Fundamentals of Compressible Fluid Mechanics

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‘We are like dwarfs sitting on the shoulders of giants”

from The Metalogicon by John in 1159

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About This Author

Genick Bar-Meir holds a Ph.D. in Mechanical Engineering from University of Minnesota and a Master in Fluid Mechanics from Tel Aviv University. Dr. Bar-Meir was the last student of the late Dr. R.G.E. Eckert. Much of his time has been spend doing research in the field of heat and mass transfer (this includes fluid mechanics) related to manufacturing processes and design. Currently, he spends time writing books and software for the POTTO project (see Potto Epilogue). The author enjoys to encourages his students to understand the material beyond the basic requirements of exams.

In his early part of his professional life, Bar-Meir was mainly interested in elegant models whether they have or not a practical applicability. Now, this author's views had changed and the virtue of the practical part of any model becomes the essential part of his ideas, books and softwares.

He developed models for Mass Transfer in high concentration that became a building blocks for many other models. These models are based on analytical solution to a family of equations[1]. As the change in the view occurred, Bar-Meir developed models that explained several manufacturing processes such the rapid evacuation of gas from containers, the critical piston velocity in a partially filled chamber (related to hydraulic jump), supply and demand to rapid change power system and etc. All the models have practical applicability.

These models have been extended by several research groups (needless to say with large research grants). For example, the Spanish Comision Interministerial provides grants TAP97-0489 and PB98-0007, and the CICYT and the European Commission provides 1FD97-2333 grants for minor aspects of that models. Moreover, these models were used in numerical works, in GM, British industry, and even Iran.

The author believes that this book, as in the past, will promote new re-

[1]Where the mathematicians were able only to prove that the solution exists.

search. More than that, this author believes that the book will blaze a trail of new understanding.

The author lives with his wife and three children. A past project of his was building a four stories house, practically from scratch. While he writes his programs and does other computer chores, he often feels clueless about computers and programing. While he known to look like he know about many things, the author just know to learn quickly. The author spent years working on the sea (ships) as a engine sea officer but now the author prefers to remain on solid ground.

Prologue for the POTTO Project

This series of books was born out of frustrations in two respects. The first issue is the enormous price of college textbooks. It is unacceptable that the price of the college books will be over $150 per book (over 10 hours of work for an average student in The United States).

The second issue that prompted the writing of this book is the fact that we as the public have to deal with a corrupted judicial system. As individuals we have to obey the law, particularly the copyright law with the "infinite[2]" time with the copyright holders. However, when applied to "small" individuals who are not able to hire a large legal firm, judges simply manufacture facts to make the little guy lose and pay for the defense of his work. On one hand, the corrupted court system defends the "big" guys and on the other hand, punishes the small "entrepreneur" who tries to defend his or her work. It has become very clear to the author and founder of the POTTO Project that this situation must be stopped. Hence, the creation of the POTTO Project. As R. Kook, one of this author's sages, said instead of whining about arrogance and incorrectness, one should increase wisdom. This project is to increase wisdom and humility.

The POTTO Project has far greater goals than simply correcting an abusive Judicial system or simply exposing abusive judges. It is apparent that writing textbooks especially for college students as a cooperation, like an open source, is a new idea[3]. Writing a book in the technical field is not the same as writing a novel. The writing of a technical book is really a collection of information and practice. There is always someone who can add to the book. The study of technical

[2]After the last decision of the Supreme Court in the case of Eldred v. Ashcroff (see `http://cyber.law.harvard.edu/openlaw/eldredvashcroft` for more information) copyrights practically remain indefi nitely with the holder (not the creator).

[3]In some sense one can view the encyclopedia Wikipedia as an open content project (see `http://en.wikipedia.org/wiki/Main_Page`). The wikipedia is an excellent collection of articles which are written by various individuals.

material isn't only done by having to memorize the material, but also by coming to understand and be able to solve related problems. The author has not found any technique that is more useful for this purpose than practicing the solving of problems and exercises. One can be successful when one solves as many problems as possible. To reach this possibility the collective book idea was created/adapted. While one can be as creative as possible, there are always others who can see new aspects of or add to the material. The collective material is much richer than any single person can create by himself.

The following example explains this point: The army ant is a kind of carnivorous ant that lives and hunts in the tropics, hunting animals that are even up to a hundred kilograms in weight. The secret of the ants' power lies in their collective intelligence. While a single ant is not intelligent enough to attack and hunt large prey, the collective power of their networking creates an extremely powerful intelligence to carry out this attack (see for information `http://www.ex.ac.uk/bugclub/raiders.html`)[4]. So when an insect which is blind can be so powerful by networking, so can we in creating textbooks by this powerful tool.

Why would someone volunteer to be an author or organizer of such a book? This is the first question the undersigned was asked. The answer varies from individual to individual. It is hoped that because of the open nature of these books, they will become the most popular books and the most read books in their respected field. In a way, the popularity of the books should be one of the incentives for potential contributors. The desire to be an author of a well-known book (at least in his/her profession) will convince some to put forth the effort. For some authors, the reason is the pure fun of writing and organizing educational material. Experience has shown that in explaining to others any given subject, one also begins to better understand the material. Thus, contributing to this book will help one to understand the material better. For others, the writing of or contributing to this kind of book will serve as a social function. The social function can have at least two components. One component is to come to know and socialize with many in the profession. For others the social part is as simple as a desire to reduce the price of college textbooks, especially for family members or relatives and those students lacking funds. For some contributors/authors, in the course of their teaching they have found that the textbook they were using contains sections that can be improved or that are not as good as their own notes. In these cases, they now have an opportunity to put their notes to use for others. Whatever the reasons, the undersigned believes that personal intentions are appropriate and are the author's/organizer's private affair.

If a contributor of a section in such a book can be easily identified, then that contributor will be the copyright holder of that specific section (even within question/answer sections). The book's contributor's names could be written by their sections. It is not just for experts to contribute, but also students who happened to be doing their homework. The student's contributions can be done by

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[4]see also in Franks, Nigel R.; "Army Ants: A Collective Intelligence," American Scientist, 77:139, 1989

adding a question and perhaps the solution. Thus, this method is expected to accelerate the creation of these high quality books.

These books are written in a similar manner to the open source software process. Someone has to write the skeleton and hopefully others will add "flesh and skin." In this process, chapters or sections can be added after the skeleton has been written. It is also hoped that others will contribute to the question and answer sections in the book. But more than that, other books contain data[5] which can be typeset in LaTeX. These data (tables, graphs and etc.) can be redone by anyone who has the time to do it. Thus, the contributions to books can be done by many who are not experts. Additionally, contributions can be made from any part of the world by those who wish to translate the book.

It is hoped that the book will be error-free. Nevertheless, some errors are possible and expected. Even if not complete, better discussions or better explanations are all welcome to these books. These books are intended to be "continuous" in the sense that there will be someone who will maintain and improve the book with time (the organizer).

These books should be considered more as a project than to fit the traditional definition of "plain" books. Thus, the traditional role of author will be replaced by an organizer who will be the one to compile the book. The organizer of the book in some instances will be the main author of the work, while in other cases This may merely be the person who decides what will go into the book and what will not (gate keeper). Unlike a regular book, these works will have a version number because they are alive and continuously evolving.

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Statics	not started yet		0.0.0
Dynamics	not started yet		0.0.0
Strength of Material	not started yet		0.0.0
Compressible Flow	early beta		0.4
Fluid Mechanics	alpha		0.1
Thermodynamics	early alpha		0.0.01
Heat Transfer	not started yet	Based on Eckert	0.0.0
Open Channel Flow	not started yet		0.0.0
Two/Multi phases flow	not started yet	Tel-Aviv'notes	0.0.0

The meaning of the progress is as:

- The Alpha Stage is when some of the chapters are already in rough draft;

[5] Data are not copyrighted.

- In Beta Stage is when all or almost all of the chapters have been written and are at least in a draft stage; and
- In Gamma Stage is when all the chapters are written and some of the chapters are in a mature form.
- The Advanced Stage is when all of the basic material is written and all that is left are aspects that are active, advanced topics, and special cases.

The mature stage of a chapter is when all or nearly all of the sections are in a mature stage and have a mature bibliography as well as mature and numerous examples for every section. The mature stage of a section is when all of the topics in the section are written, and all of the examples and data (tables, figures, etc.) are already presented. While some terms are defined in a relatively clear fashion, other definitions give merely a hint on the status. But such a thing is hard to define and should be enough for this stage.

The idea that a book can be created as a project has mushroomed from the open source software concept, but it has roots in the way science progresses. However, traditionally books have been improved by the same author(s), a process in which books have a new version every a few years. There are book(s) that have continued after their author passed away, i.e., the *Boundary Layer Theory* originated[6] by Hermann Schlichting but continues to this day. However, projects such as the Linux Documentation project demonstrated that books can be written as the cooperative effort of many individuals, many of whom volunteered to help.

Writing a textbook is comprised of many aspects, which include the actual writing of the text, writing examples, creating diagrams and figures, and writing the LaTeX macros[7] which will put the text into an attractive format. These chores can be done independently from each other and by more than one individual. Again, because of the open nature of this project, pieces of material and data can be used by different books.

[6] Originally authored by Dr. Schlichting, who passed way some years ago. A new version is created every several years.

[7] One can only expect that open source and readable format will be used for this project. But more than that, only LaTeX, and perhaps troff, have the ability to produce the quality that one expects for these writings. The text processes, especially LaTeX, are the only ones which have a cross platform ability to produce macros and a uniform feel and quality. Word processors, such as OpenOffi ce, Abiword, and Microsoft Word software, are not appropriate for these projects. Further, any text that is produced by Microsoft and kept in “Microsoft” format are against the spirit of this project In that they force spending money on Microsoft software.

Prologue For This Book

0.1 Version 0.4.3

The title of this section is change to reflect that it moved to beginning of the book. While it move earlier but the name was not changed. Dr. Menikoff pointed to this inconsistency, and the author is apologizing for this omission.

Several sections were add to this book with many new idea for example on the moving shock tables. However, this author cannot add all the things that he was asked and want to the book in instant fashion. For example, one of the reader ask why not one of the example of oblique shock was not turn into the explanation of von Neumann paradox. The author was asked by a former client why he didn't insert his improved tank filling and evacuating models (the addtion of the energy equation instead of isentropic model). While all these requests are important, the time is limited and they will be inserted as time permitted.

The moving shock issues are not completed and more work is needed also in the shock tube. Nevertheless, the idea of moving will reduced the work for many student of compressible flow. For example solving homework problem from other text books became either just two mouse clicks away or just just looking that the tables in this book. I also got request from a India to write the interface for Microsoft. I am sorry will not be entertaining work for non Linux/Unix systems, especially for Microsoft. If one want to use the software engine it is okay and permitted by the license of this work.

0.2 Version 0.4.2

It was surprising to find that over 14,000 downloaded and is encouraging to receive over 200 thank you eMail (only one from U.S.A./Arizona) and some other reactions.

This textbook has sections which are cutting edge research[8].

The additions of this version focus mainly on the oblique shock and related issues as results of questions and reactions on this topic. However, most readers reached to www.potto.org by searching for either terms “Rayleigh flow” (107) and “Fanno flow” ((93). If the total combined variation search of terms “Fanno” and “Rayleigh” (mostly through google) is accounted, it reaches to about 30% (2011). This indicates that these topics are highly is demanded and not many concerned with the shock phenomena as this author believed and expected. Thus, most additions of the next version will be concentrated on Fanno flow and Rayleigh flow. The only exception is the addition to Taylor–Maccoll flow (axisymmetricale conical flow) in Prandtl -Meyer function (currently in a note form).

Furthermore, the questions that appear on the net will guide this author on what is really need to be in a compressible flow book. At this time, several questions were about compressibility factor and two phase flow in Fanno flow and other kind of flow models. The other questions that appeared related two phase and connecting several chambers to each other. Also, an individual asked whether this author intended to write about the unsteady section, and hopefully it will be near future.

0.3 Version 0.4

Since the last version (0.3) several individuals sent me remarks and suggestions. In the introductory chapter, extensive description of the compressible flow history was written. In the chapter on speed of sound, the two phase aspects were added. The isothermal nozzle was combined with the isentropic chapter. Some examples were added to the normal shock chapter. The fifth chapter deals now with normal shock in variable area ducts. The sixth chapter deals with external forces fields. The chapter about oblique shock was added and it contains the analytical solution. At this stage, the connection between Prandtl–Meyer flow and oblique is an note form. The a brief chapter on Prandtl–Meyer flow was added.

0.4 Version 0.3

In the traditional class of compressible flow it is assumed that the students will be aerospace engineers or dealing mostly with construction of airplanes and turbomachinery. This premise should not be assumed. This assumption drives students from other fields away from this knowledge. This knowledge should be spread to other fields because it needed there as well. This “rejection” is especially true when students feel that they have to go through a “shock wave” in their understanding.

This book is the second book in the series of POTTO project books. POTTO project books are open content textbooks. The reason the topic of Com-

[8]A reader asked this author to examine a paper on Triple Shock Entropy Theorem and Its Consequences by Le Roy F. Henderson and Ralph Menikoff. This led to comparison between maximum to ideal gas model to more general model.

pressible Flow was chosen, while relatively simple topics like fundamentals of strength of material were delayed, is because of the realization that manufacture engineering simply lacks fundamental knowledge in this area and thus produces faulty designs and understanding of major processes. Unfortunately, the undersigned observed that many researchers who are dealing with manufacturing processes are lack of understanding about fluid mechanics in general but particularly in relationship to compressible flow. In fact one of the reasons that many manufacturing jobs are moving to other countries is because of the lack of understanding of fluid mechanics in general and compressible in particular. For example, the lack of competitive advantage moves many of the die casting operations to off shore[9]. It is clear that an understanding of Compressible Flow is very important for areas that traditionally have ignored the knowledge of this topic[10].

As many instructors can recall from their time as undergraduates, there were classes during which most students had a period of confusion, and then later, when the dust settled, almost suddenly things became clear. This situation is typical also for Compressible Flow classes, especially for external compressible flow (e.g. flow around a wing, etc.). This book offers a more balanced emphasis which focuses more on internal compressible flow than the traditional classes. The internal flow topics seem to be common for the “traditional” students and students from other fields, e.g., manufacturing engineering.

This book is written in the spirit of my adviser and mentor E.R.G. Eckert. Who, aside from his research activity, wrote the book that brought a revolution in the heat transfer field of education. Up to Eckert’s book, the study of heat transfer was without any dimensional analysis. He wrote his book because he realized that the dimensional analysis utilized by him and his adviser (for the post doc), Ernst Schmidt, and their colleagues, must be taught in engineering classes. His book met strong criticism in which some called to burn his book. Today, however, there is no known place in world that does not teach according to Eckert’s doctrine. It is assumed that the same kind of individuals who criticized Eckert’s work will criticize this work. This criticism will not change the future or the success of the ideas in this work. As a wise person says “don’t tell me that it is wrong, show me what is wrong”; this is the only reply. With all the above, it must be emphasized that this book will not revolutionize the field even though considerable new materials that have never been published are included. Instead, it will provide a new emphasis and new angle to Gas Dynamics.

Compressible flow is essentially different from incompressible flow in mainly two respects: discontinuity (shock wave) and choked flow. The other issues, while important, are not that crucial to the understanding of the unique phenomena of compressible flow. These unique issues of compressible flow are to be emphasized and shown. Their applicability to real world processes is to be

[9]Please read the undersigned’s book “Fundamentals of Die Casting Design,” which demonstrates how ridiculous design and research can be.

[10]The fundamental misunderstanding of choking results in poor models (research) in the area of die casting, which in turn results in many bankrupt companies and the movement of the die casting industry to offshore.

demonstrated[11].

The book is organized into several chapters which, as a traditional textbook, deals with a basic introduction of thermodynamics concepts (under construction). The second chapter deals with speed of sound. The third chapter provides the first example of choked flow (isentropic flow in a variable area). The fourth chapter deals with a simple case of discontinuity (a simple shock wave in a nozzle). The next chapter is dealing with isothermal flow with and without external forces (the moving of the choking point), again under construction. The next three chapters are dealing with three models of choked flow: Isothermal flow[12], Fanno flow and Rayleigh flow. First, the Isothermal flow is introduced because of the relative ease of the analytical treatment. Isothermal flow provides useful tools for the pipe systems design. These chapters are presented almost independently. Every chapter can be "ripped" out and printed independently. The topics of filling and evacuating of gaseous chambers are presented, normally missed from traditional textbooks. There are two advanced topics which included here: oblique shock wave, and properties change effects (ideal gases and real gases) (under construction). In the oblique shock, for the first time analytical solution is presented, which is excellent tool to explain the strong, weak and unrealistic shocks. The chapter on one-dimensional unsteady state, is currently under construction.

The last chapter deals with the computer program, Gas Dynamics Calculator (CDC-POTTO). The program design and how to use the program are described (briefly).

Discussions on the flow around bodies (wing, etc), and Prandtl–Meyer expansion will be included only after the gamma version unless someone will provide discussion(s) (a skeleton) on these topics.

It is hoped that this book will serve the purposes that was envisioned for the book. It is further hoped that others will contribute to this book and find additional use for this book and enclosed software.

[11] If you have better and different examples or presentations you are welcome to submit them.
[12] It is suggested to referred to this model as Shapiro flow

How This Book Was Written

This book started because I needed an explanation for manufacturing engineers. Apparently many manufacturing engineers and even some researchers in manufacturing engineering were lack of understanding about fluid mechanics in particularly about compressible flow. Therefore, I wrote to myself some notes and I converted one of the note to a chapter in my first book, “Fundamentals Of Die Casting Design.” Later, I realized that people need down to earth book about compressible flow and this book was born.

The free/open content of the book was created because the realization that open content accelerated the creation of books and reaction to the corruption of the court implementing the copyright law by manufacturing facts and laws. It was farther extended by the allegation of free market and yet the academic education cost is sky rocketing without a real reason and real competition. There is no reason why a text book which cost leas than 10$ to publish/produce will cost about 150 dollars. If a community will pull together, the best books can be created. Anyone can be part of it. For example, even my 10 years old son, Eliezer made me change the chapter on isothermal flow. He made me realized that the common approach to supersonic branch of isothermal as non–existent is the wrong approach. It should be included because this section provides the explanation and direction on what Fanno flow model will approach if heat transfer is taken into account[13].

I realized that books in compressible flow are written in a form that is hard for non fluid mechanic engineer to understand. Therefore, this book is designed to be in such form that is easy to understand. I wrote notes and asked myself what materials should be included in such a book so when I provide consultation to a company, I do not need to explain the fundamentals. Therefore, there are some chapters in this book which are original materials never published before. The presentation of some of the chapters is different from other books. The book

[13]Still in untyped note form.

does not provide the old style graphical solution methods yet provide the graphical explanation of things.

Of course, this book was written on Linux (MicrosoftLess book). This book was written using the vim editor for editing (sorry never was able to be comfortable with emacs). The graphics were done by TGIF, the best graphic program that this author experienced so far. The old figures where done by grap (part the old Troff). Unfortunately, I did not have any access to grap and switched to Grace. Grace is a problematic program but is the best I have found. The spell checking was done by gaspell, a program that cannot be used on new system and I had to keep my old Linux to make it work[14]. I hope someone will write a new spell check so I can switch to a new system.

The figure in cover page was created by Michael Petschauer, graphic designer, and is open/free content copyright by him (happy_circle@yahoo.com).

[14]If you would like to to help me to write a new spell check user interface, please contact me.

About Gas Dynamics Calculator

Gas Dynamic Calculator, (Potto–GDC) was created to generate various tables for the book either at end the chapters or for the exercises. This calculator was given to several individuals and they found Potto–GDC to be very useful. So, I decided to include Potto–GDC to the book.

Initially, the Potto-GDC was many small programs for specific tasks. For example, the stagnation table was one such program. Later, the code became a new program to find the root of something between the values of the tables e.g. finding parameters for a given $\frac{4fL}{D}$. At that stage, the program changed to contain a primitive interface to provide parameters to carry out the proper calculations. Yet, then, every flow model was a different program.

When it become cumbersome to handle several programs, the author utilized the object oriented feature of C++ and assigned functions to the common tasks to a base class and the specific applications to the derived classes. Later, a need to intermediate stage of tube flow model (the PipeFlow class) was created and new classes were created.

The graphical interface was created only after the engine was written. The graphical interface was written to provide a filter for the unfamiliar user. It also remove the need to recompile the code everytime.

0.1 The new version

Version 4.1.7 had several bug fixes and add two angle calculations to the oblique shock. Change the logtable to tabular environment for short tables.

Preface

"In the beginning, the POTTO project was without form, and void; and emptiness was upon the face of the bits and files. And the Fingers of the Author moved upon the face of the keyboard. And the Author said, Let there be words, and there were words."[15].

This book, Fundamentals of Compressible Flow, describes the fundamentals of compressible flow phenomena for engineers and others. This book is designed to replace the book(s) or instructor's notes for the compressible flow in (mostly) undergraduate classes for engineering/science students. It is hoped that the book could be used as a reference book for people who have at least some knowledge of the basics of fundamental fluid mechanics, and basic science such as calculus, physics, etc. It is hoped that the computer program enclosed in the book will take on a life of its own and develop into an open content or source project.

The structure of this book is such that many of the chapters could be usable independently. For example, if you need information about, say, Fanno flow, you can read just chapter 9. I hope this makes the book easier to use as a reference manual. However, this manuscript is first and foremost a textbook, and secondly a reference manual only as a lucky coincidence.

I have tried to describe why the theories are the way they are, rather than just listing "seven easy steps" for each task. This means that a lot of information is presented which is not necessary for everyone. These explanations have been marked as such and can be skipped.[16] Reading everything will, naturally, increase your understanding of the fundamentals of compressible fluid flow.

This book is written and maintained on a volunteer basis. Like all volunteer work, there is a limit on how much effort I was able to put into the book and its organization. Moreover, due to the fact that English is my third language and time limitations, the explanations are not as good as if I had a few years to

[15] To the power and glory of the mighty God. This book is only to explain his power.

[16] At the present, the book is not well organized. You have to remember that this book is a work in progress.

perfect them. Nevertheless, I believe professionals working in many engineering fields will benefit from this information. This book contains many original models, and explanations never published before.

I have left some issues which have unsatisfactory explanations in the book, marked with a Mata mark. I hope to improve or to add to these areas in the near future. Furthermore, I hope that many others will participate of this project and will contribute to this book (even small contributions such as providing examples or editing mistakes are needed).

I have tried to make this text of the highest quality possible and am interested in your comments and ideas on how to make it better. Incorrect language, errors, ideas for new areas to cover, rewritten sections, more fundamental material, more mathematics (or less mathematics); I am interested in it all. If you want to be involved in the editing, graphic design, or proofreading, please drop me a line. You may contact me via Email at “barmeir@gmail.com”.

Naturally, this book contains material that never was published before. This material never went through a peer review. While peer review and publication in a professional publication is excellent idea in theory. In practice, this process leaves a large room to blockage of novel ideas and plagiarism. If you would like be “peer reviews” or critic to my new ideas please send me your idea(s). Even reaction/comments from individuals like David Marshall[17]

Several people have helped me with this book, directly or indirectly. I would like to especially thank to my adviser, Dr. E. R. G. Eckert, whose work was the inspiration for this book. I also would like to thank Amy Ross for her advice ideas, and assistance.

The symbol META was added to provide typographical conventions to blurb as needed. This is mostly for the author’s purposes and also for your amusement. There are also notes in the margin, but those are solely for the author’s purposes, ignore them please. They will be removed gradually as the version number advances.

I encourage anyone with a penchant for writing, editing, graphic ability, LaTeX knowledge, and material knowledge and a desire to provide open content textbooks and to improve them to join me in this project. If you have Internet e-mail access, you can contact me at “barmeir@gmail.com”.

[17]Dr. Marshall wrote to this author that the author should review other people work before he write any thing new (well, literature review is always good?). Over ten individuals wrote me about this letter. I am asking from everyone to assume that his reaction was innocent one. While his comment looks like unpleasant reaction, it brought or cause the expansion the oblique shock chapter. However, other email that imply that someone will take care of this author isn’t appreciated.

To Do List and Road Map

This book is not complete and probably never will be completed. There will always new problems to add or to polish the explanations or include more new materials. Also issues that associated with the book like the software has to be improved. It is hoped the changes in TEX and LATEX related to this book in future will be minimal and minor. It is hoped that the style file will be converged to the final form rapidly. Nevertheless, there are specific issues which are on the “table” and they are described herein.

At this stage, several chapters are missing. The effects of the deviations from the ideal gas model on the properties should be included. Further topics related to non-ideal gas such as steam and various freons are in the process of being added to this book especially in relationship to Fanno flow.

One of the virtue of this book lay in the fact that it contains a software that is extensible. For example, the Fanno module can be extended to include effects of real gases. This part will be incorporated in the future hopefully with the help of others.

Specific missing parts from every chapters are discussed below. These omissions, mistakes, approach problems are sometime appears in the book under the Meta simple like this

Meta

sample this part.

Meta End

Questions/problems appear as a marginal note. On occasions a footnote was used to point out for a need of improvement. You are always welcome to add a new material: problem, question, illustration or photo of experiment. Material can be further illuminate. Additional material can be provided to give a different angle on the issue at hand.

0.0.1 Speed of Sound

Discussion about the movement in medium with variation in speed of sound. This concept in relation of the wind tunnel and atmosphere with varied density and temperature.

More problems in relationship to two phase.
Speed of sound in wet steam.

0.0.2 Stagnation effects

extend the applicability with examples
C_p as a function of temperature (deviation of ideal gas model)
"real gas"' like water
History – on the teaching (for example when the concept of stagnation was first taught.

0.0.3 Nozzle

The effect of external forces (add problems).
History specifically, who and when the important of part De Level nozzle were developed.
Real gases effects (only temperature effects)
Flow with "tabulated gases" calculations
Phase change and two phase flow (multi choking points) effects (after 1.0 version).

The dimensional analysis of the flow when the flow can be considered as isothermal.
The combined effect of isentropic nozzle with heat transfer (especially with relationship to the program.).

0.0.4 Isothermal Flow

Classification of Problems
work on the software
Comparison of results with Fanno flow
Pipes Network calculation.

0.0.5 Fanno Flow

More examples: various categories
some improvement on the software (clean up)
real gas effects (compressible factor)

0.0.6 Rayleigh Flow

To mature the chapter: discussion on the "dark" corners of this model.
Provide discussion on variations of the effecting parameters.
Examples: provide categorization

0.0.7 Evacuation and filling semi rigid Chambers

To construct the Rayleigh flow in the tube (thermal chocking)
Examples classifications
Software (converting the FORTRAN program to c++)

0.0.8 Evacuating and filling chambers under external forces

Comparison with chemical reaction case
Examples
Software transformation from FORTRAN to c++. The FORTRAN version will not be included.

0.0.9 Oblique shock

Add application to design problems
To add the note on the relation ship between Prandtl–Meyer and the weak oblique shock. (almost finished)
Example on the above relationship

0.0.10 Prandtl–Meyer

The limitations (Prandtl-Meyer). Application
Marcell–Taylor (from the notes)
Examples

0.0.11 Transient problem

CHAPTER 1

Introduction

1.1 What is Compressible Flow ?

This book deals with an introduction[1] to the flow of compressible substances (gases). The main difference between compressible flow and almost incompressible flow is not the fact that compressibility has to be considered. Rather, the difference is in two phenomena that do not exist in incompressible flow[2]. The first phenomenon is the very sharp discontinuity (jump) in the flow in properties. The second phenomenon is the choking of the flow. Choking is when downstream variations don't effect the flow[3]. Though choking occurs in certain pipe flows in astronomy, there also are situations of choking in general (external) flow[4]. Choking is referred to as the situation where downstream conditions which are beyond a critical value do not affect the flow.

The shock wave and choking are not intuitive for most people. However, one has to realize that intuition is really a condition where one uses his past experiences to predict other situations. Here one has to learn to use his intuition as a tool for future use. Thus, not only aeronautic engineers, but other engineers, and even manufacturing engineers will be able use this "intuition" in design and even research.

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[1] This book gradually skipping to include more material that isn't so introductory. But attempt is made to present the material in introductory level.

[2] It can be argued that in open channel flow there is a hydraulic jump (discontinuity) and in some ranges no effect of downstream conditions on the flow. However, the uniqueness of the phenomena in the gas dynamics provides spectacular situations of a limited length (see Fanno model) and thermal choking, etc. Further, there is no equivalent to oblique shock wave. Thus, this richness is unique to gas dynamics.

[3] The thermal choking is somewhat different but similarity exists.

[4] This book is intended for engineers and therefore no discussion about astronomical conditions is presented.

1.2 Why Compressible Flow is Important?

Compressible flow appears in many natural and many technological processes. Compressible flow deals with more than air, including steam, natural gas, nitrogen and helium, etc. For instance, the flow of natural gas in a pipe system, a common method of heating in the u.s., should be considered a compressible flow. These processes include the flow of gas in the exhaust system of an internal combustion engine, and also gas turbine, a problem that led to the Fanno flow model. The above flows that were mentioned are called internal flows. Compressible flow also includes flow around bodies such as the wings of an airplane, and is considered an external flow.

These processes include situations not expected to have a compressible flow, such as manufacturing process such as the die casting, injection molding. The die casting process is a process in which liquid metal, mostly aluminum, is injected into a mold to obtain a near final shape. The air is displaced by the liquid metal in a very rapid manner, in a matter of milliseconds, therefore the compressibility has to be taken into account.

Clearly, Aero Engineers are not the only ones who have to deal with some aspect of compressible flow. For manufacturing engineers there are many situations where the compressibility or compressible flow understating is essential for adequate design. For instance, the control engineers who are using pneumatic systems use compressed substances. The cooling of some manufacturing systems and design of refrigeration systems also utilizes compressed air flow knowledge. Some aspects of these systems require consideration of the unique phenomena of compressible flow.

Traditionally, most gas dynamics (compressible flow) classes deal mostly with shock waves and external flow and briefly teach Fanno flows and Rayleigh flows (two kind of choking flows). There are very few courses that deal with isothermal flow. In fact, many books on compressible flow ignore the isothermal flow[5].

In this book, a greater emphasis is on the internal flow. This doesn't in any way meant that the important topics such as shock wave and oblique shock wave should be neglected. This book contains several chapters which deal with external flow as well.

1.3 Historical Background

In writing this book it became clear that there is more unknown and unwritten about the history of compressible fluid than known. While there are excellent books about the history of fluid mechanics (hydraulic) see for example book by Rouse[6]. There are numerous sources dealing with the history of flight and airplanes (aeronau-

[5]Any search on the web on classes of compressible flow will show this fact and the undersigned can testify that this was true in his first class as a student of compressible flow.

[6]Hunter Rouse and Simon Inc, History of Hydraulics (Iowa City: Institute of Hydraulic Research, 1957)

tic)[7]. Aeronautics is an overlapping part of compressible flow, however these two fields are different. For example, the Fanno flow and isothermal flow, which are the core of gas dynamics, are not part of aerodynamics. Possible reasons for the lack of written documentation are one, a large part of this knowledge is relatively new, and two, for many early contributors this topic was a side issue. In fact, only one contributor of the three main models of internal compressible flow (Isothermal, Fanno, Rayleigh) was described by any text book. This was Lord Rayleigh, for whom the Rayleigh flow was named. The other two models were, to the undersigned, unknown. Furthermore, this author did not find any reference to isothermal flow model earlier to Shapiro's book. There is no book[8] that describes the history of these models. For instance, the question, who was Fanno, and when did he live, could not be answered by any of the undersigned's colleagues in University of Minnesota or elsewhere.

At this stage there are more questions about the history of compressible flow needing to be answered. Sometimes, these questions will appear in a section with a title but without text or with only a little text. Sometimes, they will appear in a footnote like this[9]. For example, it obvious that Shapiro published the erroneous conclusion that all the chocking occurred at $M = 1$ in his article which contradicts his isothermal model. Additional example, who was the first to "conclude" the "all" the chocking occurs at $M = 1$? Is it Shapiro?

Orientally, there was no idea there are special effects and phenomena of compressible flow. Some researchers even have suggested that compressibility can be "swallowed" into the ideal flow (Euler's equation's flow is sometimes referred to as ideal flow). Even before Prandtl's idea of boundary layer appeared, the significant and importance of compressibility emerged.

In the first half of nineteen century there was little realization that the compressibility is important because there were very little applications (if any) that required the understanding of this phenomenon. As there were no motivations to investigate the shock wave or choked flow both were treated as the same, taking compressible flow as if it were incompressible flow.

It must be noted that researchers were interested in the speed of sound even long before applications and knowledge could demand any utilization. The research and interest in the speed of sound was a purely academic interest. The early application in which compressibility has a major effect was with fire arms. The technological improvements in fire arms led to a gun capable of shooting bullets at speeds approaching to the speed of sound. Thus, researchers were aware that the speed of sound is some kind of limit.

In the second half of the nineteen century, Mach and Flinger "stumbled" over the shock wave and choking, respectively. Mach observed shock and Flinger

[7] Anderson, J. D., Jr. 1997. A History of Aerodynamics: And Its Impact on Flying Machines, Cambridge University Press, Cambridge, England.

[8] The only remark found about Fanno flow that it was taken from the Fanno Master thesis by his adviser. Here is a challenge: find any book describing the history of the Fanno model.

[9] Who developed the isothermal model? The research so far leads to Shapiro. Perhaps this flow should be named after the Shapiro. Is there any earlier reference to this model?

measured the choking but theoretical science did not provide explanation to it (or was award that there is explanation for it.).

In the twentieth century the flight industry became the pushing force. Understandably, aerospace engineering played a significant role in the development of this knowledge. Giants like Prandtl and his students like Van Karman, as well as others like Shapiro, dominated the field. During that time, the modern basic classes became "solidified." Contributions by researchers and educators from other fields were not as dominant and significant, so almost all text books in this field are written from an aerodynamic prospective.

To add history from the work. Topics that should be included in this history review but that are not yet added to this section are as follows: Multi Phase flow, capillary flow and phase change.

1.3.1 Early Developments

The compressible flow is a subset of fluid mechanics/hydraulics and therefore the knowledge development followed the understanding of incompressible flow. Early contributors were motivated from a purely intellectual curiosity, while most later contributions were driven by necessity. As a result, for a long time the question of the speed of sound was bounced around.

1.3.1.1 Speed of Sound

The idea that there is a speed of sound and that it can be measured is a major achievement. A possible explanation to this discovery lies in the fact that mother nature exhibits in every thunder storm the difference between the speed of light and the speed of sound. There is no clear evidence as to who came up with this concept, but some attribute it to Galileo Galilei: 166x. Galileo, an Italian scientist, was one of the earliest contributors to our understanding of sound. Dealing with the difference between the two speeds (light, sound) was a major part of Galileo's work. However, once there was a realization that sound can be measured, people found that sound travels in different speeds through different mediums. The early approach to the speed of sound was by the measuring of the speed of sound.

Other milestones in the speed of sound understanding development were by Leonardo Da Vinci, who discovered that sound travels in waves (1500). Marin Mersenne was the first to measure the speed of sound in air (1640). Robert Boyle discovered that sound waves must travel in a medium (1660) and this lead to the concept that sound is a pressure change. Newton was the first to formulate a relationship between the speed of sound in gases by relating the density and compressibility in a medium (by assuming isothermal process). Newton's equation is missing the heat ratio, k (late 1660's). Maxwell was the first to derive the speed of sound for gas as $c = \sqrt{kRT}$ from particles (statistical) mechanics. Therefore some referred to coefficient $\sqrt{k}$ as Maxwell's coefficient.

1.3.2 The shock wave puzzle

Here is where the politics of science was a major obstacle to achieving an advancement[10]. In the early 18xx, conservation of energy was a concept that was applied only to mechanical energy. On the other side, a different group of scientists dealt with calorimetry (internal energy). It was easier to publish articles about the second law of thermodynamics than to convince anyone of the first law of thermodynamics. Neither of these groups would agree to "merge" or "relinquish" control of their "territory" to the other. It took about a century to establish the first law[11].

At first, Poisson found a "solution" to the Euler's equations with certain boundary conditions which required discontinuity[12] which had obtained an implicit form in 1808. Poisson showed that solutions could approach a discontinuity by using conservation of mass and momentum. He had then correctly derived the jump conditions that discontinuous solutions must satisfy. Later, Challis had noticed contradictions concerning some solutions of the equations of compressible gas dynamics[13]. Again the "jumping" conditions were redeveloped by two different researchers independently: Stokes and Riemann. Riemann, in his 1860 thesis, was not sure whether or not discontinuity is only a mathematical creature or a real creature. Stokes in 1848 retreated from his work and wrote an apology on his "mistake." [14] Stokes was convinced by Lord Rayleigh and Lord Kelvin that he was mistaken on the grounds that energy is conserved (not realizing the concept of internal energy).

At this stage some experimental evidence was needed. Ernst Mach studied several fields in physics and also studied philosophy. He was mostly interested in experimental physics. The major breakthrough in the understanding of compressible flow came when Ernest Mach "stumbled" over the discontinuity. It is widely believed that Mach had done his research as purely intellectual research. His research centered on optic aspects which lead him to study acoustic and therefor supersonic flow (high speed, since no Mach number was known at that time). However, it is logical to believe that his interest had risen due to the

[10]Amazingly, science is full of many stories of conflcts and disputes. Aside from the conflcts of scientists with the Catholic Church and Muslim religion, perhaps the most famous is that of Newton's netscaping (stealing and embracing) Leibniz['s] invention of calculus. There are even conflcts from not giving enough credit, like Moody not giving the due credit to Rouse. Even the undersigned encountered individuals who have tried to ride on his work. The other kind of problem is "hijacking" by a sector. Even on this subject, the Aeronautic sector "took over" gas dynamics as did the emphasis on mathematics like perturbations methods or asymptotic expansions instead on the physical phenomena. Major material like Fanno fbw isn't taught in many classes, while many of the mathematical techniques are currently practiced. So, these problems are more common than one might be expected.

[11]This recognition of the fi rst law is today the most "obvious" for engineering students. Yet for many it was still debatable up to the middle of the nineteen century.

[12]Siméon Denis Poisson, French mathematician, 1781-1840 worked in Paris, France. "M'emoire sur la th'eorie du son," J. Ec. Polytech. 14 (1808), 319-392. From Classic Papers in Shock Compression Science, 3-65, High-press. Shock Compression Condens. Matter, Springer, New York, 1998.

[13]James Challis, English Astronomer, 1803-1882. worked at Cambridge, England UK. "On the velocity of sound," Philos. Mag. XXXII (1848), 494-499

[14]Stokes George Gabriel Sir, Mathematical and Physical Papers, Reprinted from the original journals and transactions, with additional notes by the author. Cambridge, University Press, 1880-1905.

need to achieve powerful/long–distance shooting rifles/guns. At that time many inventions dealt with machine guns which where able to shoot more bullets per minute. At the time, one anecdotal story suggests a way to make money by inventing a better killing machine for the Europeans. While the machine gun turned out to be a good killing machine, defense techniques started to appear such as sand backs. A need for bullets that could travel faster to overcome these obstacles was created. Therefore, Mach's paper from 1876 deals with the flow around bullets. Nevertheless, no known[15] equations or explanations results in of this experiments.

Mach used his knowledge in Optics to study the flow around bullets. What makes Mach's achievement all the more remarkable was the technique he used to take the historic photograph: He employed an innovative approach called the shadowgraph. He was the first to photograph the shock wave. In his paper discussing "Photographische Fixierung der durch Projektile in der Luft eingeleiten Vorgange" he showed a picture of a shock wave (see figure 1.7). He utilized the variations of the air density to clearly show shock line at the front of the bullet. Mach had good understanding of the fundamentals of supper sonic flow and the effects on bullet movement (super sonic flow). Mach's paper from 1876 demonstrated shock wave (discontinuity) and suggested the importance of the ratio of the velocity to the speed of sound. He also observed the existence of a conical shock wave (oblique shock wave).

Mach's contributions can be summarized as providing an experimental proof to discontinuity. He further showed that the discontinuity occurs at $M = 1$ and realized that the velocity ratio (Mach number), and not the velocity, is the important parameter in the study of the compressible flow. Thus, he brought confidence to the theoreticians to publish their studies. While Mach proved shock wave and oblique shock wave existence, he was not able to analyze it (neither was he aware of Poission's work or the works of others.).

Back to the pencil and paper, the jump conditions were redeveloped and now named after Rankine[16] and Hugoniot[17]. Rankine and Hugoniot, redeveloped independently the equation that governs the relationship of the shock wave. Shock was assumed to be one dimensional and mass, momentum, and energy equations[18] lead to a solution which ties the upstream and downstream properties. What they could not prove or find was that shock occurs only when upstream is supersonic, i.e., direction of the flow. Later, others expanded Rankine-Hugoniot's

[15] The words "no known" refer to the undersigned. It is possible that some insight was developed but none of the documents that were reviewed revealed it to the undersigned.

[16] William John Macquorn Rankine, Scottish engineer, 1820-1872. He worked in Glasgow, Scotland UK. "On the thermodynamic theory of waves of finite longitudinal disturbance," Philos. Trans. 160 (1870), part II, 277-288. Classic papers in shock compression science, 133-147, High-press. Shock Compression Condens. Matter, Springer, New York, 1998

[17] Pierre Henri Hugoniot, French engineer, 1851-1887. "Sur la propagation du mouvement dans les corps et sp'ecialement dans les gaz parfaits, I, II" J. Ec. Polytech. 57 (1887), 3-97, 58 (1889), 1-125. Classic papers in shock compression science, 161-243, 245-358, High-press. Shock Compression Condens. Matter, Springer, New York, 1998

[18] Today it is well established that shock has three dimensions but small sections can be treated as one dimensional.

conditions to a more general form[19].

Here, the second law has been around for over 40 years and yet the significance of it was not was well established. Thus, it took over 50 years for Prandtl to arrive at and to demonstrate that the shock has only one direction[20]. Today this equation/condition is known as Prandtl's equation or condition (1908). In fact Prandtl is the one who introduced the name of Rankine-Hugoniot's conditions not aware of the earlier developments of this condition. Theodor Meyer (Prandtl's student) derived the conditions for oblique shock in 1908[21] as a byproduct of the expansion work.

It was probably later that Stodola (Fanno's adviser) realized that the shock is the intersection of the Fanno line with the Rayleigh line. Yet, the supersonic branch is missing from his understanding (see Figure 1.1). In fact, Stodola suggested the graphical solution utilizing the Fanno line.

The fact that the conditions and direction were known did not bring the solution to the equations. The "last nail" of understanding was put by Landau, a Jewish scientist who worked in Moscow University in the 1960's during the Communist regimes. A solution was found by Landau & Lifshitz and expanded by Kolosnitsyn & Stanyukovich (1984).

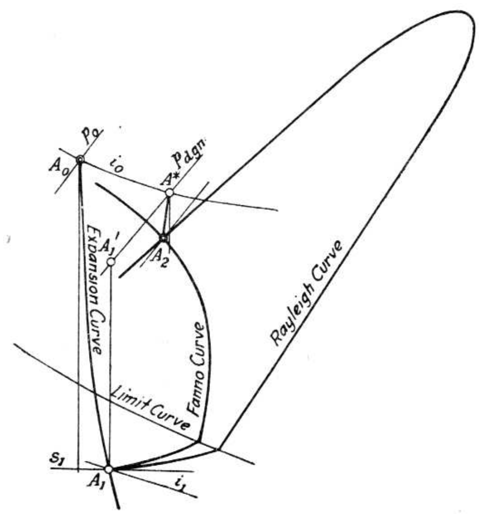

FIG. 51.—"Condition loci" for compression shock.

Fig. 1.1: The shock as connection of Fanno and Rayleigh lines after Stodola, Steam and Gas Turbine

to be add to oblique shock chapter.

Since early of the 1950 the relationship between the the oblique shock deflection angle and shock angle and Mach number was described as not possible to obtained. There were up recently (version 0.3 of this book) several equations that tied various properties/quantities. The first analytical solution connecting the angles with upstream Mach number was published in this book version 0.3. The probable reason that analytical solution was not published because the claim in the

[19]To add discussion about the general relationships.
[20]Some view the work of G. I. Taylor from England as the proof (of course utilizing the second law)
[21]Theodor Meyer in Mitteil. üb. Forsch-Arb. Berlin, 1908, No. 62, page 62.

famous report of NACA 1135 that there is not explicit analytical solution[22].

The question whether the oblique shock is stable or which root is stable was daunting since the early discovery that there are more than one possible solution. It is amazing that early research concluded that only the weak solution is possible or stable as oppose the reality. The first that attempt this question where in 1931 by Epstein[23]. His analysis was based on Hamilton's principle when he ignore the boundary condition. The results of that analysis was that strong shock is unstable. The researchers understood that flow after a strong shock governed by elliptic equation while the flow after weak shock governed by hyperbolic equations. This difference probably results in not recognizing that The boundary conditions play important role in the stability of the shock[24]. In fact analysis based on Hamilton's principle isn't suitable for stability because entropy creation was recognized 1955 by Herivel[25].

Carrier[26] was first to recognize that strong and weak shock stable. If fact the confusion on this issue was persistent until now. Even all books that published recently claimed that no strong shock ever was observed in flow around cone (Taylor–Maccoll flow). In fact, even this author sinned in this erroneous conclusion. The real question isn't if they exist rather under what conditions these shocks exist which was suggested by Courant and Friedrichs in their book "Supersonic Flow and Shock Waves," published by Interscience Publishers, Inc. New York, 1948, p. 317.

The effect of real gases was investigated very early since steam was used move turbines. In general the mathematical treatment was left to numerical investigation and there is relatively very little known on the difference between ideal gas model and real gas. For example, recently, Henderson and Menikoff[27] dealt with only the procedure to find the maximum of oblique shock, but no comparison between real gases and ideal gas is offered there.

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[22]Since writing this book, several individuals point out that a solution was found in book "Analytical Fluid Dynamics" by Emanuel, George, second edition, December 2000 (US$ 124.90). That solution is based on a transformation of $\sin\theta$ to $\tan\beta$. It is interesting that transformation result in one of root being negative. While the actual solution all the roots are real and positive for the attached shock. The presentation was missing the condition for the detachment or point where the model collapse. But more surprisingly, similar analysis was published by Briggs, J. "Comment on Calculation of Oblique shock waves," AIAA Journal Vol 2, No 5 p. 974, 1963. Hence, Emanuel just redone 36 years work (how many times works have to be redone in this fi eld). In a way part of analysis of this book redoing old work. Yet, what is new in this work is completeness of all the three roots and the analytical condition for detached shock and breaking of the model.

[23]Epstein, P. S., "On the air resistance of Projectiles," Proceedings of the National Academy of Science, Vol. 17, 1931, pp. 532-547.

[24]In study this issue this author realized only after examining a colleague experimental Picture 13.4 that it was clear that the Normal shock along with strong shock and weak shock "live" together peacefully and in stable conditions.

[25]Herivel, J. F., "The Derivation of The Equations of Motion On an Ideal Fluid by Hamilton's Principle,," Proceedings of the Cambridge philosophical society, Vol. 51, Pt. 2, 1955, pp. 344-349.

[26]Carrier, G.F., "On the Stability of the supersonic Flows Past as a Wedge," Quarterly of Applied Mathematics, Vol. 6, 1949, pp. 367–378.

[27]Henderson and Menikoff, "Triple Shock Entropy Theorem," Journal of Fluid Mechanics 366 (1998) pp. 179–210.

1.3.3 Choking Flow

The choking problem is almost unique to gas dynamics has many different forms. The choking wasn't clear to be observed, even researcher stumbled over it. No one was looking or expecting for the choking to occur, when it was found the significance of the choking phenomenon was not clear. The first experimental choking phenomenon was discovered by Fliegner's experiments which were conducted some time in the middle of 186x[28] on air flow through converging nozzle. As result the invention of deLavel's nozzle was invented by Carl Gustaf Patrik de Laval in 1882 and first successful operation by another inventor (Curtis) 1896 used in steam turbine. Yet, there was no realization that the flow is choked just that the flow moves faster than speed of sound.

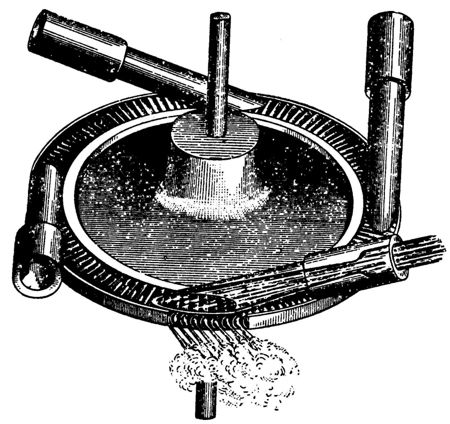


Fig. 1.2: The schematic of deLavel's turbine after Stodola, Steam and Gas Turbine

The introduction of the steam engine and other thermodynamics cycles led to the choking problem. The problem was introduced because people wanted to increase the output of the Engine by increasing the flames (larger heat transfer or larger energy) which failed, leading to the study and development of Rayleigh flow. According the thermodynamics theory (various cycles) the larger heat supply for a given temperature difference (larger higher temperature) the larger the output, but after a certain point it did matter (because the steam was choked). The first to discover (try to explain) the choking phenomenon was Rayleigh[29].

After the introduction of the deLavel's converging–diverging nozzle the theoretical work started by Zeuner[30]. Later continue by Prandtl's group[31] starting 1904. In 1908 Meyer has extend this work to make two dimensional calculations[32]. Experimental work by Parenty[33] and other measure the pressure along the converging-diverging nozzle.

[28]Fliegner Schweizer Bauztg., Vol 31 1898, p. 68–72. The theorical first work on this issue was done by Zeuner, "Theorie die Turbinen," Leipzig 1899, page 268 f.

[29]Rayleigh was the first to develop the model that bears his name. It is likely that others had noticed that flow is choked, but did not produce any model or conduct successful experimental work.

[30]Zeuner, "Theorie der Turbinen, Leipzig 1899 page 268 f.

[31]Some of the publications were not named after Prandtl but rather by his students like Meyer, Theodor. In the literature appeared reference to article by Lorenz in the Physik Zeitshr., as if in 1904. Perhaps, there are also other works that this author did not come crossed.

[32]Meyer, Th., Über zweidimensionals Bewegungsvordange eines Gases, Dissertation 1907, erschienen in den Mitteilungen über Forsch.-Arb. Ing.-Wes. heft 62, Berlin 1908.

[33]Parenty, Comptes R. Paris, Vol. 113, 116, 119; Ann. Chim. Phys. Vol. 8. 8 1896, Vol 12, 1897.

It was commonly believed[34] that the choking occurs only at $M = 1$. The first one to analyzed that choking occurs at $1/\sqrt{k}$ for isothermal flow was Shapiro (195x). It is so strange the giant like Shapiro did not realize his model on isothermal contradict his conclusion from his own famous paper. Later Romer at el extended it to isothermal variable area flow (1955). In this book, this author adapts E.R.G. Ecert's idea of dimensionless parameters control which determines where the reality lay between the two extremes. Recently this concept was proposed (not explicitly) by Dutton and Converdill (1997)[35]. Namely, in many cases the reality is somewhere between the adiabatic and the isothermal flow. The actual results will be determined by the modified Eckert number to which model they are closer.

1.3.3.1 Nozzle flow

The first "wind tunnel" was not a tunnel but a rotating arm attached at the center. At the end of the arm was the object that was under observation and study. The arm's circular motion could reach a velocity above the speed of sound at its end. Yet, in 1904 the Wright brothers demonstrated that results from the wind tunnel and spinning arm are different, due to the circular motion. As a result, the spinning arm was no longer used in testing. Between the turn of the century and 1947-48, when the first supersonic wind tunnel was built, models that explained choking at the throat have been built.

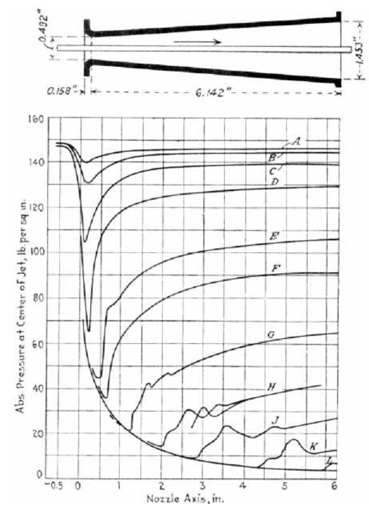


Fig. 1.3: The measured pressure in a nozzle taken from Stodola 1927 Steam and Gas Turbines

A different reason to study the converging-diverging nozzle was the Venturi meter which was used in measuring the flow rate of gases. Bendemann [36] carried experiments to study the accuracy of these flow meters and he measured and refound that the flow reaches a critical value (pressure ratio of 0.545) that creates the maximum flow rate.

There are two main models or extremes that describe the flow in the nozzle: isothermal and adiabatic.

[34] The personal experience of this undersigned shows that even instructors of Gas Dynamics are not aware that the chocking occurs at different Mach number and depends on the model.

[35] These researchers demonstrate results between two extremes and actually proposed this idea. However, that the presentation here suggests that topic should be presented case between two extremes.

[36] Bendemann Mitteil über Forschungsarbeiten, Berlin, 1907, No. 37.

1.3.3.2 Nozzle flow

Romer et al[37] analyzed the isothermal flow in a nozzle. It is remarkable that choking was found as $1/\sqrt{k}$ as opposed to one (1). In general when the model assumed to be isothermal the choking occurs at $1/\sqrt{k}$.

The concept that the choking point can move from the throat was was introduced by[38] the unknown name.

It is very interesting that the isothermal nozzle was proposed by Romer at el 1955 (who was behind the adviser or the student?). These researchers where the first one to realized that choking can occurs in different Mach number ($1/\sqrt{k}$ aside to the isothermal pipe.

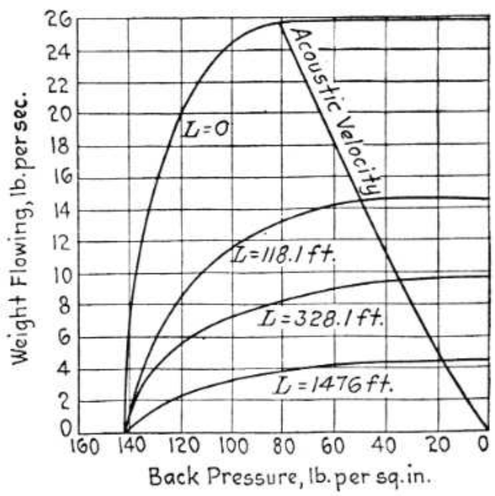


Fig. 1.4: Flow rate as a function of the back pressure taken from Stodola 1927 Steam and Gas Turbines

to insert the isothermal nozzle with external forces like gravity and to show that choking location can move depending on the direction of the force.

1.3.3.3 Rayleigh Flow

Rayleigh was probably[39], the first to suggest a model for frictionless flow with a constant heat transfer. Rayleigh's work was during the time when it was debatable as to whether there are two forms of energies (mechanical, thermal), even though Watt and others found and proved that they are the same. Therefore, Rayleigh looked at flow without mechanical energy transfer (friction) but only thermal energy transfer. In Rayleigh flow, the material reaches choking point due to heat transfer, hence term "thermally choked" is used; no additional flow can occur.

To find where Rayleigh did understand that his model leads to $1/\sqrt{k}$ point flow and graphical representation of the flow. The $1/\sqrt{k}$ question.

to insert information about the detonation wave and relationship to Rayleigh line.

1.3.3.4 Fanno Flow

The most important model in compressible flow was suggested by Gino Fanno in his Master's thesis (1904). The model bears his name. Yet, according to Dr. Rudolf

[37] Romer, I Carl Jr., and Ali Bulent Cambel, "Analysis of Isothermal Variable Area Flow," Aircraft Eng. vol. 27 no 322, p. 398 December 1955.

