k \in \mathbb{Z}$

c. $F_kA.F_\ell M \subset F_{k+\ell}M \quad ; \quad k, \ell \in \mathbb{Z}$

here we will always also assume that $F_kM = 0$ for $k \ll 0$.

The order of a non zero element of M is the smallest integer $k \in \mathbb{Z}$ such that $a \in F_kM$. We denote it by $\mathrm{ord}(m)$. By convention $\mathrm{ord}(0) = -\infty$. Of course, $\mathrm{ord}(m.n) \leq \mathrm{ord}(m) + \mathrm{ord}(n)$ and $\mathrm{ord}(m + n) \leq \sup(\mathrm{ord}(n), \mathrm{ord}(n))$.

A morphism

$$Fu : FM \longrightarrow FN$$

of FA-modules is a morphism $u : M \longrightarrow N$ of the underlying A-modules such that

$$u(F_kM) \subset F_kN.$$

We denote by F_ku the morphism $u_{|F_kM} : F_kM \longrightarrow F_kN$.

Morphisms from an FA-module FM to an FA-module FN form a group which is denoted by $\mathrm{Hom}_{FA}(FM, FN)$. With this notion of morphisms the category of FA-modules is an additive category. Consider an arbitrary morphism

$$Fu : FM \longrightarrow FN$$

having $u : M \longrightarrow N$ as underlying morphism. The *kernel* of Fu is the kernel of u filtered by the family $(\ker u \cap F_k M)_{k\in\mathbb{Z}}$. The *cokernel* of Fu is the cokernel of u filtered by the family $(p(F_k N))_{k\in\mathbb{Z}}$ where $p : N \longrightarrow \operatorname{coker} u$ is the canonical projection. As usual, the *image* of Fu is the kernel of its cokernel. Hence, it is the A-module $\operatorname{im} u$ filtered with the filtration $(\operatorname{im} u \cap F_k N)_{k\in\mathbb{Z}}$. As for the *coimage* of Fu, it is the cokernel of the kernel of Fu. Hence, it is the A-module $\operatorname{im} u$ endowed with the filtration $(u(F_k M))_{k\in\mathbb{Z}}$.

Note that the canonical filtered morphism

$$\operatorname{coim} Fu \longrightarrow \operatorname{im} Fu$$

is a bijection but is not in general an isomorphism.

The morphisms for which this map is an isomorphism are called *strict*.

The category of filtered FA-modules is *not* Abelian. However, one can show that it is an exact category in the sense of Quillen [14]. Hence, it is suitable for homological algebra. For the sake of brevity we will avoid this point of view here and refer the interested reader to Laumon [13].

An *exact sequence* of FA-modules is a sequence

$$FM \xrightarrow{Fu} FN \xrightarrow{Fv} FP$$

such that $\ker F_k v = \operatorname{im} F_k u$. It follows from this definition that Fu is strict. If moreover Fv is strict we say that it is a *strict exact sequence*.

For any $r \in \mathbb{Z}$ and any FA-module FM, we define the shifted module $FM(r)$ as the module M endowed with the filtration $(F_{k+r} M)_{k\in\mathbb{Z}}$.

Finally, we introduce for any couple FM, FN of FA-modules the filtered group

$$\mathrm{FHom}_{FA}(FM, FN)$$

by setting

$$\mathrm{F}_k\mathrm{Hom}_{FA}(FM, FN) = \mathrm{Hom}_{FA}(FM, FN(k)).$$

To any filtered ring FA we associate a graded ring GA defined by setting

$$G_k A = F_k A / F_{k-1} A$$

the multiplication being induced by that of FA.

To a filtered FA-module FN we associate a graded GA-module GM defined by setting

$$G_k M = F_k M / F_{k-1} M$$

the action of GA on GM being induced by that of FA on FM.

The image of $m \in F_kM$ (resp. $a \in F_kA$) in G_kM (resp. G_kA) is its symbol of order k. We denote it by $\sigma_k(m)$ (resp. $\sigma_k(a)$). The principal symbol of m (resp. a) is its symbol of order $\mathrm{ord}(m)$ (resp. $\mathrm{ord}(a)$).

A filtered morphism of FA-modules

$$Fu : FM \longrightarrow FN$$

induces a graded morphism of GA-modules

$$Gu : GM \longrightarrow GN.$$

The preceding construction clearly defines a functor from the category of FA-modules to the category of GA-modules. We denote this functor by Gr.

The last part of this section will be devoted to the study of this functor. In particular, we will investigate how finiteness and dimensionality properties of GA induces similar properties on FA and on A.

Proposition 10.3.1 *Let FA be a filtered ring and consider two morphisms*

$$Fu : FM \longrightarrow FN, \quad Fv : FN \longrightarrow FP$$

such that $Fv \circ Fu = 0$. Then, the sequence

$$FM \xrightarrow{Fu} FN \xrightarrow{Fv} FP$$

is a strict exact sequence if and only if the sequence

$$GM \xrightarrow{Gu} GN \xrightarrow{Gv} GP$$

is exact.

Proof: The condition is necessary.