[38] This undersign didn't find the actual trace to the source of proposing this effect. However, some astronomy books showing this effect in a dimensional form without mentioning the original researcher. In dimensionless form, this phenomenon produces a dimensionless number similar to Ozer number and therefor the name Ozer number adapted in this book.

[39] As most of the history research has shown, there is also a possibility that someone found it earlier. For example, Piosson was the first one to realize the shock wave possibility.

Mumenthaler from UTH University, no copy of the thesis can be found in the original University and perhaps only in the personal custody of the Fanno family[40]. Fanno attributes the main pressure reduction to friction. Thus, flow that is dominantly adiabatic could be simplified and analyzed. The friction factor is the main component in the analysis as Darcy f[41] was already proposed in 1845. The arrival of the Moody diagram, which built on Hunter Rouse's (194x) work made Darcy–Weisbach's equation universally useful. Without the existence of the friction factor data, the Fanno model wasn't able to produce a prediction useful for the industry. Additionally an understating of the supersonic branch of the flow was unknown (The idea of shock in tube was not raised at that time.). Shapiro organized all the material in a coherent way and made this model useful.

Meta

Did Fanno realize that the flow is choked? It appears at least in Stodola's book that choking was understood in 1927 and even earlier. The choking was assumed only to be in the subsonic flow. But because the actual Fanno's thesis is not available, the question cannot be answered yet. When was Gas Dynamics (compressible flow) as a separate class started? Did the explanation for the combination of diverging-converging nuzzle with tube for Fanno flow first appeared in Shapiro's book?

Meta End

expanding model by others

1.3.3.5 Isothermal Flow

If it turned out that no one had done it before Shapiro, this flow model should be called Shapiro's flow. Call for others to help in this information.

The earliest reference to isothermal flow was found in Shapiro's Book. The model suggests that the choking occurs at $1/\sqrt{k}$ and it appears that Shapiro was the first one to realize this difference compared to the other models. In reality, the flow is choked somewhere between $1/\sqrt{k}$ to one for cases that are between Fanno (adiabatic) and isothermal flow. This fact was evident in industrial applications where the expectation of the choking is at Mach one, but can be explained by choking at a lower Mach number. No experimental evidence, known by the undersigned, was ever produced to verify this finding.

1.3.4 External flow

When the flow over an external body is about .8 Mach or more the flow must be considered to be compressible flow. However at a Mach number above 0.8 (relative of velocity of the body to upstream velocity) a local Mach number (local velocity) can reach $M = 1$. At that stage, a shock wave occurs which increases the resistance. The Navier-Stokes equations which describe the flow (or even

[40]This material is very important and someone should find it and make it available to researchers.

[41]Fanning f based radius is only one quarter of the Darcy f which is based on diameter

Euler equations) were considered unsolvable during the mid 18xx because of the high complexity. This problem led to two consequences. Theoreticians tried to simplify the equations and arrive at approximate solutions representing specific cases. Examples of such work are Hermann von Helmholtz's concept of vortex filaments (1858), Lanchester's concept of circulatory flow (1894), and the Kutta-Joukowski circulation theory of lift (1906). Practitioners like the Wright brothers relied upon experimentation to figure out what theory could not yet tell them.

Ludwig Prandtl in 1904 explained the two most important causes of drag by introducing the boundary layer theory. Prandtl's boundary layer theory allowed various simplifications of the Navier-Stokes equations. Prandtl worked on calculating the effect of induced drag on lift. He introduced the *lifting line theory*, which was published in 1918-1919 and enabled accurate calculations of induced drag and its effect on lift[42].

During World War I, Prandtl created his thin–airfoil theory that enabled the calculation of lift for thin, cambered airfoils. He later contributed to the Prandtl-Glauert rule for subsonic airflow that describes the compressibility effects of air at high speeds. Prandtl's student, Von Karman reduced the equations for supersonic flow into a single equation.

After First World War aviation became important and in 1920s a push of research focused on what was called the *compressibility problem*. Airplanes could not yet fly fast, but the propellers (which are also airfoils) did exceed the speed of sound, especially at the propeller tips, thus exhibiting inefficiency. Frank Caldwell and Elisha Fales demonstrated in 1918 that at a critical speed (later renamed the *critical Mach number*) airfoils suffered dramatic increases in drag and decreases in lift. Later, Briggs and Dryden showed that the problem was related to the shock wave. Meanwhile in Germany, one of Prandtl's assistants, J. Ackeret, simplified the shock equations so that they became easy to use. After World War Two, the research had continued and some technical solutions were found. Some of the solutions lead to tedious calculations which lead to the creation of Computational Fluid Dynamics (CFD). Today these methods of perturbations and asymptotic are hardly used in wing calculations[43]. That is the "dinosaur[44]" reason that even today some instructors are teaching mostly the perturbations and asymptotic methods in Gas Dynamics classes.

More information on external flow can be found in , John D. Anderson's Book "History of Aerodynamics and Its Impact on Flying Machines," Cambridge University Press, 1997

[42]The English call this theory the Lanchester-Prandtl theory. This is because the English Astronomer Frederick Lanchester published the foundation for Prandtl's theory in his 1907 book *Aerodynamics*. However, Prandtl claimed that he was not aware of Lanchester's model when he had begun his work in 1911. This claim seems reasonable in the light that Prandtl was not ware of earlier works when he named erroneously the conditions for the shock wave. See for the full story in the shock section.

[43]This undersigned is aware of only one case that these methods were really used to calculations of wing.

[44]It is like teaching using slide ruler in today school. By the way, slide rule is sold for about 7.5$ on the net. Yet, there is no reason to teach it in a regular school.

1.3.4.1 Filling and Evacuating Gaseous Chambers

It is remarkable that there were so few contributions made in the area of a filling or evacuation gaseous chamber. The earlier work dealing with this issue was by Giffen, 1940, and was republished by Owczarek, J. A., the model and solution to the nozzle attached to chamber issue in his book "Fundamentals of Gas Dynamics."[45]. He also extend the model to include the unchocked case. Later several researchers mostly form University in Illinois extend this work to isothermal nozzle (chock and unchecked).

The simplest model of nozzle, is not sufficient in many cases and a connection by a tube (rather just nozzle or orifice) is more appropriated. Since World War II considerable works have been carried out in this area but with very little progress[46]. In 1993 the first reasonable models for forced volume were published by the undersigned. Later, that model was extended by several research groups, The analytical solution for forced volume and the "*balloon*" problem (airbag's problem) model were published first in this book (version 0.35) in 2005. The classification of filling or evacuating the chamber as external control and internal control (mostly by pressure) was described in version 0.3 of this book.

1.3.5 Biographies of Major Figures

In this section a short summary of major figures that influenced the field of gas dynamics is present. There are many figures that should be included and a biased selection was required. Much information can be obtained from other resources, such as the Internet. In this section there is no originality and none should be expected.

Fig. 1.5: Portrait of Galileo Galilei

1.3.5.1 Galileo Galilei

Galileo was born in Pisa, Italy on February 15, 1564 to musician Vincenzo Galilei and Giulia degli Ammannati. The oldest of six children, Galileo moved with his family in early 1570 to Florence. Galileo started his studying at the University of Pisa in 1581. He then became a

[45]International Textbook Co., Scranton, Pennsylvania, 1964.

[46]In fact, the emergence of the CFD gave the illusion that there are solutions at hand, not realizing that garbage in is garbage out, i.e., the model has to be based on scientifi c principles and not detached from reality. As anecdotal story explaining the lack of progress, in die casting conference there was a discussion and presentation on which turbulence model is suitable for a *complete* still liquid. Other "strange" models can be found in the undersigned's book "Fundamentals of Die Casting Design.

professor of mathematics at the University of Padua in 1592. During the time after his study, he made numerous discoveries such as that of the pendulum clock, (1602). Galileo also proved that objects fell with the same velocity regardless of their size.

Galileo had a relationship with Marina Gamba (they never married) who lived and worked in his house in Padua, where she bore him three children. However, this relationship did not last and Marina married Giovanni Bartoluzzi and Galileo's son, Vincenzio, joined him in Florence (1613).

Galileo invented many mechanical devices such as the pump and the telescope (1609). His telescopes helped him make many astronomic observations which proved the Copernican system. Galileo's observations got him into trouble with the Catholic Church, however, because of his noble ancestry, the church was not harsh with him. Galileo was convicted after publishing his book *Dialogue*, and he was put under house arrest for the remainder of his life. Galileo died in 1642 in his home outside of Florence.

1.3.5.2 Ernest Mach (1838-1916)

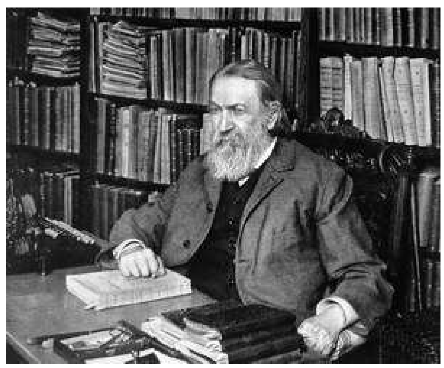


Fig. 1.6: Photo of Ernest Mach

Ernst Mach was born in 1838 in Chrlice (now part of Brno), when Czechia was still a part of the Austro–Hungary empire. Johann, Mach's father, was a high school teacher who taught Ernst at home until he was 14, when he studied in the gymnasium before he entered the university of Vienna. He graduated from Vienna in 1860. There Mach wrote his thesis "On Electrical Discharge and Induction." At first he received a professorship position at Graz in mathematics (1864) and was then offered a position as a professor of surgery at the university of Salzburg, but he declined. He then turned to physics, and in 1867 he received a position in the Technical University in Prague[47] where he taught experimental physics for the next 28 years.

Mach was also a great thinker/philosopher and influenced the theory of relativity dealing with frame of reference. In 1863, Ernest Mach (1836 - 1916) published Die Machanik in which he formalized this argument. Later, Einstein was greatly influenced by it, and in 1918, he named it *Mach's Principle*. This was one of the primary sources of inspiration for Einstein's theory of General Relativity.

[47] It is interesting to point out that Prague provided us two of the top influential researchers[:] E. Mach and E.R.G. Eckert.

Mach's revolutionary experiment demonstrated the existence of the shock wave as shown in Figure 1.7. It is amazing that Mach was able to photograph the phenomenon using the spinning arm technique (no wind tunnel was available at that time and most definitely nothing that could take a photo at supersonic speeds. His experiments required exact timing. He was not able to attach the camera to the arm and utilize the remote control (not existent at that time). Mach's shadowgraph technique and a related method called *Schlieren Photography* are still used today.

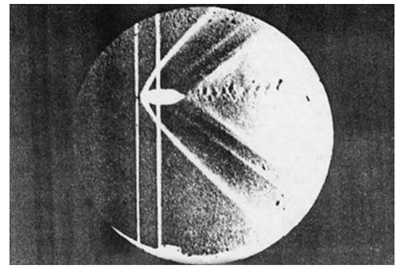


Fig. 1.7: The Photo of the bullet in a supersonic flow that Mach made. Note it was not taken in a wind tunnel

Yet, Mach's contributions to supersonic flow were not limited to experimental methods alone. Mach understood the basic characteristics of external supersonic flow where the most important variable affecting the flow is the ratio of the speed of the flow[48] (U) relative to the speed of sound (c). Mach was the first to note the transition that occurs when the ratio U/c goes from being less than 1 to greater than 1. The name Mach Number (M) was coined by J. Ackeret (Prandtl's student) in 1932 in honor of Mach.

1.3.5.3 John William Strutt (Lord Rayleigh)

A researcher with a wide interest, started studies in compressible flow mostly from from a mathematical approach. At that time there wasn't the realization that the flow could be choked. It seems that Rayleigh was the first who realized that flow with chemical reactions (heat transfer) can be choked.

Lord Rayleigh was a British physicist born near Maldon, Essex, on November 12, 1842. In 1861 he entered Trinity College at Cambridge, where he commenced reading mathematics. His exceptional abilities soon enabled him to overtake his colleagues. He graduated in the Mathematical Tripos in 1865 as Senior Wrangler and Smith's Prizeman. In 1866 he obtained a fellowship at Trinity which he held un-

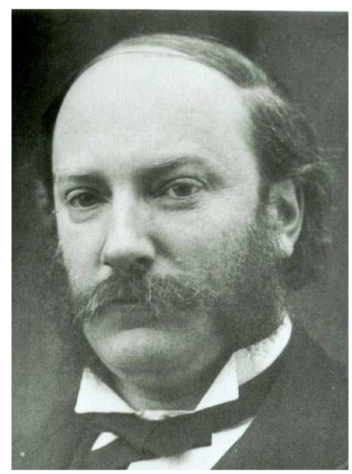


Fig. 1.8: Photo of Lord Rayleigh

[48]Mach dealt with only air, but it is reasonable to assume that he understood that this ratio was applied to other gases.

til 1871, the year of his marriage. He served for six years as the president of the government committee on explosives, and from 1896 to 1919 he acted as Scientific Adviser to Trinity House. He was Lord Lieutenant of Essex from 1892 to 1901.

Lord Rayleigh's first research was mainly mathematical, concerning optics and vibrating systems, but his later work ranged over almost the whole field of physics, covering sound, wave theory, color vision, electrodynamics, electromagnetism, light scattering, flow of liquids, hydrodynamics, density of gases, viscosity, capillarity, elasticity, and photography. Rayleigh's later work was concentrated on electric and magnetic problems. Rayleigh was considered to be an excellent instructor. His Theory of Sound was published in two volumes during 1877-1878, and his other extensive studies are reported in his Scientific Papers, six volumes issued during 1889-1920. Rayleigh was also a contributer to the Encyclopedia Britannica. He published 446 papers which, reprinted in his collected works, clearly show his capacity for understanding everything just a little more deeply than anyone else. He intervened in debates of the House of Lords only on rare occasions, never allowing politics to interfere with science. Lord Rayleigh, a Chancellor of Cambridge University, was a Justice of the Peace and the recipient of honorary science and law degrees. He was a Fellow of the Royal Society (1873) and served as Secretary from 1885 to 1896, and as President from 1905 to 1908. He received the Nobel Prize in 1904. Lord Rayleigh died on June 30, 1919, at Witham, Essex.

In 1871 he married Evelyn, sister of the future prime minister, the Earl of Balfour (of the famous Balfour declaration of the Jewish state). They had three sons, the eldest of whom was to become a professor of physics at the Imperial College of Science and Technology, London.

As a successor to James Clerk Maxwell, he was head of the Cavendish Laboratory at Cambridge from 1879-1884, and in 1887 became Professor of Natural Philosophy at the Royal Institute of Great Britain. Rayleigh died on June 30, 1919 at Witham, Essex.

Fig. 1.9: Portrait of Rankine

1.3.5.4 William John Macquorn Rankine

William John Macquorn Rankine (July 2, 1820 - December 24, 1872) was a Scottish engineer and physicist. He was a founding contributor to the science of thermodynamics (Rankine Cycle). Rankine developed a theory of the steam engine. His steam engine manuals were used for many decades.

Rankine was born in Edinburgh to British Army lieutenant David Rankine and Barbara Grahame, Rankine.

1.3.5.5 Gino Girolamo Fanno

Fig. 1.10: The photo of Gino Fanno approximately in 1950

Fanno a Jewish Engineer was born on November 18, 1888. he studied in a technical institute in Venice and graduated with very high grades as a mechanical engineer. Fanno was not as lucky as his brother, who was able to get into academia. Faced with anti–semitism, Fanno left Italy to Zurich, Switzerland in 1900 to attend graduate school for his master's degree. In this new place he was able to pose as a Roman Catholic, even though for short time he went to live in a Jewish home, Isaak Baruch Weil's family. As were many Jews at that time, Fanno was fluent in several languages including Italian, English, German, and French. He likely had a good knowledge of Yiddish and possibly some Hebrew. Consequently, he did not have a problem studying in a different language. In July 1904 he received his diploma (master). When one of Professor Stodola's assistants attended military service this temporary position was offered to Fanno. "Why didn't a talented guy like Fanno keep or obtain a position in academia after he published his model?" The answer is tied to the fact that somehow rumors about his roots began to surface. Additionally, the fact that his model was not a "smashing[49] success" did not help.

Later Fanno had to go back to Italy to find a job in industry. Fanno turned out to be a good engineer and he later obtained a management position. He married, and like his brother, Marco, was childless. He obtained a Ph.D. from Regian Istituto Superiore d'Ingegneria di Genova. However, on February 1939 Fanno was degraded (denounced) and he lost his Ph.D. (is this the first case in history) because his of his Jewish nationality[50]. During the War (WWII), he had to be under house arrest to avoid being sent to the "vacation camps." To further camouflage himself, Fanno converted to Catholicism. Apparently, Fanno had a cache of old Italian currency (which was apparently still highly acceptable) which helped him and his wife survive the war. After the war, Fanno was only able to work in agriculture and agricultural engineering. Fanno passed way in 1960 without world recognition for his model.

Fanno older brother, mentioned earlier Marco Fanno is a famous economist who later developed fundamentals of the supply and demand theory.

[49]Missing data about friction factor

[50]In some places, the ridicules claims that Jews persecuted only because their religion. Clearly, Fanno was not part of the Jewish religion (see his picture) only his nationality was Jewish.

1.3.5.6 Ludwig Prandtl

Perhaps Prandtl's greatest achievement was his ability to produce so many great scientists. It is mind boggling to look at the long list of those who were his students and colleagues. There is no one who educated as many great scientists as Prandtl. Prandtl changed the field of fluid mechanics and is called the modern father of fluid mechanics because of his introduction of boundary layer, turbulence mixing theories etc.

Ludwig Prandtl was born in Freising, Bavaria, in 1874. His father was a professor of engineering and his mother suffered from a lengthy illness. As a result, the young Ludwig spent more time with his father which made him interested in his father's physics and machinery books. This upbringing fostered the young Prandtl's interest in science and experimentation.

Prandtl started his studies at the age of 20 in Munich, Germany and he graduated at the age of 26 with a Ph.D. Interestingly, his Ph.D. was focused on solid mechanics. His interest changed when, in his first job, he was required to design factory equipment that involved problems related to the field of fluid mechanics (a suction device). Later he sought and found a job as a professor of mechanics at a technical school in Hannover, Germany (1901). During this time Prandtl developed his boundary layer theory and studied supersonic fluid flows through nozzles. In 1904, he presented the revolutionary paper "Flussigkeitsbewegung Bei Sehr Kleiner Reibung" (Fluid Flow in Very Little Friction), the paper which describes his boundary layer theory.

Fig. 1.11: Photo of Prandtl

His 1904 paper raised Prandtl's prestige. He became the director of the Institute for Technical Physics at the University of Göttingen. He developed the Prandtl-Glauert rule for subsonic airflow. Prandtl, with his student Theodor Meyer, developed the first theory for calculating the properties of shock and expansion waves in supersonic flow in 1908 (two chapters in this book). As a byproduct they produced the theory for *oblique shock*. In 1925 Prandtl became the director of the Kaiser Wilhelm Institute for Flow Investigation at Göttingen. By the 1930s, he was known worldwide as the leader in the science of fluid dynamics. Prandtl also contributed to research in many areas, such as meteorology and structural mechanics.

Ludwig Prandtl worked at Göttingen until his death on August 15, 1953. His work and achievements in fluid dynamics resulted in equations that simplified understanding, and many are still used today. Therefore many referred to him as the father of modern fluid mechanics. Ludwig Prandtl died in Göttingen, Germany on August 15th 1953.

Prandtl's other contributions include: the introduction of the Prandtl number in fluid mechanics, airfoils and wing theory (including theories of aerodynamic interference, wing-fuselage, wing-propeller, biplane, etc); fundamental studies in the wind tunnel, high speed flow (correction formula for subsonic compressible flows), theory of turbulence. His name is linked to the following:

- Prandtl number (heat transfer problems)
- Prandtl-Glauert compressibility correction
- Prandtl's boundary layer equation
- Prandtl's lifting line theory
- Prandtl's law of friction for smooth pipes
- Prandtl-Meyer expansion fans (supersonic flow)
- Prandtl's Mixing Length Concept (theory of turbulence)

1.3.5.7 E.R.G. Eckert

Eckert was born in 1904 in Prague, where he studied at the German Institute of Technology. During World War II, he developed methods for jet engine turbine blade cooling at a research laboratory in Prague. He emigrated to the United States after the war, and served as a consultant to the U.S. Air Force and the National Advisory Committee for Aeronautics before coming to Minnesota.

Fig. 1.12: The photo of Ernst Rudolf George Eckert with the author's family

Eckert developed the understanding of heat dissipation in relation to kinetic energy, especially in compressible flow. Hence, the dimensionless group has been designated as the Eckert number, which is associated with the Mach number. Schlichting suggested this dimensionless group in honor of Eckert. In addition to being named to the National Academy of Engineering in 1970, He authored more than 500 articles and received several medals for his contributions to science. His book ”Introduction to the Transfer of Heat and Mass,” published in 1937, is still considered a fundamental text in the field.

Eckert was an excellent mentor to many researchers (including this author), and he had a reputation for being warm and kindly. He was also a leading figure in bringing together engineering in the East and West during the Cold War years.

1.3.5.8 Ascher Shapiro

MIT Professor Ascher Shapiro[51], the Eckert equivalent for the compressible flow, was instrumental in using his two volume book "The Dynamics of Thermodynamics of the Compressible Fluid Flow," to transform the gas dynamics field to a coherent text material for engineers. Furthermore, Shapiro's knowledge of fluid mechanics enabled him to "sew" the missing parts of the Fanno line with Moody's diagram to create the most useful model in compressible flow. While Shapiro viewed gas dynamics mostly through aeronautic eyes, The undersigned believes that Shapiro was the first one to propose an isothermal flow model that is not part of the aeronautic field. Therefore it is proposed to call this model Shapiro's Flow.

In his first 25 years Shapiro focused primarily on power production, high-speed flight, turbomachinery and propulsion by jet engines and rockets. Unfortunately for the field of Gas Dynamics, Shapiro moved to the field of biomedical engineering where he was able to pioneer new work. Shapiro was instrumental in the treatment of blood clots, asthma, emphysema and glaucoma.

Shapiro grew up in New York City and received his S.B. in 1938 and the Sc.D. (It is M.I.T.'s equivalent of a Ph.D. degree) in 1946 in mechanical engineering from MIT. He was assistant professor in 1943, three years before receiving his Sc.D. In 1965 he become the head of the Department of Mechanical Engineering until 1974. Shapiro spent most of his active years at MIT. Asher Shapiro passed way in November 2004

[51]Parts taken from Sasha Brown, MIT

CHAPTER 2

Fundamentals of Basic Fluid Mechanics

2.1 Introduction

In this chapter a review of the fundamtatals that are expected from the student to know. The basic principles are related from the basic consrvation pricible. Several terms would be review like stream lines. In addition the basic Bernoulli's equation will drived of incompressible flow and later for compressible flow. Several application of the fluid mechanics will demonstraed. This material is not covered in the history chapter.

2.2 Fluid Properties

2.3 Control Volume

2.4 Reynold's Transport Theorem

For simplification the discuation will focused one dimensional control volume and we generalzed later. The flow throgh a stream tube is assumed one-dimentinal so that there isn't any flow expet the tube opening. At the initial time the mass that was in the tube was m_0. The mass after very shot time of dt is dm. For simplisity, it is assume the control volume is a fixed boundry. The flow through the opening on the left is assumed to enter the stream tube while the flow is assumed to leave the stream tube.

Supposed that the fluid has a property η

$$\left.\frac{dN_s}{dt}\right) = \lim_{\Delta t \to 0} \frac{N_s(t_0 + \Delta t) - N_s(t_0)}{\Delta t} \tag{2.1}$$

CHAPTER 3

Speed of Sound

3.1 Motivation

In tradition compressible flow classes there is very little discussion about speed of sound outside the ideal gas. This author think that approach has many shortcomings. In recent consultation an engineer[1] A design of industrial system that contains converting diverging nozzle with filter to remove small particles from air. The engineer was well aware about the calculation of the nozzle. Thus, engineer was able to predict that was a chocking point. Yet, the engineer was not ware what is the effect of particles on the speed of sound. Hence, the actual flow rate was only half of his prediction. As it will shown in this chapter, the particles reduces the speed of sound by almost half. With the "new" knowledge of the consultation the calculations were within the range of acceptable results.

The above situation is not unique in the industry. It should be expected that engineers know how to manged this situation of non pure substances (like clean air). The fact that the engineer know about the chocking is great but it is enough for today sophisticated industry[2]. In this chapter an introductory discussion about different situations that can appear the industry in regards to speed of sound.

3.2 Introduction

[1]Aerospace engineer that alumni of University of Minnesota.

[2]Pardon, but a joke is must in this situation. A cat is pursuing a mouse and the mouse escape and hide in the hole. Suddenly, the mouse hear a barking dog and a cat yelling. The mouse go out to investigate, and cat is catching the mouse. The mouse ask the cat I thought I hear a dog. The cat reply, yes you right. My teacher was right, one language is not enough today.

The people had recognized for several hundred years that sound is a variation of pressure. The ears sense the variations by frequency and magnitude to transfer to the brain which translates to voice. Thus, it raises the question: what is the speed of the small disturbance travel in a "quiet" medium. This velocity is referred to as the speed of sound.

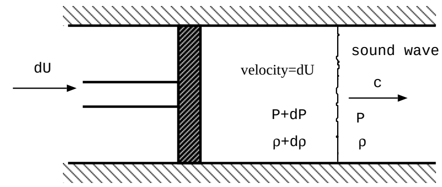


Fig. 3.1: A very slow moving piston in a still gas

To answer this question consider a piston moving from the right to the left at a relatively small velocity (see Figure 3.1). The information that the piston is moving passes thorough a single "pressure pulse." It is assumed that if the velocity of the piston is infinitesimally small, the pules will be infinitesimally small. Thus, the pressure and density can be assumed to be continuous.

In the control volume it is convenient to look at a control volume which is attached to a pressure pulse. Applying the mass balance yields

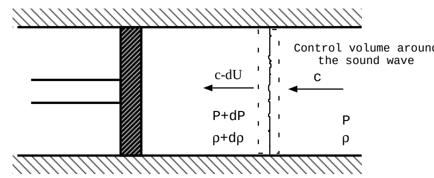


Fig. 3.2: stationary sound wave and gas move relative to the pulse

$$\rho c = (\rho + d\rho)(c - dU) \tag{3.1}$$

or when the higher term $dU d\rho$ is neglected yields

$$\rho dU = c d\rho \Longrightarrow dU = \frac{c d\rho}{\rho} \tag{3.2}$$

From the energy equation (Bernoulli's equation), assuming isentropic flow and neglecting the gravity results

$$\frac{(c - dU)^2 - c^2}{2} + \frac{dP}{\rho} = 0 \tag{3.3}$$

neglecting second term (dU^2) yield

$$-c dU + \frac{dP}{\rho} = 0 \tag{3.4}$$

Substituting the expression for dU from equation (3.2) into equation (3.4) yields

$$c^2 \left(\frac{d\rho}{\rho}\right) = \frac{dP}{\rho} \Longrightarrow c^2 = \frac{dP}{d\rho} \tag{3.5}$$

An expression is needed to represent the right hand side of equation (3.5). For ideal gas P is a function of two independent variables. Here, it is considered that $P = P(\rho, s)$ where s is the entropy. The full differential of the pressure can be expressed as follows:

$$dP = \left.\frac{\partial P}{\partial \rho}\right|_s d\rho + \left.\frac{\partial P}{\partial s}\right|_\rho ds \tag{3.6}$$

In the derivations for the speed of sound it was assumed that the flow is isentropic, therefore it can be written

$$\frac{dP}{d\rho} = \left.\frac{\partial P}{\partial \rho}\right|_s \tag{3.7}$$

Note that the equation (3.5) can be obtained by utilizing the momentum equation instead of the energy equation.

Example 3.1:
Demonstrate that equation (3.5) *can be derived from the momentum equation.*

SOLUTION
The momentum equation written for the control volume shown in Figure (3.2) is

$$\overbrace{(P + dP) - P}^{\sum F} = \overbrace{(\rho + d\rho)(c - dU)^2 - \rho c^2}^{\int_{cs} U(\rho U dA)} \tag{3.8}$$

Neglecting all the relative small terms results in

$$dP = (\rho + d\rho)\left(c^2 - \cancelto{\sim 0}{2cdU +} \cancelto{\sim 0}{dU^2}\right) - \rho c^2 \tag{3.9}$$

$$dP = c^2 d\rho \tag{3.10}$$

This yields the same equation as (3.5).

3.3 Speed of sound in ideal and perfect gases

The speed of sound can be obtained easily for the equation of state for an ideal gas (also perfect gas as a sub set) because of a simple mathematical expression. The pressure for ideal gas can be expressed as a simple function of density, ρ, and a function "molecular structure" or ratio of specific heats, k namely

$$P = constant \times \rho^k \tag{3.11}$$

and hence

$$c = \sqrt{\frac{dP}{d\rho}} = k \times constant \times \rho^{k-1} = k \times \frac{\overbrace{constant \times \rho^k}^{P}}{\rho}$$
$$= k \times \frac{P}{\rho} \qquad (3.12)$$

Remember that P/ρ is defined for ideal gas as RT, and equation (3.12) can be written as

$$c = \sqrt{kRT} \qquad (3.13)$$

Example 3.2:
Calculate the speed of sound in water vapor at $20[bar]$ and $350°\mathrm{C}$, (a) utilizes the steam table (b) assuming ideal gas.

SOLUTION
The solution can be estimated by using the data from steam table[3]

$$c = \sqrt{\frac{\Delta P}{\Delta \rho}}_{s=constant} \qquad (3.14)$$

At $20[bar]$ and $350°\mathrm{C}$: s = 6.9563 $\left[\frac{kJ}{K\ kg}\right]$ ρ = 6.61376 $\left[\frac{kg}{m^3}\right]$
At $18[bar]$ and $350°\mathrm{C}$: s = 7.0100 $\left[\frac{kJ}{K\ kg}\right]$ ρ = 6.46956 $\left[\frac{kg}{m^3}\right]$
At $18[bar]$ and $300°\mathrm{C}$: s = 6.8226 $\left[\frac{kJ}{K\ kg}\right]$ ρ = 7.13216 $\left[\frac{kg}{m^3}\right]$

After interpretation of the temperature:
At $18[bar]$ and $335.7°\mathrm{C}$: s $\sim$ 6.9563 $\left[\frac{kJ}{K\ kg}\right]$ $\rho \sim$ 6.94199 $\left[\frac{kg}{m^3}\right]$
and substituting into the equation yields

$$c = \sqrt{\frac{200000}{0.32823}} = 780.5 \left[\frac{m}{sec}\right] \qquad (3.15)$$

for ideal gas assumption (data taken from Van Wylen and Sontag, Classical Thermodynamics, table A 8.)

$$c = \sqrt{kRT} \sim \sqrt{1.327 \times 461 \times (350 + 273)} \sim 771.5 \left[\frac{m}{sec}\right]$$

Note that a better approximation can be done with a steam table, and it will be part of the future program (potto–GDC).

[3] This data is taken form Van Wylen and Sontag “Fundamentals of Classical Thermodynamics” 2nd edition

Example 3.3:
The temperature in the atmosphere can be assumed to be a linear function of the height for some distances. What is the time it take for sound to travel from point "A" to point "B" under this assumption.?

SOLUTION
The temperature is denoted at "A" as T_A and temperature in "B" is T_B. The distance between "A" and "B" is denoted as h.

$$T = (T_B - T_A)\frac{x}{h} + T_A$$

Where the distance x is the variable distance. It should be noted that velocity is provided as a function of the distance and not the time (another reverse problem). For an infinitesimal time dt is equal to

$$dt = \frac{dx}{\sqrt{kRT(x)}} = \frac{dx}{\sqrt{kRT_A\left(\frac{(T_B - T_A)x}{T_A h} + 1\right)}}$$

integration of the above equation yields

$$t = \frac{2hT_A}{3\sqrt{kRT_A}\,(T_B - T_A)}\left(\left(\frac{T_B}{T_A}\right)^{\frac{3}{2}} - 1\right) \tag{3.16}$$

For assumption of constant temperature the time is

$$t = \frac{h}{\sqrt{kR\bar{T}}} \tag{3.17}$$

Hence the correction factor

$$\frac{t_{corrected}}{t} = \sqrt{\frac{T_A}{\bar{T}}}\,\frac{2}{3}\,\frac{T_A}{(T_B - T_A)}\left(\left(\frac{T_B}{T_A}\right)^{\frac{3}{2}} - 1\right) \tag{3.18}$$

This correction factor approaches one when $T_B \longrightarrow T_A$.

is it reasonable to put a discussion here about atmosphere and other affects on the air?

3.4 Speed of Sound in Real Gas

The ideal gas model can be improved by introducing the compressibility factor. The compressibility factor represent the deviation from the ideal gas.

Thus, a real gas equation can be expressed in many cases as

$$P = z\rho RT \tag{3.19}$$

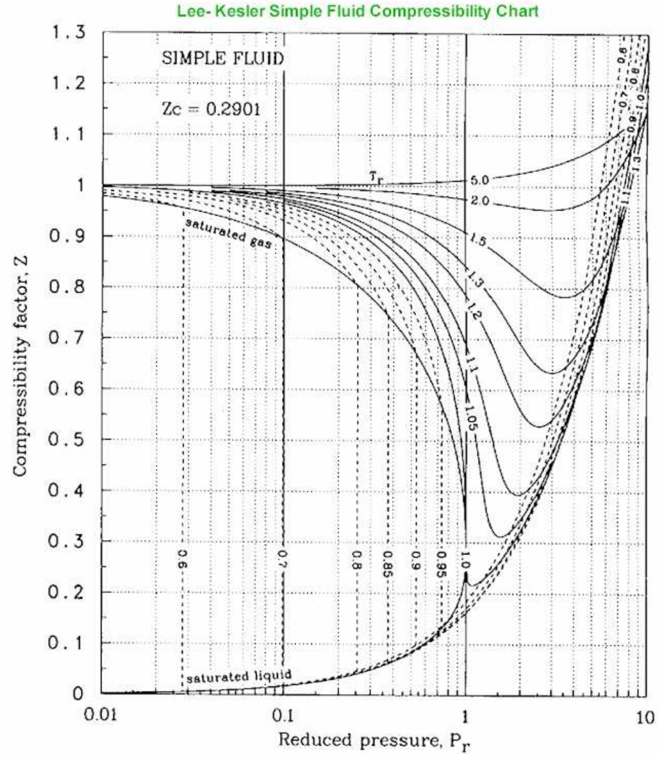


Fig. 3.3: The Compressibility Chart

The speed of sound of any gas is (3.7). To obtain the expression for gas that obey the law expressed by (3.19) some mathematical expression are needed. Recalling from thermodynamics, the Gibbs function (3.20) is used to obtain

$$T ds = dh - \frac{dP}{\rho} \tag{3.20}$$

The definition of pressure specific heat for a pure substance is

$$C_p = \left(\frac{\partial h}{\partial T}\right)_P = T\left(\frac{\partial s}{\partial T}\right)_P \tag{3.21}$$

The definition of volumetric specific heat for a pure substance is

$$C_v = \left(\frac{\partial u}{\partial T}\right)_\rho = T\left(\frac{\partial s}{\partial T}\right)_\rho \tag{3.22}$$

From thermodynamics, it can be shown [4]

$$dh = C_p dT + \left[v - T \left(\frac{\partial v}{\partial T} \right)_P \right] \tag{3.23}$$

The specific volumetric is the inverse of the density as $v = zRT/P$ and thus

$$\left(\frac{\partial v}{\partial T} \right)_P = \left(\frac{\partial \left(\frac{zRT}{P} \right)}{\partial T} \right)_P = \frac{RT}{P} \left(\frac{\partial z}{\partial T} \right)_P + \frac{zR}{P} \cancelto{1}{\left(\frac{\partial T}{\partial T} \right)_P} \tag{3.24}$$

Substitute the equation (3.24) into equation (3.23) results

$$dh = C_p dT + \left[v - T \left(\overbrace{\frac{RT}{P}}^{\frac{v}{z}} \left(\frac{\partial z}{\partial T} \right)_P + \overbrace{\frac{zR}{P}}^{\frac{v}{T}} \right) \right] dP \tag{3.25}$$

Simplifying equation (3.25) to became

$$dh = C_p dT - \left[\frac{Tv}{z} \left(\frac{\partial z}{\partial T} \right)_P \right] dP = C_p dT - \frac{T}{z} \left(\frac{\partial z}{\partial T} \right)_P \frac{dP}{\rho} \tag{3.26}$$

Utilizing Gibbs equation (3.20)

$$\begin{aligned} T ds = & \overbrace{C_p dT - \frac{T}{z} \left(\frac{\partial z}{\partial T} \right)_P \frac{dP}{\rho}}^{dh} - \frac{dP}{\rho} = C_p dT - \frac{dP}{\rho} \left[\frac{T}{z} \left(\frac{\partial z}{\partial T} \right)_P + 1 \right] \\ = & C_p dT - \frac{dP}{P} \overbrace{\frac{P}{\rho}}^{zRT} \left[\frac{T}{z} \left(\frac{\partial z}{\partial T} \right)_P + 1 \right] \end{aligned} \tag{3.27}$$

Letting $ds = 0$ for isentropic process results in

$$\frac{dT}{T} = \frac{dP}{P} \frac{R}{C_p} \left[z + T \left(\frac{\partial z}{\partial T} \right)_P \right] \tag{3.28}$$

Equation (3.28) can be integrated by parts. However, it is more convenient to express dT/T in terms of C_v and $d\rho/\rho$ as following

$$\frac{dT}{T} = \frac{d\rho}{\rho} \frac{R}{C_v} \left[z + T \left(\frac{\partial z}{\partial T} \right)_\rho \right] \tag{3.29}$$

[4] See Van Wylen p. 372 SI version, perhaps to insert the discussion here.

Equating the right hand side of equations (3.28) and (3.29) results in

$$\frac{d\rho}{\rho}\frac{R}{C_v}\left[z+T\left(\frac{\partial z}{\partial T}\right)_\rho\right]=\frac{dP}{P}\frac{R}{C_p}\left[z+T\left(\frac{\partial z}{\partial T}\right)_P\right] \tag{3.30}$$

Rearranging equation (3.30) yields

$$\frac{d\rho}{\rho}=\frac{dP}{P}\frac{C_v}{C_p}\left[\frac{z+T\left(\frac{\partial z}{\partial T}\right)_P}{z+T\left(\frac{\partial z}{\partial T}\right)_\rho}\right] \tag{3.31}$$

If the terms in the square parentheses are constant in the range under the interest in this study equation (3.31) can be integrated. For short hand writing convenience, n is defined as

$$n=\overbrace{\frac{C_p}{C_v}}^{k}\left(\frac{z+T\left(\frac{\partial z}{\partial T}\right)_\rho}{z+T\left(\frac{\partial z}{\partial T}\right)_P}\right) \tag{3.32}$$

Note that n approach to k when $z\rightarrow 1$ and when z is constant. The integration of equation (3.31) yields

$$\left(\frac{\rho_1}{\rho_2}\right)^n=\frac{P_1}{P_2} \tag{3.33}$$

Equation (3.33) the similar to equation (3.11). What is different in these derivation the relationship between coefficient n to k was established. The relationship (3.33) isn't new, and in–fact any thermodynamics book show this relationship. But with the definition of n in equation (3.32) provide a tool to estimate n In the same manner as the ideal gas speed of sound the speed of sound for real gas can be obtained.

$$\frac{dP}{d\rho}=nzRT \tag{3.34}$$

Example 3.4:
Calculate the speed of sound of air at $30°\mathrm{C}$ and atmospheric pressure $\sim 1[bar]$. The specific heat for air is $k=1.407$, $n=1.403$, and $z=0.995$.
Make the calculation based on the ideal gas model and compare these calculation to real gas model (compressibility factor). Assume that $R=287[j/kg/K]$.

SOLUTION
According to the ideal gas model the speed of sound should be

$$c=\sqrt{kRT}=\sqrt{1.407\times 287\times 300}\sim 348.1[m/sec]$$

For the real gas first the coefficient $n=1.403$ has

$$c=\sqrt{znRT}=\sqrt{1.403\times 0.995 times 287\times 300}=346.7[m/sec]$$

The correction of speed of sound of air in normal condition (atmospheric condition plus even increase of pressure) is minimal on the speed of sound. However, the change of temperature can have dramatical change in the speed of sound. For example, at relative moderate pressure but low temperature common in atmosphere, the compressibility factor, $z = 0.3$ and $n \sim 1$ which means that speed of sound is only $\sqrt{\frac{0.3}{1.4}}$ factor (0.5) to calculated by ideal gas model.

3.5 Speed of Sound in Almost Incompressible Liquid

Even liquid *normally* is assumed to be incompressible in reality has a small and important compressible aspect. The ratio of the change in the fractional volume to pressure or compression is referred to as the bulk modulus of the material. For example, the average bulk modulus for water is $2.2 \times 10^9\ N/m^2$. At a depth of about 4,000 meters, the pressure is about $4 \times 10^7\ N/m^2$. The fractional volume change is only about 1.8% even under this pressure nevertheless it is a change.

The compressibility of the substance is the reciprocal of the bulk modulus. The amount of compression of almost all liquids is seen to be very small as given in Table (3.5). The mathematical definition of bulk modulus as following

$$B = \rho \frac{dP}{d\rho} \tag{3.35}$$

In physical terms can be written as

$$c = \sqrt{\frac{elastic\ property}{inertial\ property}} = \sqrt{\frac{B}{\rho}} \tag{3.36}$$

For example for water

$$c = \sqrt{\frac{2.2 \times 10^9 N/m^2}{1000 kg/m^3}} = 1493 m/s$$

This agrees well with the measured speed of sound in water, 1482 m/s at 20°C. Many researchers have looked at this velocity, and for purposes of comparison it is given in Table (3.5)

The effect of impurity and temperature is relatively large, as can be observed from the equation (3.37). For example, with an increase of 34 degrees from 0°C there is an increase in the velocity from about 1430 m/sec to about 1546 [m/sec]. According to Wilson[5], the speed of sound in sea water depends on temperature, salinity, and hydrostatic pressure.

Wilson's empirical formula appears as follows:

$$c(S,T,P) = c_0 + c_T + c_S + c_P + c_{STP}, \tag{3.37}$$

[5] J. Acoust. Soc. Amer., 1960, vol.32, N 10, p. 1357. Wilson's formula is accepted by the National Oceanographic Data Center (NODC) USA for computer processing of hydrological information.

Remark	reference	Value [m/sec]
Fresh Water (20 °C)	Cutnell, John D. & Kenneth W. Johnson. Physics. New York: Wiley, 1997: 468.	1492
Distilled Water at (25 °C)	The World Book Encyclopedia. Chicago: World Book, 1999. 601	1496
Water distilled	Handbook of Chemistry and Physics. Ohio: Chemical Rubber Co., 1967-1968: E37	1494

Table 3.1: Water speed of sound from different sources

where $c_0 = 1449.14$ is about clean/pure water, c_T is a function temperature, and c_S is a function salinity, c_P is a function pressure, and c_{STP} is a correction factor between coupling of the different parameters.

material	reference	Value [m/sec]
Glycerol		1904
Sea water	25 °C	1533
Mercury		1450
Kerosene		1324
Methyl alcohol		1143
Carbon tetrachloride		926

Table 3.2: Liquids speed of sound, after Aldred, John, Manual of Sound Recording, London: Fountain Press, 1972

In summary, the speed of sound in liquids is about 3 to 5 relative to the speed of sound in gases.

3.6 Speed of Sound in Solids

The situation with solids is considerably more complicated, with different speeds in different directions, in different kinds of geometries, and differences between transverse and longitudinal waves. Nevertheless, the speed of sound in solids is larger than in liquids and definitely larger than in gases.

Young's Modulus for a representative value for the bulk modulus for steel is 160 10^9 N /m^2.

Speed of sound in solid of steel, using a general tabulated value for the bulk modulus, gives a sound speed for structural steel of

material	reference	Value [m/sec]
Diamond		12000
Pyrex glass		5640
Steel	longitudinal wave	5790
Steel	transverse shear	3100
Steel	longitudinal wave (extensional wave)	5000
Iron		5130
Aluminum		5100
Brass		4700
Copper		3560
Gold		3240
Lucite		2680
Lead		1322
Rubber		1600

Table 3.3: Solids speed of sound, after Aldred, John, Manual of Sound Recording, London:Fountain Press, 1972

$$U = \sqrt{\frac{E}{\rho}} = \sqrt{\frac{160 \times 10^9 N/m^2}{7860 Kg/m^3}} = 4512 m/s \tag{3.38}$$

Compared to one tabulated value the example values for stainless steel lays between the speed for longitudinal and transverse waves.

3.7 Sound Speed in Two Phase Medium

The gas flow in many industrial situations contains other particles. In actuality, there could be more than one speed of sound for two phase flow. Indeed there is double chocking phenomenon in two phase flow. However, for homogeneous and under certain condition a single velocity can be considered. There can be several models that approached this problem. For simplicity, it assumed that two materials are homogeneously mixed. Topic for none homogeneous mixing are beyond the scope of this book. It further assumed that no heat and mass transfer occurs between the particles. In that case, three extreme cases suggest themselves: the flow is mostly gas with drops of the other phase (liquid or solid), about equal parts of gas and the liquid phase, and liquid with some bubbles. The first case is analyzed.

The equation of state for the gas can be written as

$$P_a = \rho_a R T_a \tag{3.39}$$

The average density can be expressed as

$$\frac{1}{\rho_m} = \frac{\xi}{\rho_a} + \frac{1-\xi}{\rho_b} \tag{3.40}$$

where $\xi = \frac{\dot{m}_b}{\dot{m}}$ is the mass ratio of the materials.

For small value of ξ equation (3.40) can be approximated as

$$\frac{\rho}{\rho_a} = 1 + m \tag{3.41}$$

where $m = \frac{\dot{m}_b}{\dot{m}_a}$ is mass flow rate per gas flow rate.

The gas density can be replaced by equation (3.39) and substituted into equation (3.41)

$$\frac{P}{\rho} = \frac{R}{1+m} T \tag{3.42}$$

A approximation of addition droplets of liquid or dust (solid) results in reduction of R and yet approximate equation similar to ideal gas was obtained. It must noticed that $m = constant$. If the droplets (or the solid particles) can be assumed to have the same velocity as the gas with no heat transfer or fiction between the particles isentropic relation can be assumed as

$$\frac{P}{{\rho_a}^k} = constant \tag{3.43}$$

Assuming that partial pressure of the particles is constant and applying the second law for the mixture yields

$$0 = \overbrace{mC\frac{dT}{T}}^{droplets} + \overbrace{C_p\frac{dT}{T} - R\frac{dP}{P}}^{gas} = \frac{(C_p + mC)dT}{T} - R\frac{dP}{P} \tag{3.44}$$

Therefore, the mixture isentropic relationship can be expressed as

$$\frac{P^{\frac{\gamma-1}{\gamma}}}{T} = constant \tag{3.45}$$

where

$$\frac{\gamma-1}{\gamma} = \frac{R}{C_p + mC} \tag{3.46}$$

Recalling that $R = C_p - C_v$ reduces equation (3.46) into

$$\gamma = \frac{C_p + mC}{C_v + mC} \tag{3.47}$$

In a way the definition of γ was so chosen that effective specific pressure heat and effective specific volumetric heat are $\frac{C_p+mC}{1+m}$ and $\frac{C_v+mC}{1+m}$ respectively. The correction factors for the specific heat is not linear.

Since the equations are the same as before hence the familiar equation for speed of sound can be applied as

$$c = \sqrt{\gamma R_{mix} T} \tag{3.48}$$

It can be noticed that R_{mix} and γ are smaller than similar variables in a pure gas. Hence, this analysis results in lower speed of sound compared to pure gas. Generally, the velocity of mixtures with large gas component is smaller of of the pure gas. For example, the velocity of sound in slightly wed steam can be about one third of the pure steam speed of sound.

Insert example with small steel particles with air up to 20%

At this stage the other models for two phase are left for next version (0.6).

Meta

For a mixture of two phases, speed of sound can be expressed as

$$c^2 = \frac{\partial P}{\partial \rho} = \frac{\partial P[f(X)]}{\partial \rho} \tag{3.49}$$

where X is defined as

$$X = \frac{s - s_f(P_B}{s_{fg}(P_B)} \tag{3.50}$$

Meta End

CHAPTER 4

Isentropic Variable Area Flow

In this Chapter a discussion on steady state flow though a smooth and continuous area flow rate (steady state) is presented. A discussion about the flow through a converging–diverging nozzle is also part of this Chapter. The isentropic flow models are important because two main reasons: one, it provides the information about the trends and important parameters, two, the correction factors later can be introduced to account for deviations from the ideal state.

4.1 Stagnation State for Ideal Gas Model

4.1.1 General Relationship

It is assumed that the flow is one–dimensional. Figure (4.1) describes a gas flows through a diverging–converging nozzle. It has been found that a theoretical state known as the stagnation state is very useful in simplifying the solution and treatment of the flow. The stagnation state is a theoretical state in which the flow is brought into a complete motionless condition in isentropic process without other forces (e.g. gravity force). Several properties that can be represented by this theoretical process which include temperature, pressure, and density et cetera and de-

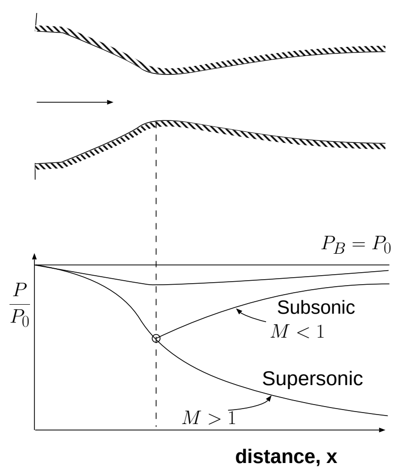


Fig. 4.1: Flow of a compressible substance (gas) thorough a converging diverging nozzle

noted by the subscript "0."

First, the stagnation temperature is calculated. The energy conservation can be written as

$$h + \frac{U^2}{2} = h_0 \tag{4.1}$$

Perfect gas is an ideal gas with a constant heat capacity, C_p. For perfect gas equation (4.1) is simplified into

$$C_p T + \frac{U^2}{2} = C_p T_0 \tag{4.2}$$

Again it common to denote T_0 as the stagnation temperature. Recalling from thermodynamic the relationship for perfect gas

$$R = C_p - C_v \tag{4.3}$$

and denoting $k \equiv C_p \div C_v$ than the thermodynamics relationship obtains the form

$$C_p = \frac{kR}{k-1} \tag{4.4}$$

and where R is a specific constant. Dividing equation (4.2) by $(C_p T)$ yield

$$1 + \frac{U^2}{2C_p T} = \frac{T_0}{T} \tag{4.5}$$

Now, substituting $c^2 = kRT$ or $T = c^2/kR$ equation (4.5) changes into

$$1 + \frac{kRU^2}{2C_p c^2} = \frac{T_0}{T} \tag{4.6}$$

Utilizing the definition of k by equation (4.4) and inserting it into equation (4.6) yields

$$1 + \frac{k-1}{2}\,\frac{U^2}{c^2} = \frac{T_0}{T} \tag{4.7}$$

It very useful to convert equation (4.6) into a dimensionless form and denote Mach number as the ratio of velocity to speed of sound as

$$M \equiv \frac{U}{c} \tag{4.8}$$

Inserting the definition of Mach number, (4.8) into equation (4.7) reads

$$\frac{T_0}{T} = 1 + \frac{k-1}{2} M^2 \tag{4.9}$$

The usefulness of Mach number and equation (4.9) can be demonstrated by this following simple example. In this example a gas flows through a tube (see Figure 4.2) of any shape can be expressed as a function of only the stagnation temperature as oppose to the function of the temperatures and velocities.

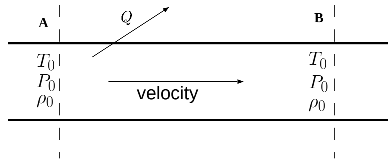


Fig. 4.2: Perfect gas flows through a tube

The definition of the stagnation provides the advantage of a compact writing. For example, writing the energy equation for the tube when by an external forces or energy exchange shown in Figure (4.2). In that case, the energy question is reduced to

$$\dot{Q} = C_p (T_{0B} - T_{0A}) \dot{m} \tag{4.10}$$

The ratio of stagnation pressure to the static pressure can be expressed as a function of the temperature ratio because of the isentropic relationship as

$$\frac{P_0}{P} = \left(\frac{T_0}{T}\right)^{\frac{k}{k-1}} = \left(1 + \frac{k-1}{2} M^2\right)^{\frac{k}{k-1}} \tag{4.11}$$

In the same manner the relationship for the density

$$\frac{\rho_0}{\rho} = \left(\frac{T_0}{T}\right)^{\frac{1}{k-1}} = \left(1 + \frac{k-1}{2} M^2\right)^{\frac{1}{k-1}} \tag{4.12}$$

A new useful definition is introduced for the case when $M = 1$ and denoted by superscript $*$. The special case of ratio of the star values to stagnation values are depend only on the heat ratio as following:

$$\frac{T^*}{T_0} = \frac{{c^*}^2}{{c_0}^2} = \frac{2}{k+1} \tag{4.13}$$

$$\frac{P^*}{P_0} = \left(\frac{2}{k+1}\right)^{\frac{k}{k-1}} \tag{4.14}$$

$$\frac{\rho^*}{\rho_0} = \left(\frac{2}{k+1}\right)^{\frac{1}{k-1}} \tag{4.15}$$

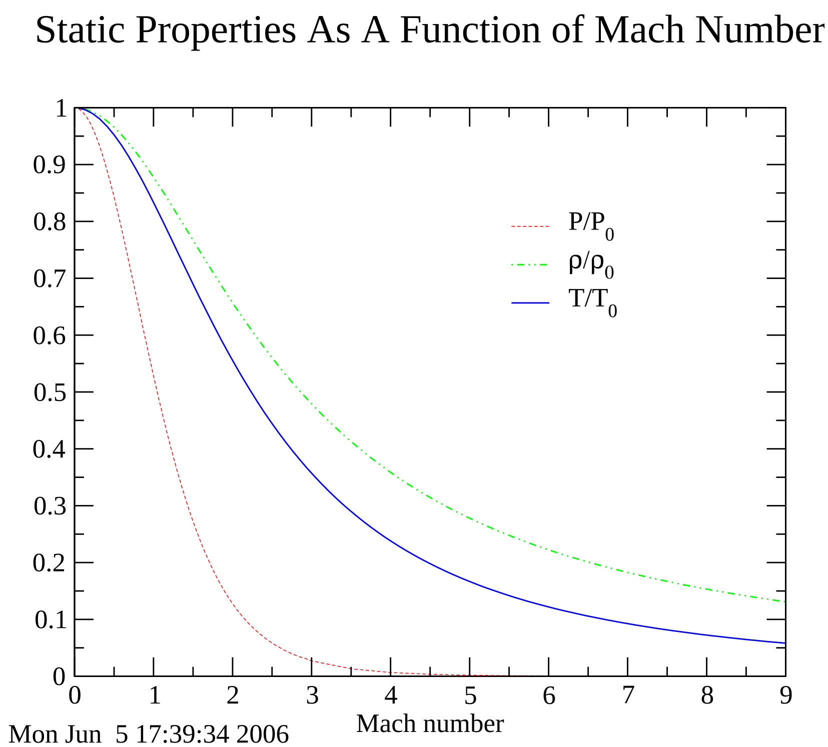


Fig. 4.3: The stagnation properties as a function of the Mach number, $k = 1.4$

4.1.2 Relationships for Small Mach Number

Even with today computers a simplified method can reduce the tedious work involve in computational work. In particular, the trends can be examined with analytical methods. It further will be used in the book to examine trends in derived models. It can be noticed that the Mach number involved in the above equations is in a square power. Hence, if an acceptable error is of about one percent than $M < 0.1$ provides the desire range. Further, if higher power is used, much smaller error results. First it can be noticed that the ratio of temperature to stagnation temperature, $\frac{T}{T_0}$ is provided in power series. Expanding the equations results according to binomial expansion to

$$(1+x)^n = 1 + nx + \frac{n(n-1)x^2}{2!} + \frac{n(n-1)(n-2)x^3}{3!} + \cdots \tag{4.16}$$

In the same fashion

$$\frac{P_0}{P} = 1 + \frac{(k-1)M^2}{4} + \frac{kM^4}{8} + \frac{2(2-k)M^6}{48} \cdots \tag{4.17}$$

$$\frac{\rho_0}{\rho} = 1 + \frac{(k-1)M^2}{4} + \frac{kM^4}{8} + \frac{2(2-k)M^6}{48} \cdots \tag{4.18}$$

The pressure difference normalized by the velocity (kinetic energy) as correction factor is

$$\frac{P_0 - P}{\frac{1}{2}\rho U^2} = 1 + \overbrace{\frac{M^2}{4} + \frac{(2-k)M^4}{24} + \cdots}^{compressibility\ correction} \tag{4.19}$$

From above equation, it can be observed that the correction factor approaches zero when $M \longrightarrow 0$ and then the equation (4.19) approaches to the standard equation for incompressible flow.