Let $n_k \in G_kN$ be such that $G_kv(n_k) = 0$. There is $n'_k \in F_kN$ such that $n_k = \sigma_k(n'_k)$. Hence we have $v(n'_k) \in F_{k-1}P$. Since Fv is strict we find $n''_{k-1} \in F_{k-1}N$ such that $v(n''_{k-1}) = v(n'_k)$. Then $v(n'_k - n''_{k-1}) = 0$ and there is $m_k \in F_kM$ such that $u(m_k) = n'_k - n''_{k-1}$. This shows that

$$G_ku(\sigma_k(m_k)) = \sigma_k(n'_k) = n_k$$

and the conclusion follows.

The condition is sufficient.

Let us prove that Fv is strict. Assume $p_k \in F_kP \cap \mathrm{im}\, v$. Let ℓ be the smallest integer such that $p_k = v(n_\ell)$ with $n_\ell \in F_\ell N$. We need to show that $\ell \leq k$. Assume the contrary. Then

$$G_\ell v(\sigma_\ell(n_\ell)) = \sigma_\ell(p_k) = 0.$$

Hence, there is an $m_\ell \in F_\ell M$ such that

$$G_\ell u(\sigma_\ell(m_\ell)) = \sigma_\ell(n_\ell).$$

As a consequence, we get $n_\ell - u(m_\ell) \in F_{\ell-1}M$. But $v(n_\ell - u(m_\ell)) = p_k$ and we get a contradiction.

Let us prove now that $\ker F_k v = \operatorname{im} F_k u$ by increasing induction on k. For $k \ll 0$, $F_k N = 0$ so the result is obvious. Let $n_k \in \ker F_k v$. Since $G_k v(\sigma_k(n_k)) = 0$, there is $m_k \in F_k M$ such that

$$G_k u(\sigma_k(m_k)) = \sigma_k(n_k).$$

Hence $n_k - u(m_k) \in F_{k-1}N$. By the induction hypothesis there is $m_{k-1} \in F_{k-1}M$ such that

$$u(m_{k-1}) = n_k - u(m_k)$$

and the proof is complete. □

Corollary 10.3.2 *Let FA be a filtered ring and $Fu : FM \longrightarrow FN$ be a morphism of FA-modules. Then $\operatorname{Gr}\ker Fu \subset \ker \operatorname{Gr} Fu$ and $\operatorname{im}\operatorname{Gr} Fu \subset \operatorname{Gr}\operatorname{im} Fu$. Moreover, the following conditions are equivalent:*

(a) *Fu is strict,*

(b) $\operatorname{Gr}\ker Fu = \ker\operatorname{Gr} Fu$,

(c) $\operatorname{im}\operatorname{Gr} Fu = \operatorname{Gr}\operatorname{im} Fu$.

Proof: The equivalence of (a), (b) and (c) is a consequence of the preceding proposition. To conclude we just have to note that

$$\begin{aligned}\operatorname{Gr}_k \ker Fu &= \ker u \cap F_k M / \ker u \cap F_{k-1}M \\ \ker \operatorname{Gr}_k Fu &= F_k M \cap u^{-1}(F_{k-1}N)/F_{k-1}M\end{aligned}$$

and that

$$\begin{aligned}\operatorname{im}\operatorname{Gr}_k Fu &= u(F_k M)/F_{k-1}N \\ \operatorname{Gr}_k \operatorname{im} Fu &= \operatorname{im} u \cap F_k N / \operatorname{im} u \cap F_{k-1}N.\end{aligned}$$

□

An FA-module FM is *finite free* if it is isomorphic to

$$\bigoplus_{i=1}^{p} FA(-d_i)$$

where $d_1, \dots, d_p$ are integers. An FA-module FM is of finite type if there is a strict epimorphism

$$FL \longrightarrow FM$$

where FL is finite free. This means that we can find $m_1 \in F_{d_1}M, \dots, m_p \in F_{d_p}M$ such that any $m \in F_d M$ may be written as

$$m = \sum_{i=1}^{p} a_{d-d_i} m_i$$

where $a_{d-d_i} \in F_{d-d_i}A$.

10.3.1 Noether property

For a filtered ring FA, the following conditions are equivalent.

- Any filtered submodule of a finite type FA-module is of finite type.
- Any filtered ideal of FA is of finite type.
- Any increasing sequence of filtered submodules of a finite free FA-module is stationary.

Let us emphasize that in the preceding conditions we deal with *non necessarily strict* submodules and ideals.

A filtered ring FA satisfying the above equivalent conditions is said to be *(filtered) Noetherian*. From this definition, it is clear that if FA is filtered Noetherian then the underlying ring A is itself Noetherian.

Proposition 10.3.3 *Let FA be a filtered ring and denote by GA its associated graded ring. Then FA is filtered Noetherian if and only if GA is graded Noetherian.*

Proof: The condition is sufficient.

We need to prove that a filtered submodule FM' of a finitely generated FA-module FM is finitely generated.

If FM' is strict, then the associated GA-module GM' is a submodule of the GA-module GM associated to FM. Since GA is Noetherian and GM is finitely generated so is GM' and the conclusion follows.