The definition of the star Mach is ratio of the velocity and star speed of sound (the speed of sound at $M = 1$).

$$M^* = \frac{U}{c^*} = \sqrt{\frac{k+1}{2}} M \left(1 - \frac{k-1}{4} M^2 + \cdots\right) \tag{4.20}$$

$$\frac{P_0 - P}{P} = \frac{kM^2}{2}\left(1 + \frac{M^2}{4} + \cdots\right) \tag{4.21}$$

$$\frac{\rho_0 - \rho}{\rho} = \frac{M^2}{2}\left(1 - \frac{kM^2}{4} + \cdots\right) \tag{4.22}$$

The normalized mass rate becomes

$$\frac{\dot{m}}{A} = \sqrt{\frac{k{P_0}^2 M^2}{RT_0}} \left(1 + \frac{k-1}{4} M^2 + \cdots\right) \tag{4.23}$$

The ratio of the area to star area is

$$\frac{A}{A^*} = \left(\frac{2}{k+1}\right)^{\frac{k+1}{2(k-1)}} \left(\frac{1}{M} + \frac{k+1}{4} M + \frac{(3-k)(k+1)}{32} M^3 + \cdots\right) \tag{4.24}$$

4.2 Isentropic Converging-Diverging Flow in Cross Section

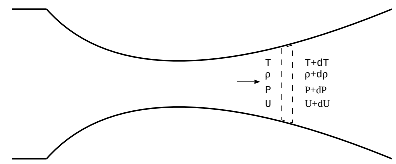


Fig. 4.4: Control volume inside of a converging-diverging nozzle

The important sub case in this Chapter is the flow in a converging–diverging nozzle is considered here. The control volume is shown in Figure (4.4). There are two models that assumed variable area flow: isentropic and adiabatic and the second is isentropic and isothermal model. Clearly, the stagnation temperature, T_0, is constant through the adiabatic flow because there isn't heat transfer. Therefore, the stagnation pressure is also constant through the flow because the flow isentropic. Conversely, in mathematical terms, equation (4.9) and equation (4.11) are the same. If the right hand side is constant for one variable is constant for the other. In the same argument, the stagnation density is constant through the flow. Thus, knowing the Mach number or the temperature provides all what is needed to find the other properties. The only properties that need to be connected are the cross section area and the Mach number. Examination of the relation between properties is carried out.

4.2.1 The Properties in The Adiabatic Nozzle

When no external work and heat transfer, the energy equation, reads

$$dh + UdU = 0 \tag{4.25}$$

Differentiation of continuity equation, $\rho A U = \dot{m} = constant$, and dividing by the continuity equation reads

$$\frac{d\rho}{\rho} + \frac{dA}{A} + \frac{dU}{U} = 0 \tag{4.26}$$

The thermodynamic relationship between the properties can be expressed as

$$Tds = dh - \frac{dP}{\rho} \tag{4.27}$$

For isentropic process $ds \equiv 0$ and combining equations (4.25) with (4.27) yields

$$\frac{dP}{\rho} + UdU = 0 \tag{4.28}$$

Differentiation of the equation state (perfect gas), $P = \rho R T$, and dividing the results by the equation of state ($\rho R T$) yields

$$\frac{dP}{P} = \frac{d\rho}{\rho} + \frac{dT}{T} \tag{4.29}$$

Obtaining an expression for dU/U from the mass balance equation (4.26) and utilizing it in equation (4.28) reads

$$\frac{dP}{\rho} - U^2 \overbrace{\left[\frac{dA}{A} + \frac{d\rho}{\rho}\right]}^{\frac{dU}{U}} = 0 \tag{4.30}$$

Rearranging equation (4.30) so that the density, $d\rho$, can be replaced by the static pressure, dP/ρ yields

$$\frac{dP}{\rho} = U^2 \left(\frac{dA}{A} + \frac{d\rho}{\rho}\frac{dP}{dP}\right) = U^2 \left(\frac{dA}{A} + \overbrace{\frac{d\rho}{dP}}^{\frac{1}{c^2}}\frac{dP}{\rho}\right) \tag{4.31}$$

Recalling that $dP/d\rho = c^2$ and substitute the speed of sound into equation (4.31) to obtain

$$\frac{dP}{\rho}\left[1 - \left(\frac{U}{c}\right)^2\right] = U^2 \frac{dA}{A} \tag{4.32}$$

Or in a dimensionless form

$$\frac{dP}{\rho}\left(1 - M^2\right) = U^2 \frac{dA}{A} \tag{4.33}$$

Equation (4.33) is a differential equation for the pressure as a function of the cross section area. It is convenient to rearrange equation (4.33) to obtain a variables separation form of

$$dP = \frac{\rho U^2}{A} \frac{dA}{1 - M^2} \tag{4.34}$$

4.2.1.1 The pressure relationship to Mach number

Before going further in the mathematical derivation it is worth in looking at the physical meaning equation (4.34). The term $\rho U^2/A$ is always positive (because all the three terms can be only positive). Now, it can be observed that dP can be positive or negative deepening on the dA and Mach number. The meaning of the sign change for the pressure differential is that the pressure can increase or decrease. It can be observed that the critical Mach number is one. If the Mach number is larger than one than dP has opposite sign of dA. If Mach number is smaller than one dP and dA have the same sign. For the subsonic branch $M < 1$ the term $1/(1 - M^2)$ is positive hence

$$dA > 0 \Longrightarrow dP > 0$$
$$dA < 0 \Longrightarrow dP < 0$$

From these observations the trends are, similar to incompressible fluid, an increase in area results in increase of the static pressure (converting the dynamic pressure to a static pressure). Conversely, if the area decrease (as a function of x) the pressure decreases. Note that the pressure decrease is larger in compressible flow compared to incompressible flow.

For the supersonic branch $M > 1$, the phenomenon is different. For $M > 1$ the term $1/1 - M^2$ is negative and change the character of the equation.

$$dA > 0 \Rightarrow dP < 0$$
$$dA < 0 \Rightarrow dP > 0$$

This behavior is opposite to incompressible flow behavior.

For the special case of $M = 1$ (sonic flow) the value of the term $1 - M^2 = 0$ thus mathematically $dP \to \infty$ or $dA = 0$. Since physically dP can increase only in a finite amount it must that $dA = 0$.It must also be noted that when $M = 1$ occurs only when $dA = 0$. However, the opposite, not necessarily means that when $dA = 0$ that $M = 1$. In that case, it is possible that $dM = 0$ thus in the diverging side is in the subsonic branch and the flow isn't choked.

The relationship between the velocity and the pressure can be observed from equation (4.28) by solving it for dU.

$$dU = -\frac{dP}{PU} \tag{4.35}$$

From equation (4.35) it is obvious that dU has an opposite sign to dP (since the term PU is positive). Hence the pressure increase when the velocity decreases and vice versa.

From the speed of sound, one can observe that the density, ρ, increases with pressure and visa versa (see equation 4.36).

$$d\rho = \frac{1}{c^2} dP \tag{4.36}$$

It can be noted that the derivations of the above equations (4.35 - 4.36), the equation of state was not used. Thus, the equations are applicable for any gas (perfect or imperfect gas).

The second law (isentropic relationship) dictates that $ds = 0$ and from thermodynamics

$$ds = 0 = C_p \frac{dT}{T} - R \frac{dP}{P}$$

and for perfect gas

$$\frac{dT}{T} = \frac{k-1}{k}\frac{dP}{P} \tag{4.37}$$

Thus, the temperature varies according in the same way that Pressure does.

The relationship between the Mach number and the temperature can be obtained by utilizing the fact that the process is assumed to be adiabatic $dT_0 = 0$. Differentiation of equation (4.9), the relationship between the temperature and the stagnation temperature, yields

$$dT_0 = 0 = dT\left(1+\frac{k-1}{2}M^2\right) + T(k-1)MdM \tag{4.38}$$

and simplifying equation (4.38) yields

$$\frac{dT}{T} = -\frac{(k-1)MdM}{1+\frac{k-1}{2}M^2} \tag{4.39}$$

4.2.1.2 Relationship Between the Mach Number and Cross Section Area

The equations used in the solution are energy (4.39), second law (4.37), state (4.29), mass (4.26)[1]. Note, equation (4.33) isn't the solution but demonstration of certain properties on the pressure.

The relationship between temperature and the cross section area can be obtained by utilizing the relationship between the pressure and temperature (4.37) and the relationship of pressure and cross section area (4.33). First stage equation (4.39) is combined with equation (4.37) and becomes

$$\frac{(k-1)}{k}\frac{dP}{P} = -\frac{(k-1)MdM}{1+\frac{k-1}{2}M^2} \tag{4.40}$$

Combining equation (4.40) with equation (4.33) yields

$$\frac{1}{k}\frac{\frac{\rho U^2}{A}\ \frac{dA}{1-M^2}}{P} = -\frac{MdM}{1+\frac{k-1}{2}M^2} \tag{4.41}$$

The following identify, $\rho U^2 = kMP$ can be proved as

$$kM^2P = k\,\overbrace{\frac{U^2}{c^2}}^{M^2}\,\overbrace{\rho RT}^{P} = k\frac{U^2}{kRT}\overbrace{\rho RT}^{P} = \rho U^2 \tag{4.42}$$

Utilizing the identify in equation (4.42) changes equation (4.41) into

$$\frac{dA}{A} = \frac{M^2-1}{M\left(1+\frac{k-1}{2}M^2\right)}dM \tag{4.43}$$

[1]The momentum equation is not used normally in isentropic process, why?

Equation (4.43) is very important because it relates the geometry (area) with the relative velocity (Mach number). In equation (4.43), the factors $M\left(1+\frac{k-1}{2}M^2\right)$ and A are positive regardless to the values of M or A. Therefore, the only factor that effects relationship between the cross area and the Mach number is M^2-1. For $M<1$ the Mach number is varied opposite to the cross section area. In the case of $M>1$ the Mach number increases with cross section area and vise versa. The special case is when $M=1$, in that case requires that $dA=0$. This condition imposes that internal[2] flow has to pass a converting diverging device to obtain supersonic velocity. This minimum area is referred to as "throat."

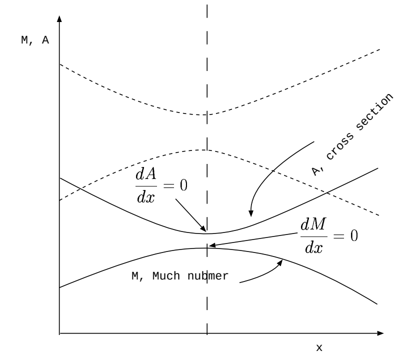


Fig. 4.5: The relationship between the cross section and the Mach number on the subsonic branch

Again, the opposite conclusion that when $dA=0$ implies that $M=1$ is not correct because possibility of $dM=0$. In subsonic flow branch, from the mathematical point of view, on one hand, decrease of the cross section increase the velocity and the Mach number. On the other hand, increase of the cross section decreases the velocity and Mach number (see Figure 4.5).

4.2.2 Examples

Example 4.1:
Air allowed to flow from a reservoir with temperature of 21°C and with pressure of 5[MPa] through a tube. It was measured that air mass flow rate is 1[kg/sec]. At some point on the tube static pressure pressure was measured to be 3[MPa]. Assume that process is isentropic and neglects the velocity at the reservoir, calculate the Mach number, velocity, and the cross section area at that point where the static pressure was measured. Assumed that the ratio of specific heats is $k=C_p/C_v=1.4$.

<u>SOLUTION</u>
The stagnation conditions at the reservoir will be maintained through out tube because the process is isentropic. Hence the stagnation temperature can be written $T_0=constant$ and $P_0=constant$ and both of them are known (the condition at the reservoir). For the point where the static pressure is known, the Mach number can

[2]This condition does not impose any restrictions for external flow. In external flow, an object can be moved in arbitrary speed.

be calculated utilizing the pressure ratio. With known Mach number, the temperature, and velocity can be calculated. Finally, the cross section can be calculated with all these information.

In the point where the static pressure known

$$\bar{P} = \frac{P}{P_0} = \frac{3[MPa]}{5[MPa]} = 0.6$$

From Table (4.2) or from Figure (4.3) or utilizing the enclosed program from Potto-GDC, or simply using the equations shows that

$\mathbf{M}$	$\frac{\mathbf{T}}{\mathbf{T_0}}$	$\frac{\rho}{\rho_0}$	$\frac{\mathbf{A}}{\mathbf{A^*}}$	$\frac{\mathbf{P}}{\mathbf{P_0}}$	$\frac{\mathbf{A\times P}}{\mathbf{A^*\times P_0}}$	$\frac{\mathbf{F}}{\mathbf{F^*}}$
0.88639	0.86420	0.69428	1.0115	0.60000	0.60693	0.53105

With these values the static temperature and the density can be calculated.

$$T = 0.86420338 \times (273 + 21) = 254.076K$$

$$\rho = \frac{\rho}{\rho_0} \overbrace{\frac{P_0}{RT_0}}^{\rho_0} = 0.69428839 \times \frac{5 \times 10^6 [Pa]}{287.0 \left[\frac{J}{kg K}\right] \times 294[K]}$$

$$= 41.1416 \left[\frac{kg}{m^3}\right]$$

The velocity at that point is

$$U = M \overbrace{\sqrt{kRT}}^{c} = 0.88638317 \times \sqrt{1.4 \times 287 \times 294} = 304[m/sec]$$

The tube area can be obtained from the mass conservation as

$$A = \frac{\dot{m}}{\rho U} = 8.26 \times 10^{-5}[m^3] \tag{4.44}$$

For a circular tube the diameter is about 1[cm].

Example 4.2:

The Mach number at point A on tube is measured to be $M^3 = 2$ and the static pressure is 2 [Bar][4]. Downstream at point B the pressure was measured to be 1.5[Bar]. Calculate the Mach number at point B under the isentropic flow assumption. Also, estimate the temperature at point B. Assume that the specific heat ratio $k = 1.4$ and assume a perfect gas model.

---

[3]Well, this question is for academic purpose, there is no known way to this author to directly measure the Mach number. The best approximation is by using inserted cone for supersonic flow and measure the oblique shock. Here it is sub sonic and that technique is not suitable.

[4]This pressure is about two atmospheres with temperature of 250[K]

SOLUTION

With known Mach number at point A all the ratios of the static properties to total (stagnation) properties can be calculated. Therefore, the stagnation pressure at point A is known and stagnation temperature can be calculated.

At $M = 2$ (supersonic flow) the ratios are

$\mathbf{M}$	$\frac{\mathbf{T}}{\mathbf{T_0}}$	$\frac{\rho}{\rho_0}$	$\frac{\mathbf{A}}{\mathbf{A^*}}$	$\frac{\mathbf{P}}{\mathbf{P_0}}$	$\frac{\mathbf{A\times P}}{\mathbf{A^*\times P_0}}$	$\frac{\mathbf{F}}{\mathbf{F^*}}$
2.0000	0.55556	0.23005	1.6875	0.12780	0.21567	0.59309

With this information the pressure at Point B expressed

$$\frac{P_A}{P_0} = \overbrace{\frac{P_B}{P_0}}^{\text{from the table 4.2 @ M = 2}} \times \frac{P_A}{P_B} = 0.12780453 \times \frac{2.0}{1.5} = 0.17040604$$

The corresponding Mach number for this pressure ratio is 1.8137788 and $T_B = 0.60315132$ $\frac{P_B}{P_0} = 0.17040879$. The stagnation temperature can be "bypassed" to calculated the temperature at point $\mathbf{B}$

$$T_B = T_A \times \overbrace{\frac{T_0}{T_A}}^{M=2} \times \overbrace{\frac{T_B}{T_0}}^{M=1.81..} = 250[K] \times \frac{1}{0.55555556} \times 0.60315132 \simeq 271.42[K]$$

Example 4.3:

Gas flows through a converging–diverging duct. At point "A" the cross section area is 50 [cm^2] and the Mach number was measured to be 0.4. At point B in the duct the cross section area is 40 [cm^2]. Find the Mach number at point B. Assume that the flow is isentropic and the gas specific heat ratio is 1.4.

SOLUTION

To obtain the Mach number at point B by finding the ratio of the area to the critical area. This relationship can be obtained by

$$\frac{A_B}{A*} = \frac{A_B}{A_A} \times \frac{A_A}{A^*} = \frac{40}{50} \times \overbrace{1.59014}^{\text{from the Table 4.2}} = 1.272112$$

With the value of $\frac{A_B}{A*}$ from the Table (4.2) or from Potto-GDC two solutions can be obtained. The two possible solutions: the first supersonic M = 1.6265306 and second subsonic M = 0.53884934. Both solution are possible and acceptable. The supersonic branch solution is possible only if there where a transition at throat where M=1.

$\mathbf{M}$	$\frac{\mathbf{T}}{\mathbf{T_0}}$	$\frac{\rho}{\rho_0}$	$\frac{\mathbf{A}}{\mathbf{A^*}}$	$\frac{\mathbf{P}}{\mathbf{P_0}}$	$\frac{\mathbf{A\times P}}{\mathbf{A^*\times P_0}}$
1.6266	0.65396	0.34585	1.2721	0.22617	0.28772
0.53887	0.94511	0.86838	1.2721	0.82071	1.0440

4.2.3 Mass Flow Rate (Number)

One of the important engineering parameters is the mass flow rate which is at any point is

$$\dot{m} = \rho U A = \frac{P}{RT} U A \tag{4.45}$$

This parameter is studied here, to examine the maximum flow rate and to see what is the effect of the compressibility on the flow rate. The area ratio as a function of the Mach number needed to be established, specifically and explicitly the relationship for the chocked flow. The area ratio is defined as the ratio of the cross section at any point to the throat area (the narrow area). It is convenient to rearrange the equation (4.45) to be expressed in terms of the stagnation properties as

$$\frac{\dot{m}}{A} = \frac{P}{P_0}\,\frac{P_0 U}{\sqrt{kRT}}\sqrt{\frac{k}{R}}\sqrt{\frac{T_0}{T}}\,\frac{1}{\sqrt{T_0}} = \frac{P_0}{\sqrt{T_0}} M \sqrt{\frac{k}{R}}\overbrace{\frac{P}{P_0}\sqrt{\frac{T_0}{T}}}^{f(M,k)} \tag{4.46}$$

Expressing the temperature in term of Mach number in equation (4.46) results in

$$\frac{\dot{m}}{A} = \left(\frac{\sqrt{k} M P_0}{\sqrt{RT_0}}\right)\left(1 + \frac{k-1}{2}M^2\right)^{-\frac{k+1}{2(k-1)}} \tag{4.47}$$

It can be noted that equation (4.47) holds everywhere in the converging-diverging duct and this statement also true for the throat. The throat area can be denoted as by A^*. It can be noticed that at the throat when the flow is chocked or in other words $M = 1$ and that the stagnation conditions (i.e. temperature, pressure) do not change. Hence equation (4.47) obtained the form

$$\frac{\dot{m}}{A^*} = \left(\frac{\sqrt{k} P_0}{\sqrt{RT_0}}\right)\left(1 + \frac{k-1}{2}\right)^{-\frac{k+1}{2(k-1)}} \tag{4.48}$$

Since the mass flow rate is constant in the duct, dividing equations (4.48) by equation (4.47) yields

$$\frac{A}{A^*} = \frac{1}{M}\left(\frac{1 + \frac{k-1}{2}M^2}{\frac{k+1}{2}}\right)^{\frac{k+1}{2(k-1)}} \tag{4.49}$$

The equation (4.49), relates the Mach number at specific point at the duct to the cross section area.

The maximum flow rate can be expressed either by taking the derivative of equation (4.49) in with respect to M and equating to zero. Carrying this calculation results at $M = 1$.

$$\left(\frac{\dot{m}}{A^*}\right)_{max} \frac{P_0}{\sqrt{T_0}} = \sqrt{\frac{k}{R}} \left(\frac{k+1}{2}\right)^{-\frac{k+1}{2(k-1)}} \tag{4.50}$$

For specific heat ratio, $k = 1.4$

$$\left(\frac{\dot{m}}{A^*}\right)_{max} \frac{P_0}{\sqrt{T_0}} \sim \frac{0.68473}{\sqrt{R}} \tag{4.51}$$

The maximum flow rate for air ($R = 287 j/kgK$) becomes,

$$\frac{\dot{m}\sqrt{T_0}}{A^* P_0} = 0.040418 \tag{4.52}$$

Equation (4.52) is known as Fliegner's Formula on the name of one of the first engineer who observed experimentally the choking phenomenon.

It can be noticed that Fliengner's equation is actually dimensionless equation and lead to definition of the Fliengener's Number.

$$\frac{\dot{m}\sqrt{T_0}}{A^* P_0} = \frac{\dot{m}\overbrace{\sqrt{kRT_0}}^{c_0}}{\sqrt{kR}A^* P_0} = \frac{\overbrace{\dot{m}c_0}^{Fn}}{A^* P_0} \frac{1}{\sqrt{kR}} \tag{4.53}$$

If the Fliengner's number as above at the every point it will be

$$Fn = kM \left(1 + \frac{k-1}{2} M^2\right)^{-\frac{k+1}{2(k-1)}} \tag{4.54}$$

and the maximum point point

$$Fn = k \left(\frac{k+1}{2}\right)^{-\frac{k+1}{2(k-1)}} \tag{4.55}$$

4.2.3.1 Flow with pressure losses

The expression for the mass flow rate (4.47) is appropriate regardless the flow is isentropic or adiabatic. That expression was derived based on the theoretical total pressure and temperature (Mach number) which does not based on the considerations whether the flow is isentropic or adiabatic. In the same manner the definition of A^* referred to the theoretical minimum area ("throat area") if the flow would continued to flow isentropic manner. Clearly, in a case where the flow isn't isentropic or adiabatic the total pressure and the total temperature change (due to friction, and heat transfer). A constant flow rate requires that $\dot{m}_A = \dot{m}_B$. Denoting subscript A for a point and subscript B or another point mass equation (4.48) can be equated.

$$\left(\frac{kP_0A^*}{RT_0}\right)\left(1+\frac{k-1}{2}M^2\right)^{-\frac{k-1}{2(k-1)}} = constant \tag{4.56}$$

From equation (4.56), it is clear that the function $f(P_0, T_0, A^*) = constant$. There two possible models that can be used to simplify the calculations. The first model for neglected heat transfer (adiabatic) flow and in which the total temperature remained constant (Fanno flow like). The second model which there is significant heat transfer but insignificant pressure loss (Rayleigh flow like).

If the mass flow rate is constant at any point on the tube (no mass loss occur) then

$$\dot{m} = A^*\sqrt{\frac{k}{RT_0}\left(\frac{2}{k+1}\right)^{\frac{k+1}{k-1}}}P_0 \tag{4.57}$$

For adiabatic flow, comparison of mass flow rate at point A and point B leads to

$$P_0A^*|_A = P_0A^*|_B$$

$$\leadsto \frac{P_0|_A}{P_0|_B} = \frac{A^*|_A}{A^*|_B} \tag{4.58}$$

And utilizing the equality of $A^* = \frac{A^*}{A}A$ leads to

$$\frac{P_0|_A}{P_0|_B} = \frac{\left.\frac{A}{A^*}\right|_{M_A}}{\left.\frac{A}{A^*}\right|_{M_B}}\frac{A|_A}{A|_B} \tag{4.59}$$

For a flow with a constant stagnation pressure (frictionless flow) and non adiabatic flow reads

$$\frac{T_0|_A}{T_0|_B} = \left[\frac{\left.\frac{B}{A^*}\right|_{M_B}}{\left.\frac{A}{A^*}\right|_{M_A}}\frac{A|_B}{A|_A}\right]^2 \tag{4.60}$$

Example 4.4:

At point A of the tube the pressure is $3[Bar]$, Mach number is 2.5, and the duct section area is 0.01$[m^2]$. Downstream at exit of tube, point B, the cross section area is 0.015$[m^2]$ and Mach number is 1.5. Assume no mass lost and adiabatic steady state flow, calculated the total pressure lost.

SOLUTION

Both Mach numbers are known, thus the area ratios can be calculated. The total pressure can be calculated because the Mach number and static pressure are known. With these information, and utilizing equation (4.59) the stagnation pressure at point B can be obtained.

$\mathbf{M}$	$\frac{\mathbf{T}}{\mathbf{T_0}}$	$\frac{\rho}{\rho_0}$	$\frac{\mathbf{A}}{\mathbf{A^\star}}$	$\frac{\mathbf{P}}{\mathbf{P_0}}$	$\frac{\mathbf{A\times P}}{\mathbf{A^*\times P_0}}$	$\frac{\mathbf{F}}{\mathbf{F^*}}$
1.5000	0.68966	0.39498	1.1762	0.27240	0.32039	0.55401
2.5000	0.44444	0.13169	2.6367	0.05853	0.15432	0.62693

First, the stagnation at point A is obtained from Table 4.2 as

$$P_0|_A = \left.\frac{P}{\underbrace{\left(\frac{P}{P_0}\right)}_{M=2.5}}\right|_A = \frac{3}{0.058527663} = 51.25781291[Bar]$$

Utilizing equation (4.59) provides

$$P_0|_B = 51.25781291 \times \frac{1.1761671}{2.6367187} \times \frac{0.01}{0.015} \approx 15.243[Bar]$$

Hence

$$P_0|_A - P_0|_B = 51.257 - 15.243 = 36.013[Bar]$$

Note that the large total pressure loss is much larger than the static pressure loss (Pressure point B the pressure is $0.27240307 \times 15.243 = 4.146$[Bar]).

4.3 Isentropic Tables

Table 4.2: Isentropic Table $k = 1.4$

$\mathbf{M}$	$\frac{\mathbf{T}}{\mathbf{T_0}}$	$\frac{\rho}{\rho_0}$	$\frac{\mathbf{A}}{\mathbf{A^\star}}$	$\frac{\mathbf{P}}{\mathbf{P_0}}$	$\frac{\mathbf{A\times P}}{\mathbf{A^*\times P_0}}$	$\frac{\mathbf{F}}{\mathbf{F^*}}$
$0.0E+0$	1.00000	1.00000	$5.8E+5$	1.0000	$5.8E+5$	$2.4E+5$
0.050	0.99950	0.99875	11.59	0.99825	11.57	4.838
0.100	0.99800	0.99502	5.822	0.99303	5.781	2.443
0.200	0.99206	0.98028	2.964	0.97250	2.882	1.268
0.300	0.98232	0.95638	2.035	0.93947	1.912	0.89699
0.400	0.96899	0.92427	1.590	0.89561	1.424	0.72632
0.500	0.95238	0.88517	1.340	0.84302	1.130	0.63535

Table 4.2: Isentropic Table $k = 1.4$ (continue)

$\mathbf{M}$	$\frac{\mathbf{T}}{\mathbf{T_0}}$	$\frac{\rho}{\rho_0}$	$\frac{\mathbf{A}}{\mathbf{A^\star}}$	$\frac{\mathbf{P}}{\mathbf{P_0}}$	$\frac{\mathbf{A\times P}}{\mathbf{A^*\times P_0}}$	$\frac{\mathbf{F}}{\mathbf{F^*}}$
0.600	0.93284	0.84045	1.188	0.78400	0.93155	0.58377
0.700	0.91075	0.79158	1.094	0.72093	0.78896	0.55425
0.800	0.88652	0.73999	1.038	0.65602	0.68110	0.53807
0.900	0.86059	0.68704	1.009	0.59126	0.59650	0.53039
1.00	0.83333	0.63394	1.000	0.52828	0.52828	0.52828
1.100	0.80515	0.58170	1.008	0.46835	0.47207	0.52989
1.200	0.77640	0.53114	1.030	0.41238	0.42493	0.53399
1.300	0.74738	0.48290	1.066	0.36091	0.38484	0.53974
1.400	0.71839	0.43742	1.115	0.31424	0.35036	0.54655
1.500	0.68966	0.39498	1.176	0.27240	0.32039	0.55401
1.600	0.66138	0.35573	1.250	0.23527	0.29414	0.56182
1.700	0.63371	0.31969	1.338	0.20259	0.27099	0.56976
1.800	0.60680	0.28682	1.439	0.17404	0.25044	0.57768
1.900	0.58072	0.25699	1.555	0.14924	0.23211	0.58549
2.000	0.55556	0.23005	1.688	0.12780	0.21567	0.59309
2.500	0.44444	0.13169	2.637	0.058528	0.15432	0.62693
3.000	0.35714	0.076226	4.235	0.027224	0.11528	0.65326
3.500	0.28986	0.045233	6.790	0.013111	0.089018	0.67320
4.000	0.23810	0.027662	10.72	0.00659	0.070595	0.68830
4.500	0.19802	0.017449	16.56	0.00346	0.057227	0.69983
5.000	0.16667	0.011340	25.00	0.00189	0.047251	0.70876
5.500	0.14184	0.00758	36.87	0.00107	0.039628	0.71578
6.000	0.12195	0.00519	53.18	0.000633	0.033682	0.72136
6.500	0.10582	0.00364	75.13	0.000385	0.028962	0.72586
7.000	0.092593	0.00261	$1.0E{+}2$	0.000242	0.025156	0.72953
7.500	0.081633	0.00190	$1.4E{+}2$	0.000155	0.022046	0.73257
8.000	0.072464	0.00141	$1.9E{+}2$	0.000102	0.019473	0.73510
8.500	0.064725	0.00107	$2.5E{+}2$	$6.90E{-}5$	0.017321	0.73723
9.000	0.058140	0.000815	$3.3E{+}2$	$4.74E{-}5$	0.015504	0.73903
9.500	0.052493	0.000631	$4.2E{+}2$	$3.31E{-}5$	0.013957	0.74058
10.00	0.047619	0.000495	$5.4E{+}2$	$2.36E{-}5$	0.012628	0.74192

4.4 Isentropic Isothermal Flow Nozzle

4.4.1 General Relationship

In this section, the other extreme case model where the heat transfer to the gas is perfect, (e.g. Eckert number is very small) is presented. Again in reality the heat transfer is somewhere in between the two extremes. So, knowing the two limits provides a tool to examine where the reality should be expected. The perfect gas model is again assumed (later more complex models can be assumed and constructed in a future versions). In isothermal process the perfect gas model reads

$$P = \rho R T \leadsto dP = d\rho R T \tag{4.61}$$

Substituting equation (4.61) into the momentum equation[5] yields

$$U dU + \frac{RT dP}{P} = 0 \tag{4.62}$$

Integration of equation (4.62) yields the Bernoulli's equation for ideal gas in isothermal process which reads

$$\leadsto \frac{{U_2}^2 - {U_1}^2}{2} + RT \ln \frac{P_2}{P_1} = 0 \tag{4.63}$$

Thus, The velocity at point 2 becomes

$$U_2 = \sqrt{2RT \ln \frac{P_2}{P_1} - {U_1}^2} \tag{4.64}$$

The velocity at point 2 for stagnation point, $U_1 \approx 0$ reads

$$U_2 = \sqrt{2RT \ln \frac{P_2}{P_1}} \tag{4.65}$$

Or in explicit terms of the stagnation properties the velocity is

$$U = \sqrt{2RT \ln \frac{P}{P_0}} \tag{4.66}$$

Transform from equation (4.63) to a dimensionless form becomes

$$\leadsto \frac{kR\cancelto{constant}{T}({M_2}^2 - {M_1}^2)}{2} = R\cancelto{constant}{T} \ln \frac{P_2}{P_1} \tag{4.67}$$

Simplifying equation (4.67) yields

$$\leadsto \frac{k({M_2}^2 - {M_1}^2)}{2} = \ln \frac{P_2}{P_1} \tag{4.68}$$

Or in terms of the pressure ratio equation (4.68) reads

$$\frac{P_2}{P_1} = e^{\frac{k({M_1}^2 - {M_2}^2)}{2}} = \left(\frac{e^{{M_1}^2}}{e^{{M_2}^2}}\right)^{\frac{k}{2}} \tag{4.69}$$

---

[5]The one dimensional momentum equation for steady state is $UdU/dx = -dP/dx + 0$(other effects) which are neglected here.

As oppose to the adiabatic case ($T_0 = constant$) in the isothermal flow the stagnation temperature ratio can be expressed

$$\frac{T_{01}}{T_{02}} = \cancelto{1}{\frac{T_1}{T_2}} \frac{\left(1+\frac{k-1}{2}{M_1}^2\right)}{\left(1+\frac{k-1}{2}{M_2}^2\right)} = \frac{\left(1+\frac{k-1}{2}{M_1}^2\right)}{\left(1+\frac{k-1}{2}{M_2}^2\right)} \tag{4.70}$$

Utilizing conservation of the mass $A\rho M = constant$ to yield

$$\frac{A_1}{A_2} = \frac{M_2 P_2}{M_1 P_1} \tag{4.71}$$

Combing equation (4.71) and equation (4.69) yields

$$\frac{A_2}{A_1} = \frac{M_1}{M_2}\left(\frac{\mathrm{e}^{{M_2}^2}}{\mathrm{e}^{{M_1}^2}}\right)^{\frac{k}{2}} \tag{4.72}$$

The change in the stagnation pressure can be expressed as

$$\frac{P_{02}}{P_{01}} = \frac{P_2}{P_1}\left(\frac{1+\frac{k-1}{2}{M_2}^2}{1+\frac{k-1}{2}{M_1}^2}\right)^{\frac{k}{k-1}} = \left[\frac{\mathrm{e}^{{M_1}^2}}{\mathrm{e}^{{M_1}^2}}\right]^{\frac{k}{2}} \tag{4.73}$$

The critical point, at this stage, is unknown (at what Mach number the nozzle is choked is unknown) so there are two possibilities: the choking point or $M = 1$ to normalize the equation. Here the critical point defined as the point where $M = 1$ so results can be compared to the adiabatic case and denoted by star. Again it has to emphasis that this critical point is not really related to physical critical point but it is arbitrary definition. The true critical point is when flow is choked and the relationship between two will be presented.

The critical pressure ratio can be obtained from (4.69) to read

$$\frac{P}{P^*} = \frac{\rho}{\rho^*} = \mathrm{e}^{\frac{(1-M^2)k}{2}} \tag{4.74}$$

Equation (4.72) is reduced to obtained the critical area ratio writes

$$\frac{A}{A^*} = \frac{1}{M}\mathrm{e}^{\frac{(1-M^2)k}{2}} \tag{4.75}$$

Similarly the stagnation temperature reads

$$\frac{T_0}{{T_0}^*} = \frac{2\left(1+\frac{k-1}{2}{M_1}^2\right)}{k+1}^{\frac{k}{k-1}} \tag{4.76}$$

Finally, the critical stagnation pressure reads

$$\frac{P_0}{{P_0}^*} = e^{\frac{(1-M)k}{2}} \frac{2\left(1+\frac{k-1}{2}{M_1}^2\right)}{k+1}^{\frac{k}{k-1}} \tag{4.77}$$

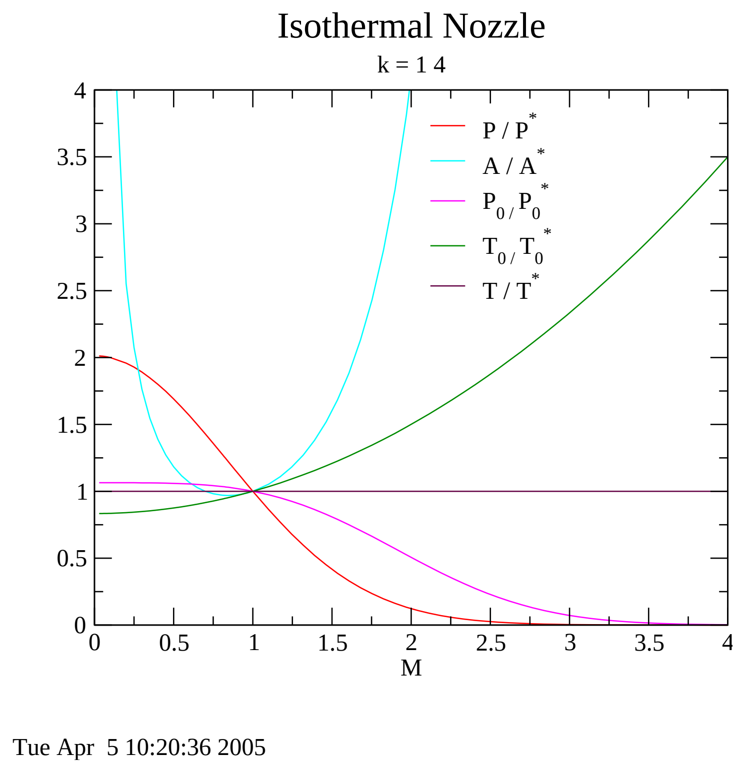


Fig. 4.6: Various ratios as a function of Mach number for isothermal Nozzle

Of course in isothermal process $T = T^*$. All these equations are plotted in Figure (4.6). From the Figure 4.3 it can be observed that minimum of the curve A/A^* isn't on $M = 1$. The minimum of the curve is when area is minimum and at the point where the flow is choked. It should be noted that the stagnation temperature is not constant as in the adiabatic case and the critical point is the only one constant.

The mathematical procedure to find the minimum is simply taking the derivative and equating to zero as following

$$\frac{d\left(\frac{A}{A^*}\right)}{dM} = \frac{kM^2 exp^{\frac{k(M^2-1)}{2}} - exp^{\frac{k(M^2-1)}{2}}}{M^2} = 0 \tag{4.78}$$

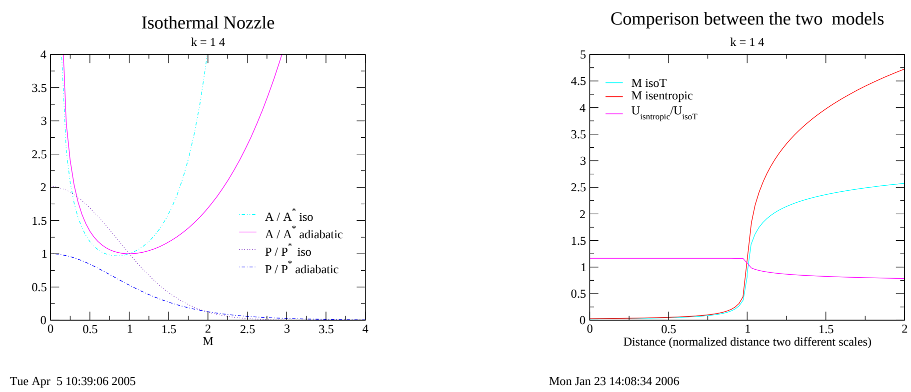
Equation (4.78) simplified to

$$kM^2 - 1 = 0 \leadsto M = \frac{1}{\sqrt{k}} \tag{4.79}$$

It can be noticed that a similar results are obtained for adiabatic flow. The velocity at the throat of isothermal model is smaller by a factor of $\sqrt{k}$. Thus, dividing the critical adiabatic velocity by $\sqrt{k}$ results in

$$U_{throat_{max}} = \sqrt{RT} \tag{4.80}$$

On the other hand, the pressure loss in adiabatic flow is milder as can be seen in Figure 4.7(a).



(a) Comparison between the isothermal nozzle and adiabatic nozzle in various variables

(b) The comparison of the adiabatic model and isothermal model

Fig. 4.7: The comparison of nozzle flow

It should be emphasis that the stagnation pressure decrees. It is convenient to find expression for the ratio of the initial stagnation pressure (the stagnation pressure before entering the nozzle) to the pressure at the throat. Utilizing equation (4.74) the following relationship can be obtained

$$\frac{P_{throat}}{P_{0_{initial}}} = \frac{P^*}{P_{0_{initial}}} \frac{P_{throat}}{P^*} =$$

$$\frac{1}{e^{\frac{(1-0^2)k}{2}}} e^{\left(1-\left(\frac{1}{\sqrt{k}}\right)^2\right)\frac{k}{2}} =$$

$$e^{-\frac{1}{2}} = 0.60653 \tag{4.81}$$

Notice that the critical pressure is independent of the specific heat ratio, k, as oppose to the adiabatic case. It also has to be emphasized that the stagnation values of the isothermal model are not constant. Again, the heat transfer is expressed as

$$Q = C_p \left(T_{0_2} - T_{0_2}\right) \tag{4.82}$$

For comparison between the adiabatic model and the isothermal a simple profile of nozzle area as a function of the distance is assumed. This profile isn't ideal profile but rather a simple sample just to examine the difference between the two models so in actual situation can be bounded. To make senses and eliminate unnecessary details the distance from the entrance to the throat is normalized (to one). In the same fashion the distance from the throat to the exit is normalized (to one) (it isn't mean that these distances are the same). In this comparison the entrance area ratio and the exit area ratio are the same and equal to 20. The Mach number was computed for the two models and plotted in the Figure (4.7(b)). In this comparison it has to be remembered that critical area for the two models are different by about 3% (for $k = 1.4$). As can be observed from the Figure (4.7(b)). The Mach number for the isentropic is larger for the supersonic branch but the velocity is lower. The ratio of the velocities can be expressed as

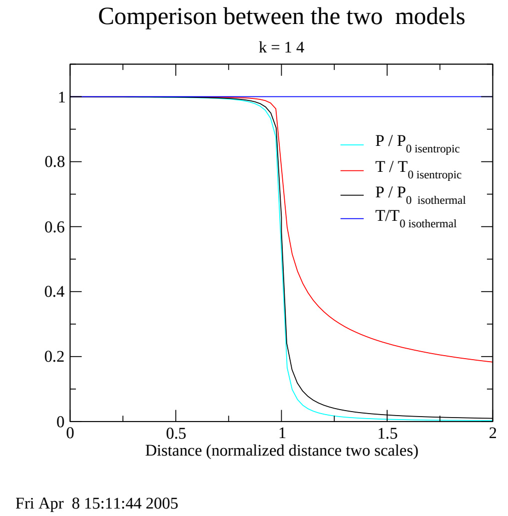


Fig. 4.8: Comparison of the pressure and temperature drop as a function of the normalized length (two scales)

$$\frac{U_s}{U_T} = \frac{M_s\sqrt{kRT_s}}{M_T\sqrt{kRT_s}} \tag{4.83}$$

It can be noticed that temperature in the isothermal model is constant while temperature in the adiabatic model can be expressed as a function of the stagnation temperature. The initial stagnation temperatures are almost the same and can be canceled out to obtain

$$\frac{U_s}{U_T} \sim \frac{M_s}{M_T\sqrt{1 + \frac{k-1}{2}{M_s}^2}} \tag{4.84}$$

Utilizing equation (4.84) the velocity ratio was obtained is plotted in Figure 4.7(b).

Thus, using the isentropic model results in under prediction of the actual results for the velocity in the supersonic branch. While, the isentropic for the subsonic branch will be over prediction. The prediction of the Mach number are similar shown in the Figure 4.7(b).

Two other ratios need to be examined: temperature and pressure. The initial stagnation temperature is denoted as T_{0int}. The temperature ratio of T/T_{0int} can be obtained via the isentropic model as

$$\frac{T}{T_{0int}} = \frac{1}{1 + \frac{k-1}{2} M^2} \tag{4.85}$$

While the temperature ratio of the isothermal model is constant and equal to one (1). The pressure ratio for the isentropic model is

$$\frac{P}{P_{0int}} = \frac{1}{\left(1 + \frac{k-1}{2} M^2\right)^{\frac{k-1}{k}}} \tag{4.86}$$

and for the isothermal process the stagnation pressure varies and had to be taken into account as following

$$\frac{P_z}{P_{0int}} = \frac{{P_0}^*}{P_{0int}} \frac{P_{0z}}{{P_0}^*} \overbrace{\frac{P_z}{P_{0z}}}^{isentropic} \tag{4.87}$$

where the z is an arbitrary point on the nozzle. Utilizing equations (4.73) and the isentropic relationship provides the sought ratio.

Figure 4.8 shows that the range between the predicted temperatures of the two models is very large. While the range between the predicted pressure by the two models is relatively small. The meaning of this analysis is that transfered heat affects the temperature in larger degree but the effect on the pressure much less significant.

To demonstrate relativity of the approached of advocated in this book consider the following example.

Example 4.5:
Consider a diverging–converging nozzle made out wood (low conductive material) with exit area equal entrance area. The throat area ratio to entrance area 1:4 respectively. The stagnation pressure is 5$[Bar]$ and the stagnation temperature is 27°C. Assume that the back pressure is low enough to have supersonic flow without shock and $k = 1.4$. Calculate the velocity at the exit using the adiabatic model? If the nozzle was made from copper (a good heat conductor) a larger heat transfer occurs, should the velocity increase or decrease? what is the maximum possible increase?

SOLUTION

The first part of the question deals with the adiabatic model i.e. the conservation of the stagnation properties. Thus, with known area ratio and known stagnation the GDC-Potto provides the following table:

$\mathbf{M}$	$\frac{\mathbf{T}}{\mathbf{T_0}}$	$\frac{\rho}{\rho_0}$	$\frac{\mathbf{A}}{\mathbf{A^\star}}$	$\frac{\mathbf{P}}{\mathbf{P_0}}$	$\frac{\mathbf{A\times P}}{\mathbf{A^*\times P_0}}$
0.14655	0.99572	0.98934	4.0000	0.98511	3.9405
2.9402	0.36644	0.08129	4.0000	0.02979	0.11915

With known Mach number, and temperature at the exit the velocity can be calculated. The exit temperature is $0.36644 \times 300 = 109.9K$. The exit velocity, then, is

$$U = M\sqrt{kRT} = 2.9402\sqrt{1.4 \times 287 \times 109.9} \sim 617.93[m/sec]$$

Even for the isothermal model, the initial stagnation temperature is given as $300K$. With the area ratio by using the Figure 4.6 or using the Potto–GDC obtains the following table is obtained

$\mathbf{M}$	$\frac{\mathbf{T}}{\mathbf{T_0}}$	$\frac{\rho}{\rho_0}$	$\frac{\mathbf{A}}{\mathbf{A^\star}}$	$\frac{\mathbf{P}}{\mathbf{P_0}}$	$\frac{\mathbf{A\times P}}{\mathbf{A^*\times P_0}}$
1.9910	1.4940	0.51183	4.0000	0.12556	0.50225

The exit Mach number is known and the initial temperature to the throat temperature ratio can be calculated as following:

$$\frac{T_{0_{ini}}}{{T_0}^*} = \frac{1}{1+\frac{k-1}{2}\frac{1}{k}} = \frac{1}{1+\frac{k-1}{k}} = 0.777777778 \tag{4.88}$$

Thus the stagnation temperature at the exit is

$$\frac{T_{0_{ini}}}{T_{0_{exit}}} = 1.4940/0.777777778 = 1.921$$

The exit stagnation temperature is $1.92 \times 300 = 576.2K$. The exit velocity determined by utilizing the following equation

$$U_{exit} = M\sqrt{kRT} = 1.9910\sqrt{1.4 \times 287 \times 300.0} = 691.253[m/sec]$$

As it was discussed before the velocity in copper nozzle will be larger than velocity in the wood nozzle. However, the maximum velocity can not exceed the $691.253[m/sec]$

4.5 The Impulse Function

4.5.1 Impulse in Isentropic Adiabatic Nozzle

One of the function that used in calculation of the forces is the Impulse function. The Impulse Function is denoted here as F, but in the literature some denote this function as I. To explain the motivation for using this definition consider calculation of the net forces that acting on section shown in Figure (4.9). To calculate the net forces that acting on the section shown in the Figure in the x direction the momentum equation has to be applied as

$$F_{net} = \dot{m}(U_2 - U_1) + P_2A_2 - P_1A_1 \tag{4.89}$$

The net force is denoted here as F_{net}. The mass conservation also can be applied to our control volume

$$\dot{m} = \rho_1 A_1 U_1 = \rho_2 A_2 U_2 \tag{4.90}$$

Combining equations (4.89) with equation (4.90) and utilizing the identity in equation (**??**) results in

$$F_{net} = kP_2 A_2 {M_2}^2 - kP_1 A_1 {M_1}^2 - P_2 A_2 - P_1 A_1 \tag{4.91}$$

Rearranging equation (4.91) and dividing it by $P_0 A^*$ results in

$$\frac{F_{net}}{P_0 A^*} = \overbrace{\frac{P_2 A_2}{P_0 A^*}}^{f(M_2)} \overbrace{\left(1 + k{M_2}^2\right)}^{f(M_2)} - \overbrace{\frac{P_1 A_1}{P_0 A^*}}^{f(M_1)} \overbrace{\left(1 + k{M_1}^2\right)}^{f(M_1)} \tag{4.92}$$

Examining equation (4.92) shows that right hand side is only function of Mach number and specific heat ratio, k. Hence, if the right hand side is only function of the Mach number and k than the left hand side must be function of only the same parameters, M and k. Defining a function that depends only on the Mach number creates the convenience for calculation the net forces action on any device. Thus, defining the Impulse function as

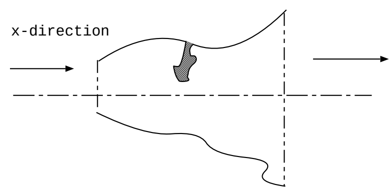


Fig. 4.9: Schematic to explain the significances of the Impulse function

$$F = PA\left(1 + k{M_2}^2\right) \tag{4.93}$$

In the Impulse function when $F\,(M=1)$ is denoted as F^*

$$F^* = P^* A^* \left(1 + k\right) \tag{4.94}$$

The ratio of the impulse function is defined as

$$\frac{F}{F^*} = \frac{P_1 A_1}{P^* A^*} \frac{\left(1 + k{M_1}^2\right)}{(1+k)} = \frac{1}{\underbrace{\frac{P^*}{P_0}}_{\left(\frac{2}{k+1}\right)^{\frac{k}{k-1}}}} \overbrace{\frac{P_1 A_1}{P_0 A^*}\left(1 + k{M_1}^2\right)}^{\text{see function (4.92)}} \frac{1}{(1+k)} \tag{4.95}$$

This ratio is different only in a coefficient from the ratio defined in equation (**??**) which make the ratio function of k and the Mach number. Hence, the net force is

$$F_{net} = P_0 A^* (1+k) \left(\frac{k+1}{2}\right)^{\frac{k}{k-1}} \left(\frac{F_2}{F^*} - \frac{F_1}{F^*}\right) \tag{4.96}$$

To demonstrate the usefulness of the this function consider the simple situation of the flow through a converging nozzle

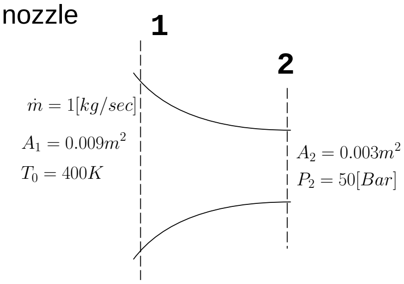


Fig. 4.10: *Schematic of a flow of a compressible substance (gas) thorough a converging nozzle for example* (4.6)

Example 4.6:

Consider a flow of gas into a converging nozzle with a mass flow rate of $1[kg/sec]$ *and the entrance area is* $0.009[m^2]$ *and the exit area is* $0.003[m^2]$. *The stagnation temperature is* $400K$ *and the pressure at point 2 was measured as* $5[Bar]$ *Calculate the the net force acting on the nozzle and pressure at point 1.*

SOLUTION

The solution is obtained by getting the data for the Mach number. To obtained the Mach number, the ratio of P_1A_1/A^*P_0 is needed to be calculated. To obtain this ratio the denominator is needed to be obtained. Utilizing Fliegner's equation (4.52), provides the following

$$A^*P_0 = \frac{\dot{m}\sqrt{RT}}{0.058} = \frac{1.0 \times \sqrt{400 \times 287}}{0.058} \sim 70061.76[N]$$

and

$$\frac{A_2P_2}{A^*P_0} = \frac{500000 \times 0.003}{70061.76} \sim 2.1$$

$\mathbf{M}$	$\frac{\mathbf{T}}{\mathbf{T_0}}$	$\frac{\rho}{\rho_0}$	$\frac{\mathbf{A}}{\mathbf{A^\star}}$	$\frac{\mathbf{P}}{\mathbf{P_0}}$	$\frac{\mathbf{A\times P}}{\mathbf{A^*\times P_0}}$	$\frac{\mathbf{F}}{\mathbf{F^*}}$
0.27353	0.98526	0.96355	2.2121	0.94934	2.1000	0.96666

With the area ratio of $\frac{A}{A^\star} = 2.2121$ the area ratio of at point 1 can be calculated.

$$\frac{A_1}{A^\star} = \frac{A_2}{A^\star}\frac{A_1}{A_2} = 2.2121 \times \frac{0.009}{0.003} = 5.2227$$

And utilizing again Potto-GDC provides

$\mathbf{M}$	$\frac{\mathbf{T}}{\mathbf{T_0}}$	$\frac{\rho}{\rho_0}$	$\frac{\mathbf{A}}{\mathbf{A^\star}}$	$\frac{\mathbf{P}}{\mathbf{P_0}}$	$\frac{\mathbf{A\times P}}{\mathbf{A^*\times P_0}}$	$\frac{\mathbf{F}}{\mathbf{F^*}}$
0.11164	0.99751	0.99380	5.2227	0.99132	5.1774	2.1949

The pressure at point 1 is

$$P_1 = P_2\frac{P_0}{P_2}\frac{P_1}{P_0} = 5.0 times 0.94934/0.99380 \sim 4.776[Bar]$$

The net force is obtained by utilizing equation (4.96)

$$
\begin{aligned}
F_{net} &= P_2 A_2 \frac{P_0 A^*}{P_2 A_2}(1+k)\left(\frac{k+1}{2}\right)^{\frac{k}{k-1}}\left(\frac{F_2}{F^*}-\frac{F_1}{F^*}\right) \\
&= 500000 \times \frac{1}{2.1} \times 2.4 \times 1.2^{3.5} \times (2.1949 - 0.96666) \sim 614[kN]
\end{aligned}
$$

4.5.2 The Impulse Function in Isothermal Nozzle

Previously Impulse function was developed in the isentropic adiabatic flow. The same is done here for the isothermal nozzle flow model. As previously, the definition of the Impulse function is reused. The ratio of the impulse function for two points on the nozzle is

$$
\frac{F_2}{F_1} = \frac{P_2 A_2 + \rho_2 {U_2}^2 A_2}{P_1 A_1 + \rho_1 {U_1}^2 A_1} \tag{4.97}
$$