To prove the general case, we may thus assume that the image of the inclusion

$$FM' \longrightarrow FM$$

is equal to FM. In this case, using a finite system of generators of FM, it is easy to find an integer ℓ such that:

$$F_k M' \subset F_k M \subset F_{k+\ell} M'.$$

We will prove the result by increasing induction on ℓ.

For $\ell = 1$, let us introduce the auxiliary GA-modules

$$\begin{aligned} GK_0 &= \bigoplus_{k\in\mathbb{Z}} F_k M'/F_{k-1}M \\ GK_1 &= \bigoplus_{k\in\mathbb{Z}} F_k M/F_k M'. \end{aligned}$$

These modules satisfy the exact sequences

$$0 \longrightarrow GK_0 \longrightarrow GM \longrightarrow GK_1 \longrightarrow 0$$

$$0 \longrightarrow GK_1(-1) \longrightarrow GM' \longrightarrow GK_0 \longrightarrow 0.$$

Since GM is a finite type GA-module, so are GK_0 and GK_1. Hence GM' is also finitely generated and the conclusion follows.

For $\ell > 1$, we define the auxiliary FA-module FM'' by setting

$$F_kM'' = F_{k-1}M + F_kM'.$$

Since we have

$$F_kM'' \subset F_kM \subset F_{k+1}M'',$$

the preceding discussion shows that FM'' is finitely generated. Moreover,

$$F_kM' \subset F_kM'' \subset F_{k+(\ell-1)}M',$$

and the conclusion follows from the induction hypothesis.

The condition is necessary.

To FA we associate the graded ring

$$\Sigma FA = \bigoplus_{k\in\mathbb{Z}} F_kA.$$

Since $F_0A \subset F_1A$ the identity $1 \in F_1A$ and defines a central element T of degree 1 in ΣFA. To any FA-module FM we associate the ΣFA-module

$$\Sigma FM = \bigoplus_{k\in\mathbb{Z}} F_kM$$

Obviously, any morphism $Fu : FM \longrightarrow FN$ induces a morphism

$$\Sigma Fu : \Sigma FM \longrightarrow \Sigma FN$$

of graded ΣFA-modules. Hence, we get a functor Σ from the category of filtered FA-modules to the category of graded ΣFA-modules. One checks easily that this functor is fully faithful and that it transforms exact sequences into exact sequences. Moreover, for any inclusion

$$M' \xrightarrow{u'} L'$$

of ΣFA-modules where L' is finite free there is a strict monomorphism

$$FM \xrightarrow{Fu} FL$$

of FA-modules where FL is finite free and a commutative diagram

$$\begin{array}{ccc} M' & \xrightarrow{u'} & L' \\ \uparrow & & \uparrow \\ \Sigma FM & \xrightarrow{\Sigma Fu} & \Sigma FL \end{array}$$

It is thus clear that ΣFA is graded Noetherian if and only if FA is filtered Noetherian. The conclusion follows from the formula

$$\Sigma FA/\Sigma FA.T = GA.$$

$\square$

Proposition 10.3.4 *Assume FA is Noetherian and FM is an FA-module. Then*

(a) FM is an FA-module of finite type if and only if GM is a GA-module of finite type,

(b) FM is a finite free FA-module if and only if GM is a finite free GA-module,

(c) any FA-module of finite type has an infinite resolution by finite free FA-modules (i.e. there is an exact sequence

$$\cdots \longrightarrow FL_s \longrightarrow FL_{s-1} \longrightarrow \cdots \longrightarrow FL_0 \longrightarrow FM \longrightarrow 0$$

where each FL_s is a finite free FA-module.

Proposition 10.3.5 *Assume FA is Noetherian. Then, for any FA-module of finite type FM and any FA-module FN,*

$$\mathrm{GExt}^j_{GA}(GM, GN) = 0 \Rightarrow \mathrm{Ext}^j_A(M, N) = 0.$$

In particular, $\mathrm{hd}_A(M) \leq \mathrm{hd}_{GA}(GM)$ *and* $\mathrm{glhd}(A) \leq \mathrm{glhd}(GA)$.

Proof: Let

$$\cdots \longrightarrow FL_n \longrightarrow FL_{n-1} \longrightarrow \cdots \longrightarrow FL_0 \longrightarrow FM \longrightarrow 0$$

be a filtered resolution of FM by finite free FA-modules. Applying the graduation functor we get a resolution

$$\cdots \longrightarrow GL_n \longrightarrow GL_{n-1} \longrightarrow \cdots \longrightarrow GL_0 \longrightarrow GM \longrightarrow 0$$

of GM by finite free GA-modules. Assuming $\mathrm{GExt}^j_{GA}(GM, GN) = 0$ means that the sequence

$$\mathrm{GHom}_{GA}(GL_{j-1}, GN) \longrightarrow \mathrm{GHom}_{GA}(GL_j, GN) \longrightarrow \mathrm{GHom}_{GA}(GL_{j+1}, GN)$$

is an exact sequence of GA-modules. The natural map

$$\mathrm{GrFHom}_{FA}(FL, FN) \longrightarrow \mathrm{GHom}_{GA}(GL, GN)$$

is an isomorphism when FL is finite free. Hence the sequence

$$\mathrm{FHom}_{FA}(FL_{j-1}, FN) \longrightarrow \mathrm{FHom}_{FA}(FL_j, FN) \longrightarrow \mathrm{FHom}_{FA}(FL_{j+1}, FN)$$

is a strict exact sequence of FA-modules. When FL is finite free the underlying module of $\mathrm{FHom}_{FA}(FL, FN)$ is $\mathrm{Hom}_A(L, N)$. Hence the conclusion. □