Utilizing the ideal gas model for density and some rearrangement results in

$$
\frac{F_2}{F_1} = \frac{P_2 A_2}{P_1 A_1} \frac{1 + \frac{{U_2}^2}{RT}}{1 + \frac{{U_1}^2}{RT}} \tag{4.98}
$$

Since $U^2/RT = kM^2$ and the ratio of equation (4.71) transformed equation into (4.98)

$$
\frac{F_2}{F_1} = \frac{M_1}{M_2} \frac{1 + k{M_2}^2}{1 + k{M_1}^2} \tag{4.99}
$$

At the star condition ($M = 1$) (not the minimum point) results in

$$
\frac{F_2}{F^*} = \frac{1}{M_2} \frac{1 + k{M_2}^2}{1 + k} \tag{4.100}
$$

4.6 Isothermal Table

Table 4.3: Isothermal Table

$\mathbf{M}$	$\frac{\mathbf{T_0}}{\mathbf{T_0}^\star}$	$\frac{\mathbf{P_0}}{\mathbf{P_0}^\star}$	$\frac{\mathbf{A}}{\mathbf{A}^\star}$	$\frac{\mathbf{P}}{\mathbf{P}^\star}$	$\frac{\mathbf{A \times P}}{\mathbf{A^* \times P_0}}$	$\frac{\mathbf{F}}{\mathbf{F}^*}$
0.00	0.52828	1.064	$5.0E+5$	2.014	$1.0E+6$	$4.2E+5$
0.05	0.52921	1.064	9.949	2.010	20.00	8.362
0.1	0.53199	1.064	5.001	2.000	10.00	4.225
0.2	0.54322	1.064	2.553	1.958	5.000	2.200

Table 4.3: Isothermal Table (continue)

$\mathbf{M}$	$\frac{\mathbf{T_0}}{\mathbf{T_0}^{\star}}$	$\frac{\mathbf{P_0}}{\mathbf{P_0}^{\star}}$	$\frac{\mathbf{A}}{\mathbf{A}^{\star}}$	$\frac{\mathbf{P}}{\mathbf{P}^{\star}}$	$\frac{\mathbf{A}\times\mathbf{P}}{\mathbf{A}^{*}\times\mathbf{P_0}}$	$\frac{\mathbf{F}}{\mathbf{F}^{*}}$
0.3	0.56232	1.063	1.763	1.891	3.333	1.564
0.4	0.58985	1.062	1.389	1.800	2.500	1.275
0.5	0.62665	1.059	1.183	1.690	2.000	1.125
0.6	0.67383	1.055	1.065	1.565	1.667	1.044
0.7	0.73278	1.047	0.99967	1.429	1.429	1.004
0.8	0.80528	1.036	0.97156	1.287	1.250	0.98750
0.9	0.89348	1.021	0.97274	1.142	1.111	0.98796
1.00	1.000	1.000	1.000	1.000	1.000	1.000
1.10	1.128	0.97376	1.053	0.86329	0.90909	1.020
1.20	1.281	0.94147	1.134	0.73492	0.83333	1.047
1.30	1.464	0.90302	1.247	0.61693	0.76923	1.079
1.40	1.681	0.85853	1.399	0.51069	0.71429	1.114
1.50	1.939	0.80844	1.599	0.41686	0.66667	1.153
1.60	2.245	0.75344	1.863	0.33554	0.62500	1.194
1.70	2.608	0.69449	2.209	0.26634	0.58824	1.237
1.80	3.035	0.63276	2.665	0.20846	0.55556	1.281
1.90	3.540	0.56954	3.271	0.16090	0.52632	1.328
2.00	4.134	0.50618	4.083	0.12246	0.50000	1.375
2.50	9.026	0.22881	15.78	0.025349	0.40000	1.625
3.000	19.41	0.071758	90.14	0.00370	0.33333	1.889
3.500	40.29	0.015317	$7.5E+2$	0.000380	0.28571	2.161
4.000	80.21	0.00221	$9.1E+3$	$2.75E-5$	0.25000	2.438
4.500	$1.5E+2$	0.000215	$1.6E+5$	$1.41E-6$	0.22222	2.718
5.000	$2.8E+2$	$1.41E-5$	$4.0E+6$	0.0	0.20000	3.000
5.500	$4.9E+2$	0.0	$1.4E+8$	0.0	0.18182	3.284
6.000	$8.3E+2$	0.0	$7.3E+9$	0.0	0.16667	3.569
6.500	$1.4E+3$	0.0	$5.3E+11$	0.0	0.15385	3.856
7.000	$2.2E+3$	0.0	$5.6E+13$	0.0	0.14286	4.143
7.500	$3.4E+3$	0.0	$8.3E+15$	0.0	0.13333	4.431
8.000	$5.2E+3$	0.0	$1.8E+18$	0.0	0.12500	4.719
8.500	$7.7E+3$	0.0	$5.4E+20$	0.0	0.11765	5.007
9.000	$1.1E+4$	0.0	$2.3E+23$	0.0	0.11111	5.296
9.500	$1.6E+4$	0.0	$1.4E+26$	0.0	0.10526	5.586
10.00	$2.2E+4$	0.0	$1.2E+29$	0.0	0.100000	5.875

4.7 The effects of Real Gases

To obtained expressions for non–ideal gas it is communally done by reusing the ideal gas model and introducing a new variable which is a function of the gas properties like the critical pressure and critical temperature. Thus, a real gas equation can be expressed in equation (3.19). Differentiating equation (3.19) and dividing by equation (3.19) yields

$$\frac{dP}{P} = \frac{dz}{z} + \frac{d\rho}{\rho} + \frac{dT}{T} \tag{4.101}$$

Again, Gibb's equation (4.27) is reused to related the entropy change to the change in thermodynamics properties and applied on non-ideal gas. Since $ds = 0$ and utilizing the equation of the state $dh = dP/\rho$. The enthalpy is a function of the temperature and pressure thus, $h = h(T, P)$ and full differential is

$$dh = \left(\frac{\partial h}{\partial T}\right)_P dT + \left(\frac{\partial h}{\partial P}\right)_T dP \tag{4.102}$$

The definition of pressure specific heat is $C_p \equiv \frac{\partial h}{\partial T}$ and second derivative is Maxwell relation hence,

$$\left(\frac{\partial h}{\partial P}\right)_T = v - T\left(\frac{\partial s}{\partial T}\right)_P \tag{4.103}$$

First, the differential of enthalpy is calculated for real gas equation of state as

$$dh = C_p dT - \left(\frac{T}{Z}\right)\left(\frac{\partial z}{\partial T}\right)_P \frac{dP}{\rho} \tag{4.104}$$

Equations (4.27) and (3.19) are combined to form

$$\frac{ds}{R} = \frac{C_p}{R}\frac{dT}{T} - z\left[1 + \left(\frac{T}{Z}\right)\left(\frac{\partial z}{\partial T}\right)_P\right]\frac{dP}{P} \tag{4.105}$$

The mechanical energy equation can be expressed as

$$\int d\left(\frac{U^2}{2}\right) = -\int \frac{dP}{\rho} \tag{4.106}$$

At the stagnation the definition requires that the velocity is zero. To carry the integration of the right hand side the relationship between the pressure and the density has to be defined. The following power relationship is assumed

$$\frac{\rho}{\rho_0} = \left(\frac{P}{P_0}\right)^{\frac{1}{n}} \tag{4.107}$$

Notice, that for perfect gas the n is substituted by k. With integration of equation (4.106) when using relationship which is defined in equation (4.107) results

$$\frac{U^2}{2} = \int_{P_0}^{P_1} \frac{dP}{\rho} = \int_{P_0}^{P} \frac{1}{\rho_0}\left(\frac{P_0}{P}\right)^{\frac{1}{n}} dP \tag{4.108}$$

Substituting relation for stagnation density (3.19) results

$$\frac{U^2}{2} = \int_{P_0}^{P} \frac{z_0 R T_0}{P_0} \left(\frac{P_0}{P}\right)^{\frac{1}{n}} dP \tag{4.109}$$

For $n > 1$ the integration results in

$$U = \sqrt{z_0 R T_0 \frac{2n}{n-1} \left[1 - \left(\frac{P}{P_0}\right)^{\left(\frac{n-1}{n}\right)}\right]} \tag{4.110}$$

For $n = 1$ the integration becomes

$$U = \sqrt{2 z_0 R T_0 \ln\left(\frac{P_0}{P}\right)} \tag{4.111}$$

It must be noted that n is a function of the critical temperature and critical pressure. The mass flow rate is regardless to equation of state as following

$$\dot{m} = \rho^* A^* U^* \tag{4.112}$$

Where ρ^* is the density at the throat (assuming the chocking condition) and A^* is the cross area of the throat. Thus, the mass flow rate in our properties

$$\dot{m} = A^* \overbrace{\frac{P_0}{z_0 R T_0} \left(\frac{P}{P_0}\right)^{\frac{1}{n}}}^{\rho^*} \overbrace{\sqrt{z_0 R T_0 \frac{2n}{n-1} \left[1 - \left(\frac{P}{P_0}\right)^{\left(\frac{n-1}{n}\right)}\right]}}^{U^*} \tag{4.113}$$

For the case of $n = 1$

$$\dot{m} = A^* \overbrace{\frac{P_0}{z_0 R T_0} \left(\frac{P}{P_0}\right)^{\frac{1}{n}}}^{\rho^*} \overbrace{\sqrt{2 z_0 R T_0 \ln\left(\frac{P_0}{P}\right)}}^{U**} \tag{4.114}$$

The Mach number can be obtained by utilizing equation (3.34) to defined the Mach number as

$$M = \frac{U}{\sqrt{z n R T}} \tag{4.115}$$

Integrating equation (4.105) when $ds = 0$ results

$$\int_{T_1}^{T_2} \frac{C_p}{R} \frac{dT}{T} = \int_{P_1}^{P_2} z \left(1 + \left(\frac{T}{Z}\right) \left(\frac{\partial z}{\partial T}\right)_P \frac{dP}{P}\right) \tag{4.116}$$

To carryout the integration of equation (4.116) looks at Bernnolli's equation which is

$$\int \frac{dU^2}{2} = -\int \frac{dP}{\rho} \tag{4.117}$$

After integration of the velocity

$$\frac{dU^2}{2} = -\int_1^{P/P_0} \frac{\rho_0}{\rho} d\left(\frac{P}{P_0}\right) \tag{4.118}$$

It was shown in Chapter (3) that (3.33) is applicable some ranges of relative temperature and pressure (relative to critical temperature and pressure not the stagnation conditions).

$$U = \sqrt{z_0 R T_0 \left(\frac{2n}{n-1}\right)\left[1-\left(\frac{P}{P_0}\right)^{\frac{n-1}{n}}\right]} \tag{4.119}$$

When $n = 1$ or when $n \to 1$

$$U = \sqrt{2 z_0 R T_0 \ln\left(\frac{P_0}{P}\right)} \tag{4.120}$$

The mass flow rate for the real gas $\dot{m} = \rho^* U^* A^*$

$$\dot{m} = \frac{A^* P_0}{\sqrt{z_0 R T_0}} \sqrt{\frac{2n}{n-1}} \left(\frac{P^*}{P_0}\right)^{\frac{1}{n}} \left[1 - \frac{P^*}{P_0}\right] \tag{4.121}$$

And for $n = 1$

$$\dot{m} = \frac{A^* P_0}{\sqrt{z_0 R T_0}} \sqrt{\frac{2n}{n-1}} \sqrt{2 z_0 R T_0 \ln\left(\frac{P_0}{P}\right)} \tag{4.122}$$

The Fliengner number is in this case is

$$Fn = \frac{\dot{m} c_0}{A^* P_0} \sqrt{\frac{2n}{n-1}} \left(\frac{P^*}{P_0}\right)^{\frac{1}{n}} \left[1 - \frac{P^*}{P_0}\right] \tag{4.123}$$

The Fliengner number is in $n = 1$ is

$$Fn = \frac{\dot{m} c_0}{A^* P_0} = 2\left(\frac{P^*}{P_0}\right)^2 - \ln\left(\frac{P^*}{P_0}\right) \tag{4.124}$$

The critical ratio of the pressure is

$$\frac{P^*}{P_0} = \left(\frac{2}{n+1}\right)^{\frac{n}{n-1}} \tag{4.125}$$

When $n = 1$ or more generally when $n \to 1$ this ratio approach

$$\frac{P^*}{P_0} = \sqrt{\mathrm{e}} \tag{4.126}$$

To obtain the relationship between the temperature and pressure, equation (4.116) can be integrated

$$\frac{T_0}{T} = \left(\frac{P_0}{P}\right)^{\frac{R}{C_p}\left[z+T\left(\frac{\partial z}{\partial T}\right)_P\right]} \tag{4.127}$$

The power of the pressure ratio is approaching $\frac{k-1}{k}$ when z approach 1. Note that

$$\frac{T_0}{T} = \left(\frac{z_0}{z}\right)\left(\frac{P_0}{P}\right)^{\frac{1-n}{n}} \tag{4.128}$$

The Mach number at every point at the nozzle can be expressed as

$$M = \sqrt{\left(\frac{2}{n-1}\right)\frac{z_0}{z}\frac{T_0}{T}\left[1 - \left(\frac{P-0}{P}\right)^{\frac{1-n}{n}}\right]} \tag{4.129}$$

For $n = 1$ the Mach number is

$$M = \sqrt{2\frac{z_0}{z}\frac{T_0}{T}\ln\frac{P_0}{P}} \tag{4.130}$$

The pressure ratio at any point can be expressed as function of the Mach number as

$$\frac{T_0}{T} = \left[1 + \frac{n-1}{2}M^2\right]^{\left(\frac{n-1}{n}\right)\left[z+T\left(\frac{\partial z}{\partial T}\right)_P\right]} \tag{4.131}$$

for $n = 1$

$$\frac{T_0}{T} = \mathrm{e}^{M^2\left[z+T\left(\frac{\partial z}{\partial T}\right)_P\right]} \tag{4.132}$$

The critical temperature is given by

$$\frac{T^*}{T_0} = \left(\frac{1+n}{2}\right)^{\left(\frac{n}{1-n}\right)\left[z+T\left(\frac{\partial z}{\partial T}\right)_P\right]} \tag{4.133}$$

and for $n = 1$

$$\frac{T^*}{T_0} = \sqrt{e^{-\left[z + T\left(\frac{\partial z}{\partial T}\right)_P\right]}} \tag{4.134}$$

The mass flow rate as function of the Mach number is

$$\dot{m} = \frac{P_0 n}{c_0} M \sqrt{\left(1 + \frac{n-1}{2} M^2\right)^{\frac{n+1}{n-1}}} \tag{4.135}$$

For the case of $n = 1$ the mass flow rate is

$$\dot{m} = \frac{P_0 A^* n}{c_0} \sqrt{e^{M^2}} \sqrt{\left(1 + \frac{n-1}{2} M^2\right)^{\frac{n+1}{n-1}}} \tag{4.136}$$

Example 4.7:
A design is required that at specific point the Mach number should be $M = 2.61$, the pressure $2[Bar]$, and temperature $300K$.

i. Calculate area ratio between the point and the throat.

ii. The stagnation pressure and the stagnation temperature.

iii. Is the stagnation pressure and temperature at the entrance are different from the point? You can assume that $k = 1.405$.

SOLUTION

1. The solution is simplified by using Potto-GDC for $M = 2.61$ the results are

$\mathbf{M}$	$\frac{\mathbf{T}}{\mathbf{T_0}}$	$\frac{\rho}{\rho_0}$	$\frac{\mathbf{A}}{\mathbf{A^\star}}$	$\frac{\mathbf{P}}{\mathbf{P_0}}$	$\frac{\mathbf{A\times P}}{\mathbf{A^*\times P_0}}$
2.6100	0.42027	0.11761	2.9066	0.04943	0.14366

2. The stagnation pressure is obtained from

$$P_0 = \frac{P_0}{P} P = \frac{2.61}{0.04943} \sim 52.802[Bar]$$

The stagnation temperature is

$$T_0 = \frac{T_0}{T} T = \frac{300}{0.42027} \sim 713.82K$$

3. Of course, the stagnation pressure is constant for isentropic flow.

CHAPTER 5

Normal Shock

In this chapter the relationships between the two sides of normal shock are presented. In this discussion, the flow is assumed to be in a steady state, and the thickness of the shock to be very small. A discussion on the shock thickness will be presented in a forthcoming section[1].

A shock can occur at least in two different mechanisms. The first is when a large difference (above critical value) between the two sides of a membrane exists, and the membrane is burst (see the discussion about the shock tube). Of course, the shock travels from the high pressure to the low pressure side. The second is when many sound waves "run into" each other and accumulate (some referred to it as "coalescing") into a large difference, which is the shock wave (piston relatively fast moving). In fact, the sound wave can be viewed as extremely weak shock. In the speed of sound analysis, it was assumed the medium to be continuous, without any abrupt changes. This assumption is no longer valid in the case of shock. Here, the relationship for a perfect gas is constructed.

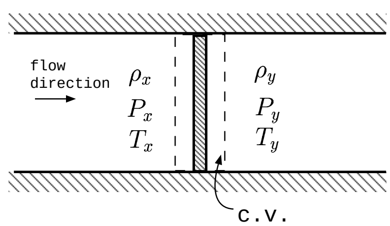


Fig. 5.1: A shock wave inside of a tube, but it also can viewed as a one dimensional shock wave

In Figure (5.1) a control volume for this analysis is shown, and the gas flows from left to right. The conditions, left and right of the shock, are assumed to

[1]Currently under construction.

be uniform[2]. The conditions to the right of the shock wave are uniform, but different from the left side. The transition in the shock is abrupt and in a very narrow width.

The chemical reactions (even condensation) are neglected, and the shock occurs at a very narrow section. Clearly the isentropic transition assumption is not appropriate in this case because the shock wave is a discontinued area. Therefore the increase of the entropy is fundamental to the phenomenon and understanding.

It is further assumed that there is no friction or heat loss at the shock (because the heat transfer is negligible due to the fact that it occurs at a relatively small surface.). It is customary in this field to denote x as the upstream condition and y as the downstream condition.

The mass flow rate is constant from the two sides of the shock and therefore the mass balance reduced to

$$\rho_x U_x = \rho_y U_y \tag{5.1}$$

In the shock wave, the momentum is the quantity that remained constant because there are no external forces. Thus, it can be written that

$$P_x - P_y = \left(\rho_x {U_y}^2 - \rho_y {U_x}^2\right) \tag{5.2}$$

The process is adiabatic, or nearly adiabatic, and therefore the energy equation can be written

$$C_p T_x + \frac{{U_x}^2}{2} = C_p T_y + \frac{{U_y}^2}{2} \tag{5.3}$$

The equation of state for perfect gas reads

$$P = \rho R T \tag{5.4}$$

If the conditions upstream are known, then there are four unknown conditions downstream. A system of four unknowns and four equations is solvable. Nevertheless, one can note that there are two solutions (because of the quadratic of equation (5.3). These two possible solutions refer to the direction of the flow. Physics dictates that there is only one possible solution. One cannot deduce the direction of flow from the pressure on both sides of the shock wave. The only tool that brings us to the direction of flow is the second law of thermodynamics. This law dictates the direction of the flow, and as it will be shown, the gas flows from supersonic flow to subsonic flow. Mathematically the second law is expressed by the entropy. For the adiabatic process, the entropy must increase. In mathematical terms, it can be written as follows:

$$s_y - s_x > 0 \tag{5.5}$$

[2]Clearly the change in the shock is so significant compared to the changes in medium before and after the shock that the changes in the mediums (flow) can be considered uniform.

Note that the greater–equal signs were not used. The reason is that the process is irreversible, and therefore no equality can exist. Mathematically the parameters are P, T, U, and ρ, which are needed to be solved. For ideal gas, equation (5.5) is

$$\ln \frac{T_y}{T_x} - (k-1)\,\frac{P_y}{P_x} > 0 \tag{5.6}$$

It can be also noticed that entropy, s, can be expressed as a function of the other parameters. Now one can view these equations as two different subsets of equations. The first set is the energy, continuity and state equations, and the second set is the momentum, continuity and state equations. The solution of every set of these equations produces one additional degree of freedom, which will produce a range of possible solutions. Thus, one can instead have a whole range of solutions. In the first case, the energy equation is used, producing various resistance to the flow. This case is called Fanno flow, and the Chapter (9) deals extensively with this topic. The mathematical explanation is given there in greater detail. Instead of solving all the equations that have been presented, one can solve only 4 equations (including the second law), which will require additional parameters. If the energy, continuity and state equations will be solved for the arbitrary value of the T_y, a parabola in the T–s diagram will be obtained. On the other hand, when the momentum equation is solved instead of the energy equation, the degree of freedom now is energy i.e., the energy amount "added" to the shock. This situation is similar to frictionless flow with the addition of heat, and this flow is known as Rayleigh flow. This flow is dealt with in greater detail in chapter 10.

Since the shock has no heat transfer (a special case of Rayleigh flow) and there isn't essentially momentum transfer (a special case of Fanno flow), the intersection of these two curves is what really happened in the shock. In Figure (5.2) the intersection is shown and two solutions are obtained. Clearly the increase of the entropy determines the direction of flow. The entropy increases from point x to point y. It is also worth noting that the temperature at $M = 1$ on Rayleigh flow is larger than that on the Fanno line.

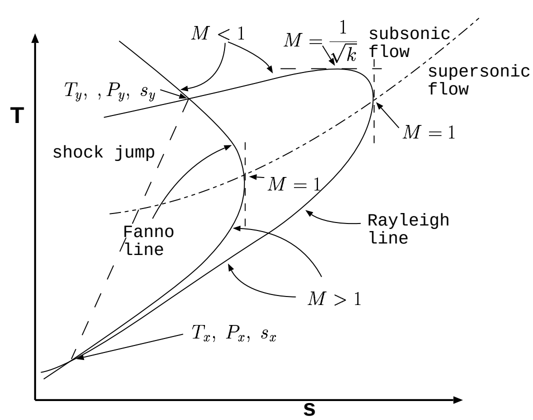


Fig. 5.2: The intersection of Fanno flow and Rayleigh flow produces two solutions for the shock wave

5.1 Solution of the Governing Equations

5.1.1 Informal model

Accepting the fact that the shock is adiabatic or nearly adiabatic requires that total energy is conserved, $T_{0x} = T_{0y}$. The relationship of the temperature to the stagnation temperature provides the relationship of the temperature for both sides of the shock.

$$\frac{T_y}{T_x} = \frac{\frac{T_y}{T_{0y}}}{\frac{T_x}{T_{0x}}} = \frac{1 + \frac{k-1}{2}{M_x}^2}{1 + \frac{k-1}{2}{M_y}^2} \tag{5.7}$$

All the other derivations are essentially derived from this equation. The only issue that is left to derived is the relationship between M_x and M_y. Note that Mach number is function of temperature, thus for known M_x all the other quantities can be determined at least numerically. As it will be seen momentarily there is analytical solution which is discussed in the next section.

5.1.2 Formal Model

The equations (5.1, 5.2, and 5.3) can be converted into a dimensionless form. The reason that dimensionless forms are heavily used in this book is because by doing so simplifies and clarifies the solution. It can also be noted that in many cases the dimensionless questions set is more easily solved.

From the continuity equation (5.1) substituting for density, ρ, the equation of state yields

$$\frac{P_x}{RT_x}U_x = \frac{P_y}{RT_y}U_y \tag{5.8}$$

Squaring equation (5.8) results

$$\frac{{P_x}^2}{R^2{T_x}^2}{U_x}^2 = \frac{{P_y}^2}{R^2{T_y}^2}{U_y}^2 \tag{5.9}$$

Multiplying the two sides by ratio of the specific heat, k provide a way to obtained the speed of sound definition/equation for perfect gas, $c^2 = kRT$ to be used for Mach definition as following,

$$\frac{{P_x}^2}{T_x \underbrace{kRT_x}_{{c_x}^2}}{U_x}^2 = \frac{{P_y}^2}{T_y \underbrace{kRT_y}_{{c_y}^2}}{U_y}^2 \tag{5.10}$$

Note that the speed of sound at different sides of the shock is different. Utilizing the definition of Mach number results in

$$\frac{{P_x}^2}{T_x}{M_x}^2 = \frac{{P_y}^2}{T_y}{M_y}^2 \tag{5.11}$$

Rearranging equation (5.11) reads

$$\frac{T_y}{T_x} = \left(\frac{P_y}{P_x}\right)^2 \left(\frac{M_y}{M_x}\right)^2 \tag{5.12}$$

Energy equation (5.3) converted to a dimensionless form as

$$T_y \left(1 + \frac{k-1}{2} {M_y}^2\right) = T_x \left(1 + \frac{k-1}{2} {M_x}^2\right) \tag{5.13}$$

It can be also noticed that equation (5.13) means that the stagnation temperature is the same, ${T_0}_y = {T_0}_x$. Under perfect gas model ρU^2 is identical to kPM^2 because

$$\rho U^2 = \overbrace{\frac{P}{RT}}^{\rho} \overbrace{\left(\frac{U^2}{\underbrace{kRT}_{c^2}}\right)}^{M^2} kRT = kPM2 \tag{5.14}$$

Using the identity (5.14) transforms the momentum equation (5.2) into

$$P_x + kP_x {M_x}^2 = P_y + kP_y {M_y}^2 \tag{5.15}$$

Rearranging the equation (5.15) yields

$$\frac{P_y}{P_x} = \frac{1 + k{M_x}^2}{1 + k{M_y}^2} \tag{5.16}$$

The pressure ratio in equation (5.16) can be interpreted as the loss of the static pressure. The loss of the total pressure ratio can be expressed by utilizing the relationship between the pressure and total pressure (see equation (4.11)) as

$$\frac{{P_0}_y}{{P_0}_x} = \frac{P_y \left(1 + \frac{k-1}{2}{M_y}^2\right)^{\frac{k}{k-1}}}{P_x \left(1 + \frac{k-1}{2}{M_x}^2\right)^{\frac{k}{k-1}}} \tag{5.17}$$

The relationship between the M_x and M_y is needed to be solved from the above equations set. This relationship can be obtained from the combination of mass, momentum, and energy equations. From the equations (5.13) (energy) and equation (5.12) (mass) the temperature ratio can be eliminated.

$$\left(\frac{P_y M_y}{P_x M_x}\right)^2 = \frac{1 + \frac{k-1}{2}{M_x}^2}{1 + \frac{k-1}{2}{M_y}^2} \tag{5.18}$$

Combining the results (5.18) with equation (5.16) results

$$\left(\frac{1 + k{M_x}^2}{1 + k{M_y}^2}\right)^2 = \left(\frac{M_x}{M_y}\right)^2 \frac{1 + \frac{k-1}{2}{M_x}^2}{1 + \frac{k-1}{2}{M_y}^2} \tag{5.19}$$

Equation (5.19) is a symmetrical equation in the sense that if M_y is substituted by M_x and M_x substituted by M_y the equation is remains the same. Thus, one solution is

$$M_y = M_x \tag{5.20}$$

It can be noticed that equation (5.19) is biquadratic. According the Gauss Biquadratic Reciprocity Theorem this kind of equation has a real solution in a certain range[3] which be discussed later. The solution can be obtained by rewriting equation (5.19) as polynomial (fourth order). It also possible to cross multiply equation (5.19) and divided it by $\left({M_y}^2 - {M_y}^2\right)$

$$1 + \frac{k-1}{2}\left({M_y}^2 + {M_y}^2\right) - k{M_y}^2{M_y}^2 = 0 \tag{5.21}$$

Equation (5.21) becomes

$$ {M_y}^2 = \frac{{M_x}^2 + \frac{2}{k-1}}{\frac{2k}{k-1}{M_x}^2 - 1} \tag{5.22}$$

The first solution (5.20) is the trivial solution in which the two sides are identical and no shock wave occurs. Clearly in this case, the pressure and the temperature from both sides of non–existent shock are the same i.e. $T_x = T_y,\ P_x = P_y$. The second solution is the case where the shock wave occurs.

The pressure ratio between the two sides can be now as a function of only single Mach number, for example, M_x. Utilizing equation (5.16) and equation (5.22) provides the pressure ratio as only function of upstream Mach number as

$$\begin{aligned} \frac{P_y}{P_x} &= \frac{2k}{k+1}{M_x}^2 - \frac{k-1}{k+1} \\ \frac{P_y}{P_x} &= 1 + \frac{2k}{k+1}\left({M_x}^2 - 1\right) \end{aligned} \tag{5.23}$$

The density and upstream Mach number relationship can be obtained in the same fashioned to became

$$\frac{\rho_y}{\rho_x} = \frac{U_y}{U_y} = \frac{(k+1){M_x}^2}{2 + (k-1){M_x}^2} \tag{5.24}$$

Utilizing the fact that the pressure ratio is a function of the upstream Mach number, M_x, provides additional way to obtain additional useful relationship. And the temperature ratio as a function of pressure ratio is transfered into

$$\frac{T_y}{T_x} = \left(\frac{P_y}{P_x}\right)\left(\frac{\frac{k+1}{k-1} + \frac{P_y}{P_x}}{1 + \frac{k+1}{k-1}\frac{P_y}{P_x}}\right) \tag{5.25}$$

[3] Ireland, K. and Rosen, M. "Cubic and Biquadratic Reciprocity." Ch. 9 in A Classical Introduction to Modern Number Theory, 2nd ed. New York: Springer-Verlag, pp. 108-137, 1990.

In the same way the relationship between the density ratio and pressure ratio is

$$\frac{\rho_x}{\rho_y} = \frac{1 + \left(\frac{k+1}{k-1}\right)\left(\frac{P_y}{P_x}\right)}{\left(\frac{k+1}{k-1}\right) + \left(\frac{P_y}{P_x}\right)} \tag{5.26}$$

5.1.3 Speed of Sound Definition

There can be several definitions of speed of sound that associated with the shock wave.

5.1.3.1 The Maximum Conditions

The maximum speed of sound is when the highest temperature is achieved. The maximum temperature that can be achieved is the stagnation temperature

$$U_{max} = \sqrt{\frac{2k}{k-1}RT_0} \tag{5.27}$$

The stagnation speed of sound is

$$c_0 = \sqrt{kRT_0} \tag{5.28}$$

Based on this definition a new Mach number can be defined

$$M_0 = \frac{U}{c_0} \tag{5.29}$$

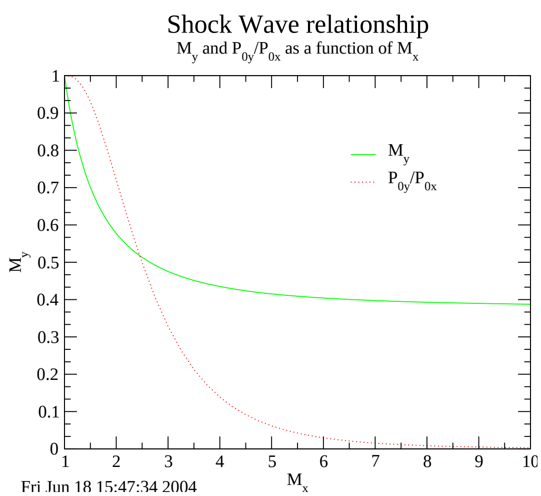


Fig. 5.3: The exit Mach number and the stagnation pressure ratio as a function of upstream Mach number

5.1.3.2 The Star Conditions

The speed of the sound at the critical condition also can be a good reference velocity. The speed of sound at that velocity

$$c^* = \sqrt{kRT^*} \tag{5.30}$$

In the same manner additional Mach number can be defined as

$$M^* = \frac{U}{c^*} \tag{5.31}$$

5.1.4 Prandtl's condition

It can be easily noticed that the temperature from both sides of the shock wave is discontinuous. Therefore the speed of sound is different in these adjoining mediums. It is therefore convenient to define the star Mach number that will be independent of the specific Mach number (independent of the temperature).

$$M^* = \frac{U}{c^*} = \frac{c}{c^*}\frac{U}{c} = \frac{c}{c^*}M \tag{5.32}$$

The jump condition across the shock must satisfy the constant energy.

$$\frac{c^2}{k-1} + \frac{U^2}{2} = \frac{{c^*}^2}{k-1} + \frac{{c^*}^2}{2} = \frac{k+1}{2(k-1)}{c^*}^2 \tag{5.33}$$

Dividing of the mass equation by momentum equation and combining with the perfect gas model yields

$$\frac{{c_1}^2}{kU_1} + U_1 = \frac{{c_2}^2}{kU_2} + U_2 \tag{5.34}$$

Combining equation (5.33) and (5.34) results in

$$\frac{1}{kU_1}\left[\frac{k+1}{2}{c^*}^2 - \frac{k-1}{2}U_1\right] + U_1 = \frac{1}{kU_2}\left[\frac{k+1}{2}{c^*}^2 - \frac{k-1}{2}U_2\right] + U_2 \tag{5.35}$$

After rearranging and diving equation (5.35) gives

$$U_1 U_2 = {c^*}^2 \tag{5.36}$$

Or in dimensionless form

$${M^*}_1 {M^*}_2 = {c^*}^2 \tag{5.37}$$

5.2 Operating Equations and Analysis

In Figure (5.3), the Mach number after the shock, M_y and the Ratio of the total pressure, P_{0y}/P_{0x} are plotted as a function of the entrance Mach number. The working equations are presented earlier. Note that the M_y has minimum value which depends on the specific heat ratio. It also can be noticed that density ratio (velocity ratio) also have a finite value regardless to the upstream Mach number.

The typical situations in which these equations can be used include also the moving shocks. The questions that appear are what should be the Mach number (upstream or downstream) for given pressure ratio or density ratio (velocity ratio). This kind of question requires examining the Table (5.1) for $k = 1.4$ or utilizing

Potto-GDC for or value of the specific heat ratio. For finding the Mach number for pressure ratio of 8.30879 and $k = 1.32$ is only a few mouse clicks way to following table.

$\mathbf{M_x}$	$\mathbf{M_y}$	$\mathbf{\frac{T_y}{T_x}}$	$\frac{\rho_y}{\rho_x}$	$\mathbf{\frac{P_y}{P_x}}$	$\mathbf{\frac{P_{0_y}}{P_{0_x}}}$
2.7245	0.47642	2.1110	3.9360	8.3088	0.38109

To illustrate the usage of the above equations, an example is provided.

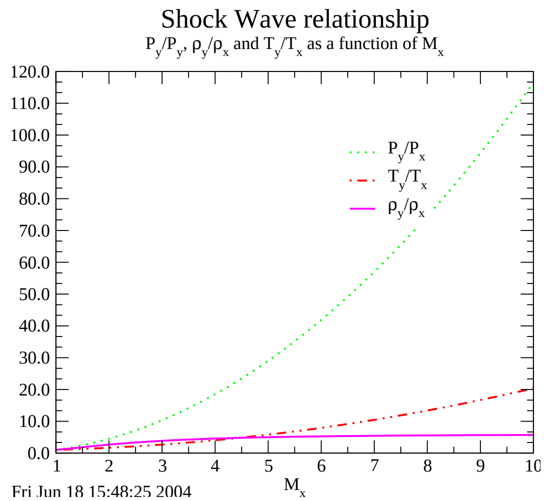


Fig. 5.4: The ratios of the static properties of the two sides of the shock

Example 5.1:
Air flows with a Mach number of $M_x = 3$, at pressure of 0.5 [bar] and temperature 0°C goes through a normal shock. Calculate the temperature, pressure, total pressure and velocity downstream of the shock.

SOLUTION
Analysis:
First, the known information $M_x = 3$, $p_x = 1.5[bar]$ and $T_x = 273K$. Using these data, the total pressure can be obtained (through an isentropic relationship Table (4.2), i.e. P_{0x} is known). Also with the temperature, T_x the velocity can readily be calculated. The relationship that was calculated will be utilized to obtain the ratios for downstream of the normal shock. $\frac{P_x}{P_{0x}} = 0.0272237 \Longrightarrow P_{0x} = 1.5/0.0272237 = 55.1[bar]$

$c_x = \sqrt{kRT_x} = \sqrt{1.4 \times 287 \times 273} = 331.2m/sec$

$\mathbf{M_x}$	$\mathbf{M_y}$	$\mathbf{\frac{T_y}{T_x}}$	$\frac{\rho_y}{\rho_x}$	$\mathbf{\frac{P_y}{P_x}}$	$\mathbf{\frac{P_{0_y}}{P_{0_x}}}$
3.0000	0.47519	2.6790	3.8571	10.3333	0.32834

$U_x = M_x \times c_x = 3 \times 331.2 = 993.6[m/sec]$

Now the velocity downstream is determined by the inverse ratio of $\rho_y/\rho_x = U_x/U_y = 3.85714$.

$$U_y = 993.6/3.85714 = 257.6[m/sec]$$

$$P_{0y} = \left(\frac{P_{0y}}{P_{0x}}\right) \times P_{0x} = 0.32834 \times 55.1[bar] = 18.09[bar]$$

5.2.1 The Limitations of The Shock Wave

When the upstream Mach number becomes very large, the downstream Mach number (see equation (5.22) is limited by

$$
{M_y}^2 = \frac{1 + \cancelto{\sim 0}{\frac{2}{(k-1)\,{M_x}^2}}}{\frac{2k}{k-1} - \cancelto{\sim 0}{\frac{1}{{M_x}^2}}} = \frac{k-1}{2k} \tag{5.38}
$$

This results is shown in Figure (5.3). The limits of of the pressure ratio can be obtained by looking at equation (5.16) and utilizing the limit that was obtained in equation (5.38).

5.2.2 Small Perturbation Solution

The Small perturbation solution referred to an analytical solution in a case where only small change occurs. In this case, it refers to a case where only a "small shock" occurs, which is up to $M_x = 1.3$. This approach had major significance and usefulness at a time when personal computers were not available. Now, during the writing of this version of the book, this technique mostly has usage in obtaining analytical expressions for simplified models. This technique also has academic value, and therefore will be described in the next version (0.5 series).

The strength of the shock wave defined as

$$
\hat{\mathcal{P}} = \frac{P_y - P_x}{P_x} = \frac{P_y}{P_x} - 1 \tag{5.39}
$$

Using the equation (5.23) transformed equation (5.39) into

$$
\hat{\mathcal{P}} = \frac{2k}{k+1}\left({M_x}^2 - 1\right) \tag{5.40}
$$

Or utilizing equation

$$
\hat{\mathcal{P}} = \frac{\frac{2k}{k-1}\left(\frac{\rho_y}{\rho x} - 1\right)}{\frac{2}{k-1} - \left(\frac{\rho_y}{\rho_x} - 1\right)} \tag{5.41}
$$

5.2.3 Shock Thickness

The issue of the shock thickness is presented (will be presented in version 0.45) here for completeness. This issue has very limited practical application for most students, however, to have the student convinced that indeed the assumption of very thin shock is validated by analytical and experimental study, the issue must be presented.

The shock thickness has several way to defined it. The most common definition is passing a tangent to the velocity at the center and finding where is the theoretical upstream and downstream condition are meet.

5.3 The Moving Shocks

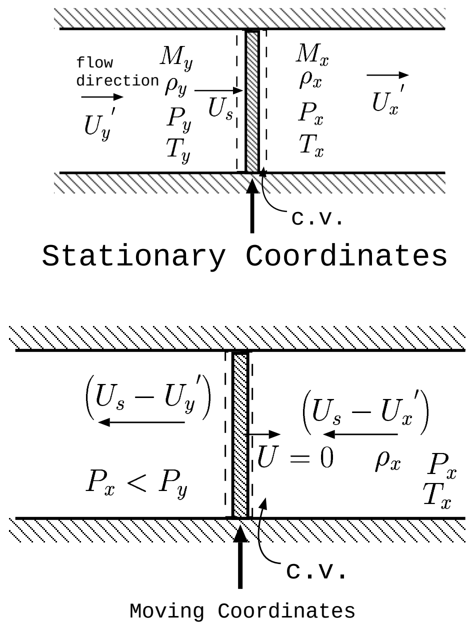


Fig. 5.5: Comparison between stationary shock and moving shock in ducts

In some situations, the shock wave isn't stationary. This kind of situation is arisen in many industrial applications. For example, when a valve is suddenly [4] closed and a shock is propagating upstream. On the other extreme when suddenly valve is opened or a membrane is ruptured a shock occurs and propagates downstream (the opposite direction of the previous case). In some industrial applications a liquid (metal) is pushed in two rapid stages to a cavity through a pipe system. This liquid (metal) is pushing gas (mostly) air which creates two shock stages. As a general rule, the shock can move downstream or upstream. The last situation is the most general case which this section will be dealing with. There is further more generalized cases where the moving shock is created which include a change in the physical properties which this book will not be dealing (at this stage). The reluctance to deal with the most general case because it is rather specialized and complicated beyond what even early graduate students work. In these changes (for examples, opening value and closing valve on the other side) create situations where different shocks are moving in the tube. In case where two shocks collide into one shock which can move upstream or downstream is the general case. As specific example, which is common in die casting process, After the first shock moving a second shock is created in which its velocity is dictated by the upstream and downstream velocity.

In cases when the shock velocity can be approximated as a constant (the majority of the cases) or as a nearly constant, the previous analysis equations, and the tools developed in this chapter can be employed. In these cases, the problem can be reduced to the previously studied shock, i.e., to the stationary case when the coordinate are attached to shock front. In such case, the steady state is obtained in the moving control value. It has to be mentioned that the direction of the shock alone doesn't determine the "upstream" side of the shock. The determining factor is the relative velocity of the flow to the shock.

For this analysis, the coordinates move with the shock. Here, the prime ' will be denoting the values of the static coordinates. Note that this notation is contrary to the conversion notation in the literature. The reason the deviation from

[4]Later on the dimensional analysis what is suddenly

the common is that this choice reduces the programing work (especially for object oriented programing like C++) and still use the notation that were used before. Observer moving with the shock will notice that the pressure in the shock is

$$P_x{}' = P_x \qquad P_y{}' = P_y \tag{5.42}$$

The temperature measured by the observer is

$$T_x{}' = T_x \qquad T_y{}' = T_y \tag{5.43}$$

Assuming that shock is moving to the right, (refer to Figure (5.6)) the velocity measured by the observer is

$$U_x = U_s - U_x{}' \tag{5.44}$$

Where U_s is the shock velocity which is moving to the right. The "downstream" velocity is

$$U_y{}' = U_s - U_y \tag{5.45}$$

The speed of sound on both sides of shock depends only the temperature and it is assumed constant. The upstream prime Mach number can be defined as

$$M_x{}' = \frac{U_s - U_x}{c_x} = \frac{U_s}{c_x} - M_x = M_{sx} - M_x \tag{5.46}$$

It can be noted that the additional definition was introduced for the shock upstream Mach number, $M_{sx} = \frac{U_s}{c_x}$. The downstream prime Mach number obtained the form

$$M_y{}' = \frac{U_s - U_y}{c_y} = \frac{U_s}{c_y} - M_y = M_{sy} - M_y \tag{5.47}$$

Similarly to previous case, additional definition was introduced of the shock downstream Mach number, M_{sy}. The relation between the two new shock Mach numbers is

$$\frac{U_s}{c_x} = \frac{c_y}{c_x}\frac{U_s}{c_y}$$

$$M_{sx} = \sqrt{\frac{T_y}{T_x}} M_{sy} \tag{5.48}$$

The "upstream" stagnation temperature of the fluid is

$$T_{0x} = T_x\left(1 + \frac{k-1}{2}{M_x}^2\right) \tag{5.49}$$

and the "upstream" prime stagnation pressure is

$$P_{0x} = P_x\left(1 + \frac{k-1}{2}{M_x}^2\right)^{\frac{k}{k-1}} \tag{5.50}$$

The same can be said for the "downstream" side of the shock. The difference between the stagnation temperature is in the moving coordinates is

$$T_{0y} - T_{0x} = 0 \tag{5.51}$$

It should be noted that the stagnation temperature (in the stationary coordinates) rises as opposed to the stationary normal shock. The rise of total temperature is due to the fact that new material has entered the c.v. at a very high velocity, and is "converted" or added into the total temperature.

$$\begin{aligned}
T_{0y} - T_{0x} =& T_y \left(1 + \frac{k-1}{2}\left(M_{sy} - {M_y}'\right)^2\right) - T_x \left(1 + \frac{k-1}{2}\left(M_{sx} - {M_x}'\right)^2\right) \\
0 =& \overbrace{T_y \left(1 + \frac{k-1}{2} {M_y}'^2\right)}^{{T_{0y}}'} + T_y M_{sy} \frac{k-1}{2}\left(M_{sy} - 2M_y\right) \\
& - \overbrace{T_x \left(1 + \frac{k-1}{2} {M_x}'^2\right)}^{{T_{0x}}'} - T_x M_{sx} \frac{k-1}{2}\left(M_{sx} - 2M_x\right)
\end{aligned} \tag{5.52}$$

And according to equation (5.51) leads to

$$ {T_{0y}}' - {T_{0x}}' = U_s \left(\frac{T_x}{c_x}\frac{k-1}{2}\left(M_{sx} - 2M_x\right) - \frac{T_y}{c_y}\frac{k-1}{2}\left(M_{sy} - 2M_y\right)\right) \tag{5.53}$$

Again, this difference in the moving shock is expected due to fact that moving material velocity (kinetic energy) converted into internal energy. This difference can also view are results of unsteady state of the shock.

5.3.1 Shock Result From A Sudden and Complete Stop

The general discussion can be simplified in the extreme case where the shock is moving from a still medium. This situation arises in many cases in the industry, for example, a sudden and complete closing of a valve. The sudden closing of the valve must result in a zero velocity of the gas. This shock is viewed by some as a reflective shock. This information propagates upstream in which the gas velocity is converted into temperature. In many such cases the steady state is established quite rapidly. In such case, the shock velocity "downstream" is U_s. The equations (5.42) to (5.56) can be transformed into simpler equations when M_x is zero and U_s is positive values.

The "upstream" Mach number reads

$$M_x = \frac{U_s + U_x}{c_x} = M_{sx} + M_x \tag{5.54}$$

The “downstream” Mach number reads

$$M_y = \frac{|U_s|}{c_y} = M_{sy} \tag{5.55}$$

Again, the shock is moving to the left but in the moving coordinates. The observer (with the shock) sees the flow moving from the left to the right. The upstream is on the left of the shock. The stagnation temperature increases by

$$T_{0y} - T_{0x} = U_s \left(\frac{T_x}{c_x} \frac{k-1}{2} \left(M_{sx} + 2M_x \right) - \frac{T_y}{c_y} \frac{k-1}{2} \left(M_{sy} \right) \right) \tag{5.56}$$

The prominent question in this situation is what the shock wave velocity for a given fluid velocity, ${U_x}'$ and for a given specific heat ratio. The “upstream” or the “downstream” Mach number are not known even if the pressure and the temperature downstream are given. The difficulty lays in the jump from the stationary coordinates to the moving coordinates. it turned out that it is very useful to use the dimensionless parameter M_{sx}, or M_{sy} instead the velocity because it combines the temperature and velocity into one.

The relationship between the Mach number on two sides of the shock are tied through equations (5.54) and (5.55) by

$$\left(M_y\right)^2 = \frac{\left({M_x}' + M_{sx}\right)^2 + \frac{2}{k-1}}{\frac{2k}{k-1}\left({M_x}' + M_{sx}\right)^2 - 1} \tag{5.57}$$

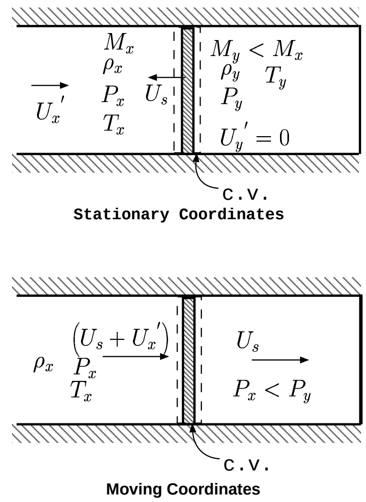


Fig. 5.6: Comparison between stationary shock and moving shock in a stationary medium in ducts

And substituting equation (5.57) into (5.48) results

$$M_x = \overbrace{\sqrt{\frac{T_x}{T_y}}}^{f(M_{sx})} \sqrt{\frac{\left({M_x}' + M_{sx}\right)^2 + \frac{2}{k-1}}{\frac{2k}{k-1}\left({M_x}' + M_{sx}\right)^2 - 1}} \tag{5.58}$$

The temperature ratio in equation (5.58)and the rest of the right hand side show clearly that M_{sx} has four possible solutions (fourth order polynomial for M_{sx} has four solutions). Only one real solution is possible. The solution to equation (5.58) can be obtained by several numerical methods. Note, that analytical solution can be obtained to (5.58) but it seems very simple to utilize numerical methods. The typical methods is of “smart” guessing of Msx. For very small values of upstream

Mach number, $M_x{}' \sim \epsilon$ equation (5.58) provides that $M_{sx} \sim 1+\frac{1}{2}\epsilon$ and $M_{sy} = 1-\frac{1}{2}\epsilon$ (the coefficient is only approximated as 0.5) as it shown in Figure (5.7). From Figure also it can be noted that high velocity can results in much larger velocity of the reflective shock. For example, for Much number close to one, which easily can be obtained in Fanno flow, result in about double sonic velocity of reflective shock. Some times this phenomenon can have tremendous significance in industrial applications.

Note, that to achieve supersonic velocity (in stationary coordinate) requires diverging–converging nozzle. Here no such device is needed! Luckily and hopefully, engineers who are dealing supersonic flow when installing the nozzle and pipe systems for gaseous medium understand the importance of the reflective shock wave.

Two numerical methods and the algorithm employed to solve this problem is provided herein:

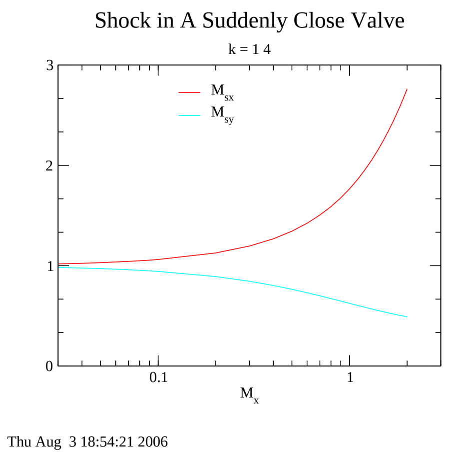


Fig. 5.7: The moving shock Mach numbers as results of sudden and complete stop

(a) Guess $M_x > 1$,

(b) Using shock table or Potto GDC to calculate temperature ratio and M_y,

(c) Calculate the $M_x = M_x{}' - \sqrt{\frac{T_x}{T_y}} M_y$

(d) Compare to the calculated $M_x{}'$ to the given $M_x{}'$. and adjust the new guess $M_x > 1$ accordingly.

The second method is successive substitution which has better convergence to the solution initial in most ranges but less effective to higher accuracy.

(a) Guess $M_x = 1 + M_x{}'$,

(b) Using shock table or Potto GDC to calculate temperature ratio and M_y,

(c) Calculate the $M_x = M_x{}' - \sqrt{\frac{T_x}{T_y}} M_y$

(d) Check if new M_x approach the old M_x if not satisfactory is the new $M_x{}'$ to calculate $M_x = 1 + M_x{}'$ return to part (b).

5.3.2 Moving Shock Into Stationary Medium

5.3.2.1 General Velocities Issues

When valve or membrane is suddenly opened a shock is created and propagate the downstream. With the exception of a close proximity to the valve, the shock is moving in a constant velocity (see Figure (5.8)). Converting the moving shock to a coordinates system that attached to the shock results in a stationary shock when the flow is moving to left. The “upstream” is on the right (see Figure (5.8)).

(a) The reflective shock into close end pipe in stationary coordinates

(b) The reflective shock into close end pipe in moving coordinates

Fig. 5.8: The reflective shock into close end pipe coordinates

Similar definitions of the right and the left side shock Mach numbers for are utilized. It has to be noticed that the “upstream” and “downstream” are the reverse from the previous case. The “upstream” Mach number is

$$M_x = \frac{U_s}{c_x} = M_{sx} \tag{5.59}$$

The “downstream” Mach number is

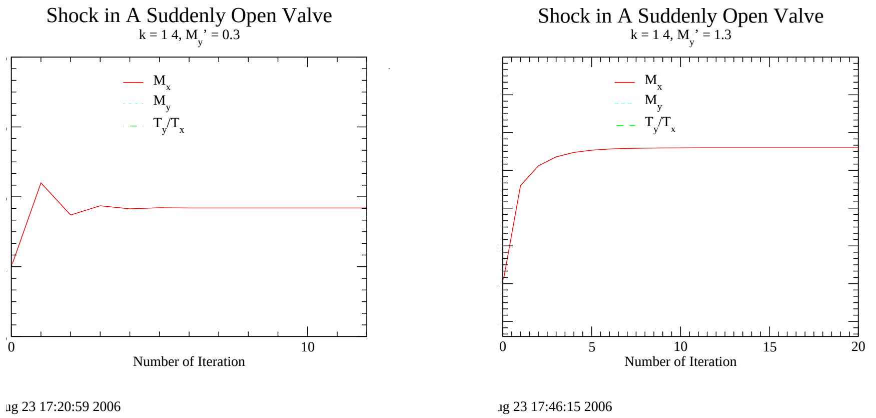
$$M_y = \frac{U_s - U_y{}'}{c_y} = M_{sy} - M_y \tag{5.60}$$

Note that in this case the stagnation temperature in stationary coordinate changes (as in the previous case) where the thermal energy (due to pressure difference) is converted into velocity. The stagnation temperature (of moving coordinates) is

$$T_{0y} - T_{0x} = T_y \left(1 + \frac{k-1}{2} \left(M_{sy} - M_y\right)^2\right) - T_x \left(1 + \frac{k-1}{2} \left(M_x\right)^2\right) = 0 \tag{5.61}$$

After similar rearrangement as in the previous case results in

$$T_{0y}{}' - T_{0x}{}' = T_y \left(1 + \frac{k-1}{2} \left(-2M_{sy}M_y + {M_y}^2\right)^2\right) \tag{5.62}$$



(a) The number of iterations to achieve the results for $M_y{}' = 0.3$

(b) The number of iterations to achieve the results for $M_y{}' = 1.3$

The same question that was prominent in the previous case appears now. what will be the shock velocity for given upstream Mach number? The relationship again between the two sides is

$$M_{sy} = M_y + \sqrt{\frac{(M_{sx})^2 + \frac{2}{k-1}}{\frac{2k}{k-1}(M_{sx})^2 - 1}} \tag{5.63}$$

Since M_{sx} can be transformed to M_{sy} theatrically equation (5.63) can be solved. It is common practice to solve this equation by numerical methods. One of such method is successive substitutions. This method is done by the following algorithm:

(a) Assume that $M_x = 1$.

(b) Calculate the Mach number M_y utilizing the tables or Potto–GDC

(c) Utilizing

$$M_x = \sqrt{\frac{T_y}{T_x}}\left(M_y + M_y{}'\right)$$

calculate the new "improved" M_x

(d) Check the new and improved M_x against the old one. If it satisfactory stop or return to stage (b).

To illustrate the convergence of the procedure consider the case of $M_y{}' = 0.3$ and $M_y{}' = 1.3$. The results show that the convergence occurs very rapidly (see Figure (5.9(b))). The larger value of the $M_y{}'$ the larger number of iterations is required to achieve the same accuracy. Yet, for most practical purpose sufficient results can be achieved after 3-4 iterations.