10.3.2 Syzygies

Proposition 10.3.6 *Assume FA is a filtered ring and GA is syzygic then any FA-module of finite type FM has a finite free resolution*

$$0 \longrightarrow FL_n \longrightarrow FL_{n-1} \longrightarrow \cdots \longrightarrow FL_0 \longrightarrow FM \longrightarrow 0$$

of length $n = \sup(1, \mathrm{hd}_{GA} GM)$. In particular, A is syzygic.

Proof: Let

$$\cdots \longrightarrow FL_n \xrightarrow{d_n} FL_{n-1} \xrightarrow{d_{n-1}} \cdots \longrightarrow FL_0 \longrightarrow FM \longrightarrow 0$$

be a finite free resolution of FM. Set $FP_n = \ker d_{n-1}$. Since the sequence

$$0 \longrightarrow GP_n \longrightarrow GL_{n-1} \longrightarrow \cdots \longrightarrow GL_0 \longrightarrow GM \longrightarrow 0$$

is exact, $\mathrm{hd}_{GA}(GP_n) \leq 0$ and GP_n is projective. It is thus stably finite free. Let GL'_n be a finite free GA-module such that $GL'_n \oplus GP_n$ is finite free. Let FL'_n be a finite free FA-module which has GL'_n as associated graded GA-module. We have the exact sequence

$$0 \longrightarrow FP_n \oplus FL'_n \longrightarrow FL_{n-1} \oplus FL_n \longrightarrow \cdots \longrightarrow FL_0 \longrightarrow FM \longrightarrow 0$$

and $FP_n \oplus FL'_n$ is finite free since its associated GA-module $GP_n \oplus GL'_n$ is finite free.

To conclude, note that any finite type A-module M is the underlying A-module of a finite type FA-module FM. □

10.3.3 Gabber's theorem

Let FA be a filtered ring and denote by GA its associated graded ring.

For any $a_k \in F_k A$, $b_\ell \in F_\ell A$ we define the *commutator* $[a_k, a_\ell]$ of a_k and a_ℓ as usual by setting

$$[a_k, a_\ell] = a_k \cdot a_\ell - a_\ell \cdot a_k.$$

Obviously, GA is commutative if and only if

$$[F_k A, F_\ell A] \subset F_{k+\ell-1} A.$$

In this case, one checks easily that

$$\sigma_{k+\ell-1}[a_k, a_\ell] = \sigma_{k+\ell-1}[a'_k, a'_\ell]$$

if $a_k, a'_k \in F_k A$ (resp. $a_\ell, a'_\ell \in F_\ell A$) have the same symbol of order k (resp. ℓ). This property allows us to define the *Poisson bracket* of two symbols $s_k \in G_k A$ and $s_l \in G_\ell A$ by setting

$$\{s_k, s_\ell\} = \sigma_{k+\ell-1}([a_k, a_\ell])$$

where $a_k \in F_k A$ and $a_\ell \in F_\ell A$ are such that $\sigma_k(a_k) = s_k$ and $\sigma_\ell(a_\ell) = s_\ell$. The Poisson bracket satisfies the following easily checked identities:

$$\begin{aligned}
\{s, s'\} &= -\{s', s\} \\
\{s, s's''\} &= \{s, s'\}s'' + s'\{s, s''\} \\
\{s, \{s', s''\}\} &= \{\{s, s'\}, s''\} + \{s', \{s, s''\}\}
\end{aligned}$$

Now, let FM be an FA-module and denote by FI its annihilating ideal. Obviously, if $a_k \in F_k I$ and $a_\ell \in F_\ell I$ we have

$$[a_k, a_\ell] FM = 0$$

and $[a_k, a_\ell] \in F_{k+\ell-1} I$. Hence the graded ideal GI associated to FI is stable by $\{\cdot, \cdot\}$. However, it is not obvious at all that $\sqrt{GI}$ is also stable by $\{\cdot, \cdot\}$. This problem was solved by Gabber in [5]. His proof is rather technical, so we skip it here and refer the interested reader to the original paper.

Theorem 10.3.7 *Let FA be a filtered ring and assume GA is a commutative Noetherian $\mathbb{Q}$-algebra. Then, for any finite type FA-module FM we have*

$$\{\sqrt{\operatorname{Ann} GM}, \sqrt{\operatorname{Ann} GM}\} \subset \sqrt{\operatorname{Ann} GM}$$

where GM is the GA-module associated with FM.

10.4 Transition to sheaves

The reader will easily extend the definition of graded and filtered rings and modules given above to the case of filtered and graded sheaves of rings and modules on a topological space X. The notion of graded or filtered morphisms are also easily extended to sheaves as is the notion of strict filtered morphism and of graded or filtered exact sequences.

Let $\mathcal{GA}$ (resp. $\mathcal{FA}$) be a graded (resp. filtered) sheaf of rings and $\mathcal{GM}, \mathcal{GN}$ (resp. $\mathcal{FM}, \mathcal{FN}$) two $\mathcal{GA}$ (resp. $\mathcal{FA}$) modules. The functors

$$U \longrightarrow \operatorname{Hom}_{\mathcal{GA}_{|U}}(\mathcal{GM}_{|U}, \mathcal{GN}_{|U}); \quad U \mapsto \operatorname{GHom}_{\mathcal{GA}_{|U}}(\mathcal{GM}_{|U}, \mathcal{GN}_{|U})$$

(resp.

$$U \longrightarrow \operatorname{Hom}_{\mathcal{FA}_{|U}}(\mathcal{FM}_{|U}, \mathcal{FN}_{|U}); \quad U \mapsto \operatorname{FHom}_{\mathcal{FA}_{|U}}(\mathcal{FM}_{|U}, \mathcal{FN}_{|U})$$

) clearly define sheaves of groups and graded (resp. filtered) groups on X. We denote then by

$$\mathcal{H}om_{\mathcal{GA}}(\mathcal{GM}, \mathcal{GN}); \quad \mathcal{GH}om_{\mathcal{GA}}(\mathcal{GM}, \mathcal{GN})$$

(resp.

$$\mathcal{H}om_{\mathcal{FA}}(\mathcal{FM}, \mathcal{FN}); \quad \mathcal{FH}om_{\mathcal{FA}}(\mathcal{FM}, \mathcal{FN})$$

)

The definition of the graduation functor will also be easily extended to sheaves of filtered rings and modules. Finally, the reader will generalize easily to sheaves the notion of finite free and finite type modules and that of a finite free s-presentation.

The finiteness conditions require more attention since the category of modules of finite type over a sheaf of rings is almost never well-behaved. To get satisfactory finiteness conditions on $\mathcal{A}$ (resp. $\mathcal{GA}$, $\mathcal{FA}$) we need to go a step further and consider modules having locally finite free 1-presentations.

Definition 10.4.1 The ring $\mathcal{A}$ (resp. $\mathcal{GA}$, $\mathcal{FA}$) is *coherent* if the category of modules which admits locally finite free 1-presentations is stable by kernels, cokernels and extensions as a subcategory of the category of modules. A *module* on a coherent ring (resp. graded ring, filtered ring) is *coherent* if it has locally a finite free 1-presentation.

A sheaf of rings $\mathcal{A}$ is coherent if and only if for any open subset U of X the kernel of a morphism of free $\mathcal{A}_U$-modules is locally of finite type.

An analogous criterion holds for sheaves of graded rings.

For a sheaf of filtered rings $\mathcal{FA}$, the criterion is that for any open subset U of X the filtered kernel and the filtered image of a morphism of finite free $\mathcal{FA}_U$-modules are both locally of finite type.

Remark 10.4.2 Let $\mathcal{FA}$ be a filtered ring on X and denote by $\mathcal{GA}$ the associated graded ring. Then, an $\mathcal{FA}$-module $\mathcal{FM}$ is locally of finite type (resp. finite free), if and only if the $\mathcal{GA}$-module $\mathcal{GM}$ is locally of finite type (resp. finite free).

Proposition 10.4.3 *Let $\mathcal{FA}$ be a coherent filtered ring on X and denote by $\mathcal{GA}$ the associated graded ring and by $\mathcal{A}$ the underlying ring. Then*

(a) the ring $\mathcal{A}$ and the graded ring $\mathcal{GA}$ are coherent,

(b) an $\mathcal{FA}$-module $\mathcal{FM}$ is coherent if and only if $\mathcal{GM}$ is $\mathcal{GA}$ coherent,

(c) a strict submodule $\mathcal{FN}$ of a coherent $\mathcal{FA}$-module $\mathcal{FM}$ is coherent if and only if the underlying $\mathcal{A}$-module $\mathcal{N}$ is locally of finite type,

(d) an $\mathcal{FA}$-module locally of finite type $\mathcal{FM}$ is coherent if and only if the underlying $\mathcal{A}$-module $\mathcal{M}$ is coherent.

Proof: (a) Let us prove that $\mathcal{A}$ is coherent. Let U be an open subset of X and consider a morphism

$$\mathcal{A}^p_{|U} \xrightarrow{u} \mathcal{A}^q_{|U}$$

This is the underlying morphism of a filtered morphism

$$\bigoplus_{i=1}^{p} \mathcal{FA}(-d_i) \longrightarrow \mathcal{FA}^q$$

We know that the kernel of this morphism is an $\mathcal{FA}$-module locally of finite type. Since its underlying $\mathcal{A}$-module $\mathcal{N}$ is the kernel of u, the conclusion follows.

To prove that $\mathcal{GA}$ is coherent, we proceed as in Proposition 10.3.3.

(b) Proceed as in Proposition 10.3.4.