5.3.2.2 Supersonic Issues of Moving Shock

Assuming that gas velocity is supersonic (in stationary coordinates) before the shock moves, what is the maximum velocity that can be approached before this model fails. In other words is there point where the moving shock is fast enough to reduce the "upstream" relative velocity below the speed of sound. This is the point where no matter what the pressure difference be, the shock velocity cannot be increased.

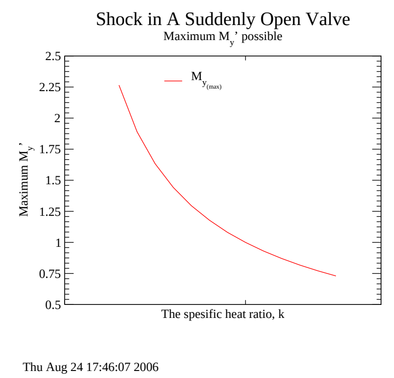


Fig. 5.9: The Maximum of Mach number of "downstream" as function of the specifi c heat

This shock chocking phenomenon is similar to the chocking phenomenon that will be discussed earlier in nozzle flow and will appear in the nozzle flow and in other pipe flows models (later chapters). It must be noted that in the previous case of suddenly and complete closing of valve results in no limit (at least from the model point of view). To explain this phenomenon look at the normal shock. Consider when the shock wave approaches infinity, $M_x = M_{sx} \rightarrow \infty$ the downstream Mach number, according to equation (5.38), is approaching to $(k-1)/2k$. One can view this as the source for the shock chocking phenomenon.

To study this limit consider that the maximum Mach number is obtained when the pressure ratio is approaching infinity $\frac{P_y}{P_x} \rightarrow \infty$. Applying equation (5.23) to this situation yields

$$M_{sx} = \sqrt{\frac{k+1}{2k}\left(\frac{P_x}{P_y} - 1\right) + 1} \tag{5.64}$$

From the mass conservation leads into

$$U_y \rho_y = U_s \rho_x$$
$$\left(U_s - U_y{}'\right) \rho_y = U_s \rho_x$$
$$M_y{}' = \sqrt{\frac{T_y}{T_x}} \left(1 - \frac{\rho_x}{\rho_y}\right) M_{sx} \tag{5.65}$$

Substituting equations (5.26) and (5.25) into equation (5.65) results in

$$M_y{}' = \frac{1}{k}\left(1 - \frac{P_y}{P_x}\right)\sqrt{\frac{\frac{2k}{k+1}}{\frac{P_y}{P_x} + \frac{k-1}{k+1}}} \times \sqrt{\frac{1 + \left(\frac{k+1}{k-1}\right)\left(\frac{P_y}{P_x}\right)}{\left(\frac{k+1}{k-1}\right) + \left(\frac{P_y}{P_x}\right)}} \tag{5.66}$$

When the pressure ratio is approaching infinity (extremely strong pressure ratio)

$$M_y{}' = \sqrt{\frac{2}{k(k-1)}} \tag{5.67}$$

What happened when a gas with Mach number larger than the maximum Mach number possible is flowing in the tube? Obviously the semi steady state described by moving shock cannot be sustained. And similar phenomenon to the choking in nozzle and later in internal pipe flow is obtained. The Mach number is reduced to the maximum value very rapidly. Or, a stationary shock occurs as it will shown in chapters on internal flow.

$\mathbf{k}$	$\mathbf{M_x}$	$\mathbf{M_y}$	$\mathbf{M_y}{}'$	$\mathbf{\frac{T_y}{T_x}}$
1.3000	1073.25	0.33968	2.2645	169842.29
1.4000	985.85	0.37797	1.8898	188982.96
1.5000	922.23	0.40825	1.6330	204124.86
1.6000	873.09	0.43301	1.4434	216507.05
1.7000	833.61	0.45374	1.2964	226871.99
1.8000	801.02	0.47141	1.1785	235702.93
1.9000	773.54	0.48667	1.0815	243332.79
2.0000	750.00	0.50000	1.00000	250000.64
2.1000	729.56	0.51177	0.93048	255883.78
2.2000	711.62	0.52223	0.87039	261117.09
2.3000	695.74	0.53161	0.81786	265805.36
2.4000	681.56	0.54006	0.77151	270031.44
2.5000	668.81	0.54772	0.73029	273861.85

Table of maximum values of the shock-chocking phenomenon.

Example 5.2:
A shock is moving at a speed of 450 [m/sec] in a stagnated gas at pressure of 1 [Bar] and temperature of 27°C. Compute the pressure and temperature behind the shock. Assume the specific heat ratio is 1.3.

<u>SOLUTION</u>

It can be noticed that the gas behind the shock is moving while the gas ahead the shock is still. Thus it is the case of shock moving into still medium (suddenly open valve case). First the Mach velocity ahead the shock has to calculated.

$$M_y{}' = \frac{U}{\sqrt{kRT}} = \frac{450}{\sqrt{1.3 \times 287 \times 300}} \sim 1.296$$

Utilizing POTTO-GDC or that Table (5.4) one can obtain the following table

$\mathbf{M_x}$	$\mathbf{M_y}$	$\mathbf{M_x}'$	$\mathbf{M_y}'$	$\frac{\mathbf{T_y}}{\mathbf{T_x}}$	$\frac{\mathbf{P_y}}{\mathbf{P_x}}$	$\frac{\mathbf{P_{0y}}}{\mathbf{P_{0x}}}$
2.1206	0.54220	0.0	1.132	1.604	4.953	0.63955

Using the above table, the temperature behind the shock is

$$T_y = T_y{}' = \frac{T_y}{T_x} T_x = 1.604 \times 300 \sim 481.2K$$

In same can be done for the pressure ratio as following

$$P_y = P_y{}' = \frac{P_y}{P_x} P_x = 4.953 \times 1.0 \sim 4.953[Bar]$$

The velocity behind the shock wave is obtain by utilizing the mass balance as

$$U_y{}' = M_x{}' c_x = 1.132 \times \sqrt{1.3 \times 287 \times 300} \sim 378.72 \left[\frac{m}{sec}\right]$$

Example 5.3:

Gas flows at supersonic velocity in a tube with velocity of $450[m/sec]$. The static pressure at the tube is $2Bar$ and (static) temperature of $300K$. The gas is brought into complete stop by a sudden by closing a value. Calculate the velocity and the pressure behind the reflecting shock. The specific heat ratio can be assumed to be $k = 1.4$.

SOLUTION

The first thing which is needed to be done is to find the prime Mach number $M_x{}' = 1.2961$. Then, the prime properties can be found. At this stage the reflecting shock velocity is unknown.

Simply using the Potto-GDC provides for the temperature and velocity the following table:

$\mathbf{M_x}$	$\mathbf{M_y}$	$\mathbf{M_x}'$	$\mathbf{M_y}'$	$\frac{\mathbf{T_y}}{\mathbf{T_x}}$	$\frac{\mathbf{P_y}}{\mathbf{P_x}}$	$\frac{\mathbf{P_{0y}}}{\mathbf{P_{0x}}}$
2.0445	0.56995	1.2961	0.0	1.724	4.710	0.70009

Or if you insist on doing the steps yourself find the upstream prime Mach, $M_x{}'$ to be 1.2961. Then using the Table (5.2) you can find the the proper M_x. If this detail is not sufficient enough that simply utilize the iteration procedure described earlier and obtain

$\mathbf{i}$	$\mathbf{M_x}$	$\mathbf{M_y}$	$\frac{\mathbf{T_y}}{\mathbf{T_x}}$	$\mathbf{M_y}'$
$0.00E+00$	2.296	0.53487	1.943	0.0
1.000	2.042	0.57040	1.722	0.0
2.000	2.045	0.56994	1.724	0.0
3.000	2.044	0.56995	1.724	0.0
4.000	2.044	0.56995	1.724	0.0

The about table for obtained utilizing Potto-GDC with the iteration request.

Example 5.4:
What should be the prime Mach number (or the combination velocity with the temperature, those who like extra step) in order to double temperature when the valve is suddenly and totally close?

<u>SOLUTION</u>
The ratio can be obtained from from Table (5.3). It also can be obtained from the stationary normal shock wave table. Potto-GDC provides for this temperature ratio the following table

$\mathbf{M_x}$	$\mathbf{M_y}$	$\frac{\mathbf{T_y}}{\mathbf{T_x}}$	$\frac{\rho_\mathbf{y}}{\rho_\mathbf{x}}$	$\frac{\mathbf{P_y}}{\mathbf{P_x}}$	$\frac{\mathbf{P_{0_y}}}{\mathbf{P_{0_x}}}$
2.3574	0.52778	2.0000	3.1583	6.3166	0.55832

which means that the required $M_x = 2.3574$ using this number in the moving shock table provides

$\mathbf{M_x}$	$\mathbf{M_y}$	$\mathbf{M_x}'$	$\mathbf{M_y}'$	$\frac{\mathbf{T_y}}{\mathbf{T_x}}$	$\frac{\mathbf{P_y}}{\mathbf{P_x}}$	$\frac{\mathbf{P_{0_y}}}{\mathbf{P_{0_x}}}$
2.3574	0.52778	0.78928	0.0	2.000	6.317	0.55830

Example 5.5:
Open question to be addressed in the next version.
A gas is flowing in a pipe with Mach number of 0.4. Calculate the speed of the shock when a valve is close in such a way that the Mach number is reduced to half. Hint, this is semi close valve case in which the ratio of prime Mach number is half (the new parameter that add in the general case).

<u>SOLUTION</u>

Example 5.6:
Open question to be addressed in the next version.
A piston is pushing air flowing in tube with Mach number of $M = 0.5$. The piston

is accelerated in very rapid way and air adjoint to piston is flowing in Mach number $M = 0.8$. Calculate what is the velocity of the shock created by the piston in the air? Calculate the time the shock it take to reach to the end of the tube of $1.0m$ length. Assume that there is no friction and Fanno flow model is not applicable.

SOLUTION

5.4 Shock Tube

The Shock tube is a study tool with very little practical purposes which is used in many cases to understand certain phenomena. Other situations can be examined and extended from this phenomena. A cylinder with two chambers connected by a diaphragm. In one side the pressure is high while the pressure in the other side is low. When the diaphragm is rupture the gas from the high pressure is flowing into the low pressure section. When the pressure is high enough shock is created that is travels to the low pressure chamber. This is the same case as was study in suddenly open valve described before. In the back of the shock expansion waves occur with reduction of the pressure. The temperature has known to reach several thousands for very brief time. The high pressure chamber referred in the literature as the *driver section and the low section is referred as the* expansion section.

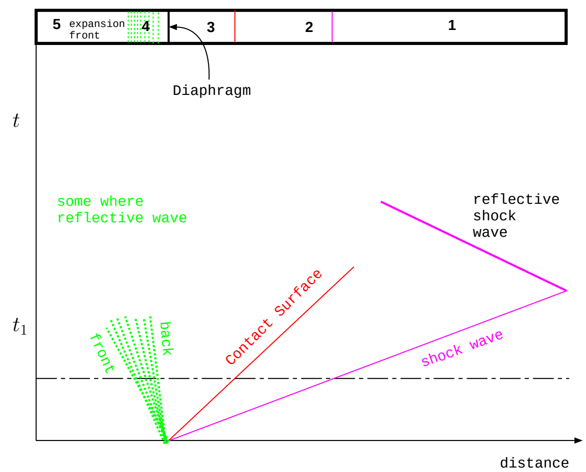


Fig. 5.10: The shock tube schematic with pressure "diagram"

Initially the gas from the driver section is coalescing of small shock waves into a large shock wave. In this analysis, it is assumed that this time is essentially zero. Zone 1 is undisturbed gas and zone 2 is area where the shock already passed. Due to the assumption that the shock is very sharp with zero width. On the other side, the explanation waves are moving into the high pressure chamber i.e. driver section. The shock is moving in supersonic speed (depend on the definition i.e what reference temperature is used used) and the medium behind shock is also moving but in velocity, U_2 which can be supersonic or subsonic in stationary coordinates. The velocities in the expansion chamber are varied between three zones. In zone 3 is original material that was in high pressure chamber but is in the same pressure as zone 2. Zone 4 is where the gradual transition between original high pressure to the low pressure occurs. The boundaries of the zone 4 are defined by initial conditions. The expansion front is moving at the local speed of sound in the high pressure section. The expansion back front is moving at the local speed of sound velocity but the actual gas is moving in the opposite direction in U_2. In fact, in the expansion chamber and the front are moving to the left while the actual flow of the gas is moving to the right (refer to Figure (5.10)). In zone 5 the velocity is zero and the pressure is in its original value.

The properties in the different zones have different relationship. The relationship between zone 1 and 2 zones is of the moving shock in to still medium (again this is the case of suddenly open chamber that was discussed in the previous section). The material in zone 2 and 3 is moving in the same velocity (speed) but the temperature and the entropy are different, while the pressure in the two zone is the same. The pressure and the temperature amount other properties in zone 4 isn't constant and are continuous between the conditions at zone 3 to conditions at zone zone 5. The expansion front wave velocity is larger then the velocity at the back front expansion wave velocity. The zone 4 is expanding during initial stage (until the expansion reach to the wall).

The shock tube has relatively small length $1-2[m]$ and the typical velocity is in the range of speed of sound, $c \sim \sqrt{340}$ thus the whole process take only a few milliseconds or less. Thus, this kind of experiments require fast recoding devises (relatively fast camera and fast data acquisitions devises.). A typical design problem of shock tube is to find the pressure to a achieve the desired temperature or Mach number. The relationship of the different properties were discussed earlier and because it is a common problem and it provides the a review of the material so far.

The following equations were developed earlier and are repeated here to clarifies the derivations. The pressure ratio between the two sides of the shock is

$$\frac{P_2}{P_1} = \frac{k-1}{k+1}\left(\frac{2k}{k-1}{M_{s1}}^2 - 1\right) \tag{5.68}$$

Rearranging equation (5.68) becomes

$$M_{s1} = \sqrt{\frac{k-1}{2k} + \frac{k+1}{2k}\frac{P_2}{P_1}} \tag{5.69}$$

Or expressing the velocity is

$$U_s = M_{s1} c_1 = c_1 \sqrt{\frac{k-1}{2k} + \frac{k+1}{2k}\frac{P_2}{P_1}} \tag{5.70}$$

And the velocity ratio between the two sides of the shock is

$$\frac{U_1}{U_2} = \frac{\rho_2}{\rho_2} = \frac{1 + \frac{k+1}{k-1}\frac{P_2}{P_1}}{\frac{k+1}{k-1}\frac{P_2}{P_1}} \tag{5.71}$$

The fluid velocity in zone 2 is the same

$$U_2{}' = U_s - U_2 = U_s \left(1 - \frac{U_2}{U_s}\right) \tag{5.72}$$

From the mass conservation, it follows that

$$\frac{U_2}{U_s} = \frac{\rho_1}{\rho_2} \tag{5.73}$$

$$U_2{}' = c_1 \sqrt{\frac{k-1}{2k} + \frac{k+1}{2k}\frac{P_2}{P_1}} \sqrt{1 - \frac{\frac{k+1}{k-1} + \frac{P_2}{P_1}}{1 + \frac{k+1}{k-1}\frac{P_2}{P_1}}} \tag{5.74}$$

After some rearrangement equation (5.74)

$$U_2{}' = \frac{c_1}{k}\left(\frac{P_2}{P_1} - 1\right)\sqrt{\frac{\frac{2k}{k+1}}{\frac{P_2}{P_1}\frac{k-1}{1+k}}} \tag{5.75}$$

On the isentropic side, on zone 4, taking the derivative of the continuity equation, $d(\rho U) = 0$ and diving by continuity equation results in

$$\frac{d\rho}{\rho} = -\frac{dU}{c} \tag{5.76}$$

Since the process in zone 4 is isentropic, applying the isentropic relationship ($T \propto \rho^{k-1}$) yields

$$\frac{c}{c_5} = \sqrt{\frac{T}{T_5}} = \left(\frac{\rho}{\rho_5}\right)^{\frac{k-1}{2}} \tag{5.77}$$

From equation (5.76) it follows that

$$dU = -c\frac{d\rho}{\rho} = c_5\left(\frac{\rho}{\rho_5}\right)^{\frac{k-1}{2}} d\rho \tag{5.78}$$

Equation (5.78) can be integrated as

$$\int_{U_5=0}^{U_3} dU = \int_{\rho_5}^{\rho_3} c_5\left(\frac{\rho}{\rho_5}\right)^{\frac{k-1}{2}} d\rho \tag{5.79}$$

The results of the integration are

$$U_3 = \frac{2c_5}{k-1}\left(1 - \left(\frac{\rho_3}{\rho_5}\right)^{\frac{k-1}{2}}\right) \tag{5.80}$$

Or in-terms of the pressure ratio is

$$U_3 = \frac{2c_5}{k-1}\left(1-\left(\frac{P_3}{P_5}\right)^{\frac{k-1}{2k}}\right) \tag{5.81}$$

As it was mentioned earlier the velocity at points $2'$ and 3 are identical, hence equation (5.81) and equation (5.75) can be combined to yield

$$\frac{2c_5}{k-1}\left(1-\left(\frac{P_3}{P_5}\right)^{\frac{k-1}{2k}}\right) = \frac{c_1}{k}\left(\frac{P_2}{P_1}-1\right)\sqrt{\frac{\frac{2k}{k+1}}{\frac{P_2}{P_1}\frac{k-1}{1+k}}} \tag{5.82}$$

After some rearrangement equation (5.82) transformed into

$$\frac{P_5}{P_1} = \frac{P_2}{P_1}\left(1-\frac{(k-1)\frac{c_1}{c_5}\left(\frac{P_5}{P_3}-1\right)}{\sqrt{2k}\sqrt{2k+(k+1)\left(\frac{P_2}{P_1}-1\right)}}\right)^{-\frac{2k}{k-1}} \tag{5.83}$$

Or in terms of Mach number of the M_{s1}

$$\frac{P_5}{P_1} = \frac{k_1-1}{k+1+1}\left(\frac{2k}{k_1-1}{M_{s1}}^2-1\right)\left[1-\frac{\frac{k-1}{k+1}\frac{c_1}{c_5}\left({M_{s1}}^2-1\right)}{M_{s1}}\right]^{-\frac{2k}{k-1}} \tag{5.84}$$

Utilizing the Rankine–Hugoniot relationship and the perfect gas model results in

$$\frac{T_2}{T_1} = \frac{1+\frac{k_1-1}{k_1+1}\frac{P_2}{P_1}}{1+\frac{k_1-1}{k_1+1}\frac{P_1}{P_2}} \tag{5.85}$$

Utilizing the isentropic relationship for zone 3-5 results in

$$\frac{T_3}{T_5} = \left(\frac{P_3}{P_5}\right)^{\frac{k_5-1}{k_5}} = \left(\frac{\frac{P_2}{P_1}}{\frac{P_5}{P_1}}\right)^{\frac{k_5-1}{k_5}} \tag{5.86}$$

Example 5.7:
A shock tube with initial pressure ratio of $\frac{P_5}{P_1=20}$ *and the initial temperature of* $300K$*. Find what is the shock velocity, temperature behind the shock. what will be temperature behind the shock if the pressure ratio is* $\frac{P_5}{P_1=40}$*?*

<u>SOLUTION</u>

5.5 Shock with Real Gases

5.6 Shock in Wet Steam

5.7 Normal Shock in Ducts

The flow in ducts is related to boundary layer issues. For a high Reynolds number, the assumption of uniform flow in the duct is closer to reality. It is normal to have a large Mach number with a large Re number. In that case, the assumptions in construction of these models are acceptable and reasonable.

5.8 Tables of Normal shocks, $k = 1.4$ Ideal Gas

Table 5.1: The shock wave Table for k = 1.4

$\mathbf{M_x}$	$\mathbf{M_y}$	$\frac{\mathbf{T_y}}{\mathbf{T_x}}$	$\frac{\rho_y}{\rho_x}$	$\frac{\mathbf{P_y}}{\mathbf{P_x}}$	$\frac{\mathbf{P_{0y}}}{\mathbf{P_{0x}}}$
1.00	1.00000	1.00000	1.00000	1.00000	1.00000
1.05	0.95313	1.03284	1.08398	1.11958	0.99985
1.10	0.91177	1.06494	1.16908	1.24500	0.99893
1.15	0.87502	1.09658	1.25504	1.37625	0.99669
1.20	0.84217	1.12799	1.34161	1.51333	0.99280
1.25	0.81264	1.15938	1.42857	1.65625	0.98706
1.30	0.78596	1.19087	1.51570	1.80500	0.97937
1.35	0.76175	1.22261	1.60278	1.95958	0.96974
1.40	0.73971	1.25469	1.68966	2.12000	0.95819
1.45	0.71956	1.28720	1.77614	2.28625	0.94484
1.50	0.70109	1.32022	1.86207	2.45833	0.92979
1.55	0.68410	1.35379	1.94732	2.63625	0.91319
1.60	0.66844	1.38797	2.03175	2.82000	0.89520
1.65	0.65396	1.42280	2.11525	3.00958	0.87599
1.70	0.64054	1.45833	2.19772	3.20500	0.85572
1.75	0.62809	1.49458	2.27907	3.40625	0.83457
1.80	0.61650	1.53158	2.35922	3.61333	0.81268
1.85	0.60570	1.56935	2.43811	3.82625	0.79023
1.90	0.59562	1.60792	2.51568	4.04500	0.76736
1.95	0.58618	1.64729	2.59188	4.26958	0.74420
2.00	0.57735	1.68750	2.66667	4.50000	0.72087
2.05	0.56906	1.72855	2.74002	4.73625	0.69751
2.10	0.56128	1.77045	2.81190	4.97833	0.67420
2.15	0.55395	1.81322	2.88231	5.22625	0.65105
2.20	0.54706	1.85686	2.95122	5.48000	0.62814
2.25	0.54055	1.90138	3.01863	5.73958	0.60553

Table 5.1: The shock wave table for k = 1.4 (continue)

$\mathbf{M_x}$	$\mathbf{M_y}$	$\frac{\mathbf{T_y}}{\mathbf{T_x}}$	$\frac{\rho_y}{\rho_x}$	$\frac{\mathbf{P_y}}{\mathbf{P_x}}$	$\frac{\mathbf{P_{0y}}}{\mathbf{P_{0x}}}$
2.30	0.53441	1.94680	3.08455	6.00500	0.58329
2.35	0.52861	1.99311	3.14897	6.27625	0.56148
2.40	0.52312	2.04033	3.21190	6.55333	0.54014
2.45	0.51792	2.08846	3.27335	6.83625	0.51931
2.50	0.51299	2.13750	3.33333	7.12500	0.49901
2.75	0.49181	2.39657	3.61194	8.65625	0.40623
3.00	0.47519	2.67901	3.85714	10.33333	0.32834
3.25	0.46192	2.98511	4.07229	12.15625	0.26451
3.50	0.45115	3.31505	4.26087	14.12500	0.21295
3.75	0.44231	3.66894	4.42623	16.23958	0.17166
4.00	0.43496	4.04688	4.57143	18.50000	0.13876
4.25	0.42878	4.44891	4.69919	20.90625	0.11256
4.50	0.42355	4.87509	4.81188	23.45833	0.09170
4.75	0.41908	5.32544	4.91156	26.15625	0.07505
5.00	0.41523	5.80000	5.00000	29.00000	0.06172
5.25	0.41189	6.29878	5.07869	31.98958	0.05100
5.50	0.40897	6.82180	5.14894	35.12500	0.04236
5.75	0.40642	7.36906	5.21182	38.40625	0.03536
6.00	0.40416	7.94059	5.26829	41.83333	0.02965
6.25	0.40216	8.53637	5.31915	45.40625	0.02498
6.50	0.40038	9.15643	5.36508	49.12500	0.02115
6.75	0.39879	9.80077	5.40667	52.98958	0.01798
7.00	0.39736	10.46939	5.44444	57.00000	0.01535
7.25	0.39607	11.16229	5.47883	61.15625	0.01316
7.50	0.39491	11.87948	5.51020	65.45833	0.01133
7.75	0.39385	12.62095	5.53890	69.90625	0.00979
8.00	0.39289	13.38672	5.56522	74.50000	0.00849
8.25	0.39201	14.17678	5.58939	79.23958	0.00739
8.50	0.39121	14.99113	5.61165	84.12500	0.00645
8.75	0.39048	15.82978	5.63218	89.15625	0.00565
9.00	0.38980	16.69273	5.65116	94.33333	0.00496
9.25	0.38918	17.57997	5.66874	99.65625	0.00437
9.50	0.38860	18.49152	5.68504	105.12500	0.00387
9.75	0.38807	19.42736	5.70019	110.73958	0.00343
10.00	0.38758	20.38750	5.71429	116.50000	0.00304

Table 5.2: Table for Shock Reflecting from suddenly closed end (k=1.4)

M_x	M_y	$M_x{}'$	$M_y{}'$	$\frac{T_y}{T_x}$	$\frac{P_y}{P_x}$	$\frac{P_{0y}}{P_{0x}}$
1.0060	0.99403	0.01000	0.0	1.004	1.014	1.00000
1.012	0.98812	0.020000	0.0	1.008	1.028	1.00000
1.018	0.98227	0.030000	0.0	1.012	1.043	0.99999
1.024	0.97647	0.040000	0.0	1.016	1.057	0.99998
1.030	0.97074	0.050000	0.0	1.020	1.072	0.99997
1.037	0.96506	0.060000	0.0	1.024	1.087	0.99994
1.043	0.95944	0.070000	0.0	1.028	1.102	0.99991
1.049	0.95387	0.080000	0.0	1.032	1.118	0.99986
1.055	0.94836	0.090000	0.0	1.036	1.133	0.99980
1.062	0.94291	0.10000	0.0	1.040	1.149	0.99973
1.127	0.89128	0.20000	0.0	1.082	1.316	0.99790
1.196	0.84463	0.30000	0.0	1.126	1.502	0.99317
1.268	0.80251	0.40000	0.0	1.171	1.710	0.98446
1.344	0.76452	0.50000	0.0	1.219	1.941	0.97099
1.423	0.73029	0.60000	0.0	1.269	2.195	0.95231
1.505	0.69946	0.70000	0.0	1.323	2.475	0.92832
1.589	0.67171	0.80000	0.0	1.381	2.780	0.89918
1.676	0.64673	0.90000	0.0	1.442	3.112	0.86537
1.766	0.62425	1.00000	0.0	1.506	3.473	0.82755
1.858	0.60401	1.100	0.0	1.576	3.862	0.78652
1.952	0.58578	1.200	0.0	1.649	4.280	0.74316
2.048	0.56935	1.300	0.0	1.727	4.728	0.69834
2.146	0.55453	1.400	0.0	1.810	5.206	0.65290
2.245	0.54114	1.500	0.0	1.897	5.715	0.60761
2.346	0.52904	1.600	0.0	1.990	6.256	0.56312
2.448	0.51808	1.700	0.0	2.087	6.827	0.51996
2.552	0.50814	1.800	0.0	2.189	7.431	0.47855
2.656	0.49912	1.900	0.0	2.297	8.066	0.43921
2.762	0.49092	2.000	0.0	2.410	8.734	0.40213
3.859	0.43894	3.000	0.0	3.831	17.21	0.15637
5.000	0.41523	4.000	0.0	5.800	29.00	0.061716
6.162	0.40284	5.000	0.0	8.325	44.14	0.026517
7.336	0.39566	6.000	0.0	11.41	62.62	0.012492
8.517	0.39116	7.000	0.0	15.05	84.47	0.00639
9.703	0.38817	8.000	0.0	19.25	$1.1E{+}02$	0.00350
10.89	0.38608	9.000	0.0	24.01	$1.4E{+}02$	0.00204
12.08	0.38457	10.00	0.0	29.33	$1.7E{+}02$	0.00125

Table 5.3: Table for Shock Propagating From suddenly **open** valve (k=1.4)

$\mathbf{M_x}$	$\mathbf{M_y}$	$\mathbf{M_x}'$	$\mathbf{M_y}'$	$\frac{\mathbf{T_y}}{\mathbf{T_x}}$	$\frac{\mathbf{P_y}}{\mathbf{P_x}}$	$\frac{\mathbf{P_{0y}}}{\mathbf{P_{0x}}}$
1.0060	0.99402	0.0	0.010000	1.004	1.014	1.00000
1.012	0.98807	0.0	0.020000	1.008	1.028	1.00000
1.018	0.98216	0.0	0.030000	1.012	1.043	0.99999
1.024	0.97629	0.0	0.040000	1.016	1.058	0.99998
1.031	0.97045	0.0	0.050000	1.020	1.073	0.99996
1.037	0.96465	0.0	0.060000	1.024	1.088	0.99994
1.044	0.95888	0.0	0.070000	1.029	1.104	0.99990
1.050	0.95315	0.0	0.080000	1.033	1.120	0.99985
1.057	0.94746	0.0	0.090000	1.037	1.136	0.99979
1.063	0.94180	0.0	0.10000	1.041	1.152	0.99971
1.133	0.88717	0.0	0.20000	1.086	1.331	0.99763
1.210	0.83607	0.0	0.30000	1.134	1.541	0.99181
1.295	0.78840	0.0	0.40000	1.188	1.791	0.98019
1.390	0.74403	0.0	0.50000	1.248	2.087	0.96069
1.495	0.70283	0.0	0.60000	1.317	2.441	0.93133
1.613	0.66462	0.0	0.70000	1.397	2.868	0.89039
1.745	0.62923	0.0	0.80000	1.491	3.387	0.83661
1.896	0.59649	0.0	0.90000	1.604	4.025	0.76940
2.068	0.56619	0.0	1.00000	1.744	4.823	0.68907
2.269	0.53817	0.0	1.100	1.919	5.840	0.59699
2.508	0.51223	0.0	1.200	2.145	7.171	0.49586
2.799	0.48823	0.0	1.300	2.450	8.975	0.38974
3.167	0.46599	0.0	1.400	2.881	11.54	0.28412
3.658	0.44536	0.0	1.500	3.536	15.45	0.18575
4.368	0.42622	0.0	1.600	4.646	22.09	0.10216
5.551	0.40843	0.0	1.700	6.931	35.78	0.040812
8.293	0.39187	0.0	1.800	14.32	80.07	0.00721
8.821	0.39028	0.0	1.810	16.07	90.61	0.00544
9.457	0.38870	0.0	1.820	18.33	$1.0E{+}02$	0.00395
10.24	0.38713	0.0	1.830	21.35	$1.2E{+}02$	0.00272
11.25	0.38557	0.0	1.840	25.57	$1.5E{+}02$	0.00175
12.62	0.38402	0.0	1.850	31.92	$1.9E{+}02$	0.00101
14.62	0.38248	0.0	1.860	42.53	$2.5E{+}02$	0.000497
17.99	0.38096	0.0	1.870	63.84	$3.8E{+}02$	0.000181
25.62	0.37944	0.0	1.880	$1.3E{+}02$	$7.7E{+}02$	$3.18E{-}05$
61.31	0.37822	0.0	1.888	$7.3E{+}02$	$4.4E{+}03$	0.0
62.95	0.37821	0.0	1.888	$7.7E{+}02$	$4.6E{+}03$	0.0
64.74	0.37820	0.0	1.888	$8.2E{+}02$	$4.9E{+}03$	0.0
66.69	0.37818	0.0	1.888	$8.7E{+}02$	$5.2E{+}03$	0.0

Table 5.3: Table for Shock Propagating from suddenly **open** valve (k=1.4)

M_x	M_y	$M_x{}'$	$M_y{}'$	$\frac{T_y}{T_x}$	$\frac{P_y}{P_x}$	$\frac{P_{0_y}}{P_{0_x}}$
68.83	0.37817	0.0	1.888	$9.2E{+}02$	$5.5E{+}03$	0.0
71.18	0.37816	0.0	1.889	$9.9E{+}02$	$5.9E{+}03$	0.0
73.80	0.37814	0.0	1.889	$1.1E{+}03$	$6.4E{+}03$	0.0
76.72	0.37813	0.0	1.889	$1.1E{+}03$	$6.9E{+}03$	0.0
80.02	0.37812	0.0	1.889	$1.2E{+}03$	$7.5E{+}03$	0.0
83.79	0.37810	0.0	1.889	$1.4E{+}03$	$8.2E{+}03$	0.0

Table 5.4: Table for Shock Propagating from suddenly **open** valve (k=1.3)

M_x	M_y	$M_x{}'$	$M_y{}'$	$\frac{T_y}{T_x}$	$\frac{P_y}{P_x}$	$\frac{P_{0_y}}{P_{0_x}}$
1.0058	0.99427	0.0	0.010000	1.003	1.013	1.00000
1.012	0.98857	0.0	0.020000	1.006	1.026	1.00000
1.017	0.98290	0.0	0.030000	1.009	1.040	0.99999
1.023	0.97726	0.0	0.040000	1.012	1.054	0.99998
1.029	0.97166	0.0	0.050000	1.015	1.067	0.99997
1.035	0.96610	0.0	0.060000	1.018	1.081	0.99995
1.042	0.96056	0.0	0.070000	1.021	1.096	0.99991
1.048	0.95506	0.0	0.080000	1.024	1.110	0.99987
1.054	0.94959	0.0	0.090000	1.028	1.125	0.99981
1.060	0.94415	0.0	0.10000	1.031	1.140	0.99975
1.126	0.89159	0.0	0.20000	1.063	1.302	0.99792
1.197	0.84227	0.0	0.30000	1.098	1.489	0.99288
1.275	0.79611	0.0	0.40000	1.136	1.706	0.98290
1.359	0.75301	0.0	0.50000	1.177	1.959	0.96631
1.452	0.71284	0.0	0.60000	1.223	2.252	0.94156
1.553	0.67546	0.0	0.70000	1.274	2.595	0.90734
1.663	0.64073	0.0	0.80000	1.333	2.997	0.86274
1.785	0.60847	0.0	0.90000	1.400	3.471	0.80734
1.919	0.57853	0.0	1.00000	1.478	4.034	0.74136
2.069	0.55074	0.0	1.100	1.570	4.707	0.66575
2.236	0.52495	0.0	1.200	1.681	5.522	0.58223
2.426	0.50100	0.0	1.300	1.815	6.523	0.49333
2.644	0.47875	0.0	1.400	1.980	7.772	0.40226
2.898	0.45807	0.0	1.500	2.191	9.367	0.31281
3.202	0.43882	0.0	1.600	2.467	11.46	0.22904
3.576	0.42089	0.0	1.700	2.842	14.32	0.15495
4.053	0.40418	0.0	1.800	3.381	18.44	0.093988
4.109	0.40257	0.0	1.810	3.448	18.95	0.088718

Table 5.4: Table for Shock Propagating from suddenly **open** valve (k=1.3)

M_x	M_y	$M_x{}'$	$M_y{}'$	$\frac{T_y}{T_x}$	$\frac{P_y}{P_x}$	$\frac{P_{0_y}}{P_{0_x}}$
4.166	0.40097	0.0	1.820	3.519	19.49	0.083607
4.225	0.39938	0.0	1.830	3.592	20.05	0.078654
4.286	0.39780	0.0	1.840	3.669	20.64	0.073863
4.349	0.39624	0.0	1.850	3.749	21.25	0.069233
4.415	0.39468	0.0	1.860	3.834	21.90	0.064766
4.482	0.39314	0.0	1.870	3.923	22.58	0.060462
4.553	0.39160	0.0	1.880	4.016	23.30	0.056322
4.611	0.39037	0.0	1.888	4.096	23.91	0.053088
4.612	0.39035	0.0	1.888	4.097	23.91	0.053053
4.613	0.39034	0.0	1.888	4.098	23.92	0.053018
4.613	0.39033	0.0	1.888	4.099	23.93	0.052984
4.614	0.39031	0.0	1.888	4.099	23.93	0.052949
4.615	0.39030	0.0	1.889	4.100	23.94	0.052914
4.615	0.39029	0.0	1.889	4.101	23.95	0.052879
4.616	0.39027	0.0	1.889	4.102	23.95	0.052844
4.616	0.39026	0.0	1.889	4.103	23.96	0.052809
4.617	0.39025	0.0	1.889	4.104	23.97	0.052775

CHAPTER 6

Normal Shock in Variable Duct Areas

In the previous two chapters, the flow in a variable area duct and the normal shock (discontinuity) were discussed. A discussion of the occurrences of shock in flow in a variable is presented. As is was presented before, the shock can occur **only** when there is a supersonic flow. As it was shown in Chapter 5, the gas has to pass trough a converging–diverging nozzle to obtain a supersonic flow. Thus, the study of the normal shock in converging–diverging nozzle is presented.

In the previous Chapter, the flow in a convergent–divergent nuzzle was presented when the pressure ratio was above or below the special range. In this Chapter, the flow in this special range of pressure ratios is presented. It is interesting to note that a normal shock must occur in these situations (pressure ratios).

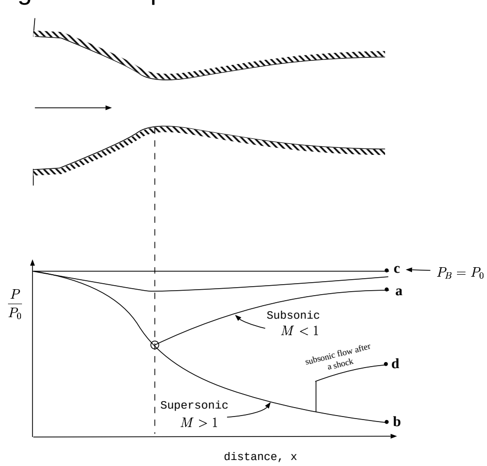


Fig. 6.1: The flow in the nozzle with different back pressures

In figure (6.1) the reduced pressure distribution in the converging–diverging nozzle is shown in whole range pressure ratios. In case, when the pressure ratio, P_B is between point "**a**" and point "**b**" the flow is different from what was discussed before. In this case, no continuous pressure possibility can exists. Only in one point where $P_{\text{B}} = P_b$ continuous pressure exist. If the back pressure, P_{B} is smaller than P_b a discontinuous point (a shock) will occur. As conclusion, once the flow becomes supersonic, only exact geometry can achieve continuous pressure flow.

In the literature, some refers to a nozzle with area ratio such point **b** is above the back pressure and it is referred to an under–expanded nozzle. In the under–expanded case, the nozzle doesn't provide the maximum thrust possible. On the other hand, when the nozzle exit area is too large a shock will occur and other phenomenon such plume will separate from the wall inside the nozzle. This nozzle is called an over-expanded nozzle. In comparison of nozzle performance for rocket and aviation, is that over-expanded nozzle is worse than in the under-expanded nozzle because the nozzle's large exit area results in extra drag.

The location of the shock is determined by geometry to achieve the right back pressure. Obviously if the back pressure, P_{B}, is lower than the critical value (the only value that achieve continuous pressure) a shock occurs outside of the nozzle if needed. If the back pressure is within the range of P_a to P_b than the exact location determined in a such location that after the shock the subsonic branch will matches the back pressure.

First example is pressed for academic reasons. It has to be recognized that the shock wave isn't easily visible (see for Mach's photography techniques). Therefore, this example provides an demonstration of the calculations for required location even it isn't realistic. Nevertheless, this example provide the fundamentals to explain the usage of the tools (equations and tables) that were developed so far.

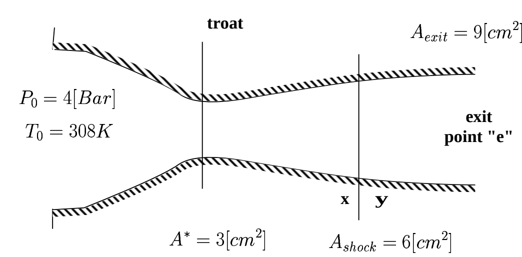


Fig. 6.2: A nozzle with normal shock

Example 6.1:
*A large tank with compressed air is attached into a converging–diverging nozzle at pressure $4[Bar]$ and temperature of $35[°\text{C}]$. Nozzle throat area is 3[cm^2] and the exit area is 9[cm^2]. The shock occurs in a location where the cross section area is 6[cm^2]. Calculate the back pressure and the temperature of the flow (It should be noted that the temperature of the surrounding is irrelevant in this case.) Also determine the critical points for the back pressure (point "**a**" and point "**b**".*

SOLUTION

Since the key word "large tank" was used that means that the stagnation temperature and pressure are known and equal to the conditions in the tank.

First, the exit Mach number has to be determined. This Mach number can be calculated by utilizing the isentropic relationship from the large tank to shock (point **"x"**). Then the relationship developed for the shock can be utilized to calculated the Mach number after the shock, (point **"y"**). From the Mach number after the shock, M_y, the Mach number at the exit can be calculated utilizing the isentropic relationship.

It has to be realized that for a large tank the inside conditions are essentially the stagnation conditions (This statement said without a a proof, but can be shown that the correction is negligible for a typical dimension ratio that is over 100. For example, in the case of ratio of 100 the Mach number is 0.00587 and the error is less than %0.1). Thus, the stagnation temperature and pressure are known $T_0 = 308K$ and $P_0 = 4[Bar]$. The star area (the throat area), A^*, before the shock is known and given as well.

$$\frac{A_x}{A^*} = \frac{6}{3} = 2$$

With this ratio $(A/A^* = 2)$ utilizing the Table (5.1) or equation (4.49) or the GDC–Potto, the Mach number, M_x is about 2.197 as shown table below:

$\mathbf{M}$	$\frac{\mathbf{T}}{\mathbf{T_0}}$	$\frac{\rho}{\rho_0}$	$\frac{\mathbf{A}}{\mathbf{A^*}}$	$\frac{\mathbf{P}}{\mathbf{P_0}}$	$\frac{\mathbf{A \times P}}{\mathbf{A^* \times P_0}}$
2.1972	0.50877	0.18463	2.0000	0.09393	0.18787

With this Mach number, $M_x = 2.1972$ the Mach number, M_y can be obtained. From equation (5.22) or from Table (5.1) $M_y \cong 0.54746$. With these values, the subsonic branch can be evaluated for the pressure and temperature ratios.

$\mathbf{M_x}$	$\mathbf{M_y}$	$\frac{\mathbf{T_y}}{\mathbf{T_x}}$	$\frac{\rho_y}{\rho_x}$	$\frac{\mathbf{P_y}}{\mathbf{P_x}}$	$\frac{\mathbf{P_0}_y}{\mathbf{P_0}_x}$
2.1972	0.54743	1.8544	2.9474	5.4656	0.62941

From Table (**??**) or from equation (4.11) the following table for the isentropic relationship is obtained

$\mathbf{M}$	$\frac{\mathbf{T}}{\mathbf{T_0}}$	$\frac{\rho}{\rho_0}$	$\frac{\mathbf{A}}{\mathbf{A^*}}$	$\frac{\mathbf{P}}{\mathbf{P_0}}$	$\frac{\mathbf{A \times P}}{\mathbf{A^* \times P_0}}$
0.54743	0.94345	0.86457	1.2588	0.81568	1.0268

Again utilizing the isentropic relationship the exit conditions can be evaluated. With known Mach number the new star area ratio, A_y/A^* is known and the exit area can be calculated as

$$\frac{A_e}{A^*} = \frac{A_e}{A_y} \times \frac{A_y}{A^*} = 1.2588 \times \frac{9}{6} = 1.8882$$

with this area ratio, $\frac{A_e}{A^*} = 1.8882$, one can obtain using the isentropic relationship as

$\mathbf{M}$	$\frac{\mathbf{T}}{\mathbf{T_0}}$	$\frac{\rho}{\rho_0}$	$\frac{\mathbf{A}}{\mathbf{A^\star}}$	$\frac{\mathbf{P}}{\mathbf{P_0}}$	$\frac{\mathbf{A\times P}}{\mathbf{A^*\times P_0}}$
0.32651	0.97912	0.94862	1.8882	0.92882	1.7538

Since the stagnation pressure is constant as well the stagnation temperature, the exit conditions can be calculated.

$$\begin{aligned} P_{exit} &= \left(\frac{P_{exit}}{P_0}\right)\left(\frac{P_0}{P_y}\right)\left(\frac{P_y}{P_x}\right)\left(\frac{P_x}{P_0}\right)P_0 \\ &= 0.92882 \times \left(\frac{1}{0.81568}\right) \times 5.466 \times 0.094 \times 4 \\ &\cong 2.34[Bar] \end{aligned}$$

The exit temperature is

$$\begin{aligned} T_{exit} &= \left(\frac{T_{exit}}{T_0}\right)\left(\frac{T_0}{T_y}\right)\left(\frac{T_y}{T_x}\right)\left(\frac{T_x}{T_0}\right)T_0 \\ &= 0.98133 \times \left(\frac{1}{0.951}\right) \times 1.854 \times 0.509 \times 308 \\ &\cong 299.9K \end{aligned}$$

For the “critical” points ”a” and ”b” are the points that the shock doesn’t occur and yet the flow achieve Mach equal 1 at the throat. In that case we don’t have to go through that shock transition. Yet we have to pay attention that there two possible back pressures that can “achieve” it or target. The area ratio for both cases, is $A/A^* = 3$ In the subsonic branch (either using equation or the isentropic Table or GDC-Potto as

$\mathbf{M}$	$\frac{\mathbf{T}}{\mathbf{T_0}}$	$\frac{\rho}{\rho_0}$	$\frac{\mathbf{A}}{\mathbf{A^\star}}$	$\frac{\mathbf{P}}{\mathbf{P_0}}$	$\frac{\mathbf{A\times P}}{\mathbf{A^*\times P_0}}$
0.19745	0.99226	0.98077	3.0000	0.97318	2.9195
2.6374	0.41820	0.11310	3.0000	0.04730	0.14190

$$P_{exit} = \left(\frac{P_{exit}}{P_0}\right)P_0 = 0.99226 \times 4 \cong 3.97[Bar]$$

For the supersonic sonic branch

$$P_{exit} = \left(\frac{P_{exit}}{P_0}\right)P_0 = 0.41820 \times 4 \cong 1.6728[Bar]$$

It should be noted that the flow rate is constant and maximum for any point beyond the point "a" even if the shock is exist. The flow rate is expressed as following

$$\dot{m} = \rho^* A^* U = \overbrace{\frac{P^*}{R T^*}}^{\rho^*} A \overbrace{c M}^{M=1} = \frac{\left(\overbrace{\frac{P^*}{P_0} P_0}^{P^*}\right)}{R \left(\underbrace{\frac{T^*}{T_0} T_0}_{T^*}\right)} A \overbrace{\sqrt{k R T^*}}^{c} = \frac{\left(\frac{P^*}{P_0} P_0\right)}{R \left(\frac{T^*}{T_0} T_0\right)} A \sqrt{k R \frac{T^*}{T_0} T_0}$$

The temperature and pressure at the throat are:

$$T^* = \left(\frac{T^*}{T_0}\right) T_0 = 0.833 \times 308 = 256.7K$$

The temperature at the throat reads

$$P^* = \left(\frac{P^*}{P_0}\right) P_0 = 0.5283 \times 4 = 2.113[Bar]$$

The speed of sound is

$$c = \sqrt{1.4 \times 287 \times 256.7} = 321.12[m/sec]$$

And the mass flow rate reads

$$\dot{m} = \frac{410^5}{287 \times 256.7} 3 \times 10^{-4} \times 321.12 = 0.13[kg/sec]$$

It is interesting to note that in this case the choking condition is obtained ($M = 1$) when the back pressure even reduced to less than 5% than original pressure (the pressure in the tank). While the pressure to achieve full supersonic flow through the nozzle the pressure has to be below the 42% the original value. Thus, over 50% of the range of pressure a shock occores some where in the nozzle. In fact in many industrial applications, these kind situations exist. In these applications a small pressure difference can produce a shock wave and a chock flow.

For more practical example[1] from industrial application point of view.

Example 6.2:
In the data from the above example (6.1) *where would be shock's location when the back pressure is* $2[Bar]$*?*

[1]The meaning of the word practical is that reality the engineer does not given the opportunity or determined the location of the shock but rather information such as pressures and temperature.

SOLUTION

A solution procedure similar to what done in previous example (6.1) can be used here. The solution process starts at the nozzle's exit and progress to the entrance.

The conditions at the tank are again the stagnation conditions. Thus, the exit pressure is between point "a" to point "b". It follows that there must exist a shock in the nozzle. Mathematically, there are two main possibles ways to obtain the solution. In the first method, the previous example information used and expanded. In fact, it requires some iterations by "smart" guessing the different shock locations. The area (location) that the previous example did not "produce" the "right" solution (the exit pressure was $2.113[Bar]$. In here, the needed pressure is only $2[Bar]$ which means that the next guess for the shock location should be with a larger area[2]. The second (recommended) method is noticing that the flow is adiabatic and the mass flow rate is constant which means that the ratio of the $P_0 \times A^* = P_{y0} \times A^*|_{@y}$ (upstream conditions are known, see also equation (4.56)).

$$\frac{P_{exit}A_{exit}}{P_{x0} \times {A_x}^*} = \frac{P_{exit}A_{exit}}{P_{y0} \times {A_y}^*} = \frac{2 \times 9}{4 \times 3} = 1.5[unitless!]$$

With the knowledge of the ratio $\frac{PA}{P_0 A^*}$ which was calculated and determines the exit Mach number. Utilizing the Table (**??**) or the GDC-Potto provides the following table is obtained

$\mathbf{M}$	$\mathbf{\frac{T}{T_0}}$	$\frac{\rho}{\rho_0}$	$\mathbf{\frac{A}{A^*}}$	$\mathbf{\frac{P}{P_0}}$	$\mathbf{\frac{A\times P}{A^*\times P_0}}$	$\mathbf{\frac{F}{F^*}}$
0.38034	0.97188	0.93118	1.6575	0.90500	1.5000	0.75158

With these values the relationship between the stagnation pressures of the shock are obtainable e.g. the exit Mach number, M_y, is known. The exit total pressure can be obtained (if needed). More importantly the pressure ratio exit is known. The ratio of the ratio of stagnation pressure obtained by

$$\frac{P_{0y}}{P_{0x}} = \overbrace{\left(\frac{P_{0y}}{P_{exit}}\right)}^{for\ M_{exit}} \left(\frac{P_{exit}}{P_{0x}}\right) = \frac{1}{0.905} \times \frac{2}{4} = 0.5525$$

Looking up in the Table (5.1) or utilizing the GDC-Potto provides

$\mathbf{M_x}$	$\mathbf{M_y}$	$\mathbf{\frac{T_y}{T_x}}$	$\frac{\rho_y}{\rho_x}$	$\mathbf{\frac{P_y}{P_x}}$	$\mathbf{\frac{P_{0y}}{P_{0x}}}$
2.3709	0.52628	2.0128	3.1755	6.3914	0.55250

With the information of Mach number (either M_x or M_y) the area where the shock (location) occurs can be found. First, utilizing the isentropic Table (**??**).

[2] Of course, the computer can be use to carry this calculations in a sophisticate way.

$\mathbf{M}$	$\mathbf{\frac{T}{T_0}}$	$\frac{\rho}{\rho_0}$	$\mathbf{\frac{A}{A^*}}$	$\mathbf{\frac{P}{P_0}}$	$\mathbf{\frac{A \times P}{A^* \times P_0}}$
2.3709	0.47076	0.15205	2.3396	0.07158	0.16747

Approaching the shock location from the upstream (entrance) yields

$$A = \frac{A}{A*} A^* = 2.3396 \times 3 \cong 7.0188 [cm^2]$$

Note, as “simple” check this value is larger than the value in the previous example.

6.1 Nozzle effi ciency

Obviously nozzles are not perfectly efficient and there are several ways to define the efficiency of the nozzle. One of the effective way is to define the efficiency as the ratio of the energy converted to kinetic energy and the total potential energy could be converted to kinetic energy. The total energy that can be converted is during isentropic process is

$$E = h_0 - h_{exits} \tag{6.1}$$

where h_{exits} is the enthalpy if the flow was isentropic. The actual energy that was used is

$$E = h_0 - h_{exit} \tag{6.2}$$

The efficiency can be defined as

$$\eta = \frac{h_0 - h_{exit}}{h_0 - h_{exits}} = \frac{{U_{actual}}^2}{{U_{ideal}}^2} \tag{6.3}$$

The typical efficiency of nozzle is ranged between 0.9 to 0.99. In the literature some define also velocity coefficient as the ratio of the actual velocity to the ideal velocity, V_c

$$V_c = \sqrt{\eta} \tag{6.4}$$

There is another less used definition which referred as the coefficient of discharge as the ratio of the actual mass rate to the ideal mass flow rate.

$$C_d = \frac{\dot{m}_{actual}}{\dot{m}_{ideal}} \tag{6.5}$$

6.1.1 Diffuser Efficiency

The efficiency of the diffuser is defined as the ratio of the enthalpy change that occurred between the entrance to exit stagnation pressure to the kinetic energy.

$$\eta = \frac{2(h_3 - h_1)}{{U_1}^2} = \frac{h_3 - h_1}{h_{01} - h_1} \tag{6.6}$$

For perfect gas equation (**??**) can be converted to

$$\eta = \frac{2C_p(T_3 - T_1)}{{U_1}^2} \tag{6.7}$$

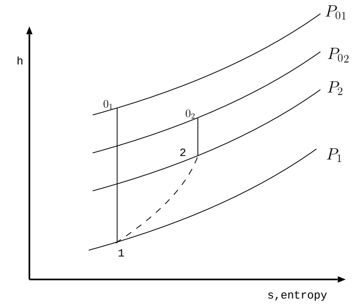


Fig. 6.3: Description to clarify the defi nition of diffuser effi ciency

And further expanding equation (6.7) results in

$$\eta = \frac{2\frac{kR}{k-1}T_1\left(\frac{T_3}{T_1} - 1\right)}{{c_1}^2{M_1}^2} = \frac{\frac{2}{k-1}\left(\frac{T_3}{T_1} - 1\right)}{{M_1}^2} = \frac{2}{{M_1}^2(k-1)}\left(\left(\frac{T_3}{T_1}\right)^{\frac{k-1}{k}} - 1\right) \tag{6.8}$$

Example 6.3:

A wind tunnel combined from a nozzle and a diffuser (actually two nozzles connected by a constant area see Figure (6.4)*) the required condition at point 3 are:* $M = 3.0$ *and pressure of* $0.7[Bar]$ *and temperature of* $250K$. *The cross section in area between the nuzzle and diffuser is* $0.02[m^2]$. *What is area of nozzle's throat and what is area of the diffuser's throat to maintain chocked diffuser with subsonic flow in the expansion section.* $k = 1.4$ *can be assumed. Assume that a shock occors in the test section.*

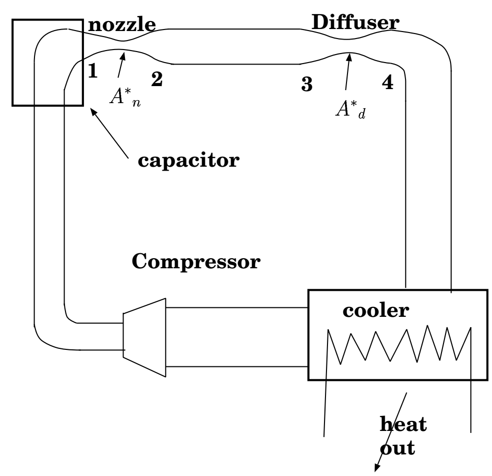


Fig. 6.4: Schematic of a supersonic tunnel in a continuous region (and also for example (6.3)

SOLUTION

The condition at $M = 3$ is summarized in following table

$\mathbf{M}$	$\frac{\mathbf{T}}{\mathbf{T_0}}$	$\frac{\rho}{\rho_0}$	$\frac{\mathbf{A}}{\mathbf{A^\star}}$	$\frac{\mathbf{P}}{\mathbf{P_0}}$	$\frac{\mathbf{A\times P}}{\mathbf{A^*\times P_0}}$	$\frac{\mathbf{F}}{\mathbf{F^*}}$
3.0000	0.35714	0.07623	4.2346	0.02722	0.11528	0.65326

The nozzle area can be calculated by

$$A^*{}_n = \frac{A^\star}{A} A = 0.02/4.2346 = 0.0047[m^2]$$

In this case, $P_0 A^*$ is constant (constant mass flow). First the stagnation behind the shock will be

$\mathbf{M_x}$	$\mathbf{M_y}$	$\frac{\mathbf{T_y}}{\mathbf{T_x}}$	$\frac{\rho_\mathbf{y}}{\rho_\mathbf{x}}$	$\frac{\mathbf{P_y}}{\mathbf{P_x}}$	$\frac{\mathbf{P_{0_y}}}{\mathbf{P_{0_x}}}$
3.0000	0.47519	2.6790	3.8571	10.3333	0.32834

$$A^*{}_d = \frac{P_{0\,n}}{P_{0\,d}} A^*{}_n \sim \frac{1}{0.32834} 0.0047 \sim 0.0143[m^3]$$

Example 6.4:
The speed of the a shock is moving at 2000 [m/sec] in pipe with gas with $k = 1.3$, pressure of 2[Bar] and temperature of $350K$. Calculate the condition after the shock.