(c) Let $\mathcal{FN}$ be a strict submodule of $\mathcal{FM}$ and assume $\mathcal{N}$ is locally of finite type. Since the local generators of $\mathcal{N}$ have locally a finite order in $\mathcal{FM}$, we may view locally $\mathcal{FN}$ as the filtered image of a morphism

$$\mathcal{FL}|_U \longrightarrow \mathcal{FN}|_U$$

where $\mathcal{FL}$ is a finite free $\mathcal{FA}$-module. Hence the conclusion.

(d) Assume that $\mathcal{FM}$ is locally of finite type and that $\mathcal{M}$ is coherent. Locally, we have a strict epimorphism

$$\mathcal{FL}|_U \longrightarrow \mathcal{FM}|_U \longrightarrow 0$$

where $\mathcal{FL}$ is finite free. Denote by $\mathcal{FN}$ its kernel. By hypothesis we know that $\mathcal{N}$ is coherent; hence the conclusion by (c). □

Theorem 10.4.4 *Assume $\mathcal{FA}$ is a filtered ring on X and denote by $\mathcal{GA}$ the associated graded ring. Then, the following conditions are equivalent:*

(a) $\mathcal{FA}$ is coherent and $\mathcal{FA}_x$ is Noetherian for every $x \in X$,

(b) $\mathcal{GA}$ is coherent and $\mathcal{GA}_x$ is Noetherian for every $x \in X$.

Proof: (a) $\Rightarrow$ (b) is a consequence of the preceding proposition and of Proposition 10.3.3.

(b) $\Rightarrow$ (a) We know already that $\mathcal{FA}_x$ is Noetherian. It remains to prove that the kernel and the image of a morphism

$$\mathcal{FL}_1 \xrightarrow{\mathcal{F}u} \mathcal{FL}_0$$

of finite free $\mathcal{FA}$-modules are locally of finite type.

Assume $\mathcal{F}u$ is strict at $x \in X$. Then $\operatorname{Gr}\ker \mathcal{F}u_x = \ker \mathcal{G}r\mathcal{F}u_x$. Since $\ker \mathcal{G}r\mathcal{F}u$ is a coherent $\mathcal{GA}$-module and $\mathcal{G}r \ker \mathcal{F}u \subset \ker \mathcal{G}r\mathcal{F}u$, $\operatorname{Gr}\ker \mathcal{F}u_y = \ker \operatorname{Gr}\mathcal{F}u_y$ for y in a neighborhood of x. Hence $\mathcal{F}u$ is strict in a neighborhood V of x and $\ker \mathcal{F}u$ is locally of finite type since $\mathcal{G}r \ker \mathcal{F}u_{|V} = \ker \mathcal{G}r\mathcal{F}u_{|V}$ is $\mathcal{GA}_{|V}$ coherent. Moreover, in a neighborhood of x, $\mathcal{G}r \operatorname{im} \mathcal{F}u_{|V} = \operatorname{im} \mathcal{G}r\mathcal{F}u_{|V}$ is a $\mathcal{GA}_{|V}$ coherent module and $\operatorname{im} \mathcal{F}u$ is locally of finite type.

Assume $\mathcal{F}u$ is not strict at $x \in X$. Since $\operatorname{im} \mathcal{F}u_x$ is a sub $\mathcal{FA}_x$-module of $(\mathcal{FL}_0)_x$ and $\mathcal{FA}_x$ is Noetherian, it is finitely generated. It is thus possible to find locally a morphism

$$\mathcal{FL}_2 \xrightarrow{\mathcal{F}v} \mathcal{FL}_0$$

of finite free $\mathcal{FA}$-modules such that $\operatorname{im} \mathcal{F}v \subset \operatorname{im} \mathcal{F}u$ and for which

$$(\mathcal{FL}_2)_x \xrightarrow{\mathcal{F}v_x} \operatorname{im} \mathcal{F}u_x \longrightarrow 0$$

is a strict epimorphism. The morphism

$$\begin{array}{rcl} \mathcal{FL}_1 \oplus \mathcal{FL}_2 & \xrightarrow{\mathcal{F}v} & \mathcal{FL}_0 \\ (\alpha, \beta) & \mapsto & (u(\alpha) + v(\beta)) \end{array}$$

is strict at x and such that locally $\operatorname{im}\mathcal{F}w = \operatorname{im}\mathcal{F}u$. By the preceding discussion, this shows that $\operatorname{im}\mathcal{F}u$ and $\ker\mathcal{F}w$ are locally of finite type at x. Since $\operatorname{im}\mathcal{F}v \subset \operatorname{im}\mathcal{F}u$, there is locally a *non filtered* morphism of $\mathcal{A}$-modules

$$f : \mathcal{L}_2 \longrightarrow \mathcal{L}_1$$

such that $u \circ f = v$. This gives us the morphism of $\mathcal{A}$-modules

$$\begin{array}{ccc} \ker w & \longrightarrow & \ker u \\ (\alpha, \beta) & \mapsto & \alpha + f(\beta) \end{array}$$