<u>SOLUTION</u>

Example 6.5:
An inventor interested in a design of tube and piston so that the pressure is double in tube when the piston is suddenly moving. The propagating piston is assumed to move in temperature of $300K$ and about atmospheric pressure of $1[Bar]$. If the steady state is achieved, what the piston velocity should be?

<u>SOLUTION</u>

Example 6.6:
A flow of gas is brought into sudden stop. The mass flow rate of the gas is 2 [kg/sec] and cross section is A = 0.002$[m^3]$. The imaginary gas conditions are temperature is $350K$ and pressure is $2[Bar]$ and $R = 143[j/kg\ K]$ and $k = 1.091$ (Butane). Calculate the condition behind the shock wave.

<u>SOLUTION</u>

CHAPTER 7

Nozzle Flow With External Forces

This chapter is under heavy construction. Please ignore. If you want to contribute and add any results of experiments, to this chapter, please do so. You can help especially if you have photos showing these effects.

In the previous chapters a simple model describing the flow in nozzle was explained. In cases where more refined calculations have to carried the gravity or other forces have to be taken into account. Flow in a vertical or horizontal nozzle are different because the gravity. The simplified models that suggests them–self are: friction and adiabatic, isothermal, seem the most applicable. These models can served as limiting cases for more realistic flow.

The effects of the gravity of the nozzle flow in two models isentropic and isothermal is analyzed here. The isothermal nozzle model is suitable in cases where the flow is relatively slow (small Eckert numbers) while as the isentropic model is more suitable for large Eckert numbers.

Insert dimensional analysis, and point out that Ec number is relatively small and, for example, Ec=0.1 is already large number

The two models produces slightly different equations. The equations results in slightly different conditions for the chocking and different chocking speed. Moreover, the working equations are also different and this author isn't aware of material in the literature which provides any working table for the gravity effect.

7.1 Isentropic Nozzle ($Q = 0$)

put cv. fi gure

The energy equation for isentropic nozzle provides

$$dh + UdU = \overbrace{f(x)dx}^{\substack{\text{external} \\ \text{work or} \\ \text{potential} \\ \text{difference,} \\ \text{i.e. } z \times g}} \tag{7.1}$$

Utilizing equation (4.27) when $ds = 0$ leads to

$$\frac{dP}{\rho} + UdU = f(x')dx' \tag{7.2}$$

For the isentropic process $dP = const \times k\rho^{k-1}d\rho$ when the $const = P/\rho^k$ at any point of the flow. The equation (7.2) becomes

$$\begin{aligned} \overbrace{\overbrace{\frac{P}{\rho^k}}^{any\ point} \quad k\frac{\rho^k}{\rho}d\rho}^{dP}\frac{1}{\rho} + UdU = k\overbrace{\frac{P}{\rho}}^{RT}\frac{d\rho}{\rho}UdU = & \quad f(x')dx' \\ \frac{kRTd\rho}{\rho} + UdU = \frac{c^2}{\rho}d\rho + UdU = & \quad f(x')dx' \end{aligned} \tag{7.3}$$

The continuity equation as developed earlier (mass conservation equation isn't effected by the gravity)

$$-\frac{d\rho}{\rho} = \frac{dA}{A} + \frac{dU}{U} = 0 \tag{7.4}$$

Substituting $d\rho/\rho$ from equation (7.4), into equation (7.2) moving $d\rho$ to the right hand side, and diving by dx' yields

$$U\frac{dU}{dx'} = c^2\left[\frac{1}{U}\frac{dU}{dx'} + \frac{1}{A}\frac{dA}{dx'}\right] + f(x') \tag{7.5}$$

Rearranging equation (7.5) yields

$$\frac{dU}{dx'} = \left[M^2\frac{dU}{dx'} + \frac{c^2}{AU}\frac{dA}{dx'}\right] + \frac{f(x')}{U} \tag{7.6}$$

And further rearranging yields

$$\left(1 - M^2\right)\frac{dU}{dx'} = \frac{c^2}{AU}\frac{dA}{dx'} + \frac{f(x')}{U} \tag{7.7}$$

Equation (7.7) can be rearranged as

$$\frac{dU}{dx'} = \frac{1}{(1-M^2)}\left[\frac{c^2}{AU}\frac{dA}{dx'} + \frac{f(x')}{U}\right] \tag{7.8}$$

Equation (7.10) dimensionless form by utilizing $x = x'/\ell$ and ℓ is the nozzle length

$$\frac{dM}{dx} = \frac{1}{(1-M^2)}\left[\frac{1}{AM}\frac{dA}{dx} + \frac{\ell f(x)}{c\,\underbrace{cM}_{U}}\right] \tag{7.9}$$

And the final form of equation (7.10) is

$$\frac{d\left(M^2\right)}{dx} = \frac{2}{(1-M^2)}\left[\frac{1}{A}\frac{dA}{dx} + \frac{\ell f(x)}{c^2}\right] \tag{7.10}$$

The term $\frac{\ell f(x)}{c^2}$ is considered to be very small ($0.1 \times 10/100000 < 0.1$%) for "standard" situations. The dimensionless number, $\frac{\ell f(x)}{c^2}$ sometimes referred as Ozer number determines whether gravity should be considered in the calculations. Nevertheless, one should be aware of value of Ozer number for large magnetic fields (astronomy) and low temperature, In such cases, the gravity effect can be considerable.

As it was shown before the transition must occur when $M = 1$. Consequently, two zones must be treated separately. First, here the Mach number is discussed and not the pressure as in the previous chapter. For $M < 1$ (the subsonic branch) the term $\frac{2}{(1-M^2)}$ is positive and the treads determined by gravity and the area function.

$$\left[\frac{1}{A}\frac{dA}{dx} + \frac{\ell f(x)}{c^2}\right] > 0 \Longrightarrow d(M^2) > 0$$

or conversely,

$$\left[\frac{1}{A}\frac{dA}{dx} + \frac{\ell f(x)}{c^2}\right] < 0 \Longrightarrow d(M^2) < 0$$

For the case of $M > 1$ (the supersonic branch) the term $\frac{2}{(1-M^2)}$ is negative and therefore

$$\left[\frac{1}{A}\frac{dA}{dx} + \frac{\ell f(x)}{c^2}\right] > 0 \Longrightarrow d(M^2) < 0$$

For the border case $M = 1$, the denominator $1 - M^2 = 0$, is zero either $d(M^2) = \infty$ or $\left[\frac{1}{A}\frac{dA}{dx} + \frac{\ell f(x)}{c^2}\right] = 0$ And the dM is indeterminate. As it was shown in chapter **??** the flow is chocked ($M = 1$) only when

$$\left[\frac{dA}{dx} + \frac{\ell f(x)}{c^2}\right] = 0. \tag{7.11}$$

It should be noticed that when $f(x)$ is zero, e.g. horizontal flow, the equation (7.11) reduced into $\frac{dA}{dx} = 0$ that was developed previously.

The ability to manipulate the location provides a mean to increase/decrease the flow rate. Yet this ability since Ozer number is relatively very small.

look for experimental picture demonstrating this point by changing the direction of the flow from experimental to vertical

Meta

To work out the example:

Calculate the location of the flow rate when the gravity is taken into account. The nozzle dimensions are shown in the attached figure.

Meta End

This condition means that the critical point can occurs in several locations that satisfies equation (7.11). Further, the critical point, sonic point is $\frac{dA}{Ax} \neq 0$ If $f(x)$ is a positive function, the critical point happen at converging part of the nozzle (before the throat) and if $f(x)$ is a negative function the critical point is diverging part of the throat. For example consider the gravity, $f(x) = -g$ a flow in a nozzle vertically the critical point will be above the throat.

7.2 Isothermal Nozzle $(T = constant)$

CHAPTER 8

Isothermal Flow

The gas flow throw a long tube has a applicability in situations which occurs in a relatively long distance and where heat transfer is relatively rapid so that the temperature can be treated, for engineering purposes, as a constant. This model is applicable when a natural gas (or other gases) flows for a large distance. Such situations are common in large cities in U.S.A. where natural gas is used for heating. It is more predominate (more applicable) in situations where the gas is pumped for a length of kilometers.

To put discussion when the "relatively fast" means.

The high speed of the gas is obtained or explained by the combination of heat transfer and the friction to the flow. For a long pipe, the pressure difference reduces the density of the gas. For instance, in a perfect gas, the density is inverse of the pressure (it has to be kept in mind that the gas undergoes a isothermal process.). To maintain conservation of mass, the velocity increases inversely to the pressure. At critical point the velocity reaches the speed of sound at the exit and hence the flow will be chocked[1].

8.1 The Control Volume Analysis/Governing equations

Figure 8.1 describes the flow of gas from the left to the right. The heat transfer up stream (or down stream) is assumed to be negligible. Hence, the energy equation can be written as following:

$$\frac{dQ}{\dot{m}} = c_p dT + d\frac{U^2}{2} = c_p dT_0 \tag{8.1}$$

[1]This explanation is not correct as it will shown later on. Close to the critical point (about, $1/\sqrt{k}$, the heat transfer, is relatively high and the isothermal flow model is not valid anymore. Therefore, the study of the isothermal flow above this point is academic discussion only.

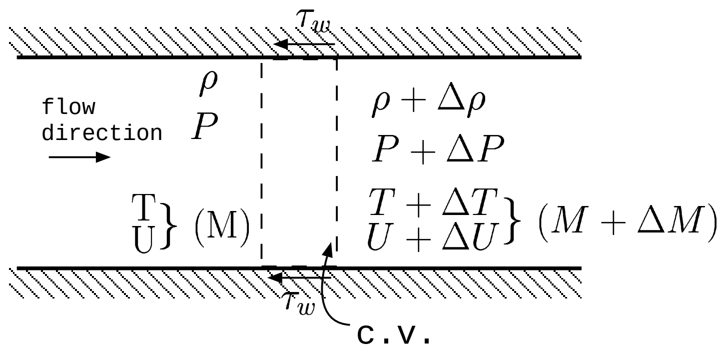


Fig. 8.1: Control volume for isothermal flow

The momentum equation written as following

$$-AdP - \tau_w dA_{\text{wetted area}} = \dot{m}dU \tag{8.2}$$

where A is the cross section area (it doesn't have to be a perfect circle a close enough shape is sufficient.). The shear stress is the force per area that acts on the fluid by the tube wall on the fluid. The $A_{wetted\ \ area}$ is the area that shear stress acts on. The second of thermodynamics reads

Perhaps more quantitative discussion about how "circular" the shape should be.

$$\frac{s_2 - s_1}{C_p} = \ln\frac{T_2}{T_1} - \frac{k-1}{k}\ln\frac{P_2}{P_1} \tag{8.3}$$

The mass conservation is reduced to

$$\dot{m} = constant = \rho U A \tag{8.4}$$

Again it is assumed that the gas is a perfect gas and therefore, equation of state expressed as following:

$$P = \rho R T \tag{8.5}$$

8.2 Dimensionless Representation

In this section the equations are transformed into the dimensionless form and presented in a dimensionless form. First it must be recalled that the temperature is constant and therefor, equation of state reads

it seems obvious to write this equation perhaps to consult with others.

$$\frac{dP}{P} = \frac{d\rho}{\rho} \tag{8.6}$$

It is convenient to define a hydraulic diameter

$$D_H = \frac{4 \times \text{Cross Section Area}}{\text{wetted perimeter}} \tag{8.7}$$

Now, the Fanning friction factor[2] is introduced, this factor is a dimensionless friction factor sometimes referred to as the friction coefficient as following:

$$f = \frac{\tau_w}{\frac{1}{2}\rho U^2} \tag{8.8}$$

Substituting equation (8.8) into momentum equation (8.2) yields

$$-dP - \frac{4dx}{D_H} f \left(\frac{1}{2}\rho U^2\right) = \overbrace{\rho U}^{\frac{\dot{m}}{A}} dU \tag{8.9}$$

Rearranging equation (8.9) and utilizing the identify for perfect gas $M^2 = \rho U^2/kP$ yield:

$$-\frac{dP}{P} - \frac{4f dx}{D_H}\left(\frac{kPM^2}{2}\right) = \frac{kPM^2 dU}{U} \tag{8.10}$$

Now the pressure, P as a function of the Mach number have to substitute along with velocity, U.

Meta

Should this material presented?

$$U^2 = kRTM^2 \tag{8.11}$$

Differentiation of equation (8.11) yields

$$d(U^2) = kR\left(M^2\ dT + T\ d(M^2)\right) \tag{8.12}$$

$$\frac{d(M^2)}{M^2} = \frac{d(U^2)}{U^2} - \frac{dT}{T} \tag{8.13}$$

Now it can be noticed that $dT = 0$ for isothermal process and therefore

$$\frac{d(M^2)}{M^2} = \frac{d(U^2)}{U^2} = \frac{2U\ dU}{U^2} = \frac{2dU}{U} \tag{8.14}$$

[2]It should be noted that Fanning factor based on hydraulic radius, instead of Diameter friction equation, thus "Fanning f" values are only 1/4th of "Darcy f" values.

Meta End

The dimensionalization of the mass conservation equation yields

$$\frac{d\rho}{\rho} + \frac{dU}{U} = \frac{d\rho}{\rho} + \frac{2U\,dU}{2U^2} = \frac{d\rho}{\rho} + \frac{d(U^2)}{2\,U^2} = 0 \tag{8.15}$$

Differentiation of the isotropic (stagnation) relationship of the pressure (4.11) yields

where is the stagnation equations? put them in a table

put explanation how to derive this expression.

$$\frac{dP_0}{P_0} = \frac{dP}{P} + \left(\frac{\frac{1}{2}kM^2}{1+\frac{k-1}{2}M^2}\right)\frac{dM^2}{M^2} \tag{8.16}$$

Differentiation of equation (4.9) yields:

$$dT_0 = dT\left(1+\frac{k-1}{2}M^2\right) + T\,\frac{k-1}{2}\,dM^2 \tag{8.17}$$

Notice that $dT_0 \neq 0$ in isothermal flow. There is no change in the actual temperature of the flow but the stagnation temperature increases or decreases depending on the Mach number (supersonic flow of subsonic flow). Substituting $\mathbf{T}$ for equation (**??**) yields:

$$dT_0 = \frac{T_0\,\frac{k-1}{2}\,d\,M^2}{\left(1+\frac{k-1}{2}M^2\right)}\,\frac{M^2}{M^2} \tag{8.18}$$

Rearranging equation (8.18) yields

$$\frac{dT_0}{T_0} = \frac{(k-1)\,M^2}{2\left(1+\frac{k-1}{2}\right)}\,\frac{dM^2}{M^2} \tag{8.19}$$

Utilizing the momentum equation also requires to obtain a relation between the pressure and density and recalling that in isothermal flow ($T=0$) yields

$$\frac{dP}{P} = \frac{d\rho}{\rho} \tag{8.20}$$

From the continuity conservation leads

$$\frac{dM^2}{M^2} = \frac{2dU}{U} \tag{8.21}$$

The four equations momentum, continuity (mass), energy, state are described above. There are 4 unknowns (M, T, P, ρ)[3] and with these four equations

[3] Assuming the upstream variables are known.

the solution is attainable. One can noticed that there are two possible solutions (because the square power). These different solutions are super sonic and subsonic solution.

The distance friction, $\frac{4fL}{D}$, is selected as the choice for the independent variable. Thus, the equations need to be obtained in the form variable as a function of $\frac{4fL}{D}$. The density is eliminated from equation (8.15) when combined with the equation (8.6) to became

$$\frac{dP}{P} = -\frac{dU}{U} \tag{8.22}$$

Substituting the velocity (8.22) into equation (8.10) and one can obtain

$$-\frac{dP}{P} - \frac{4fdx}{D_H}\left(\frac{kPM^2}{2}\right) = kPM^2\frac{dP}{P} \tag{8.23}$$

Equation (8.23) can be rearranged into

$$\frac{dP}{P} = \frac{d\rho}{\rho} = -\frac{dU}{U} = -\frac{1}{2}\frac{dM^2}{M^2} = -\frac{kM^2}{2\left(1-kM^2\right)}4f\frac{dx}{D} \tag{8.24}$$

Similarly or by other path the stagnation pressure can be expressed as a function of $\frac{4fL}{D}$

$$\frac{dP_0}{P_0} = \frac{kM^2\left(1-\frac{k+1}{2}M^2\right)}{2\left(kM^2-1\right)\left(1+\frac{k-1}{2}M^2\right)}4f\frac{dx}{D} \tag{8.25}$$

$$\frac{dT_0}{T_0} = \frac{k\left(1-k\right)M^2}{2\left(1-kM^2\right)\left(1+\frac{k-1}{2}M^2\right)}4f\frac{dx}{D} \tag{8.26}$$

The variables in equation (8.24) can be separated to obtain integrable form as follows

$$\int_0^L \frac{4fdx}{D} = \int_{M^2}^{1/k}\frac{1-kM2}{kM2}dM^2 \tag{8.27}$$

It can be noticed that at the entrance $(x=0)$ for which $M=M$ (the initial velocity to tube isn't zero.). The term $\frac{4fL}{D}$ is positive for any x, thus, the term on other side has to be positive as well. To obtain this restriction $1 = kM^2$. Thus, the value $M = \frac{1}{\sqrt{k}}$ is the limiting case where from a mathematical point of view. Mach number larger from $M > \frac{1}{\sqrt{k}}$ makes the right hand side integrate negative. The Physical meaning of this value similar to $M=1$ chocked flow which were discussed in a variable area flow Chapter 4.

Further it can be noticed from equation (8.26) that when $M \rightarrow \frac{1}{\sqrt{k}}$ the value of right hand side approached infinity (∞). Since the stagnation temperature

(T_0) has a finite value which means that $dT_0 \to \infty$. Heat transfer have a limited value therefore model of the flow must be changed. A more appropriate model is an adiabatic flow model.

Integration of equation (8.27) yields

$$\frac{4fL_{max}}{D} = \frac{1-kM^2}{kM^2} + \ln kM^2 \tag{8.28}$$

Using definition for perfect gas of $M^2 = U^2/kRT$ and noticing that $T = constant$ can be used to describe the relation of the properties at $M = 1/\sqrt{k}$. Denote the supper script of symbol $*$ for the above condition and one can obtain that

$$\frac{M^2}{U^2} = \frac{1/k}{{U^*}^2} \tag{8.29}$$

Rearranging equation (8.29) transfered into

$$\frac{U}{U^*} = \sqrt{k}M \tag{8.30}$$

Utilizing the continuity equation provides

$$\rho U = \rho^* U^*; \Longrightarrow \frac{\rho}{\rho^*} = \frac{1}{\sqrt{k}M} \tag{8.31}$$

Reusing the perfect–gas relationship

$$\frac{P}{P^*} = \frac{\rho}{\rho^*} = \frac{1}{\sqrt{k}M} \tag{8.32}$$

Now utilizing the relation for stagnated isotropic pressure one can obtain

$$\frac{P_0}{P_0^*} = \frac{P}{P^*}\left[\frac{1+\frac{k-1}{2}M^2}{1+\frac{k-1}{2k}}\right]^{\frac{k}{k-1}} \tag{8.33}$$

Substituting for $\frac{P}{P^*}$ equation (8.32) and rearranging yields

$$\frac{P_0}{P_0^*} = \frac{1}{\sqrt{k}}\left(\frac{2k}{3k-1}\right)^{\frac{k}{k-1}}\left(1+\frac{k-1}{2}M^2\right)^{\frac{k}{k-1}}\frac{1}{M} \tag{8.34}$$

And the stagnation temperature at the critical point can be expressed as

$$\frac{T_0}{T_0^*} = \frac{T}{T^*}\frac{1+\frac{k-1}{2}M^2}{1+\frac{k-1}{2k}} = \frac{2k}{3k-1}\left(1+\frac{k-1}{2}\right)M^2 \tag{8.35}$$

These equations (8.30)-(8.35) are represented on in Figure 8.2

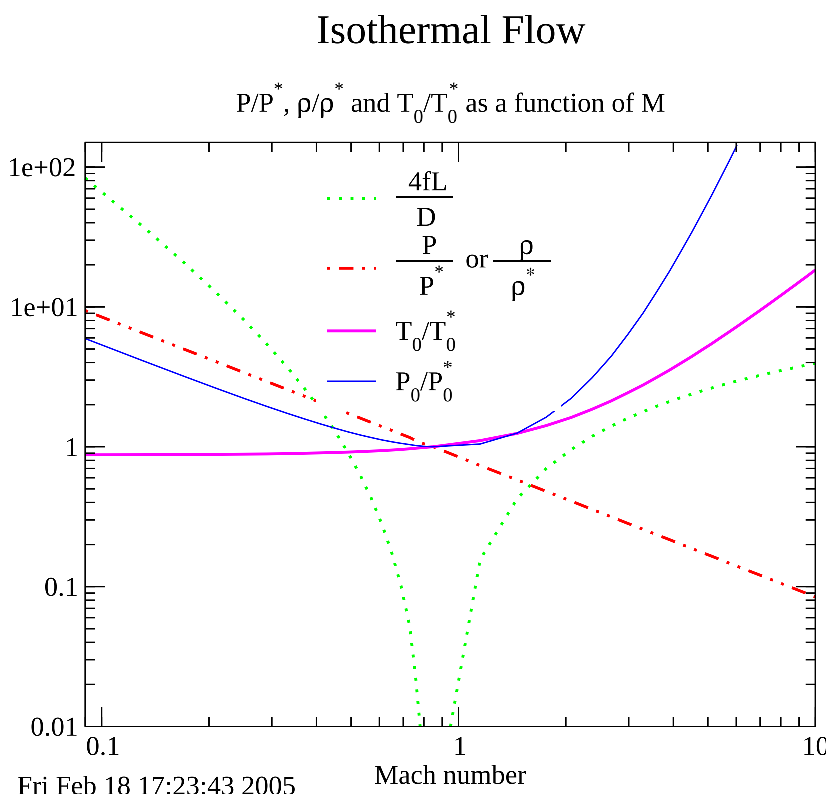


Fig. 8.2: Description of the pressure, temperature relationships as a function of the Mach number for isothermal flow

8.3 The Entrance Limitation Of Supersonic Brach

Meta

Situations where the condition at the tube exit have not arrived to the critical conditions are discussed here. It is very useful to obtain the relationship between the entrance and exit condition in this case. The denotation of 1 and 2 represented the condition at the inlet and exit respectably. From equation (8.24)

$$\frac{4fL}{D} = \left.\frac{4fL_{max}}{D}\right|_1 - \left.\frac{4fL_{max}}{D}\right|_2 = \frac{1 - k{M_1}^2}{k{M_1}^2} - \frac{1 - k{M_2}^2}{k{M_2}^2} + \ln\left(\frac{M_1}{M_2}\right)^2 \quad (8.36)$$

Meta End

8.4 Comparison with Incompressible Flow

Meta

The Mach number of the flow in some instances is relatively small. In these cases, one should expect that the isothermal flow to have similar characters as the flow of incompressible flow. For incompressible flow, the pressure loss is expressed as follows

$$P_1 - P_2 = \frac{4fL}{D}\frac{U^2}{2} \tag{8.37}$$

Now note that for incompressible flow $U_1 = U_2 = U$ and $\frac{4fL}{D}$ represent the ratio of the traditional h_{12}. To obtain similar expression for the isothermal flow, a relation between M_2 and M_1 and pressures has to be derived. From equation (8.37) one can obtained that

$$M_2 = M_1 \frac{P_1}{P_2} \tag{8.38}$$

Substituting this expression into (8.38) yield

$$\frac{4fL}{D} = \frac{1}{k{M_1}^2}\left(1 - \left(\frac{P_2}{P_1}\right)^2\right) - \ln\left(\frac{P_2}{P_1}\right)^2 \tag{8.39}$$

discuss why the pressure ratio must be greater then kM2

Because f is always positive there is only one solution to above equation even though M2. In case the no solution and in case only one solution to M2.

Expending the solution for small pressure ratio drop, $P_1 - P_2/P_1$, by some mathematics.

denote

$$\chi = \frac{P_1 - P_2}{P_1} \tag{8.40}$$

Now equation (8.39) can be transformed into

$$\frac{4fL}{D} = \frac{1}{k{M_1}^2}\left(1 - \left(\frac{P_2 - P_1 + P_1}{P_1}\right)^2\right) - \ln\left(\frac{1}{\frac{P_2}{P_1}}\right)^2 \tag{8.41}$$

$$\frac{4fL}{D} = \frac{1}{k{M_1}^2}\left(1 - (1-\chi)^2\right) - \ln\left(\frac{1}{1-\chi}\right)^2 \tag{8.42}$$

$$\frac{4fL}{D} = \frac{1}{k{M_1}^2}\left(2\chi - \chi^2\right) - \ln\left(\frac{1}{1-\chi}\right)^2 \tag{8.43}$$

now we have to expend into series around $\chi = 0$ and remember that

$$f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2} + 0\left(x^3\right) \tag{8.44}$$

and for example the first derivative of

$$\begin{aligned} \frac{d}{d\chi} \ln\left(\frac{1}{1-\chi}\right)^2 \Bigg|_{\chi=0} &= \\ (1-\chi)^2 \times \left[(-2)(1-\chi)^{-3}\right](-1)\Big|_{\chi=0} &= 2 \end{aligned} \tag{8.45}$$

similarly it can be shown that $f''(\chi = 0) = 1$ equation (8.43) now can be approximated as

$$\tfrac{4fL}{D} = \frac{1}{kM_1{}^2}(2\chi - \chi^2) - \left(2\chi - \chi^2\right) + f\left(\chi^3\right) \tag{8.46}$$

rearranging equation (8.46) yield

$$\tfrac{4fL}{D} = \frac{\chi}{kM_1{}^2}\left[(2-\chi) - kM_1{}^2\left(2-\chi\right)\right] + f\left(\chi^3\right) \tag{8.47}$$

and further rearrangement yield

$$\tfrac{4fL}{D} = \frac{\chi}{kM_1{}^2}\left[2(1 - kM_1{}^2) - \left(1 + kM_1{}^2\right)\chi\right] + f\left(\chi^3\right) \tag{8.48}$$

in cases that χ is small

when chi is small provide discussion

$$\tfrac{4fL}{D} \approx \frac{\chi}{kM_1{}^2}\left[2(1 - kM_1{}^2) - \left(1 + kM_1{}^2\right)\chi\right] \tag{8.49}$$

The pressure difference can be plotted as a function of the M_1 for given value of $\frac{4fL}{D}$. Equation (8.49) can be solved explicitly to produce a solution for

$$\chi = \frac{1 - kM_1{}^2}{1 + kM_1{}^2} - \sqrt{\frac{1 - kM_1{}^2}{1 + kM_1{}^2} - \frac{kM_1{}^2}{1 + kM_1{}^2}\tfrac{4fL}{D}} \tag{8.50}$$

A few observations can be made about equation (8.50) The larger value of the solution is not physically possible because

Meta End

8.5 Supersonic Branch

Apparently, this analysis/model is over simplified for the supersonic branch and does not produced reasonable results since it neglects to take into account the heat transfer effects. A dimensionless analysis[4] demonstrate that all the common materials that this author is familiar which create a large error that the fundamental assumption of the model breaks. Nevertheless, this model can provide a better understanding so the trends and deviations from Fanno flow model can be understood.

To add figure from the programs with comparison with Fanno flow.

In the supersonic flow, the hydraulic entry length is very large as shown below. However, the feeding diverging nozzle somewhat reduces the required entry length (as opposed to converging feeding). The thermal entry length is in the order of the hydrodynamic entry length (Look at the Prandtl number, (0.7-1.0), value for the common gases.). Most of the heat transfer is hampered in the sublayer thus the core assumption of isothermal flow (not enough heat transfer so the temperature isn't constant) breaks down[5].

The flow speed at the entrance is very large, over hundred of meters per second. For example, a flow gas in a tube with for $\frac{4fL}{D} = 10$ the required Mach number is over 200. Almost all the perfect gas model substances dealt in this book the speed of sound is a function of temperature. For this illustration, most gas cases the speed of sound is about $300[m/sec]$. For example, even with low temperature like $200K$ the speed of sound of air is $283[m/sec]$. So, for even for relatively small tubes with $\frac{4fL}{D} = 10$ the inlet speed is over 56 [km/sec]. This requires that the entrance length to be larger than the actual length of the tub for air. Remember from Fluid Dynamic book

book under construction perhaps to general ref at this stage

$$L_{entrance} = 0.06 \frac{UD}{\nu} \tag{8.51}$$

The typical values of the the kinetic viscosity, ν, are 0.0000185 kg/m-sec at 300K and 0.0000130034 kg/m-sec at 200K. Combine this information with our case of $\frac{4fL}{D} = 10$

$$\frac{L_{entrance}}{D} = 250746268.7$$

On the other hand with typical value of friction coefficient $f = 0.005$ results in

$$\frac{L_{max}}{D} = \frac{10}{4 \times 0.005} = 500$$

The fact that the actual tube length is only less 1% than the entry length means that the assumption of the isothermal also breaks (as in control large time respond).

to insert the figure from Van Dreist also figure 15-4 in Kays and Crawford.

[4]This dimensional analysis is a bit tricky, and is based on estimates. Currently and ashamedly this author is looking for a more simplified explanation. The current explanation is correct but based on hands waving and definitely dose not satisfied this author.

[5]see Kays and Crawford "Convective Heat Transfer" (equation 12-12).

Now, if Mach number is changing from 10 to 1 the kinetic energy change is about $\frac{\mathbf{T_0}}{\mathbf{T_0}^*} = 18.37$ which means that maximum of the mount of energy is insufficient.

Now with limitation, this topic will be covered in the next version because it provide some insight and boundary to Fanno Flow model.

8.6 Figures and Tables

Table 8.1: The Isothermal Flow basic parameters

$\mathbf{M}$	$\frac{\mathbf{4fL}}{\mathbf{D}}$	$\frac{\mathbf{P}}{\mathbf{P}^*}$	$\frac{\mathbf{P_0}}{\mathbf{P_0}^*}$	$\frac{\rho}{\rho^*}$	$\frac{\mathbf{T_0}}{\mathbf{T_0}^*}$
0.03000	785.97	28.1718	17.6651	28.1718	0.87516
0.04000	439.33	21.1289	13.2553	21.1289	0.87528
0.05000	279.06	16.9031	10.6109	16.9031	0.87544
0.06000	192.12	14.0859	8.8493	14.0859	0.87563
0.07000	139.79	12.0736	7.5920	12.0736	0.87586
0.08000	105.89	10.5644	6.6500	10.5644	0.87612
0.09000	82.7040	9.3906	5.9181	9.3906	0.87642
0.10000	66.1599	8.4515	5.3334	8.4515	0.87675
0.20000	13.9747	4.2258	2.7230	4.2258	0.88200
0.25000	7.9925	3.3806	2.2126	3.3806	0.88594
0.30000	4.8650	2.8172	1.8791	2.8172	0.89075
0.35000	3.0677	2.4147	1.6470	2.4147	0.89644
0.40000	1.9682	2.1129	1.4784	2.1129	0.90300
0.45000	1.2668	1.8781	1.3524	1.8781	0.91044
0.50000	0.80732	1.6903	1.2565	1.6903	0.91875
0.55000	0.50207	1.5366	1.1827	1.5366	0.92794
0.60000	0.29895	1.4086	1.1259	1.4086	0.93800
0.65000	0.16552	1.3002	1.0823	1.3002	0.94894
0.70000	0.08085	1.2074	1.0495	1.2074	0.96075
0.75000	0.03095	1.1269	1.0255	1.1269	0.97344
0.80000	0.00626	1.056	1.009	1.056	0.98700
0.81000	0.00371	1.043	1.007	1.043	0.98982
0.81879	0.00205	1.032	1.005	1.032	0.99232
0.82758	0.000896	1.021	1.003	1.021	0.99485
0.83637	0.000220	1.011	1.001	1.011	0.99741
0.84515	0.0	1.000	1.000	1.000	1.000

```
To produce this table rum program with:\\
variableName =  machV;
int isRange = yes;
whatInfo = infoStandard ;
```

8.7 Examples

There can be several kind of questions aside the proof questions[6] Generally, the "engineering" or practical questions can be divided into driving force (pressure difference), resistance (diameter, friction factor, friction coefficient, etc.), and mass flow rate questions. In this model no questions about shock (should) exist[7].

The driving force questions deal with what should be the pressure difference to obtain certain flow rate. Here is an example.

Example 8.1:
A tube of 0.25 [m] diameter and 5000 [m] in length is attached to a pump. What should be the pump pressure so that a flow rate of 2 $[kg/sec]$ will be achieved. Assume that friction factor $f = 0.005$ and the exit pressure is $1[bar]$. The specific heat for the gas, $k = 1.31$, surroundings temperature $27°\mathrm{C}$, $R = 290 \left[\frac{J}{Kkg}\right]$. Hint calculate the maximum flow rate and then check if this request is reasonable.

<u>SOLUTION</u>
If the flow was incompressible then for known density, ρ, the velocity can be calculated by utilizing $\Delta P = \frac{4fL}{D}\frac{U^2}{2g}$. In incompressible flow, the density is a function of the entrance Mach number. The exit Mach number is not necessarily $1/\sqrt{k}$ i.e. the flow is not choked. First, check whether flow is choked (or even possible).

Calculating the resistance, $\frac{4fL}{D}$

$$\frac{4fL}{D} = \frac{4 \times 0.0055000}{0.25} = 400$$

Utilizing the table 8.1 or the program provides

$\mathbf{M}$	$\frac{\mathbf{4fL}}{\mathbf{D}}$	$\frac{\mathbf{P}}{\mathbf{P^*}}$	$\frac{\mathbf{P_0}}{\mathbf{P_0}^*}$	$\frac{\rho}{\rho^*}$	$\frac{\mathbf{T_0}}{\mathbf{T_0}^*}$
0.04331	400.00	20.1743	12.5921	20.1743	0.89446

The maximum flow rate (the limiting case) can be calculated by utilizing the above table. The velocity of the gas at the entrance $U = cM = 0.04331 \times \sqrt{1.31 \times 290 \times 300} \cong 14.62 \left[\frac{m}{sec}\right]$. The density reads

$$\rho = \frac{P}{RT} = \frac{2,017,450}{290 \times 300} \cong 23.19 \left[\frac{kg}{m^3}\right]$$

The maximum flow rate then reads

$$\dot{m} = \rho AU = 23.19 \times \frac{\pi \times (0.25)^2}{4} \times 14.62 \cong 16.9 \left[\frac{kg}{sec}\right]$$

[6]The proof questions are questions that ask for proof for finding a mathematical intently (normally good for mathematicians). These questions or example will be in later versions.

[7]Those who are mathematically include these kind of questions can exit but there is no real world situations with isothermal model with shock.

The maximum flow rate is larger then the requested mass rate hence the flow is not choked. It is note worthy to mention that since the isothermal model breaks around the choking point, the flow rate is really some what different. It is more appropriate to assume isothermal model hence our model is appropriate.

To solve this problem the flow rate has to accounted as

$$\dot{m} = \rho A U = 2.0 \left[\frac{kg}{sec}\right]$$

$$\dot{m} = \frac{P_1}{RT} A \frac{kU}{k} = \frac{P_1}{\sqrt{kRT}} A \frac{kU}{\sqrt{kRT}} = \frac{P_1}{c} A k M_1$$

Now combining with equation (8.38) yields

$$\dot{m} = \frac{M_2 P_2 A k}{c}$$

$$M_2 = \frac{\dot{m} c}{P_2 A k} = \frac{2 \times 337.59}{100000 \times \frac{\pi \times (0.25)^2}{4} \times 1.31} = 0.103$$

From the table 8.1 or utilizing the program provides

$\mathbf{M}$	$\frac{\mathbf{4fL}}{\mathbf{D}}$	$\frac{\mathbf{P}}{\mathbf{P^*}}$	$\frac{\mathbf{P_0}}{\mathbf{P_0}^*}$	$\frac{\rho}{\rho^*}$	$\frac{\mathbf{T_0}}{\mathbf{T_0}^*}$
0.10300	66.6779	8.4826	5.3249	8.4826	0.89567

The entrance Mach number obtained by

$$\left.\frac{4fL}{D}\right|_1 = 66.6779 + 400 \cong 466.68$$

$\mathbf{M}$	$\frac{\mathbf{4fL}}{\mathbf{D}}$	$\frac{\mathbf{P}}{\mathbf{P^*}}$	$\frac{\mathbf{P_0}}{\mathbf{P_0}^*}$	$\frac{\rho}{\rho^*}$	$\frac{\mathbf{T_0}}{\mathbf{T_0}^*}$
0.04014	466.68	21.7678	13.5844	21.7678	0.89442

The pressure should be

$$P = 21.76780 \times 8.4826 = 2.566[bar]$$

Note that table here above for this example are for $k = 1.31$

Example 8.2:

A flow of gas was considered for a distance of 0.5 [km] (500 [m]). A flow rate of 0.2 [kg/sec] is required. Due to safety concerns, the maximum pressure allowed for the gas is only 10[bar]. Assume that the flow is isothermal and k=1.4, calculate the required diameter of tube. The friction coefficient for the tube can be assumed *as 0.02[8]. Note that tubes are provided in increments of 0.5 [in][9]. You can assume that the soundings temperature to be* $27°\mathrm{C}$*.*

SOLUTION

At first, the minimum diameter will be obtained when the flow is chocked. Thus, the maximum M_1 that can be obtained when the M_2 is at its maximum and back pressure is at the atmospheric pressure.

$$M_1 = M_2 \frac{P_2}{P_1} = \overbrace{\frac{1}{\sqrt{k}}}^{M_{max}} \frac{1}{10} = 0.0845$$

Now, with the value of M_1 either utilizing table 8.1 or using the provided program yields

$\mathbf{M}$	$\mathbf{\frac{4fL}{D}}$	$\mathbf{\frac{P}{P^*}}$	$\mathbf{\frac{P_0}{{P_0}^*}}$	$\frac{\rho}{\rho^*}$	$\mathbf{\frac{T_0}{{T_0}^*}}$
0.08450	94.4310	10.0018	6.2991	10.0018	0.87625

With $\frac{4fL_{max}}{D} = 94.431$ the value of minimum diameter.

$$D = \frac{4fL}{\frac{4fL_{max}}{D}} \simeq \frac{4 \times 0.02 \times 500}{94.43} \simeq 0.42359[m] = 16.68[in]$$

However, the pipes are provided only in 0.5 increments and the next size is 17[in] or 0.4318 [m]. With this pipe size the calculations are to be repeated in reversed to produces:

$$\dot{m} = \rho A U = \rho A M c = \frac{P}{RT} A M \sqrt{kRT} = \frac{PAM\sqrt{k}}{\sqrt{RT}}$$

The usage of the above equation clearly applied to the whole pipe. The only point that must be emphasis that all properties (like Mach number, pressure and etc) have to be taken at the same point. The new $\frac{4fL}{D}$ is

$$\frac{4fL}{D} = \frac{4 \times 0.02 \times 500}{0.4318} \simeq 92.64$$

$\mathbf{M}$	$\mathbf{\frac{4fL}{D}}$	$\mathbf{\frac{P}{P^*}}$	$\mathbf{\frac{P_0}{{P_0}^*}}$	$\frac{\rho}{\rho^*}$	$\mathbf{\frac{T_0}{{T_0}^*}}$
0.08528	92.6355	9.9108	6.2423	9.9108	0.87627

[8]A relative smooth tube of cast iron.

[9]It is unfortunate, but is seem that this standard will be around in USA for some time.

To check whether the flow rate is satisfied the requirement

$$\dot{m} = \frac{10^6 \times \frac{\pi \times 0.4318^2}{4} \times 0.0853 \times \sqrt{1.4}}{\sqrt{287 \times 300}} \approx 50.3[kg/sec]$$

Since $50.3 \geq 0.2$ the mass flow rate requirements is satisfied.

It should be noted that P should be replaced by P_0 in the calculations. If not see below The speed of sound at the entrance is

$$c = \sqrt{kRT} = \sqrt{1.4 \times 287 \times 300} \cong 347.2 \left[\frac{m}{sec}\right]$$

and the density is

$$\rho = \frac{P}{RT} = \frac{1,000,000}{287 \times 300} = 11.61 \left[\frac{kg}{m^3}\right]$$

The velocity at the entrance should be

$$U = M * c = 0.08528 \times 347.2 \cong 29.6 \left[\frac{m}{sec}\right]$$

The diameter should be

$$D = \sqrt{\frac{4\dot{m}}{\pi U \rho}} = \sqrt{\frac{4 \times 0.2}{\pi \times 29.6 \times 11.61}} \cong 0.027$$

Nevertheless, for sake of the exercise the other parameters will be calculated. This situation is reversed question. The flow rate is given with the diameter of the pipe. It should be noted that the flow isn't chocked.

Example 8.3:

A flow of gas from a station (a) with pressure of 20[bar] through a pipe with 0.4[m] diameter and 4000 [m] length to a different station (b). The pressure at the exit (station (b)) is 2[bar]. The gas and the sounding temperature can be assumed to be 300 K. Assume that the flow is isothermal, k=1.4, and the average friction f=0.01. Calculate the Mach number at the entrance to pipe and the flow rate.

SOLUTION

First, the information whether the flow is choked needed to be found. Therefore, at first it will be assumed that the whole length is the maximum length.

$$\frac{4fL_{max}}{D} = \frac{4 \times 0.01 \times 4000}{0.4} = 400$$

with $\frac{4fL_{max}}{D} = 400$ the following can be written

$\mathbf{M}$	$\frac{4fL}{D}$	$\frac{\mathbf{T_0}}{\mathbf{T_0}^{*\mathbf{T}}}$	$\frac{\rho}{\rho^{*\mathbf{T}}}$	$\frac{\mathbf{P}}{\mathbf{P}^{*\mathbf{T}}}$	$\frac{\mathbf{P_0}}{\mathbf{P_0}^{*\mathbf{T}}}$
0.0419	400.72021	0.87531	20.19235	20.19235	12.66915

From the table $M_1 \approx 0.0419$,and $\frac{P_0}{{P_0}^{*T}} \approx 12.67$

$$ {P_0}^{*T} \cong \frac{28}{12.67} \simeq 2.21[bar] $$

The pressure at point (b) utilizing the isentropic relationship ($M = 1$) pressure ratio is 0.52828.

$$ P_2 = \frac{{P_0}^{*T}}{\left(\frac{P_2}{{P_0}^{*T}}\right)} = 2.21 \times 0.52828 = 1.17[bar] $$

As the pressure at point (b) is smaller the actual pressure $P^* < P_2$ than the actual pressure one must conclude that the flow is not choked. The solution is iterative process.

1. guess reasonably the value of M_1 and calculate $\frac{4fL}{D}$
2. Calculate the value of $\left.\frac{4fL}{D}\right|_2$ by subtracting $\left.\frac{4fL}{D}\right|_1 - \frac{4fL}{D}$
3. Obtain M_2 from the table ? or using the program.
4. Calculate the pressure, P_2 in mind that this isn't the real pressure but based on the assumption)
5. Compare the results of guessed pressure P_2 with the actual pressure. and chose new M_1 accordingly.

Now the process has been done for you and provide in the Figure **??** or in table resulted from the provided program.

$\mathbf{M_1}$	$\mathbf{M_2}$	$\left.\frac{4fL_{max}}{D}\right\|_{\mathbf{1}}$	$\frac{4fL}{D}$	$\frac{\mathbf{P_2}}{\mathbf{P_1}}$
0.0419	0.59338	400.32131	400.00000	0.10000

The flow rate is

$$ \dot{m} = \rho A M c = \frac{P\sqrt{k}}{\sqrt{RT}} \frac{\pi \times D^2}{4} M = \frac{2000000\sqrt{1.4}}{\sqrt{300 \times 287}} \pi \times 0.2^2 \times 0.0419 $$
$$ \simeq 42.46[kg/sec] $$

In this chapter, there are no examples on isothermal with supersonic flow.

8.8 Unchoked situation

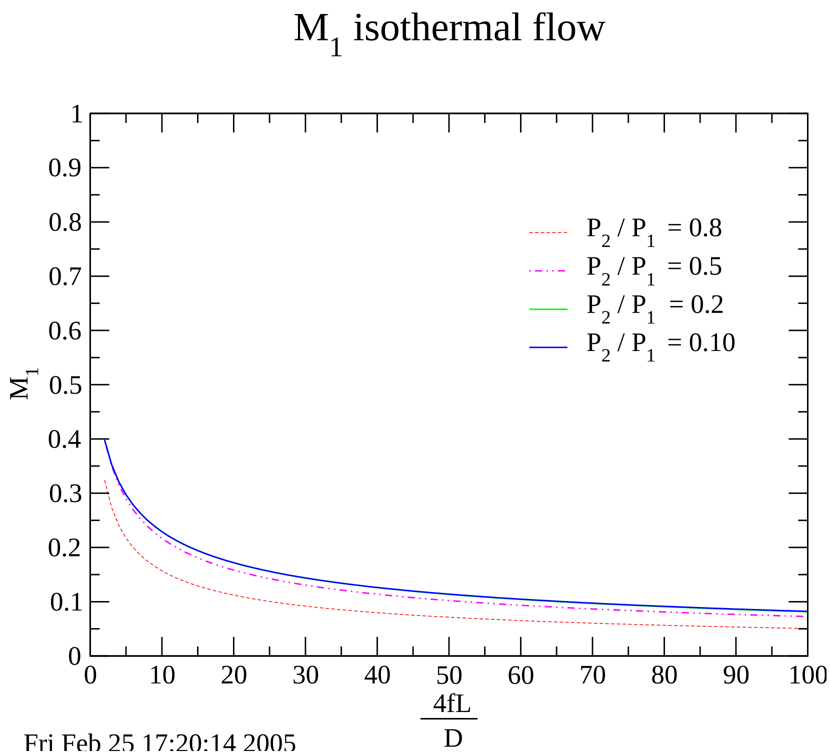


Fig. 8.3: The Mach number at the entrance to a tube under isothermal flow model as a function $\frac{4fL}{D}$

Table 8.9: The flow parameters for unchoked flow

$\mathbf{M_1}$	$\mathbf{M_2}$	$\left.\frac{4fL_{max}}{D}\right\|_{\mathbf{1}}$	$\frac{4fL}{D}$	$\frac{\mathbf{P_2}}{\mathbf{P_1}}$
0.7272	0.84095	0.05005	0.05000	0.10000
0.6934	0.83997	0.08978	0.08971	0.10000
0.6684	0.84018	0.12949	0.12942	0.10000
0.6483	0.83920	0.16922	0.16912	0.10000
0.5914	0.83889	0.32807	0.32795	0.10000
0.5807	0.83827	0.36780	0.36766	0.10000
0.5708	0.83740	0.40754	0.40737	0.10000

CHAPTER 9

Fanno Flow

Adiabatic flow with friction name after Ginno Fanno a Jewish engineer is the second model described here. The main restriction for this model is that heat transfer is negligible and can be ignored [1]. This model is applicable to flow processes which are very fast compared to heat transfer mechanisms, small Eckert number.

which are considerably slower (see more discussion about dimensionless number in the chapter about dimensional analysis (not included yet.).

This model explains many industrial flow processes which includes emptying of pressured container through relatively a short tube, exhaust system internal combustion engine, compressed air systems, etc. As this model raised from need to explain the steam flow in turbines.

9.1 Introduction

Consider a gas flows through a conduit with a friction (see Figure 9.1). It is advantages to examine the simplest situation and yet without losing the core properties of the process. Later, more general case will be examined[2].

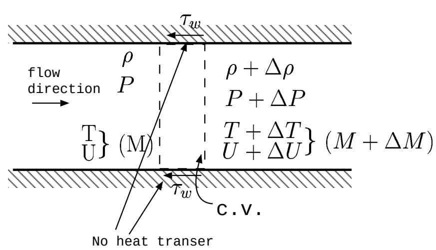


Fig. 9.1: Control volume of the gas flow in a constant cross section

[1]Even the friction does not converted into heat

[2]Not ready yet, discussed on the ideal gas model and on the ideal gas model and the entry length issues.

9.2 Model

The mass (continuity equation) balance can be written as

$$\dot{m} = \rho A U = constant \qquad (9.1)$$
$$\hookrightarrow \rho_1 U_1 = \rho_2 U_2$$

The energy conservation (under the assumption of this model as adiabatic flow and the friction in not transformed into thermal energy) reads

$$T_{01} = T_{02} \qquad (9.2)$$
$$\hookrightarrow T_1 + \frac{{U_1}^2}{2c_p} = T_2 + \frac{{U_2}^2}{2c_p} \qquad (9.3)$$

Or in a derivative form

$$C_p dT + d\left(\frac{U^2}{2}\right) = 0 \qquad (9.4)$$

Again for simplicity, the perfect gas model is assumed[3].

$$P = \rho R T \qquad (9.5)$$
$$\hookrightarrow \frac{P_1}{\rho_1 T_1} = \frac{P_2}{\rho_2 T_2}$$

It is assumed that the flow can be approximated as one dimensional. The force acting on the gas is the friction at the wall and the momentum conservation reads

$$-A dP - \tau_w dA_w = \dot{m} dU \qquad (9.6)$$

It is convenient to define a hydraulic diameter as

$$D_H = \frac{4 \times \text{Cross Section Area}}{\text{wetted perimeter}} \qquad (9.7)$$

Or in other words

$$A = \frac{\pi {D_H}^2}{4} \qquad (9.8)$$

[3]The equation of state is written again here so that all the relevant equations can be found when this chapter is printed separately.

It is convenient to substitute D for of D_H but it is referred to the same hydraulic diameter. The infinitesimal area that shear stress is acting on is

$$dA_w = \pi D dx \tag{9.9}$$

Introducing the Fanning friction factor as a dimensionless friction factor which some times referred to as friction coefficient and reads as following:

$$f = \frac{\tau_w}{\frac{1}{2}\rho U^2} \tag{9.10}$$

Utilizing equation (9.2) and substituting equation (9.10) into momentum equation (9.6) yields

$$-\overbrace{\frac{\pi D^2}{4}}^{A} dP - \pi D dx \overbrace{f\left(\frac{1}{2}\rho U^2\right)}^{\tau_w} = A \overbrace{\rho U}^{\frac{\dot{m}}{A}} dU \tag{9.11}$$

Dividing equation (9.11) by the cross section area, $\mathbf{A}$ and rearranging yields

$$-dP + \frac{4 f dx}{D}\left(\frac{1}{2}\rho U^2\right) = \rho U dU \tag{9.12}$$

The second law is the last equation to be utilized to determine only the flow direction.

$$s_2 \geq s_1 \tag{9.13}$$

9.2.1 Dimensionalization of the equations

Before solving the above equation a dimensionless process is applied. Utilizing the definition of the sound speed to produce the following identities for perfect gas

$$M^2 = \left(\frac{U}{c}\right)^2 \quad = \frac{U^2}{k \underbrace{RT}_{\frac{P}{\rho}}} \tag{9.14}$$

Utilizing the definition of the ideal gas results

$$M^2 \quad = \frac{\rho U^2}{kP} \tag{9.15}$$

Utilizing the identify in equation (9.14) and substituting it into equation (9.11) and after some rearrangement yields

$$-dP + \frac{4 f dx}{D_H}\left(\frac{1}{2} k P M^2\right) = \frac{\rho U^2}{U} dU = \overbrace{k P M^2}^{\rho U^2} \frac{dU}{U} \tag{9.16}$$

Furtherer rearranging equation (9.16) results in

$$-\frac{dP}{P} - \frac{4fdx}{D}\left(\frac{kM^2}{2}\right) = kM^2\frac{dU}{U} \tag{9.17}$$

It is convenient to relate expressions of (dP/P) and dU/U in term of the Mach number and substituting into equation (9.17). Derivative of mass conservation (9.2) results

$$\frac{d\rho}{\rho} + \overbrace{\frac{1}{2}\frac{dU^2}{U^2}}^{\frac{dU}{U}} = 0 \tag{9.18}$$

The derivation of the equation of state (9.6) and dividing the results by equation of state (9.6) results

$$\frac{dP}{P} = \frac{d\rho}{\rho} + \frac{dT}{dT} \tag{9.19}$$

Derivation of the Mach identity equation (9.14) and dividing by equation (9.14) yields

$$\frac{d(M^2)}{M^2} = \frac{d(U^2)}{U^2} - \frac{dT}{T} \tag{9.20}$$

Dividing the energy equation (9.4) by C_p and utilizing definition Mach number yields

$$\begin{aligned}
&\frac{dT}{T} + \underbrace{\frac{1}{\left(\frac{kR}{(k-1)}\right)}}_{C_p}\frac{1}{T}\frac{U^2}{U^2}d\left(\frac{U^2}{2}\right) = \\
&\hookrightarrow \frac{dT}{T} + \frac{(k-1)}{\underbrace{kRT}_{c^2}}\frac{U^2}{U^2}d\left(\frac{U^2}{2}\right) = \\
&\hookrightarrow \frac{dT}{T} + \frac{k-1}{2}M^2\frac{dU^2}{U^2} = 0
\end{aligned} \tag{9.21}$$

This equation is obtained by combining the definition of Mach number with equation of state and mass conservation. Thus, the original limitations must be applied to the resulting equation.

Equations (9.17), (9.18), (9.19), (9.20), and (9.21) need to be solved. These equations are separable so one variable is a function of only single variable (the chosen independent variable). Explicit explanation is provided only two variables, rest can be done in similar fashion. The dimensionless friction, $\frac{4fL}{D}$, is chosen as independent variable since the change in the dimensionless resistance, $\frac{4fL}{D}$, causes the change in the other variables.