Hence $\ker u$ is locally of finite type. This gives us a finite free $\mathcal{F}\mathcal{A}$-module $\mathcal{F}\mathcal{L}_3$ and a morphism

$$\mathcal{F}\mathcal{L}_3 \longrightarrow \mathcal{F}\mathcal{L}_2$$

having $\ker\mathcal{F}u$ as filtered image. Since $\mathcal{F}\mathcal{A}$ is coherent, this shows that $\ker\mathcal{F}u$ is an $\mathcal{F}\mathcal{A}$-module of finite type and the proof is complete. $\square$

Assume the sheaf of ring $\mathcal{A}$ is coherent and let $\mathcal{M}$ be a coherent $\mathcal{A}$-module then the homological dimension of $\mathcal{M}$ is the smallest integer n such that the following equivalent properties hold

- for any $\mathcal{A}$-module $\mathcal{N}$ and any $j > n$

$$\operatorname{Ext}^j_{\mathcal{A}}(\mathcal{M}, \mathcal{N}) = 0$$

- for any $\mathcal{A}$-module $\mathcal{N}$, any $x \in X$ and any $j > n$

$$\operatorname{Ext}^j_{\mathcal{A}x}(\mathcal{M}_x, \mathcal{N}_x) = 0$$

From this definition, we find that

$$\operatorname{hd}_{\mathcal{A}}(\mathcal{M}) = \sup_{x\in X} \operatorname{hd}_{\mathcal{A}x}(\mathcal{M}_x).$$

We define the global homological dimension of $\mathcal{A}$ as the supremum of

$$\{\operatorname{hd}_{\mathcal{A}}\mathcal{M} : \mathcal{M} \text{ coherent } \mathcal{A}\text{-module}\}.$$

We introduce similar definitions for a coherent graded ring X and its coherent graded modules.

Proposition 10.4.5 *Assume the filtered ring $\mathcal{F}\mathcal{A}$ on X is coherent. Then, for any coherent $\mathcal{F}\mathcal{A}$-module $\mathcal{F}\mathcal{M}$ and any $\mathcal{F}\mathcal{A}$-module $\mathcal{F}\mathcal{N}$*

$$\mathcal{G}\mathcal{E}xt^j_{\mathcal{G}\mathcal{A}}(\mathcal{G}\mathcal{M}, \mathcal{G}\mathcal{N}) = 0 \Rightarrow \operatorname{Ext}^j_{\mathcal{A}}(\mathcal{M}, \mathcal{N}) = 0$$

where $\mathcal{M}$, $\mathcal{N}$ are the underlying $\mathcal{A}$-modules of $\mathcal{F}\mathcal{M}$, $\mathcal{F}\mathcal{N}$ and $\mathcal{G}\mathcal{M}$, $\mathcal{G}\mathcal{N}$ are the corresponding graded modules. In particular, we have $\operatorname{hd}_{\mathcal{A}}(\mathcal{M}) \leq \operatorname{hd}_{\mathcal{G}\mathcal{A}}(\mathcal{G}\mathcal{M})$ and $\operatorname{glhd}(\mathcal{A}) \leq \operatorname{glhd}(\mathcal{G}\mathcal{A})$.

Proof: Proceed as for the proof of 10.3.5. □

A sheaf of rings (resp. graded rings) is *syzygic* if it is coherent, has finite global homological dimension and syzygic fibers. Since a coherent module $\mathcal{M}$ on a coherent ring $\mathcal{A}$ is finite free in a neighborhood of $x \in X$ if and only if $\mathcal{M}_x$ is finite free we get the following proposition.

Proposition 10.4.6 *Assume $\mathcal{A}$ is syzygic. Then, any coherent $\mathcal{A}$-module $\mathcal{M}$ has a finite free resolution*

$$0 \longrightarrow \mathcal{L}_n \longrightarrow \mathcal{L}_{n-1} \longrightarrow \cdots \longrightarrow \mathcal{L}_0 \longrightarrow \mathcal{M} \longrightarrow 0$$

of length $n = \sup(1, \mathrm{hd}_{\mathcal{A}}(\mathcal{M}))$.

In the same way, we get :

Proposition 10.4.7 *Let $\mathcal{FA}$ be a filtered ring on X and denote by $\mathcal{GA}$ its associated graded ring. Assume $\mathcal{GA}$ is syzygic. Then, any coherent $\mathcal{FA}$-module $\mathcal{FM}$ has locally a finite free resolution*

$$0 \longrightarrow \mathcal{FL}_n \longrightarrow \mathcal{FL}_{n-1} \longrightarrow \cdots \longrightarrow \mathcal{FL}_0 \longrightarrow \mathcal{FM} \longrightarrow 0$$

of length $\sup(1, \mathrm{hd}_{\mathcal{GA}}\mathcal{GM})$. *In particular, $\mathcal{A}$ is syzygic.*

A sheaf of rings (resp. graded rings, filtered rings) is *Noetherian* if it is coherent with Noetherian fibers and if locally any increasing sequence of coherent submodules of a coherent module is locally stationary.

Proposition 10.4.8 *Assume $\mathcal{FA}$ is a filtered ring on X and denote by $\mathcal{GA}$ and $\mathcal{A}$ its associated graded ring and its underlying ring. Then, $\mathcal{FA}$ is Noetherian if and only if $\mathcal{GA}$ is Noetherian. In this case $\mathcal{A}$ is also Noetherian.*

Proof: Assume $\mathcal{GA}$ is Noetherian. By Theorem 10.4.4 we already know that $\mathcal{FA}$ is coherent and that $\mathcal{FA}_x$ is Noetherian for any $x \in X$. Assume that $(\mathcal{FM}_k)_{k\in\mathbb{N}}$ is an increasing sequence of coherent strict submodules of the coherent $\mathcal{FA}_{|U}$ module $\mathcal{FM}_{|U}$ where U is some open subset of X. Then, $(\mathcal{GM}_k)_{k\in\mathbb{N}}$ is an increasing sequence of coherent $\mathcal{GA}$-submodules of the coherent $\mathcal{GA}$-module $\mathcal{GM}$. Since $\mathcal{GA}$ is Noetherian, every point $x \in U$ has a neighborhood $V \subset U$ for which there is an integer k_0 such that $(\mathcal{GM}_{k_0})_{|U} = (\mathcal{GM}_k)_{|U}$ for $k \geq k_0$. This implies $(\mathcal{FM}_{k_0})_{|V} = (\mathcal{FM}_k)_{|V}$ for $k \geq k_0$. We obtain the same result for an increasing sequence of general filtered submodules of $\mathcal{FM}_{|U}$ by proceeding as in 10.3.3. Hence $\mathcal{FA}$ is Noetherian.

Assume now $\mathcal{FA}$ Noetherian, working as in the proof of Proposition 10.3.3, we see easily that $\mathcal{GA}$ is Noetherian. Since any coherent $\mathcal{A}$-module may be viewed as the underlying module of a coherent $\mathcal{FA}$-module and since a submodule $\mathcal{FN}$ of a coherent $\mathcal{FA}$-module $\mathcal{FM}$ is coherent when its underlying $\mathcal{A}$-module is coherent, we get easily the Noetherianity of $\mathcal{A}$. □

Corollary 10.4.9 *A graded ring $\mathcal{GA}$ on X is Noetherian if and only if the underlying ring $\Sigma\mathcal{GA}$ is Noetherian.*

Proposition 10.4.10 *If the sheaf of rings $\mathcal{A}$ is Noetherian then so is the sheaf of rings $\mathcal{A}[x]$.*

Proof: By the preceding corollary, it is sufficient to prove that $\mathcal{A}[x]$ is Noetherian as a graded ring on X.

First we show that $\mathcal{A}[x]$ is graded coherent. Let

$$\mathcal{GL}_1 \xrightarrow{\mathcal{G}u} \mathcal{GL}_0$$

be a graded morphism of finite free graded $\mathcal{A}[x]$-modules.

Denote $\mathcal{GN}$ its kernel. Since

$$\mathcal{GL}_1 = \bigoplus_{i=1}^{p} \mathcal{A}[x](d_i).$$

$\mathcal{G}_k\mathcal{N}$ is a coherent $\mathcal{A}$-submodule of $\mathcal{G}_k\mathcal{L}_1 \simeq \mathcal{A}^p$ and since $\mathcal{GN}.x \subset \mathcal{GN}$ we get $\mathcal{G}_k\mathcal{N} \subset \mathcal{G}_{k+1}\mathcal{N}$ as submodules of $\mathcal{A}^p$. Since $\mathcal{A}$ is Noetherian, this sequence is locally stationary and for any $x \in X$ there is an open neighborhood U of x and an integer k_0 such that

$$(\mathcal{G}_{k_0}\mathcal{N})_{|U} = (\mathcal{G}_k\mathcal{N})_{|U}$$

for $k \geq k_0$. Hence $\mathcal{N}_{|U}$ is generated by $\mathcal{G}_{k_0}\mathcal{N}$ as an $\mathcal{A}_{|U}[x]$-module. The $\mathcal{GA}$-module $\mathcal{GN}$ is thus locally finitely generated on $\mathcal{A}[x]$.

To prove that $\mathcal{A}[x]$ is Noetherian, we have to prove that if $(\mathcal{GN}_\ell)_{\ell\in\mathbb{N}}$ is an increasing sequence of coherent $\mathcal{A}[x]$-submodules of a finite free $\mathcal{A}[x]$-module $\mathcal{GL}$ then $(\mathcal{GN}_\ell)_{\ell\in\mathbb{N}}$ is locally stationary. Working as above we see that

$$\mathcal{G}_k\mathcal{N}_\ell \subset \mathcal{G}_{k'}\mathcal{N}_{\ell'}$$

if $\ell \leq \ell'$, $k \leq k'$. Hence, locally, there are integers ℓ_0, k_0 such that $\mathcal{G}_{k_0}\mathcal{N}_{\ell_0} = \mathcal{G}_k\mathcal{N}_\ell$ for $k \geq k_0$, $\ell \geq \ell_0$. This implies that $\mathcal{GN}_\ell = \mathcal{GN}_{\ell_0}$ for $\ell \geq \ell_0$; hence the conclusion. □

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