Combining equations (9.19) and (9.21) when eliminating dT/T results

$$\frac{dP}{P} = \frac{d\rho}{\rho} - \frac{(k-1)M^2}{2}\frac{dU^2}{U^2} \tag{9.22}$$

The term $\frac{d\rho}{\rho}$ can be eliminated utilizing equation (9.18) and substituting into equation (9.22) and rearrangement yields

$$\frac{dP}{P} = -\frac{1+(k-1)M^2}{2}\frac{dU^2}{U^2} \tag{9.23}$$

The term dU^2/U^2 can be eliminated by using (9.23)

$$\frac{dP}{P} = -\frac{kM^2\left(1+(k-1)M^2\right)}{2(1-M^2)}\frac{4f\,dx}{D} \tag{9.24}$$

The second equation for Mach number, M variable is obtained by combining equation (9.20) and (9.21) by eliminating dT/T. Then $d\rho/\rho$ and U are eliminated by utilizing equation (9.18) and equation (9.22). and only left is P. Then dP/P can be eliminated by utilizing equation (9.24) and results in

$$\frac{4f\,dx}{D} = \frac{\left(1-M^2\right)dM^2}{kM^4(1+\frac{k-1}{2}M^2)} \tag{9.25}$$

Rearranging equation (9.25) results in

$$\frac{dM^2}{M^2} = \frac{kM^2\left(1+\frac{k-1}{2}M^2\right)}{1-M^2}\frac{4f\,dx}{D} \tag{9.26}$$

After similar mathematical manipulation one can get the relationship for the velocity to read

$$\frac{dU}{U} = \frac{kM^2}{2\left(1-M^2\right)}\frac{4f\,dx}{D} \tag{9.27}$$

and the relationship for the temperature is

$$\frac{dT}{T} = \frac{1}{2}\frac{dc}{c} = -\frac{k(k-1)M^4}{2(1-M^2)}\frac{4f\,dx}{D} \tag{9.28}$$

density is obtained by utilizing equations (9.27) and (9.18) to obtain

$$\frac{d\rho}{\rho} = -\frac{kM^2}{2\left(1-M^2\right)}\frac{4f\,dx}{D} \tag{9.29}$$

The stagnation pressure similarly obtained as

$$\frac{dP_0}{P_0} = -\frac{kM^2}{2}\frac{4f\,dx}{D} \tag{9.30}$$

The second law reads

$$ds = C_p \ln\frac{dT}{T} - R\ln\frac{dP}{P} \tag{9.31}$$

The stagnation temperature expresses as $T_0 = T(1 + (1-k)/2M^2)$. Taking derivative of this expression when M is remains constant yields $dT_0 = dT(1+(1-k)/2M^2)$ and thus when these equations are divided they yields

$$dT/T = dT_0/T_0 \tag{9.32}$$

In the similar fashion the relationship between the stagnation pressure and the pressure and substitute in the entropy equation results

$$ds = C_p \ln \frac{dT_0}{T_0} - R \ln \frac{dP_0}{P_0} \tag{9.33}$$

The first law requires that the stagnation temperature remains constant, $(dT_0 = 0)$. Therefore the entropy change

$$\frac{ds}{C_p} = -\frac{(k-1)}{k} \frac{dP_0}{P_0} \tag{9.34}$$

Utilizing the equation for stagnation pressure the entropy equation yields

$$\frac{ds}{C_p} = \frac{(k-1)M^2}{2} \frac{4f dx}{D} \tag{9.35}$$

9.3 The Mechanics and Why The Flow is Chock?

The trends of the properties can examined though looking in equations (9.24) through (9.34). For example, from equation (9.24) it can be observed that the critical point is when $M = 1$. When $M < 1$ the pressure decreases downstream as can seen from equation (9.24) because fdx and M are positive. For the same reasons, in the supersonic branch, $M > 1$, the pressure increases downstream. This pressure increase is what makes compressible flow so different than “conventional” flow. Thus the discussion will be divided into two cases; one of flow with speed above speed of sound, and, two flow with speed below of speed of sound.

9.3.0.1 Why the flow is chock?

There explanation is based on the equations developed earlier and there is no known explanation that is based on the physics. First it has to recognized that the critical point is when $M = 1$ at which show change in the trend and singular by itself. For example, $dP(@M = 1) = \infty$ and mathematically it is a singular point (see equation (9.24)). Observing from equation (9.24) that increase or decrease from subsonic just below one $M = (1-\epsilon)$ to above just above one $M = (1+\epsilon)$ requires a change in a sign pressure direction. However, the pressure has to be a monotonic function which means that flow cannot crosses over the point of $M = 1$. This constrain means that because the flow cannot “cross–over” $M = 1$ the gas has to reach to this speed, $M = 1$ at the last point. This situation called chocked flow.

9.3.0.2 The Trends

The trends or whether the variables are increasing or decreasing can be observed from looking at the equation developed. For example, the pressure can be examined by looking at equation (9.26). It demonstrates that the Mach number increases downstream when the flow is subsonic. On the other hand, when the flow is supersonic, the pressure decreases.

The summary of the properties changes on the sides of the branch

This table is only for a historical reason, is there another way to represent this information?

	Subsonic	Supersonic
Pressure, P	decrease	increase
Mach number, M	increase	decrease
Velocity, U	increase	decrease
Temperature, T	decrease	increase
Density, ρ	decrease	increase
Stagnation Temperature, T_0	decrease	increase

9.4 The working equations

Integration of equation (9.25) yields

$$\frac{4}{D}\int_{L}^{L_{max}} f dx = \frac{1}{k}\frac{1-M^2}{M^2} + \frac{k+1}{2k}\ln\frac{\frac{k+1}{2}M^2}{1+\frac{k-1}{2}M^2} \tag{9.36}$$

A representative friction factor is defined as

$$\bar{f} = \frac{1}{L_{max}}\int_{0}^{L_{max}} f dx \tag{9.37}$$

Utilizing the mean average theorem equation (9.36) yields

$$\frac{4\bar{f}L_{max}}{D} = \frac{1}{k}\frac{1-M^2}{M^2} + \frac{k+1}{2k}\ln\frac{\frac{k+1}{2}M^2}{1+\frac{k-1}{2}M^2} \tag{9.38}$$

It common to replace the $\bar{f}$ with f which is adopted in this book.

Equations (9.24), (9.27), (9.28), (9.29), (9.29), and (9.30) can be solved. For example, the pressure as written in equation 9.23 is represented by $\frac{4fL}{D}$, and Mach number. Now equation 9.24 can eliminate term $\frac{4fL}{D}$ and describe the pressure on the Mach number. Dividing equation 9.24 in equation 9.26 yields

$$\frac{\frac{dP}{P}}{\frac{dM^2}{M^2}} = -\frac{1+(k-1M^2}{2M^2\left(1+\frac{k-1}{2}M^2\right)}dM^2 \tag{9.39}$$

The symbol "*" denotes the state when the flow is choked and Mach number is equal to 1. Thus, $M = 1$ when $P = P^*$ Equation (9.39) can be integrated to yield:

$$\frac{P}{P^*} = \frac{1}{M}\sqrt{\frac{\frac{k+1}{2}}{1+\frac{k-1}{2}M^2}} \tag{9.40}$$

In the same fashion the variables ratio can be obtained

$$\frac{T}{T^*} = \frac{c^2}{{c^*}^2} = \frac{\frac{k+1}{2}}{1+\frac{k-1}{2}M^2} \tag{9.41}$$

$$\frac{\rho}{\rho^*} = \frac{1}{M}\sqrt{\frac{1+\frac{k-1}{2}M^2}{\frac{k+1}{2}}} \tag{9.42}$$

$$\frac{U}{U^*} = \left(\frac{\rho}{\rho^*}\right)^{-1} = M\sqrt{\frac{\frac{k+1}{2}}{1+\frac{k-1}{2}M^2}} \tag{9.43}$$

The stagnation pressure decreases can be expressed by

$$\frac{P_0}{{P_0}^*} = \frac{\overbrace{\frac{P_0}{P}}^{\left(1+\frac{1-k}{2}M^2\right)^{\frac{k}{k-1}}} \quad P}{\underbrace{\frac{{P_0}^*}{P^*}}_{\left(\frac{2}{k+1}\right)^{\frac{k}{k-1}}} \quad P^*} \tag{9.44}$$

Utilizing the pressure ratio in equation (9.40) and substituting into equation (9.44) yields

$$\frac{P_0}{{P_0}^*} = \left(\frac{1+\frac{k-1}{2}M^2}{\frac{k+1}{2}}\right)^{\frac{k}{k-1}} \frac{1}{M}\sqrt{\frac{1+\frac{k-1}{2}M^2}{\frac{k+1}{2}}} \tag{9.45}$$

And further rearranging equation (9.45) provides

$$\frac{P_0}{{P_0}^*} = \frac{1}{M}\left(\frac{1+\frac{k-1}{2}M^2}{\frac{k+1}{2}}\right)^{\frac{k+1}{2(k-1)}} \tag{9.46}$$

The integration of equation (9.34) yields

$$\frac{s-s^*}{c_p} = \ln M^2 \sqrt{\left(\frac{k+1}{2M^2\left(1+\frac{k-1}{2}M^2\right)}\right)^{\frac{k+1}{k}}} \tag{9.47}$$

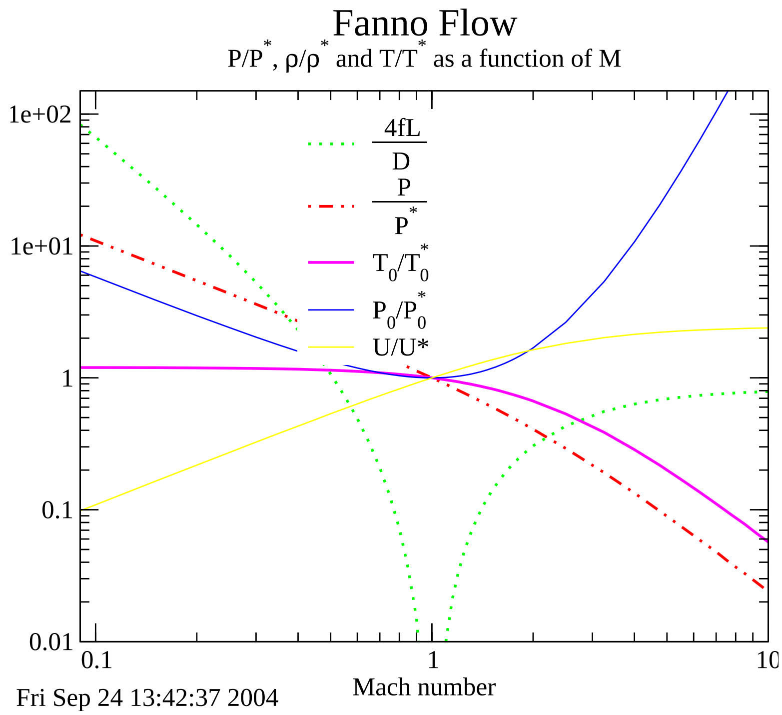


Fig. 9.2: Various parameters in Fanno flow as a function of Mach number

out Reynolds dimensionless parameter.

The results of these equations are plotted in figure 9.2 The fanno flow is in many cases shockless and therefore a relationship between two points should be derived. In most of the times, the "star" values are imaginary values that represent the value at choking. The real ratio can be obtained by ratio of two star ratios as an example

$$\frac{T_2}{T_1} = \frac{\left.\frac{T}{T^*}\right|_{M_2}}{\left.\frac{T}{T^*}\right|_{M_1}} \tag{9.48}$$

A special interest is the equation for the dimensionless friction as following

$$\int_{L_1}^{L_2} \frac{4fL}{D} dx = \int_{L_1}^{L_{max}} \frac{4fL}{D} dx - \int_{L_2}^{L_{max}} \frac{4fL}{D} dx \tag{9.49}$$

Hence,

$$\left(\frac{4fL_{max}}{D}\right)_2 = \left(\frac{4fL_{max}}{D}\right)_1 - \frac{4fL}{D} \tag{9.50}$$

9.4.1 Example

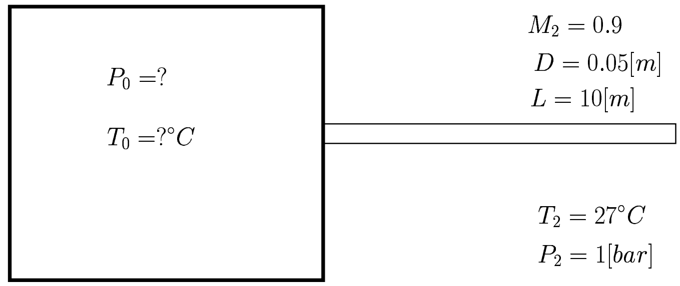


Fig. 9.3: Schematic of example 1

Example 9.1:
Air flows from a reservoir and enters a uniform pipe with a diameter of 0.05 [m] and length of 10 [m]. The air exits to the atmosphere. The following conditions prevail at the exit: $P_2 = 1[bar]$ *temperature* $T_2 = 27°\mathrm{C}$ $M_2 = 0.9$[4]*. Assume that the average friction factor to be* $f = 0.004$ *and that the flow from the reservoir up to the pipe inlet is essentially isentropic. Estimate the total temperature and total pressure in the reservoir under the Fanno flow model.*

<u>SOLUTION</u>
For isentropic flow to the pipe inlet, temperature and total pressure at the pipe inlet are the same as the those in the reservoir. Thus, finding the total pressure and temperature at the pipe inlet is the solution. With the Mach number and temperature known at the exit, the total temperature at the entrance can be obtained by knowing the $\frac{4fL}{D}$. For given Mach number ($M = 0.9$) the following is obtained.

$\mathbf{M}$	$\mathbf{\frac{4fL}{D}}$	$\mathbf{\frac{P}{P^*}}$	$\mathbf{\frac{P_0}{P_0{}^*}}$	$\frac{\rho}{\rho^*}$	$\mathbf{\frac{U}{U^*}}$	$\mathbf{\frac{T}{T^*}}$
0.90000	0.01451	1.1291	1.0089	1.0934	0.9146	1.0327

So the total temperature at the exit

$$T^*|_2 = \left.\frac{T^*}{T}\right|_2 T_2 = \frac{300}{1.0327} = 290.5[K]$$

To "move" the other side of the tube the $\frac{4fL}{D}$ is added as

$$\left.\frac{4fL}{D}\right|_1 = \frac{4fL}{D} + \left.\frac{4fL}{D}\right|_2 = \frac{4 \times 0.004 \times 10}{0.05} + 0.01451 \simeq 3.21$$

[4]This property is given only for academic purposes. There is no Mach meter.

The rest of the parameters can be obtained with the new $\frac{4fL}{D}$ either from the table by interpolations or utilizing attached program.

$\mathbf{M}$	$\mathbf{\frac{4fL}{D}}$	$\mathbf{\frac{P}{P^*}}$	$\mathbf{\frac{P_0}{{P_0}^*}}$	$\frac{\rho}{\rho^*}$	$\mathbf{\frac{U}{U^*}}$	$\mathbf{\frac{T}{T^*}}$
0.35886	3.2100	3.0140	1.7405	2.5764	0.38814	1.1699

[5]

Note that the subsonic branch is chosen. the stagnation ratios has to be added for $M = 0.35886$

$\mathbf{M}$	$\mathbf{\frac{T}{T_0}}$	$\frac{\rho}{\rho_0}$	$\mathbf{\frac{A}{A^*}}$	$\mathbf{\frac{P}{P_0}}$	$\mathbf{\frac{A\times P}{A^*\times P_0}}$
0.359	0.975	0.938	1.740	0.915	1.592

The total pressure P_{01} can be found from the combination of the ratios as follows:

$$
\begin{aligned}
P_{01} &= \overbrace{\overbrace{P_2 \left.\frac{P^*}{P}\right|_2}^{P^*} \left.\frac{P}{P^*}\right|_1}^{P_1} \left.\frac{P_0}{P}\right|_1 \\
&= 1 \times \frac{1}{1.12913} \times 3.014 \times \frac{1}{0.915} = 2.91[Bar]
\end{aligned}
$$

$$
\begin{aligned}
T_{01} &= \overbrace{\overbrace{T_2 \left.\frac{T^*}{T}\right|_2}^{T^*} \left.\frac{T}{T^*}\right|_1}^{T_1} \left.\frac{T_0}{T}\right|_1 \\
&= 300 \times \frac{1}{1.0327} \times 1.17 \times \frac{1}{0.975} \simeq 348K = 75°\mathrm{C}
\end{aligned}
$$

Another academic question:

Example 9.2:
A system compromised from a convergent-divergent nozzle followed by a tube with length of 2.5 [cm] in diameter and 1.0 [m] long. The system is supplied by a

[5]whatInfo = infoStandard ;
variableName = fldV;
k = 1.4;
FLD = 3.21 ;

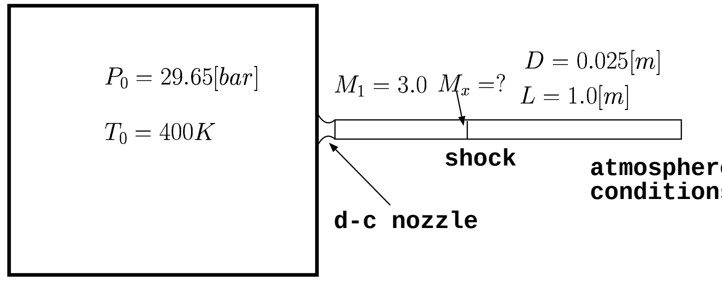


Fig. 9.4: The schematic of Example 2

vessel. The vessel conditions are is at 29.65 [Bar], 400 K. With these conditions a pipe inlet Mach number is 3.0. A normal shock wave occurs in the tube and the flow discharges to the atmosphere, determine;

(a) the mass flow rate through the system;

(b) the temperature at the pipe exit; and

(c) determine the Mach number that a normal shock wave occurs [M_x].

Take k = 1.4, R = 287 $[J/kgK]$ and $f = 0.005$.

SOLUTION

(a) Assuming that the pressure vessel very much larger than the pipe therefore, the velocity in the vessel can be assumed small enough so it can be neglected. Thus, the stagnation conditions can be approximated as the condition in the tank. It further assumed that the flow through the nozzle can be approximated as isentropic. Hence, $T_{01} = 400K$ and $P_{01} = 29.65[Par]$

The mass flow rate through the system is constant and for simplicity reason point 1 is chosen in which,

$$\dot{m} = \rho A M c$$

The density and speed of sound are unknowns and needed to be computed. With the isentropic relationship the Mach number at point one is known the following can be found either from table or program[6]

[6]It should pay attention that in the program the variable whatInfo = infoStagnation ; and not whatInfo = infoStandard ;

$\mathbf{M}$	$\frac{\mathbf{T}}{\mathbf{T_0}}$	$\frac{\rho}{\rho_0}$	$\frac{\mathbf{A}}{\mathbf{A^*}}$	$\frac{\mathbf{P}}{\mathbf{P_0}}$	$\frac{\mathbf{A\times P}}{\mathbf{A^*\times P_0}}$
3.0000	0.35714	0.07623	4.2346	0.02722	0.11528

The temperature is

$$T_1 = \frac{T_1}{T_{01}} T_{01} = 0.357 \times 400 = 142.8K$$

With the temperature the speed of sound can be calculated as

$$c_1 = \sqrt{kRT} = \sqrt{1.4 \times 287 \times 142.8} \simeq 239.54[m/sec]$$

The pressure at point 1 can be calculated as

$$P_1 = \frac{P_1}{P_{01}} P_{01} = 0.027 \times 30 \simeq 0.81[Bar]$$

The density as a function of other properties at point 1 is

$$\rho_1 = \left.\frac{P}{RT}\right|_1 = \frac{8.1 \times 10^4}{287 \times 142.8} \simeq 1.97 \left[\frac{kg}{m^3}\right]$$

The mass flow rate can be evaluated from equation (9.2)

$$\dot{m} = 1.97 \times \frac{\pi \times 0.025^2}{4} \times 3 \times 239.54 = 0.69 \left[\frac{kg}{sec}\right]$$

(b) First, a check weather the flow is shockless by comparing the flow resistance and the maximum possible resistance. From the table or by using the attached program[7], obtain the flowing

$\mathbf{M}$	$\frac{\mathbf{4fL}}{\mathbf{D}}$	$\frac{\mathbf{P}}{\mathbf{P^*}}$	$\frac{\mathbf{P_0}}{\mathbf{P_0}^*}$	$\frac{\rho}{\rho^*}$	$\frac{\mathbf{U}}{\mathbf{U^*}}$	$\frac{\mathbf{T}}{\mathbf{T^*}}$
3.0000	0.52216	0.21822	4.2346	0.50918	1.9640	0.42857

and the conditions of the tube are

$$\tfrac{4fL}{D} = \frac{4 \times 0.005 \times 1.0}{0.025} = 0.8$$

Since $0.8 > 0.52216$ the flow is chocked and with a shock wave.

7

The exit pressure determines the location of the shock, if a shock exists, by comparing “possible” P_{exit} to P_B. Two possibilities needed to be checked; one, the shock at the entrance of the tube, and two, shock at the exit and comparing the pressure ratios. First, possibility of the shock wave occurs immediately at the entrance for which the ratio for M_x are (shock wave table)

$\mathbf{M_x}$	$\mathbf{M_y}$	$\frac{\mathbf{T_y}}{\mathbf{T_x}}$	$\frac{\rho_y}{\rho_x}$	$\frac{\mathbf{P_y}}{\mathbf{P_x}}$	$\frac{\mathbf{P_0}_y}{\mathbf{P_0}_x}$
3.0000	0.47519	2.6790	3.8571	10.3333	0.32834

After shock wave the flow is subsonic with “M_1”$= 0.47519$. (fanno flow table)

$\mathbf{M}$	$\frac{\mathbf{4fL}}{\mathbf{D}}$	$\frac{\mathbf{P}}{\mathbf{P^*}}$	$\frac{\mathbf{P_0}}{\mathbf{P_0}^*}$	$\frac{\rho}{\rho^*}$	$\frac{\mathbf{U}}{\mathbf{U^*}}$	$\frac{\mathbf{T}}{\mathbf{T^*}}$
0.47519	1.2919	2.2549	1.3904	1.9640	0.50917	1.1481

The stagnation values for $M = 0.47519$ are

$\mathbf{M}$	$\frac{\mathbf{T}}{\mathbf{T_0}}$	$\frac{\rho}{\rho_0}$	$\frac{\mathbf{A}}{\mathbf{A^*}}$	$\frac{\mathbf{P}}{\mathbf{P_0}}$	$\frac{\mathbf{A\times P}}{\mathbf{A^*\times P_0}}$
0.47519	0.95679	0.89545	1.3904	0.85676	1.1912

The ratio of exit pressure to the chamber total pressure is

$$\begin{aligned}
\frac{P_2}{P_0} &= \overbrace{\left(\frac{P_2}{P^*}\right)}^{1}\left(\frac{P^*}{P_1}\right)\left(\frac{P_1}{P_{0y}}\right)\left(\frac{P_{0y}}{P_{0x}}\right)\overbrace{\left(\frac{P_{0x}}{P_0}\right)}^{1} \\
&= 1 \times \frac{1}{2.2549} \times 0.8568 \times 0.32834 \times 1 \\
&= 0.12476
\end{aligned}$$

The actual pressure ratio $1/29.65 = 0.0338$ is smaller than the case in which shock occurs at the entrance. Thus, the shock is somewhere downstream. One possible way to find the exit temperature, T_2 is by finding the location of the shock. To find the location of the shock ratio of the pressure ratio, $\frac{P_2}{P_1}$ is needed. With the location of shock, “claiming” up stream from the exit through shock to the entrance. For example, calculating the parameters for shock location with known $\frac{4fL}{D}$ in the “y” side. Then either utilizing shock table or the program to obtained the upstream Mach number.

The procedure of the calculations:

1) Calculated the entrance Mach number assuming the shock occurs at the exit:

a) set $M_2^{'} = 1$ assume the flow in the entire tube is supersonic:
b) calculated $M_1^{'}$

Note this Mach number is the high Value.

2) Calculated the entrance Mach assuming shock at the entrance.

a) set $M_2 = 1$
b) add $\frac{4fL}{D}$ and calculated M_1' for subsonic branch
c) calculated M_x for M_1'

Note this Mach number is the low Value.

3) according your root finding algorithm[8] calculated or guess the shock location and then compute as above the new M_1. To check Secant Method.

a) set $M_2 = 1$
b) for the new $\frac{4fL}{D}$ and compute the new M_y' as on the subsonic branch
c) calculated M_x' for the M_y'
d) Add the leftover of $\frac{4fL}{D}$ and calculated the M_1

4) guess new location for the shock according to your finding root procedure and according the result repeat previous stage until finding the solution.

$\mathbf{M_1}$	$\mathbf{M_2}$	$\left.\mathbf{\frac{4fL}{D}}\right\|_{\mathbf{up}}$	$\left.\mathbf{\frac{4fL}{D}}\right\|_{\mathbf{down}}$	$\mathbf{M_x}$	$\mathbf{M_y}$
3.0000	1.0000	0.22019	0.57981	1.9899	0.57910

(c) The way that numerical procedure of solving this problem is by finding $\left.\frac{4fL}{D}\right|_{up}$ that will produce $M_1 = 3$. In the process M_x and M_y must be calculated (see the chapter on the program with its algorithms.).

9.5 Supersonic Branch

In chapter **??** it was shown that the isothermal model cannot describe the adequately the situation because the thermal entry length is relatively large compared to the pipe length and the heat transfer is not sufficient to maintain constant temperature. In the Fanno model there is no heat transfer, and, furthermore, the because the very limited amount of heat transformed it closer to a adiabatic flow. The only limitation of the model is uniform of the velocity (assuming parabolic flow for laminar and different profile for turbulent flow.). The information from the wall to the tube center[9] is slower in reality. However, experiments by many starting with 1938

[8]You can use any method you which, but be-careful second order methods like Newton-Rapson method can be unstable.

[9]The word information referred into the shear stress transformed from the wall to centerer of the tube.

work by Frossel[10] since has shown that the error is not significant. Nevertheless, the comparison with shows that heat transfer cause changes to the flow and they needed to be expected. This changes includes the choking point at lower Mach number.

To insert example on the change in the flow rate between isothermal flow to Fanno Flow. Insert also example on percentage of heat transfer.

on the comparison of the maximum length of isothermal model and Fanno Model.

9.6 Maximum length for the supersonic flow

It has to be noted and recognized that as oppose to subsonic branch the supersonic branch has a limited length. It also must be recognized that there is a maximum length for which only supersonic flow can exist[11]. These results were obtained from the mathematical derivations but were verified by numerous experiments[12]. The maximum length of the supersonic can be evaluated when $M = \infty$ as follow:

$$\begin{aligned}
\frac{4fL_{max}}{D} &= \frac{1-M^2}{kM^2} + \frac{k+1}{2k} \ln \frac{\frac{k+1}{2}M^2}{2\left(1+\frac{k-1}{2}M^2\right)} = \\
\frac{4fL}{D}\left(M\to\infty\right) &\sim \frac{-\infty}{k\times\infty} + \frac{k+1}{2k}\ln\frac{(k+1)\infty}{(k-1)\infty} \\
&= \frac{-1}{k} + \frac{k+1}{2k}\ln\frac{(k+1)}{2(k-1)} \\
&= \frac{4fL}{D}(M\to\infty, k=1.4) = 0.8215
\end{aligned}$$

The maximum length of the supersonic flow is limited by the above number. From the above analysis, it can be observed that no matter how high the entrance Mach number will be the tube length is limited and depends only on specific heat ratio, k as shown in figure 9.5.

9.7 Working Conditions

It has to be recognized that there are two regimes that can occurs in Fanno flow model one of subsonic flow and supersonic flow. Even the flow in the tube starts as a supersonic parts of the tube can be transformed into the subsonic branch. A shock wave can occur and some portion of the tube will be in a subsonic flow pattern.

The discussion has to differentiate between two ways of feeding the tube: converging nozzle or a converging-diverging nozzle. Three parameters, the dimensionless friction, $\frac{4fL}{D}$, the entrance Mach number, M_1, and the pressure ratio, P_2/P_1 are controlling the flow. Only combination of two these parameters are truly independent. However, all the three parameters can be varied and there are discussed separately here.

[10] See on the web `http://naca.larc.nasa.gov/digidoc/report/tm/44/NACA-TM-844.PDF`

[11] Many in the industry have diffi culties to understand this concept. This author seeking nice explanation of this concept for the non–fluid mechanics engineers. This solicitation is about how to explain this issue to non-engineers or engineer without proper background.

[12] If you have experiments demonstrating this point, please provide to the undersign so they can be added to this book. Many of the pictures in the literature carry copyright problems.

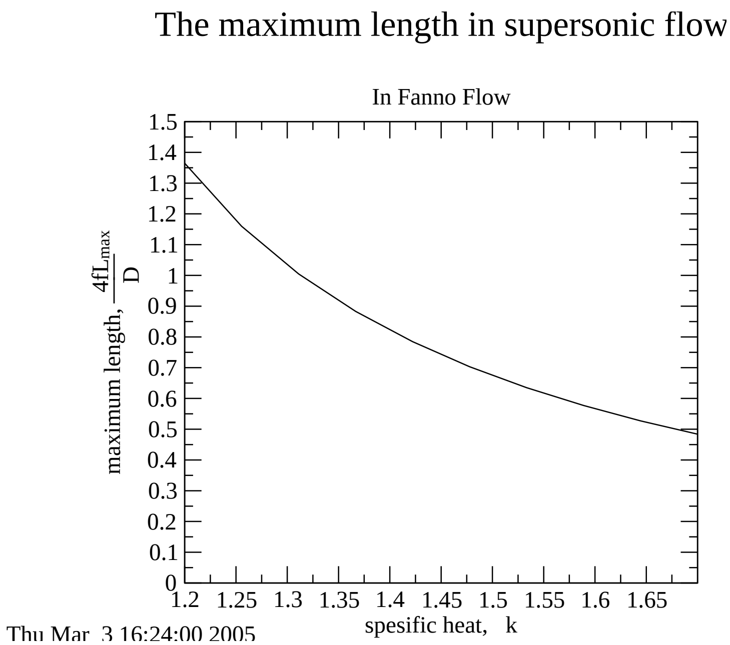


Fig. 9.5: The maximum length as a function of specifi c heat, k

9.7.1 Variations of the tube length ($\frac{4fL}{D}$) effects

In part of this analysis of this effect, it should be assumed that that back pressure is constant and/or low as possibly needed to maintain a choked flow. First, the treatment of the two branches are separated.

9.7.1.1 Subsonic branch

For converging nozzle feeding, increasing the tube length results in increasing the exit Mach number (normally denoted herein as M_2). Once the Mach number reaches maximum ($M = 1$), no further increase of the exit Mach number can be achieved. In this process, the mass flow rate decrease. It worth noting that entrance Mach number is reduced (as some might explain it to reduce the flow rate). The entrance temperature increase as can be seem from Figure 9.7. The velocity therefor must decrease because the less of the enthalpy (stagnation temperature) is "used." The density decrease because $\rho = \frac{P}{RT}$ and when Pressure is remains almost constant the the density decreases. Thus, the mass flow rate must decrease. These results applicable the converging nozzle.

In the case of the converging–diverging feeding nozzle, increase of the dimensionless friction, $\frac{4fL}{D}$, results in a similar flow pattern as in the converging nozzle. Once the flow becomes choked a different flow pattern emerged.

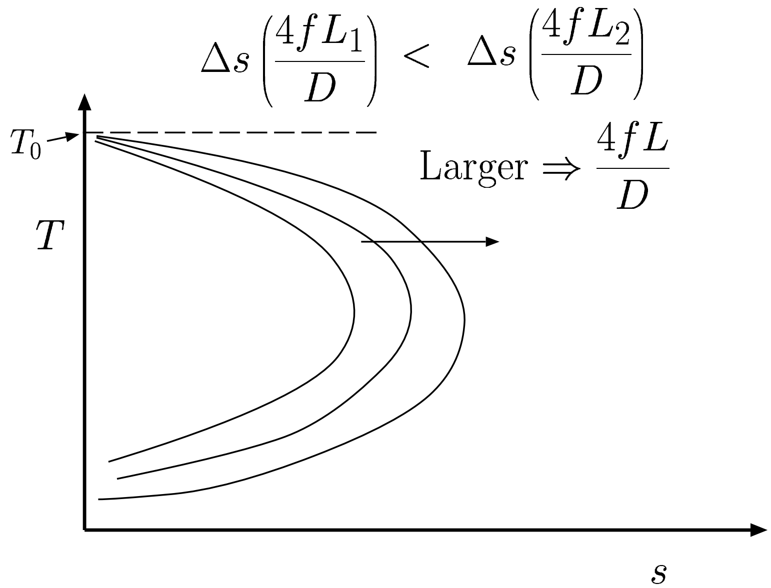


Fig. 9.6: The effects of increase of $\frac{4fL}{D}$ on the Fanno line

9.7.1.2 Supersonic Branch

There are several transitional points that change the pattern on the flow. The point $\mathbf{a}$ is the choking point (for the supersonic branch) in which the exit Mach number reaches to one. The point $\mathbf{b}$ is the maximum possible flow of supersonic flow not depend on nozzle. The next point, referred here as the critical point, $\mathbf{c}$, is the point in which no supersonic flow is possible in the tube i.e. the shock reaches to the nozzle. There is another point, $\mathbf{d}$, in which no supersonic flow is possible in the entire nozzle–tube system. Between these transitional points the effect parameters such and mass flow rate, entrance and exit Mach number are discussed.

At the starting point the flow is choked in the nozzle, (to achieve supersonic flow). The following ranges that has to be discussed which include (see figure 9.8):

0	$<$	$\frac{4fL}{D}$	$<$	$\left(\frac{4fL}{D}\right)_{choking}$	$0 \to \mathbf{a}$
$\left(\frac{4fL}{D}\right)_{choking}$	$<$	$\frac{4fL}{D}$	$<$	$\left(\frac{4fL}{D}\right)_{shockless}$	$\mathbf{a} \to \mathbf{b}$
$\left(\frac{4fL}{D}\right)_{shockless}$	$<$	$\frac{4fL}{D}$	$<$	$\left(\frac{4fL}{D}\right)_{chockless}$	$\mathbf{b} \to \mathbf{c}$
$\left(\frac{4fL}{D}\right)_{chockless}$	$<$	$\frac{4fL}{D}$	$<$	∞	$\mathbf{c} \to \infty$

The 0-$\mathbf{a}$ range, the mass flow rate is constant because the flow is choked at the

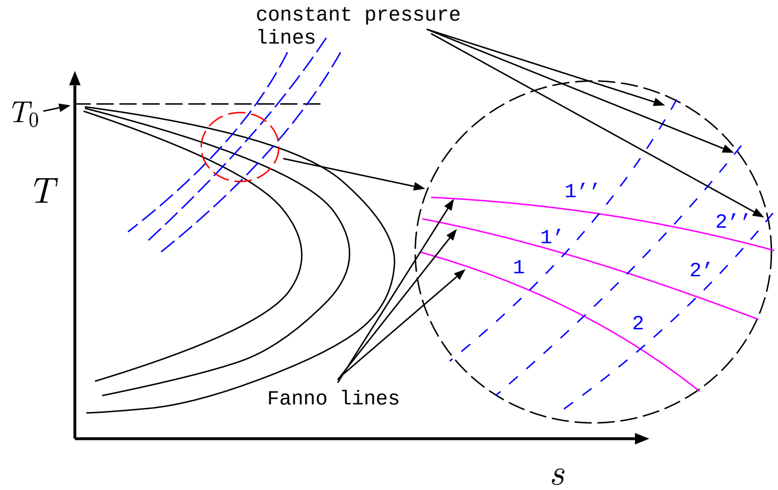


Fig. 9.7: The development properties in of converging nozzle

nozzle. The entrance Mach number, M_1 is constant because it is a function of the nuzzle design only. The exit Mach number, M_2 decreases (remember this flow is on the supersonic branch) and starts ($\frac{4fL}{D} = 0$) as $M_2 = M_1$. At end of the range $\mathbf{a}$, $M_2 = 1$. In the range of $\mathbf{a} - \mathbf{b}$ the flow is all supersonic.

In the next range $\mathbf{a} - -\mathbf{b}$ The flow is double choked and make the adjustment for the flow rate at different choking points by changing the shock location. The mass flow rate continue to be constant. The entrance Mach continues to be constant and exit Mach number is constant.

The total maximum available for supersonic flow $\mathbf{b} - -\mathbf{b}'$, $\left(\frac{4fL}{D}\right)_{max}$, is only theoretical length in which the supersonic flow can occur if nozzle will be provided with a larger Mach number (a change the nozzle area ratio which also reduces the mass flow rate.). In the range $\mathbf{b} - \mathbf{c}$, is more practical point.

In semi supersonic flow $\mathbf{b} - -\mathbf{c}$ (in which no supersonic is available in the tube but only the nozzle) the flow is still double chocked and the mass flow rate is constant. Notice that exit Mach number, M_2 is still one. However, the entrance Mach number, M_1, reduces with the increase of $\frac{4fL}{D}$.

It worth noticing that in the $\mathbf{a} - -\mathbf{c}$ the mass flow rate nozzle entrance velocity, and the exit velocity remains constant![13]

In the last range $\mathbf{c} - -\infty$ the end is really the pressure limitation or the break of the model and the isothermal model is more appropriate to describe the

[13]On personal note, this situation is rather strange to explain. On one hand, the resistance increases and on the other hand, the exit Mach number remains constant and equal to one. Is anyone have explanation for this strange behavior suitable for non–engineers or engineers without background in fluid mechanics.

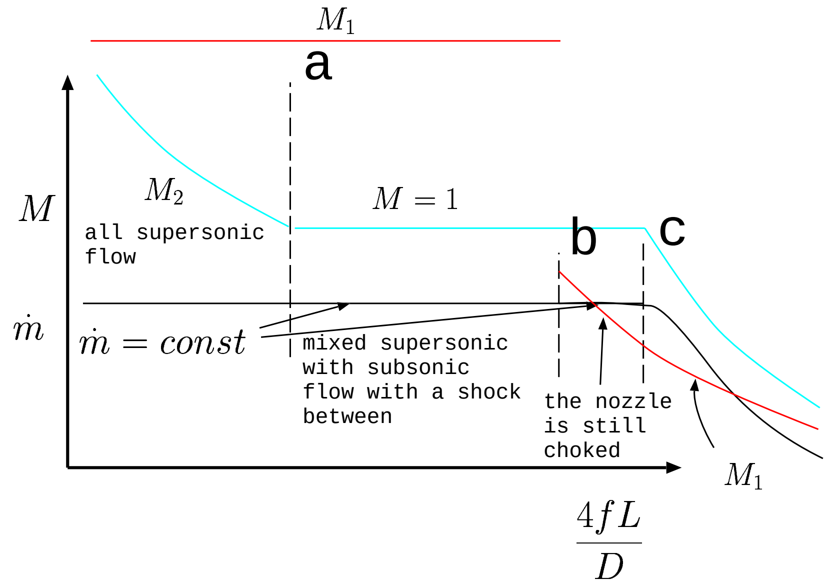


Fig. 9.8: The Mach numbers at entrance and exit of tube and mass flow rate for Fanno Flow as a function of the $\frac{4fL}{D}$

Should the mathematical derivations be inserted to demonstrate it?

flow. In this range, the flow rate decreases since $(\dot{m} \propto M_1)$[14].

To summarize the above discussion the figures 9.8 exhibits the developed of M_1, M_2 mass flow rate as a function of $\frac{4fL}{D}$. Somewhat different then the subsonic branch the the mass flow rate is constant even the flow in the tube is completely subsonic. This situation is because the "double" choked condition in the nozzle. The exit Mach M_2 is a continuous monotonic function that decreases with $\frac{4fL}{D}$. The entrance Mach M_1 is a non continuous function with a jump at point when shock occurs at the entrance "moves" into the nozzle.

Figure 9.9 exhibits the M_1 as a function of M_2. The figure was calculated by utilizing the data from figure 9.2 by obtaining the $\frac{4fL_{max}}{D}$ for M_2 and subtracting the given $\frac{4fL}{D}$ and finding the corresponding M_1.

In the figure 9.10

The figure 9.10 exhibits the entrance Mach number as a function of the M_2. Obviously there can be two extreme possibilities for the subsonic exit branch. Subsonic velocity occurs for supersonic entrance velocity, one, when the shock wave occurs at the tube exit and, two, at the tube entrance . In the figure 9.10 only for $\frac{4fL}{D} = 0.1$ and $\frac{4fL}{D} = 0.4$ two extremes are shown. For $\frac{4fL}{D} = 0.2$ shown with only shock at the exit only. Obviously, and as can be observed, the larger $\frac{4fL}{D}$ creates larger differences between exit Mach number for the different shock location. The larger $\frac{4fL}{D}$ larger M_1 must occurs even for shock at the entrance.

[14]Note that ρ_1 increases with decreases of M_1 but this effect is less significant.

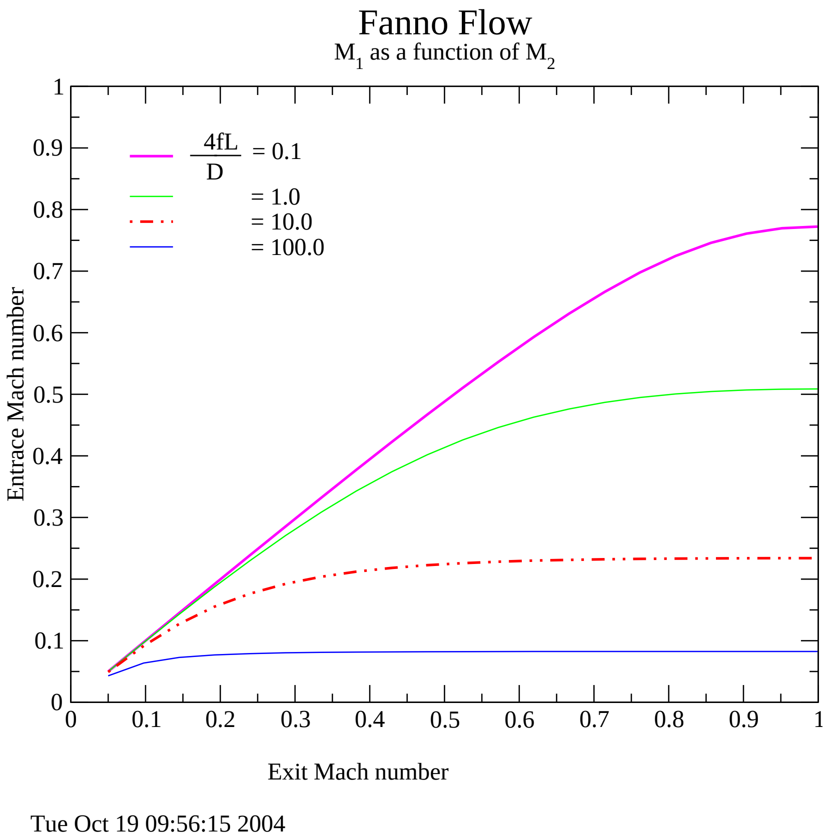


Fig. 9.9: M_1 as a function M_2 for different $\frac{4fL}{D}$

For a given $\frac{4fL}{D}$, below the maximum critical length, the entrance supersonic entrance has three different regimes which depends on the back pressure. One, shockless flow, shock at the entrance, and shock at the exit. The below the maximum critical length is mathematically

$$\frac{4fL}{D} > -\frac{1}{k} + \frac{1+k}{2k} \ln \frac{k+1}{k-1}$$

For cases $\frac{4fL}{D}$ above the maximum critical length no supersonic flow cannot be over whole tube and at some point a shock will occur and the flow becomes subsonic flow[15].

[15]See more on the discussion about changing the length of the tube.

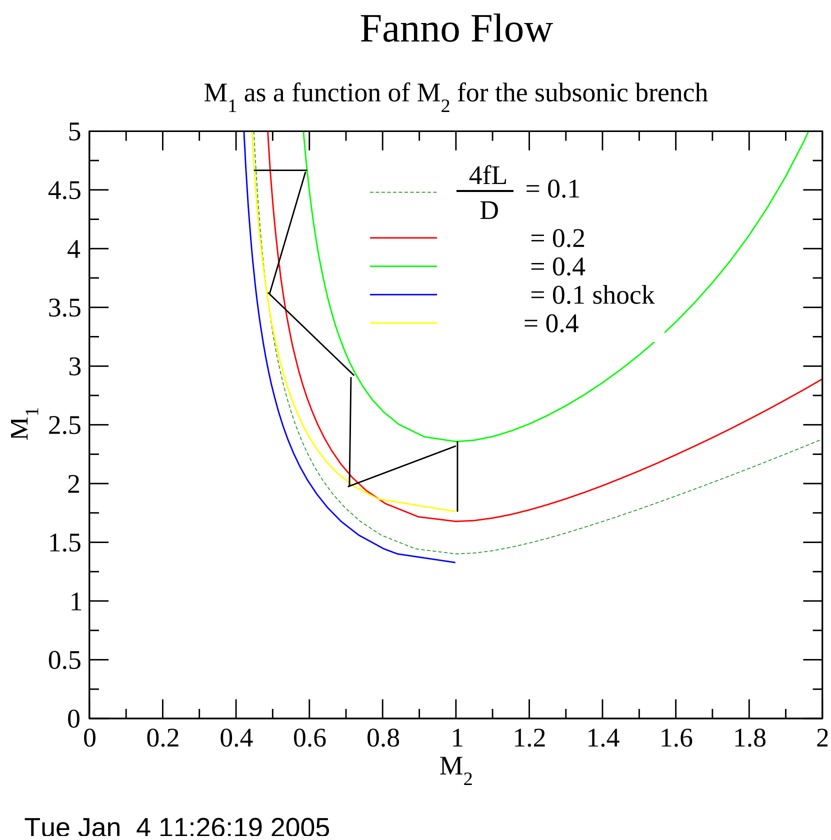


Fig. 9.10: M_1 as a function M_2 for different $\frac{4fL}{D}$ for supersonic entrance velocity

9.7.2 The Pressure Ratio, $\frac{P_2}{P_1}$, effects

In this section the studied parameter is the variation of the back pressure and thus, the pressure ratio $\frac{P_2}{P_1}$ variations. For very low pressure ratio the flow can be assumed as incompressible while exit Mach number are smaller than < 0.3. As the pressure ratio increases (smaller back pressure, P_2), the exit and entrance Mach numbers increase. According to Fanno model the value of $\frac{4fL}{D}$ is constant (friction factor, f, is independent of the parameters such as, Mach number, Reynolds number etc) thus the flow remains on the same Fanno line. For case where the supply come from a reservoir with a constant pressure, the entrance pressure decreases as well because the increase in the entrance Mach number (velocity).

Again a differentiation of the feeding is important to point out. If the feeding nozzle is converging than the flow will be only subsonic. If the nozzle is "converging–diverging" than in some part supersonic flow is possible. At first the converging nozzle is presented and later the converging-diverging nozzle is explained.

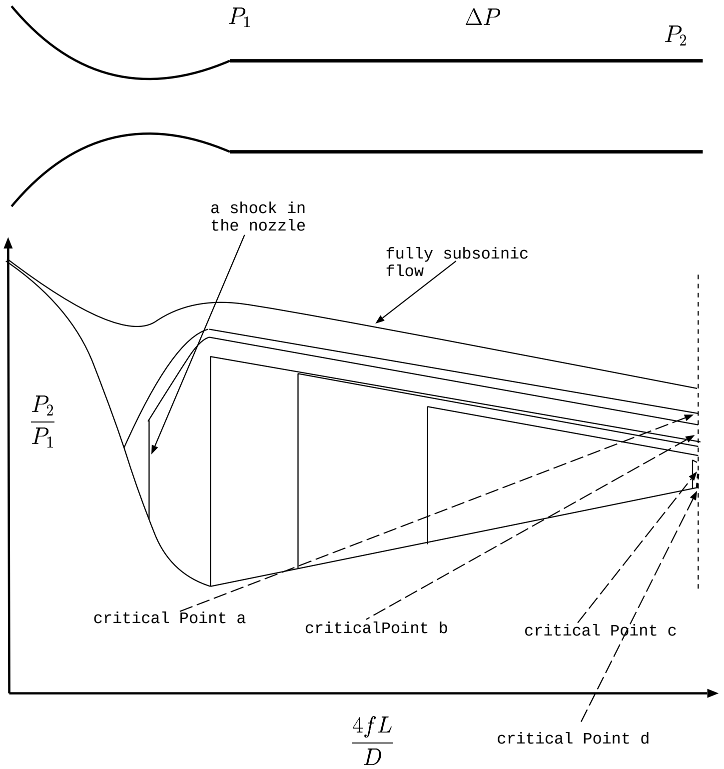


Fig. 9.11: The pressure distribution as a function of $\frac{4fL}{D}$ for a short $\frac{4fL}{D}$

Converging nozzle

9.7.2.1 Choking explanation for pressure variation/reduction

decreasing the pressure ratio or in actuality the back pressure, results in increase of the entrance and the exit velocity until a maximum is reached for the exit velocity. The maximum velocity is when exit Mach number equals one. The Much number, as it was shown in the chapter 4, can increase only if the area increase. In our model the tube area is postulated as a constant therefore the velocity cannot increase any further. However, the flow to be continuous the pressure must decrease and for that the velocity must increase. Something must break since the conflicting demands and it result in a “jump” in the flow. This jump and it is referred to as a choked flow. Any additional reduction in the back pressure will not change the situation in the tube. The only change will be at tube surroundings which are irrelevant to this discussion.

If the feeding nozzle is a "diverging-converging" then it is has to be differentiated between two cases; One case is where the $\frac{4fL}{D}$ is short or equal to the critical length. The critical length is the maximum $\frac{4fL_{max}}{D}$ that associate with entrance Mach number.

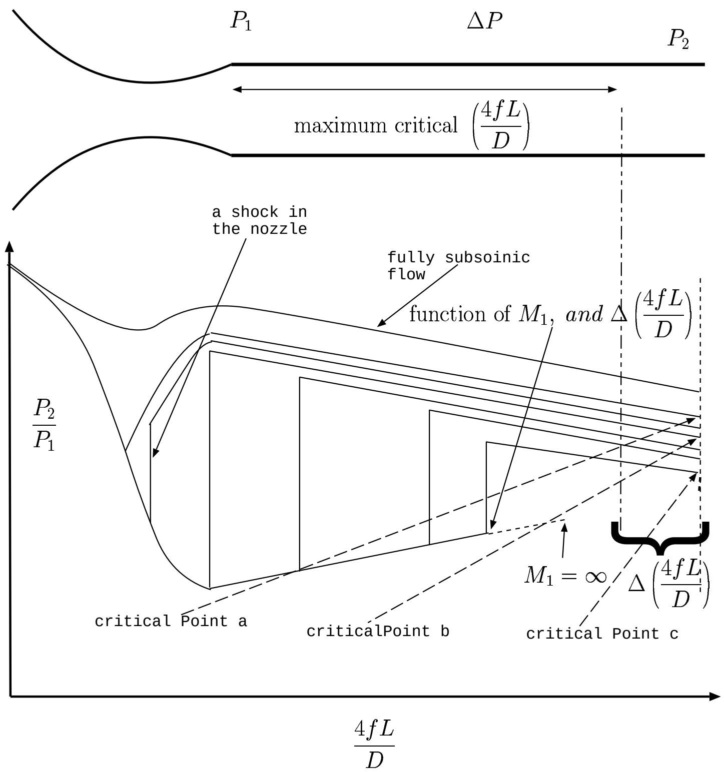


Fig. 9.12: The pressure distribution as a function of $\frac{4fL}{D}$ for a long $\frac{4fL}{D}$

9.7.2.2 Short $\frac{4fL}{D}$

Figure 9.12 shows different of pressure profiles for different back pressures. before the flow reach critical point a (in the figure) the flow is subsonic. Up to this stage the nozzle feeds the tube increases the mass flow rate (with decreasing back pressure). Between point a and point b the shock is in the nozzle. In this range and further reduction of the pressure the mass flow rate is constant no matter how low the back pressure is reduced. Once the back pressure is less the point b the supersonic reaches to the tube. Note however that exit Mach number, $M_2 < 1$ and is **not** 1. A back pressure that is at the critical point c results in a shock wave that is at the exit. When the back pressure is below point c, the tube is "clean" of any shock[16]. The back pressure below point c some a adjustment as to occurs with exceptions of point d.

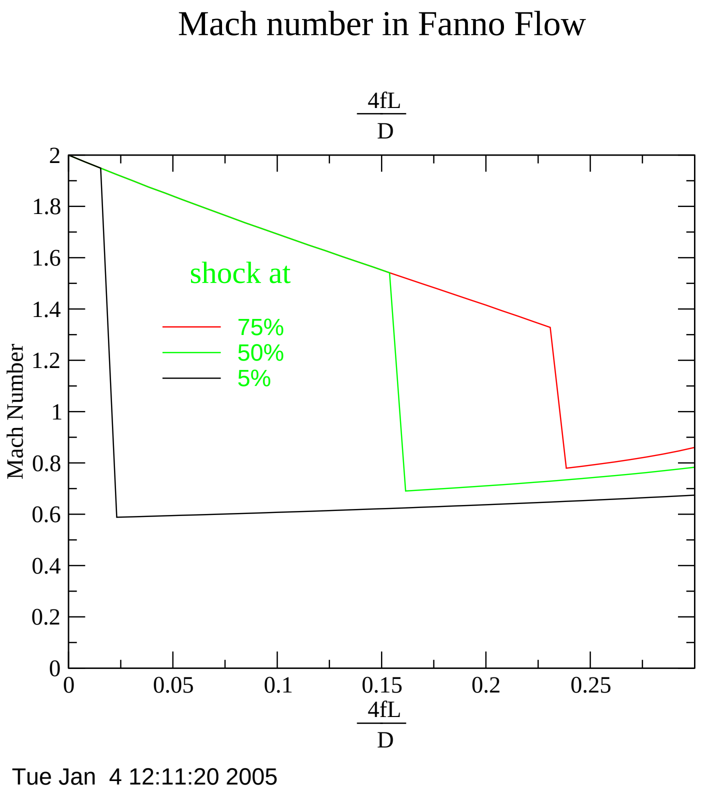


Fig. 9.13: The effects of pressure variations on Mach number profi le as a function of $\frac{4fL}{D}$ when the total resistance $\frac{4fL}{D} = 0.3$ for Fanno Flow

9.7.2.3 Long $\frac{4fL}{D}$

In the case of $\frac{4fL}{D} > \frac{4fL_{max}}{D}$ reduction of the back pressure results in the same process as explain in the short $\frac{4fL}{D}$ up to to point c. However, point c in this case is different from point c at the case of short tube $\frac{4fL}{D} < \frac{4fL_{max}}{D}$. In this point the exit Mach number is equal to 1 and the flow is double shock. Further reduction of the back pressure at this stage will not “move” the shock wave downstream the nozzle. The point c or location of the shock wave is a function entrance Mach number, M_1 and the “extra” $\frac{4fL}{D}$. The is no analytical solution for the location of this point c. The procedure is (will be) presented in later stage.

[16]It is common misconception that the back pressure has to be at point d.

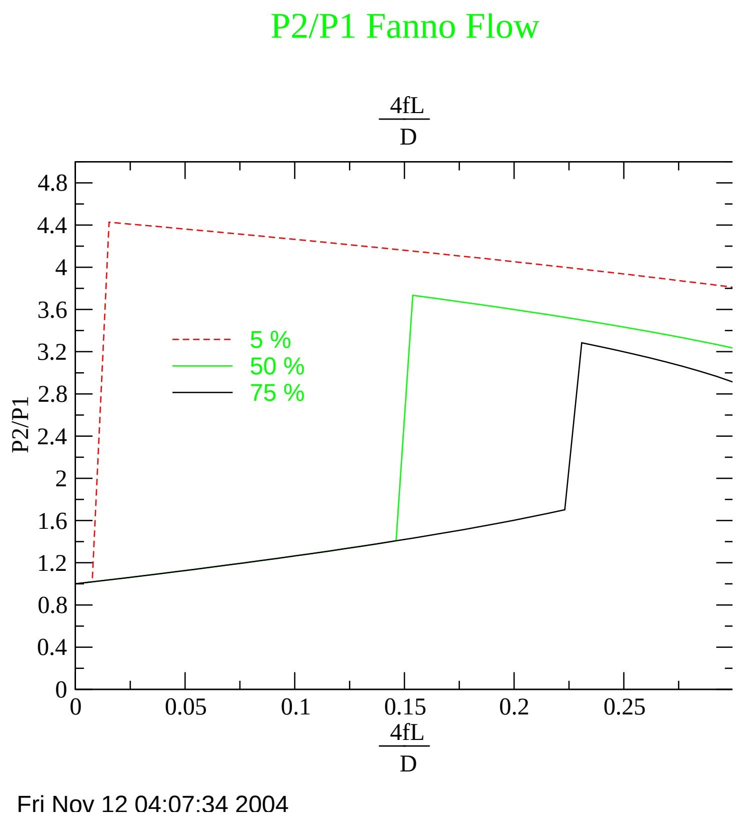


Fig. 9.14: Fanno Flow Mach number as a function of $\frac{4fL}{D}$ when the total $\frac{4fL}{D} = 0.3$

9.7.3 Entrance Mach number, M_1, effects

In this discussion, the effect of changing the throat area on the nozzle efficiency are neglected. In reality these effects have significance and needed to be accounted for in some instances. This dissection deals only when the flow reaches the supersonic branch reached otherwise the flow is subsonic with regular effects.. It is assumed that in this discussion that the pressure ratio $\frac{P_2}{P_1}$ is large enough to create a choked flow and $\frac{4fL}{D}$ is small enough to allow it to happen.

The entrance Mach number, M_1 is a function of the ratio of the nozzle's throat area to the nozzle exit area and its efficiency. This effect is the third parameter discussed here. Practically, the nozzle area ratio change by changing the throat area.

As was shown before, there are two different maximums for $\frac{4fL}{D}$; first is the total maximum $\frac{4fL}{D}$ of the supper sonic which deponent only the specific heat, k, and second the maximum depends on the entrance Mach number, M_1. This

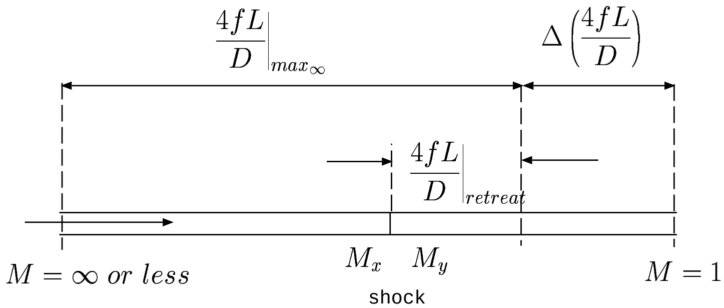


Fig. 9.15: schematic of a "long" tube in supersonic branch

analysis deals with the case where $\frac{4fL}{D}$ is shorter than total $\frac{4fL_{max}}{D}$.

Obviously, in this situation, the critical point is where $\frac{4fL}{D}$ is equal to $\frac{4fL_{max}}{D}$ as result to entrance Mach number.

The process of decreasing the diverging-converging nozzle's throat, increases the entrance[17] Mach number. If the tube contains no supersonic flow then reducing the nozzle throat area wouldn't increase the entrance Mach number.

why? make a question, is increase can increase the possibility to supersonic flow?

This part is for the case where some part of the tube under supersonic regime and there is shock as transition to subsonic branch. Decreasing the nozzle throat area moves the shock location downstream. The "payment" for increase in the supersonic length is by reducing the mass flow. Further, decrease of of the throat area results in flashing the shock out the tube. By doing so, the throat area decreases. The mass flow rate is proportional linear to throat area and therefore the mass flow rate reduces. The process of decreases the throat area also results in increasing the pressure drop of the nozzle (larger resistance in the nozzle[18])[19].

Admittedly, this author have and can explain it mathematically. The author did found any physical explanation developed one. Perhaps it is challenge to public as an open question of how to explain it. This similar to surface tension phenomenon that can be explained only heavy mathematics.

In the case of large tube $\frac{4fL}{D} > \frac{4fL_{max}}{D}$ the exit Mach number increases with the decrease of the throat area. Once the exit Mach number reaches one no further increases is possible. However, the location of the shock wave approaches to the theoretical location if entrance Mach, $M_1 = \infty$.

9.7.3.0.1 The maximum location of the shock The main point in this discussion however, to find the furtherest shock location downstream. Figure 9.16 shows the possible $\Delta\left(\frac{4fL}{D}\right)$ as function of retreat of the location of the shock wave from the maximum location. When the entrance Mach number is infinity, $M_1 = \infty$, if the shock location is at the Maximum length, than shock at $M_x = 1$ results in $M_y = 1$ and possible

[17]The word referred to the tube and not to the nozzle. The reference to the tube because it is the focus of the study.

[18]Strange? frictionless nozzle has a larger resistance when the throat area decreases

[19]It is one of the strange phenomenon that in one way increasing the resistance (changing the throat area) decreases the flow rate while in a different way (increasing the $\frac{4fL}{D}$) does not affect the flow rate.

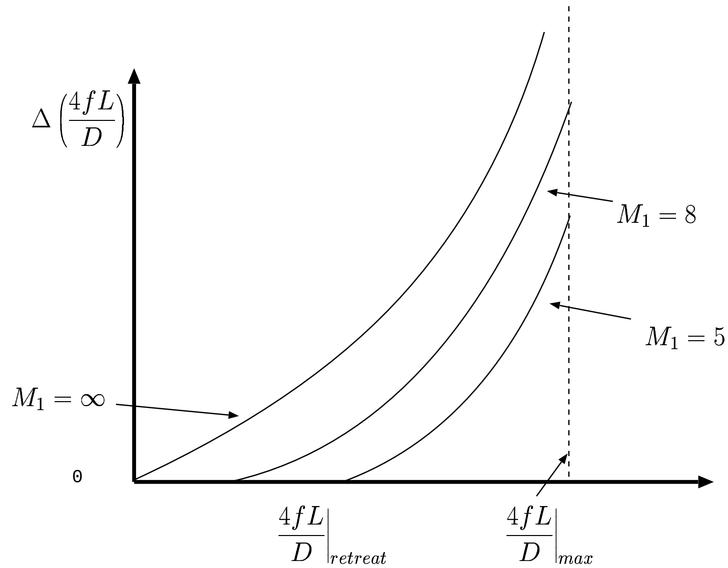


Fig. 9.16: The extra tube length as a function of the shock location, $\frac{4fL}{D}$ supersonic branch

The proposed procedure is based on figure 9.16.

i) Calculated the extra $\frac{4fL}{D}$ and subtract the actual extra $\frac{4fL}{D}$ assuming shock at the left side (at the max length).

ii) Calculated the extra $\frac{4fL}{D}$ and subtract the actual extra $\frac{4fL}{D}$ assuming shock at the right side (at the entrance).

iii) According to the positive or negative utilizes your root finding procedure.

From numerical point of view, the Mach number equal infinity when left side assume result in infinity length of possible extra (the whole flow in the tube is subsonic). To overcame this numerical problem it is suggested to start the calculation from ϵ distance from the right hand side.

Let denote

$$\Delta\left(\frac{4fL}{D}\right) = \frac{4\bar{f}L}{D}_{actual} - \frac{4fL}{D}_{sup} \tag{9.51}$$

Note that $\frac{4fL}{D}_{sup}$ is smaller than $\left.\frac{4fL}{D}\right|_{max_\infty}$. The requirement that has to satisfied is that denote $\left.\frac{4fL}{D}\right|_{retreat}$ as difference between the maximum possible of length in which the flow supersonic achieved and the actual length in which the flow is supersonic see figure 9.15. The retreating length is expressed as subsonic but

$$\left.\frac{4fL}{D}\right|_{retreat} = \left.\frac{4fL}{D}\right|_{max_\infty} - \frac{4fL}{D}_{sup} \tag{9.52}$$

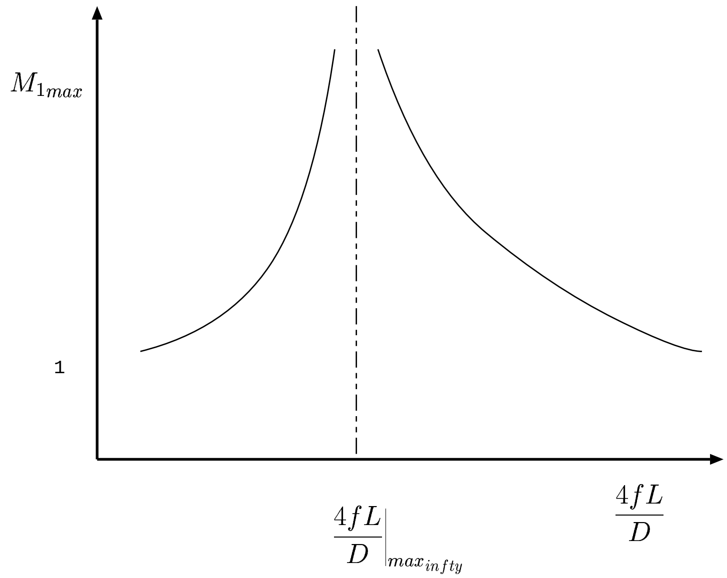


Fig. 9.17: The maximum entrance Mach number, M_1 to the tube as a function of $\frac{4fL}{D}$ supersonic branch

As the figure **??** shows the entrance Mach number, M_1 is reduces after the maximum length is excessed.

Example 9.3:
Calculate the shock location for entrance Mach number $M_1 = 8$ and for $\frac{4fL}{D} = 0.9$ assume that $k = 1.4$.

SOLUTION
The solution is obtained by iterative process. The maximum $\frac{4fL_{max}}{D}$ for $k = 1.4$ is 0.821508116. Hence, $\frac{4fL}{D}$ exceed the the maximum length $\frac{4fL}{D}$ for this entrance Mach number. The maximum for $M_1 = 8$ is $\frac{4fL}{D} = 0.76820$ thus the extra tube, $\Delta\left(\frac{4fL}{D}\right) = 0.9 - 0.76820 = 0.1318$, At the left side is when the shock occurs at $\frac{4fL}{D} = 0.76820$ (flow chocked and no any additional $\frac{4fL}{D}$). Hence, the value of left side is -0.1318. The right side is when the shock is at the entrance at which the extra $\frac{4fL}{D}$ is calculated for M_x and M_y is

$\mathbf{M_x}$	$\mathbf{M_y}$	$\frac{\mathbf{T_y}}{\mathbf{T_x}}$	$\frac{\rho_y}{\rho_x}$	$\frac{\mathbf{P_y}}{\mathbf{P_x}}$	$\frac{\mathbf{P_{0y}}}{\mathbf{P_{0x}}}$
8.000	0.3929	13.4	5.565	74.5	0.008488

With $(M_1)'$

$\mathbf{M}$	$\frac{\mathbf{4fL}}{\mathbf{D}}$	$\frac{\mathbf{P}}{\mathbf{P^*}}$	$\frac{\mathbf{P_0}}{\mathbf{P_0}^*}$	$\frac{\rho}{\rho^*}$	$\frac{\mathbf{U}}{\mathbf{U^*}}$	$\frac{\mathbf{T}}{\mathbf{T^*}}$
0.3929	2.442	2.746	1.614	2.359	0.4239	1.164

The extra $\Delta\left(\frac{4fL}{D}\right)$ is $2.442 - 0.1318 = 2.3102$ Now the solution is somewhere between the negative of left side to the positive of the right side.[20]

The solution of the above utilizing results in the following table

$\mathbf{M_1}$	$\mathbf{M_2}$	$\left.\frac{\mathbf{4fL}}{\mathbf{D}}\right\|_{\mathbf{up}}$	$\left.\frac{\mathbf{4fL}}{\mathbf{D}}\right\|_{\mathbf{down}}$	$\mathbf{M_x}$	$\mathbf{M_y}$
8.000	1.000	0.5707	0.3293	1.671	0.6483

[20]What if the right side is also negative? The flow is choked and shock must occur in the nozzle before entering the tube. or in very long tube the whole flow will be subsonic.

9.8 The Approximation of the Fanno flow by Isothermal Flow

The isothermal flow model has equation that theoreticians are easier to use compared to Fanno flow model.

One must noticed that the maximum temperature at the entrance is T_{01}. When the Mach number decreases the temperature approaches the stagnation temperature ($T \rightarrow T_0$). Hence, if one allow certain deviation of temperature, say about 1%) that flow can be assumed to be isothermal. This tolerance requires that $(T_0 - T)/T_0 = 0.99$ which requires that enough for $M_1 < 0.15$ even for large $k = 1.67$. This requirement provide that somewhere (depend) in the vicinity of $\frac{4fL}{D} = 25$ the flow can be assumed isothermal. Hence the mass flow rate is a function of $\frac{4fL}{D}$ because M_1 changes. Looking that the table or figure 9.2 or the results from computer program attached to this book shows that reduction of the mass flow is very rapid.

to insert a question or example about this issue in end

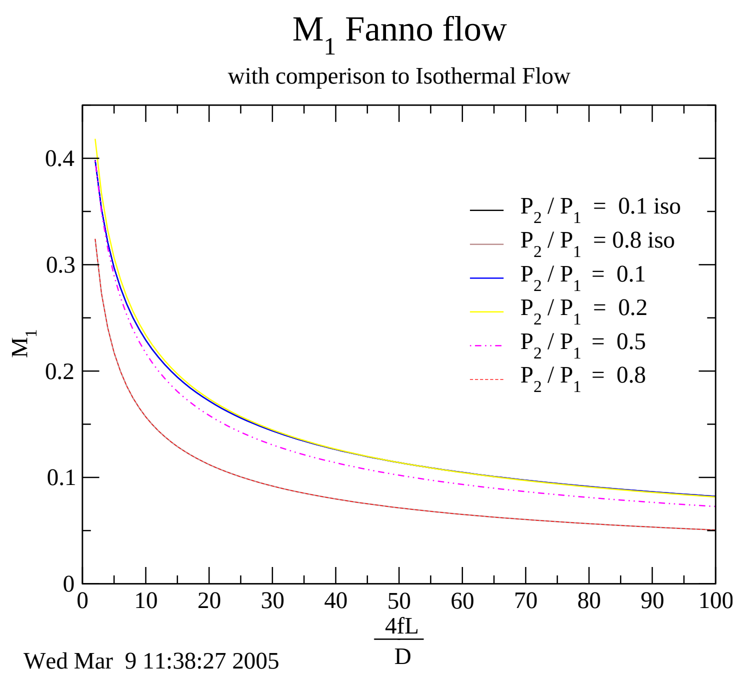


Fig. 9.18: The entrance Mach number as a function of dimensionless resistance and comparison with Isothermal Flow

As it can be seen for the figure 9.18 the dominating parameter is $\frac{4fL}{D}$. The results are very similar for isothermal flow. The only difference is in small dimensionless friction, $\frac{4fL}{D}$.

9.9 More Examples

Example 9.4:
To demonstrate the utility of the figure 9.18 consider the flowing example. Find the mass flow rate for $f = 0.05$, $L = 4[m]$, $D = 0.02[m]$ and pressure ratio $P_2/P_1 = 0.1, 0.3, 0.5, 0.8$. The stagnation conditions at the entrance are $300K$ and $3[bar]$ air.

SOLUTION
First calculate the dimensionless resistance, $\frac{4fL}{D}$.

$$\frac{4fL}{D} = \frac{4 \times 0.05 \times 4}{0.02} = 40$$

From Figure 9.18 for $P_2/P_1 = 0.1$ $M_1 \approx 0.13$ etc.
or accurately utilizing the program as in the following table.

$\mathbf{M_1}$	$\mathbf{M_2}$	$\mathbf{\frac{4fL}{D}}$	$\mathbf{\left.\frac{4fL}{D}\right\|_1}$	$\mathbf{\left.\frac{4fL}{D}\right\|_2}$	$\mathbf{\frac{P_2}{P_1}}$
0.12728	1.0000	40.0000	40.0000	0.0	0.11637
0.12420	0.40790	40.0000	42.1697	2.1697	0.30000
0.11392	0.22697	40.0000	50.7569	10.7569	0.50000
0.07975	0.09965	40.0000	107.42	67.4206	0.80000

Only for the pressure ratio of 0.1 the flow is choked.

$\mathbf{M}$	$\mathbf{\frac{T}{T_0}}$	$\frac{\rho}{\rho_0}$	$\mathbf{\frac{A}{A^\star}}$	$\mathbf{\frac{P}{P_0}}$	$\mathbf{\frac{A\times P}{A^*\times P_0}}$
0.12728	0.99677	0.99195	4.5910	0.98874	4.5393
0.12420	0.99692	0.99233	4.7027	0.98928	4.6523
0.11392	0.99741	0.99354	5.1196	0.99097	5.0733
0.07975	0.99873	0.99683	7.2842	0.99556	7.2519

Therefore, $T \approx T_0$ and for the same the pressure. Hence, the mass rate is a function of the Mach number. The Mach number is indeed function of the pressure ratio but and therefore mass flow rate is function pressure ratio only through Mach number.

The mass flow rate is

$$\dot{m} = PAM\sqrt{\frac{k}{RT}} = 300000 \times \frac{\pi \times 0.02^2}{4} \times 0.127 \times \sqrt{\frac{1.4}{287300}} \approx 0.48 \left(\frac{kg}{sec}\right)$$

and for the rest

$$\dot{m}\left(\frac{\mathbf{P_2}}{\mathbf{P_1}}=0.3\right)\sim 0.48\times\frac{0.1242}{0.1273}=0.468\left(\frac{kg}{sec}\right)$$
$$\dot{m}\left(\frac{\mathbf{P_2}}{\mathbf{P_1}}=0.5\right)\sim 0.48\times\frac{0.1139}{0.1273}=0.43\left(\frac{kg}{sec}\right)$$
$$\dot{m}\left(\frac{\mathbf{P_2}}{\mathbf{P_1}}=0.8\right)\sim 0.48\times\frac{0.07975}{0.1273}=0.30\left(\frac{kg}{sec}\right)$$

The table for Fanno Flow

Table 9.15: Fanno Flow Standard basic Table

$\mathbf{M}$	$\frac{\mathbf{4fL}}{\mathbf{D}}$	$\frac{\mathbf{P}}{\mathbf{P^*}}$	$\frac{\mathbf{P_0}}{\mathbf{P_0}^*}$	$\frac{\rho}{\rho^*}$	$\frac{\mathbf{U}}{\mathbf{U^*}}$	$\frac{\mathbf{T}}{\mathbf{T^*}}$
0.030	787.08	36.5116	19.3005	30.4318	0.03286	1.1998
0.040	440.35	27.3817	14.4815	22.8254	0.04381	1.1996
0.050	280.02	21.9034	11.5914	18.2620	0.05476	1.1994
0.060	193.03	18.2508	9.6659	15.2200	0.06570	1.1991
0.070	140.66	15.6416	8.2915	13.0474	0.07664	1.1988
0.080	106.72	13.6843	7.2616	11.4182	0.08758	1.1985
0.090	83.4961	12.1618	6.4613	10.1512	0.09851	1.1981
0.100	66.9216	10.9435	5.8218	9.1378	0.10944	1.1976
0.200	14.5333	5.4554	2.9635	4.5826	0.21822	1.1905
0.250	8.4834	4.3546	2.4027	3.6742	0.27217	1.1852
0.300	5.2993	3.6191	2.0351	3.0702	0.32572	1.1788
0.350	3.4525	3.0922	1.7780	2.6400	0.37879	1.1713
0.400	2.3085	2.6958	1.5901	2.3184	0.43133	1.1628
0.450	1.5664	2.3865	1.4487	2.0693	0.48326	1.1533
0.500	1.0691	2.1381	1.3398	1.8708	0.53452	1.1429
0.550	0.72805	1.9341	1.2549	1.7092	0.58506	1.1315
0.600	0.49082	1.7634	1.1882	1.5753	0.63481	1.1194
0.650	0.32459	1.6183	1.1356	1.4626	0.68374	1.1065
0.700	0.20814	1.4935	1.0944	1.3665	0.73179	1.0929
0.750	0.12728	1.3848	1.0624	1.2838	0.77894	1.0787
0.800	0.07229	1.2893	1.0382	1.2119	0.82514	1.0638
0.850	0.03633	1.2047	1.0207	1.1489	0.87037	1.0485
0.900	0.01451	1.1291	1.0089	1.0934	0.91460	1.0327
0.950	0.00328	1.061	1.002	1.044	0.95781	1.017
1.000	0.0	1.00000	1.00000	1.00000	1.000	1.00000
2.000	0.30500	0.40825	1.688	0.61237	1.633	0.66667
3.000	0.52216	0.21822	4.235	0.50918	1.964	0.42857
4.000	0.63306	0.13363	10.72	0.46771	2.138	0.28571

Table 9.15: Fanno Flow Standard basic Table (continue)

$\mathbf{M}$	$\frac{\mathbf{4fL}}{\mathbf{D}}$	$\frac{\mathbf{P}}{\mathbf{P^*}}$	$\frac{\mathbf{P_0}}{\mathbf{P_0}^*}$	$\frac{\rho}{\rho^*}$	$\frac{\mathbf{U}}{\mathbf{U^*}}$	$\frac{\mathbf{T}}{\mathbf{T^*}}$
5.000	0.69380	0.089443	25.00	0.44721	2.236	0.20000
6.000	0.72988	0.063758	53.18	0.43568	2.295	0.14634
7.000	0.75280	0.047619	$1.0E{+}2$	0.42857	2.333	0.11111
8.000	0.76819	0.036860	$1.9E{+}2$	0.42390	2.359	0.086957
9.000	0.77899	0.029348	$3.3E{+}2$	0.42066	2.377	0.069767
10.00	0.78683	0.023905	$5.4E{+}2$	0.41833	2.390	0.057143
20.00	0.81265	0.00609	$1.5E{+}4$	0.41079	2.434	0.014815
25.00	0.81582	0.00390	$4.6E{+}4$	0.40988	2.440	0.00952
30.00	0.81755	0.00271	$1.1E{+}5$	0.40938	2.443	0.00663
35.00	0.81860	0.00200	$2.5E{+}5$	0.40908	2.445	0.00488
40.00	0.81928	0.00153	$4.8E{+}5$	0.40889	2.446	0.00374
45.00	0.81975	0.00121	$8.6E{+}5$	0.40875	2.446	0.00296
50.00	0.82008	0.000979	$1.5E{+}6$	0.40866	2.447	0.00240
55.00	0.82033	0.000809	$2.3E{+}6$	0.40859	2.447	0.00198
60.00	0.82052	0.000680	$3.6E{+}6$	0.40853	2.448	0.00166
65.00	0.82066	0.000579	$5.4E{+}06$	0.40849	2.448	0.00142
70.00	0.82078	0.000500	$7.8E{+}06$	0.40846	2.448	0.00122

CHAPTER 10

RAYLEIGH FLOW

Rayleigh flow is (frictionless) flow with heat transfer through a pipe of constant cross sectional area. In practice Rayleigh flow is difficult to achieve. Yet Rayleigh flow is practical and useful concept in a obtaining trends and limits. The density and pressure change due to external cooling or heating. As opposed to the two previous models, the heat transfer can be in two directions not like the friction (there is no negative friction). This fact create situation different compare to the previous two models. This model applied to case where the heat transfer is significant.

10.1 Introduction

The third simple model for an one dimensional flow is for constant heat transfer for frictionless flow. This flow referred in the literature as Rayleigh Flow (see historical notes). This flow is another extreme case in which the friction effect are neglected because their relative effect is much smaller the heat transfer effect. While the isothermal flow model has heat transfer and friction the main assumption was that relative length is so the heat transfer occurs between the surrounding and tube. In contract the heat transfer in Rayleigh flow occurs either between unknown temperature to tube but the heat flux maintained constant. As before, a simple model is build around assumption of constant properties (poorer prediction to case were chemical reaction take palace).

This model usage is to a rough predict the situation involve mostly chemical reaction. In analysis the flow one has to be aware that properties do change significantly for a large range of temperature. Yet, for smaller range of temperature and length the calculations are more accurate. Nevertheless, the main characteristic of the flow such as chocking condition etc. are encapsulated in this model.

The basic physics of the flow revolves around the fact that the gas is highly compressible. The density change though the heat transfer (temperature change). As appose to the Fanno flow in which the resistance always oppose the the flow direction, in Rayleigh flow also cooling can be applied. The flow velocity acceleration change the direction when the cooling is applied.

10.2 Governing Equation

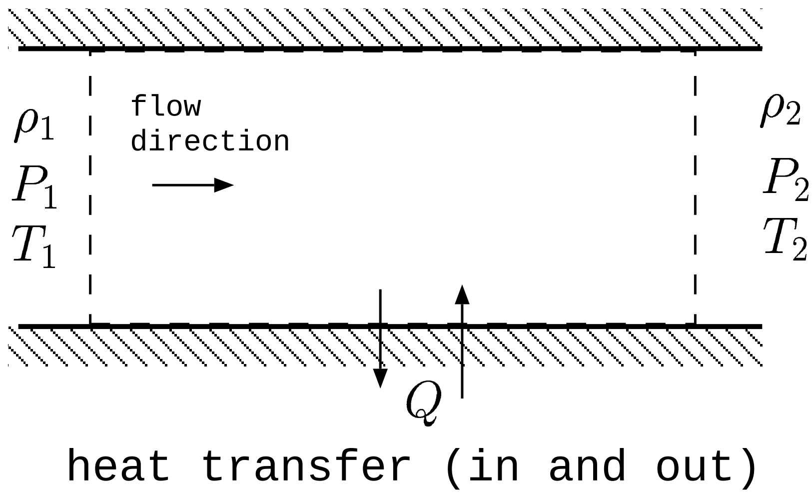


Fig. 10.1: The control volume of Rayleigh Flow

The energy balance on the control volume reads

$$Q = C_p (T_{0\,2} - T_{0\,1}) \tag{10.1}$$

the momentum balance reads

$$A(P_1 - P_2) = \dot{m}(V_2 - V_1) \tag{10.2}$$

The mass conservation reads

$$\rho_1 U_1 A = \rho_2 U_2 A = \dot{m} \tag{10.3}$$

Equation of state

$$\frac{P_1}{\rho_1 T_1} = \frac{P_2}{\rho_2 T_2} \tag{10.4}$$

There are four equations with four unknown, if the upstream conditions are known (or downstream condition are known). Thus, a solution can be obtained. One can noticed that equations ray:eq:momentum;ray:eq:mass;ray:eq:state are similar to the equations that was solved for the shock wave.

$$\frac{P_2}{P_1} = \frac{1 + k{M_1}^2}{1 + k{M_2}^2} \tag{10.5}$$

The equation of state (10.4) can further assist in obtaining the temperature ratio as

$$\frac{T_2}{T_1} = \frac{P_2}{P_1}\frac{\rho_1}{\rho_2} \tag{10.6}$$

The density ratio can be expressed in term of mass conservation as

$$\frac{\rho_1}{\rho_2} = \frac{U_2}{U_1} = \frac{\frac{U_2}{\sqrt{kRT_2}}\sqrt{kRT_2}}{\frac{U_1}{\sqrt{kRT_1}}\sqrt{kRT_1}} = \frac{M_2}{M_1}\sqrt{\frac{T_2}{T_1}} \tag{10.7}$$

Substituting equations (10.5) and (10.7) into equation (10.6) yields

$$\frac{T_2}{T_1} = \frac{1 + k{M_1}^2}{1 + k{M_2}^2}\frac{M_2}{M_1}\sqrt{\frac{T_2}{T_1}} \tag{10.8}$$

Transferring the temperature ratio to left hand side and squaring results in

$$\frac{T_2}{T_1} = \left[\frac{1 + k{M_1}^2}{1 + k{M_2}^2}\right]^2 \left(\frac{M_2}{M_1}\right)^2 \tag{10.9}$$

The Rayleigh line exhibits two possible maximums one for $dT/ds = 0$ and for $ds/dT = 0$. The second maximum can be expressed as $dT/ds = \infty$ The second law is used to find the expression for derivative.

$$\frac{s_1 - s_2}{C_p} = \ln\frac{T_2}{T_1} - \frac{k-1}{k}\ln\frac{P_2}{P_1} \tag{10.10}$$

$$\frac{s_1 - s_2}{C_p} = 2\ln\left[(\frac{1 + k{M_1}^2)}{(1 + k{M_2}^2)}\frac{M_2}{M_1}\right] + \frac{k-1}{k}\ln\left[\frac{1 + kM21^2}{1 + k{M_1}^2}\right] \tag{10.11}$$

Let the initial condition M_1, and s_1 are constant then the variable parameters are M_2, and s_2. A derivative of equation (10.11) results in

$$\frac{1}{C_p}\frac{ds}{dM} = \frac{2(1 - M^2)}{M(1 + kM^2)} \tag{10.12}$$

Take the derivative of the equation (10.9) when letting the variable parameters be T_2, and M_2 results in

$$\frac{dT}{dM} = constant \times \frac{1 - kM^2}{(1 + kM^2)^3} \tag{10.13}$$

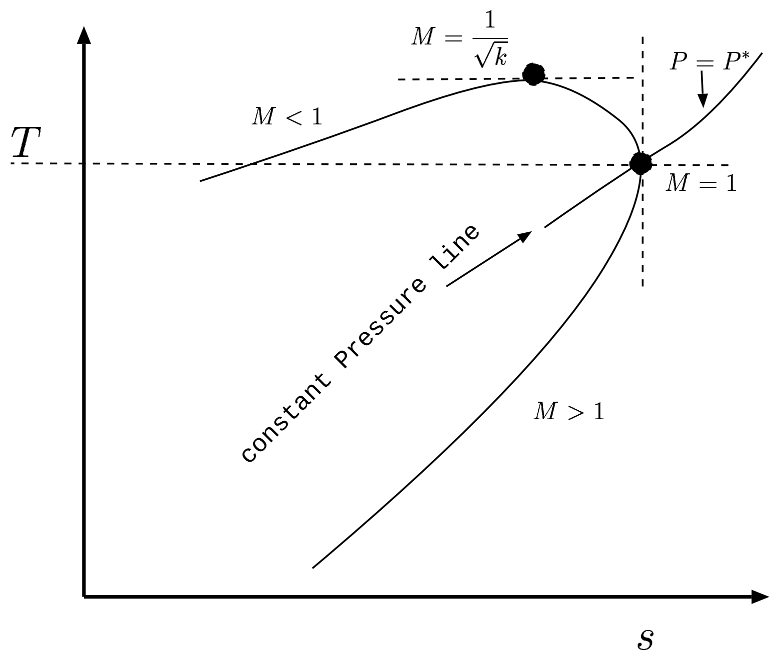


Fig. 10.2: The Temperature entropy diagram for Rayleigh line

Combining equations (10.12) and (10.13) by eliminating dM results in

$$\frac{dT}{ds} = constant \times \frac{M(1-kM^2)}{(1-M^2)(1+kM^2)^2} \tag{10.14}$$

to put a mathematical explanation how the curves are constructed.

On T-s diagram a family of curve can be drawn for a given constant. Yet for every curve several observations can be generalized. The derivative is equal to zero when $1-kM^2 = 0$ or $M = 1/\sqrt{k}$ or when $M \to 0$. The derivative is equal to infinity, $dT/ds = \infty$ when $M = 1$. From thermodynamics, increase of heating results in increase of entropy. And cooling results in reduction of entropy. Hence, when cooling applied to a tube the velocity decreases and heating applied the velocity increases. The peculiars point of $M = 1/\sqrt{k}$ when additional heat is applied the temperature is decreasing. The derivative is negative, $dT/ds < 0$, yet note this point is not the choking point. The chocking is occurred only when $M = 1$ because it violate the second law. The transition to supper sonic flow occurs when the area changes, some what similarly to Fanno flow, Yet, chocking can be explained by the fact increase of energy must accompanied by increase of entropy. But the entropy of supersonic flow is lower (see the Figure 10.2) and therefore it is not possible (the maximum entropy at $M = 1$.).

It is convent to referrers to the value of $M = 1$. These value referred as the "star"[1] values. The equation (10.5) can be written between chocking point and any point on the curve.

$$\frac{P^*}{P_1} = \frac{1 + k{M_1}^2}{1 + k} \tag{10.15}$$

The temperature ratio is

$$\frac{T^*}{T_1} = \frac{1}{M^2}\left(\frac{1 + k{M_1}^2}{1 + k}\right)^2 \tag{10.16}$$

$$\frac{\rho_1}{\rho^*} = \frac{U^*}{U_1} = \frac{\frac{U^*}{\sqrt{kRT^*}}\sqrt{kRT^*}}{\frac{U_1}{\sqrt{kRT_1}}\sqrt{kRT_1}} = \frac{1}{M_1}\sqrt{\frac{T^*}{T_1}} \tag{10.17}$$

$$\frac{T_{0\,1}}{{T_0}^*} = \frac{T_1\left(1 + \frac{k-1}{2}{M_1}^2\right)}{T^*\left(\frac{1+k}{2}\right)} = \frac{2(1+k){M_1}^2}{(1+kM^2)^2}\left(1 + \frac{k-1}{2}{M_1}^2\right) \tag{10.18}$$

The stagnation pressure ratio reads

$$\frac{P_{0\,1}}{{P_0}^*} = \frac{P_1\left(1 + \frac{k-1}{2}{M_1}^2\right)}{P^*\left(\frac{1+k}{2}\right)} = \left(\frac{1+k}{1+k{M_1}^2}\right)\left(\frac{1+k{M_1}^2}{\frac{(1+k)}{2}}\right)^{\frac{k}{k-1}} \tag{10.19}$$

The "star" values are tabulated in table 10.1. Several observations can be made in regards to the stagnation temperature.

Table 10.1: Rayleigh Flow k=1.4

$\mathbf{M}$	$\frac{\mathbf{T}}{\mathbf{T^*}}$	$\frac{\mathbf{T_0}}{\mathbf{{T_0}^*}}$	$\frac{\mathbf{P}}{\mathbf{P^*}}$	$\frac{\mathbf{P_0}}{\mathbf{{P_0}^*}}$	$\frac{\rho^*}{\rho}$
0.03000	0.00517	0.00431	2.397	1.267	0.00216
0.040000	0.00917	0.00765	2.395	1.266	0.00383
0.050000	0.014300	0.011922	2.392	1.266	0.00598
0.060000	0.020529	0.017119	2.388	1.265	0.00860
0.070000	0.027841	0.023223	2.384	1.264	0.011680
0.080000	0.036212	0.030215	2.379	1.262	0.015224
0.090000	0.045616	0.038075	2.373	1.261	0.019222
0.100000	0.056020	0.046777	2.367	1.259	0.023669
0.20000	0.20661	0.17355	2.273	1.235	0.090909

[1]The star is an asterisk.

Table 10.1: ?? (continue)

$\mathbf{M}$	$\frac{\mathbf{T}}{\mathbf{T}^*}$	$\frac{\mathbf{T_0}}{\mathbf{T_0}^*}$	$\frac{\mathbf{P}}{\mathbf{P}^*}$	$\frac{\mathbf{P_0}}{\mathbf{P_0}^*}$	$\frac{\rho^*}{\rho}$
0.25000	0.30440	0.25684	2.207	1.218	0.13793
0.30000	0.40887	0.34686	2.131	1.199	0.19183
0.35000	0.51413	0.43894	2.049	1.178	0.25096
0.40000	0.61515	0.52903	1.961	1.157	0.31373
0.45000	0.70804	0.61393	1.870	1.135	0.37865
0.50000	0.79012	0.69136	1.778	1.114	0.44444
0.55000	0.85987	0.75991	1.686	1.094	0.51001
0.60000	0.91670	0.81892	1.596	1.075	0.57447
0.65000	0.96081	0.86833	1.508	1.058	0.63713
0.70000	0.99290	0.90850	1.423	1.043	0.69751
0.75000	1.014	0.94009	1.343	1.030	0.75524
0.80000	1.025	0.96395	1.266	1.019	0.81013
0.85000	1.029	0.98097	1.193	1.011	0.86204
0.90000	1.025	0.99207	1.125	1.005	0.91097
0.95000	1.015	0.99814	1.060	1.001	0.95693
1.000	1.00000	1.000	1.00000	1.00000	1.000
1.100	0.96031	0.99392	0.89087	1.005	1.078
1.200	0.91185	0.97872	0.79576	1.019	1.146
1.300	0.85917	0.95798	0.71301	1.044	1.205
1.400	0.80539	0.93425	0.64103	1.078	1.256
1.500	0.75250	0.90928	0.57831	1.122	1.301
1.600	0.70174	0.88419	0.52356	1.176	1.340
1.700	0.65377	0.85971	0.47562	1.240	1.375
1.800	0.60894	0.83628	0.43353	1.316	1.405
1.900	0.56734	0.81414	0.39643	1.403	1.431
2.000	0.52893	0.79339	0.36364	1.503	1.455
2.100	0.49356	0.77406	0.33454	1.616	1.475
2.200	0.46106	0.75613	0.30864	1.743	1.494
2.300	0.43122	0.73954	0.28551	1.886	1.510
2.400	0.40384	0.72421	0.26478	2.045	1.525
2.500	0.37870	0.71006	0.24615	2.222	1.538
2.600	0.35561	0.69700	0.22936	2.418	1.550
2.700	0.33439	0.68494	0.21417	2.634	1.561
2.800	0.31486	0.67380	0.20040	2.873	1.571
2.900	0.29687	0.66350	0.18788	3.136	1.580
3.000	0.28028	0.65398	0.17647	3.424	1.588
3.500	0.21419	0.61580	0.13223	5.328	1.620
4.000	0.16831	0.58909	0.10256	8.227	1.641
4.500	0.13540	0.56982	0.081772	12.50	1.656

Table 10.1: ?? (continue)

M	$\frac{\mathbf{T}}{\mathbf{T^*}}$	$\frac{\mathbf{T_0}}{\mathbf{T_0}^*}$	$\frac{\mathbf{P}}{\mathbf{P^*}}$	$\frac{\mathbf{P_0}}{\mathbf{P_0}^*}$	$\frac{\rho^*}{\rho}$
5.000	0.11111	0.55556	0.066667	18.63	1.667
5.500	0.092719	0.54473	0.055363	27.21	1.675
6.000	0.078487	0.53633	0.046693	38.95	1.681
6.500	0.067263	0.52970	0.039900	54.68	1.686
7.000	0.058264	0.52438	0.034483	75.41	1.690
7.500	0.050943	0.52004	0.030094	$1.0E{+}02$	1.693
8.000	0.044910	0.51647	0.026490	$1.4E{+}02$	1.695
8.500	0.039883	0.51349	0.023495	$1.8E{+}02$	1.698
9.000	0.035650	0.51098	0.020979	$2.3E{+}02$	1.699
9.500	0.032053	0.50885	0.018846	$3.0E{+}02$	1.701
10.00	0.028972	0.50702	0.017021	$3.8E{+}02$	1.702
20.00	0.00732	0.49415	0.00428	$1.1E{+}04$	1.711
25.00	0.00469	0.49259	0.00274	$3.2E{+}04$	1.712
30.00	0.00326	0.49174	0.00190	$8.0E{+}04$	1.713
35.00	0.00240	0.49122	0.00140	$1.7E{+}05$	1.713
40.00	0.00184	0.49089	0.00107	$3.4E{+}05$	1.714
45.00	0.00145	0.49066	0.000846	$6.0E{+}05$	1.714
50.00	0.00117	0.49050	0.000686	$1.0E{+}06$	1.714
55.00	0.000971	0.49037	0.000567	$1.6E{+}06$	1.714
60.00	0.000816	0.49028	0.000476	$2.5E{+}06$	1.714
65.00	0.000695	0.49021	0.000406	$3.8E{+}06$	1.714
70.00	0.000600	0.49015	0.000350	$5.5E{+}06$	1.714

The Figure is presented in figure 10.3.

Illustrative example

The typical questions that raised in Rayleigh Flow are related to the maximum heat that can be transfered to gas (reaction heat) and to flow rate.

Example 10.1:

Air enters a pipe with pressure of $3[bar]$ *and temperature of* $27°\mathrm{C}$ *at Mach number of* $M = 0.25$*. Due internal combustion heat was released and the exit temperature was found to be* $127°\mathrm{C}$*. Calculated the exit Mach number, the exit pressure, the total exit pressure, and heat released (transfered) to the air. After what amount of energy the exit temperature will start to decrease? Assume* $C_P = 1.004\left[\frac{kJ}{kg°\mathrm{C}}\right]$

SOLUTION

The entrance Mach number and the exit temperature are given and from the Table 10.1 or from the program the initial ratio can be calculated. From the initial values the ratio at the exit can be computed as following.

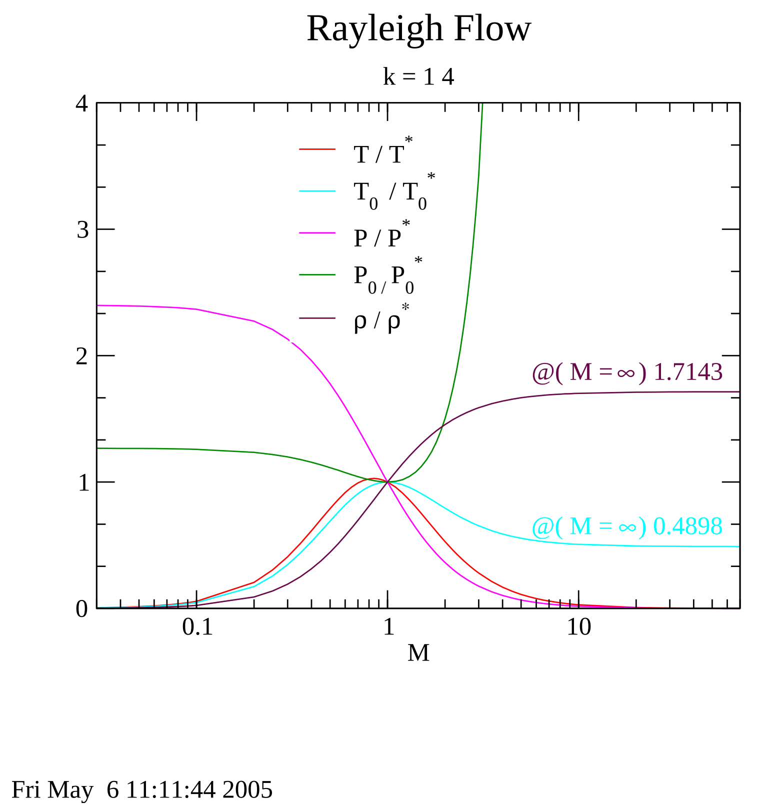


Fig. 10.3: The basic functions of Rayleigh Flow

$\mathbf{M}$	$\frac{\mathbf{T}}{\mathbf{T}^*}$	$\frac{\mathbf{T_0}}{\mathbf{T_0}^*}$	$\frac{\mathbf{P}}{\mathbf{P}^*}$	$\frac{\mathbf{P_0}}{\mathbf{P_0}^*}$	$\frac{\rho^*}{\rho}$
0.25000	0.30440	0.25684	2.2069	1.2177	0.13793

and

$$\frac{T_2}{T^*} = \frac{T_1}{T^*}\frac{T_2}{T_1} = 0.304 \times \frac{400}{300} = 0.4053$$

$\mathbf{M}$	$\frac{\mathbf{T}}{\mathbf{T}^*}$	$\frac{\mathbf{T_0}}{\mathbf{T_0}^*}$	$\frac{\mathbf{P}}{\mathbf{P}^*}$	$\frac{\mathbf{P_0}}{\mathbf{P_0}^*}$	$\frac{\rho^*}{\rho}$
0.29831	0.40530	0.34376	2.1341	1.1992	0.18991

The exit Mach number is known, the exit pressure can be calculated as

$$P_2 = P_1 \frac{P^*}{P_1}\frac{P_2}{P^*} = 3 \times \frac{1}{2.2069} \times 2.1341 = 2.901[Bar]$$

For the entrance the stagnation values are

$\mathbf{M}$	$\frac{\mathbf{T}}{\mathbf{T_0}}$	$\frac{\rho}{\rho_0}$	$\frac{\mathbf{A}}{\mathbf{A}^*}$	$\frac{\mathbf{P}}{\mathbf{P_0}}$	$\frac{\mathbf{A\times P}}{\mathbf{A^*\times P_0}}$
0.25000	0.98765	0.96942	2.4027	0.95745	2.3005

The total exit pressure, P_{0_2} can be calculated as the following:

$$P_{0_2} = P_1 \overbrace{\frac{P_{0_1}}{P_1}}^{isentropic} \frac{P_0{}^*}{P_{0_1}}\frac{P_{0_2}}{P_0{}^*} = 3 \times \frac{1}{0.95745} \times \frac{1}{1.2177} \times 1.1992 = 3.08572[Bar]$$

The heat release (heat transfer) can be calculated from obtaining the stagnation temperature form the both sides. The stagnation temperature at the entrance, T_{0_1}

$isentropic$

The exit stagnation temperature is

$$T_{0_2} = T_2 \overbrace{\frac{T_{0_2}}{T_2}}^{isentropic} = 400/0.98765 = 407.12[K]$$

The heat release becomes

$$\frac{Q}{\dot{m}} = C_p \left(T_{0_2} - T_{0_1}\right) 1 \times 1.004 \times (407.12 - 303.75) = 103.78 \left[\frac{kJ}{seckg°\text{C}}\right]$$

The maximum temperature occurs at the point the Mach number reach $1/\sqrt{k}$ and at this point

$\mathbf{M}$	$\frac{\mathbf{T}}{\mathbf{T^*}}$	$\frac{\mathbf{T_0}}{\mathbf{T_0}^*}$	$\frac{\mathbf{P}}{\mathbf{P^*}}$	$\frac{\mathbf{P_0}}{\mathbf{P_0}^*}$	$\frac{\rho^*}{\rho}$
0.84515	1.0286	0.97959	1.2000	1.0116	0.85714

The maximum heat before the temperature can be calculated as following:

$$T_{max} = T_1 \frac{T^*}{T_1} \frac{T_{max}}{T^*} \frac{300}{0.3044} \times 1.0286 = 1013.7[K]$$

$\mathbf{M}$	$\frac{\mathbf{T}}{\mathbf{T_0}}$	$\frac{\rho}{\rho_0}$	$\frac{\mathbf{A}}{\mathbf{A^*}}$	$\frac{\mathbf{P}}{\mathbf{P_0}}$	$\frac{\mathbf{A}\times\mathbf{P}}{\mathbf{A^*}\times\mathbf{P_0}}$
0.84515	0.87500	0.71618	1.0221	0.62665	0.64051

The stagnation temperature for this point is

$$T_{0_{max}} = T_{max} * \frac{T_{0_{max}}}{T_{max}} = \frac{1013.7}{0.875} = 1158.51[K]$$

The maximum heat can be calculated as

$$\frac{Q}{\dot{m}} = C_p \left(T_{0_{max}} - T_{0_1}\right) = 1 \times 1.004 \times (1158.51 - 303.75) = 858.18 \left[\frac{kJ}{kgsecK}\right]$$

Note that this point isn't the choking point.

Example 10.2:
Heat is added to the air until the flow is choked in amount of 600 [kJ/kg]. The exit temperature is 1000 [K]. Calculated the entrance temperature and the entrance Mach number.

SOLUTION

The solution involve finding the stagnation temperature at the exit and subtraction of the heat (heat equation) to obtain the entrance stagnation temperature. From the table the ratio or from the pogrom the following can be obtained.

$\mathbf{M}$	$\frac{\mathbf{T}}{\mathbf{T_0}}$	$\frac{\rho}{\rho_0}$	$\frac{\mathbf{A}}{\mathbf{A^\star}}$	$\frac{\mathbf{P}}{\mathbf{P_0}}$	$\frac{\mathbf{A\times P}}{\mathbf{A^*\times P_0}}$
1.0000	0.83333	0.63394	1.0000	0.52828	0.52828

The stagnation temperature

$$T_{0_2} = T_2 \frac{T_{0_2}}{T_2} = \frac{1000}{0.83333} = 1200.0[K]$$

The entrance temperature is

$$\frac{T_{0_1}}{T_{0_2}} = 1 - \frac{Q/\dot{m}}{T_{0_2} C_P} = 1200 - \frac{600}{1200 \times 1.004} \cong 0.5016$$

It must be noted that $T_{0_2} = {T_0}^*$. Therefore with $\frac{T_{0_1}}{{T_0}^*} = 0.5016$ either by table or the program

$\mathbf{M}$	$\frac{\mathbf{T}}{\mathbf{T^*}}$	$\frac{\mathbf{T_0}}{\mathbf{T_0}^*}$	$\frac{\mathbf{P}}{\mathbf{P^*}}$	$\frac{\mathbf{P_0}}{\mathbf{P_0}^*}$	$\frac{\rho^*}{\rho}$
0.38454	0.58463	0.50160	1.9884	1.1632	0.29403

Thus entrance Mach number is 0.38454 and the entrance temperature can be calculated as following

$$T_1 = T^* \frac{T_1}{T^*} = 1000 \times 0.58463 = 584.6[K]$$

The difference between the supersonic branch to subsonic branch

Example 10.3:
Air with Mach 3 enters a frictionless duct with heating. What is the maximum heat that can be add so there is no subsonic flow. If a shock is occurs immediately at the entrance what is the maximum heat that can be added?

<u>SOLUTION</u>

To achieve maximum heat transfer the exit Mach number has to be one, $M_2 = 1$.

$$\frac{Q}{\dot{m}} = C_p \left(T_{0_2} - T_{0_1}\right) = C_p {T_0}^* \left(1 - \frac{T_{0_1}}{{T_0}^*}\right)$$

The table for $M = 3$ as following

$\mathbf{M}$	$\frac{\mathbf{T}}{\mathbf{T}^*}$	$\frac{\mathbf{T_0}}{\mathbf{T_0}^*}$	$\frac{\mathbf{P}}{\mathbf{P}^*}$	$\frac{\mathbf{P_0}}{\mathbf{P_0}^*}$	$\frac{\rho^*}{\rho}$
3.0000	0.28028	0.65398	0.17647	3.4245	1.5882

The higher the entrance stagnation temperature the larger the heat amount that can be observed by the flow. In subsonic branch the Mach number is after the shock is

$\mathbf{M_x}$	$\mathbf{M_y}$	$\frac{\mathbf{T_y}}{\mathbf{T_x}}$	$\frac{\rho_\mathbf{y}}{\rho_\mathbf{x}}$	$\frac{\mathbf{P_y}}{\mathbf{P_x}}$	$\frac{\mathbf{P_{0y}}}{\mathbf{P_{0x}}}$
3.0000	0.47519	2.6790	3.8571	10.3333	0.32834

With Mach number of $M = 0.47519$ the maximum heat transfer requires information from Rayleigh flow as following

$\mathbf{M}$	$\frac{\mathbf{T}}{\mathbf{T}^*}$	$\frac{\mathbf{T_0}}{\mathbf{T_0}^*}$	$\frac{\mathbf{P}}{\mathbf{P}^*}$	$\frac{\mathbf{P_0}}{\mathbf{P_0}^*}$	$\frac{\rho^*}{\rho}$
0.47519	0.75086	0.65398	1.8235	1.1244	0.41176

It also must be noticed that stagnation temperature remains constant across shock wave.

$$\frac{\left.\frac{Q}{\dot{m}}\right|_{subsonic}}{\left.\frac{Q}{\dot{m}}\right|_{supersonic}} = \frac{\left(1 - \frac{{T_0}_1}{{T_0}^*}\right)_{subsonic}}{\left(1 - \frac{{T_0}_1}{{T_0}^*}\right)_{supersonic}} = \frac{1 - 0.65398}{1 - 0.65398} = 1$$

It is not surprising since the the shock wave are found on the Rayleigh flow.

To make up a example with mid range increase of temperature

CHAPTER 11

Evacuating and Filling a Semi Rigid Chambers

In some ways the next two Chapters contain materials that is new to the traditional compressible flow text books[1]. It was the undersigned experience, that in traditional classes for with compressible flow (sometimes referred to as gas dynamics) don't provide a demonstration to applicability of the class material aside to aeronautical spectrum even such as turbomachinery. In this Chapter a discussion on application of compressible flow to other fields like manufacturing is presented[2].

There is a significant importance to the "pure" models such Isothermal flow and Fanno flow which have immediate applicability. However, in many instances, the situations in life are far

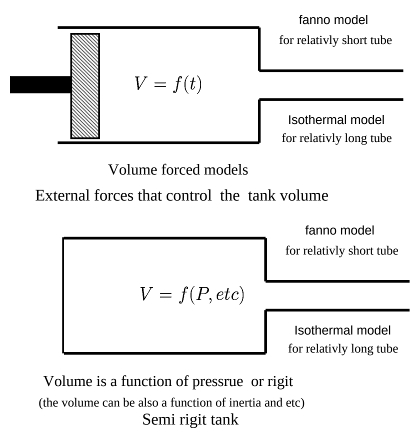


Fig. 11.1: The two different classifications of models that explain the filling or evacuating of a single chamber

[1]After completion of these Chapters, the undersigned discover two text books which to include some material related to this topic. These books are Owczarek, J. A., Fundamentals of Gas Dynamics, International Textbook Co., Scranton, Pennsylvania, 1964. and "Compressible Fluid Flow," 2nd Edition, by M. A. Saad, Prentice Hall, 1985. However, these books contained only limit discussions on the evacuation of chamber with attached nozzle.

[2]Even if the instructor feels that their students are convinced about the importance of the compressible, this example can further strength and enhance this conviction.

more complicate. Combination of gas compressibility in the chamber and flow out or through a tube post a special interest and these next two Chapters are dealing with these topics. In the first Chapter models where the chamber volume is controlled or a function of the pressure are discussed. In the second Chapter models where the chamber's volume is a function of external forces are presented (see Figure 11.1).

11.1 Governing Equations and Assumptions

The process of filing or evacuating a semi flexible (semi rigid) chamber through a tube is very common in engineering. For example, most car today equipped with an airbag. For instance, the models in this Chapter are suitable for study of the filling the airbag or filling bicycle with air. The analysis is extended to include a semi rigid tank. The term semi rigid tank referred to a tank that the volume is either completely rigid or is a function of the chamber's pressure.

As it was shown in this book the most appropriate model for the flow in the tube for a relatively fast situation is Fanno Flow. The Isothermal model is more appropriate for cases where the tube is relatively long in–which a significant heat transfer occurs keeping the temperature almost constant. As it was shown in Chapter 9 the resistance, $\frac{4fL}{D}$, should be larger than 400. Yet Isothermal flow model is used as to limiting case.

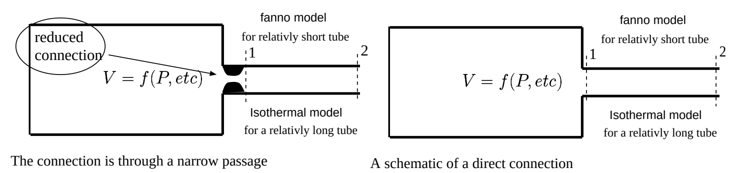


Fig. 11.2: A schematic of two possible connections of the tube to a single chamber

The Rayleigh flow model requires that a constant heat transfer supplied either by chemical reactions or otherwise. This author isn't familiar with situations in which Rayleigh flow model is applicable. And therefore, at this stage, no discussion is offered here.

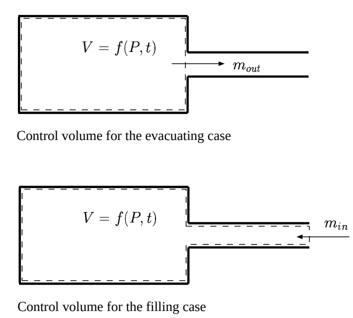


Fig. 11.3: A schematic of the control volumes used in this model

Fanno flow model is the most appropriate in the case where the filling and evacuating is relatively fast. In case the the filling is relatively slow (long $\frac{4fL}{D}$ than the Isothermal flow is appropriate model. Yet as it was stated before, here Isothermal flow

To put the dimensionless analysis to discuss what is fast enough.

and Fanno flow are used as limiting or bounding cases for the real flow. Additionally, the process in the chamber can be limited or bounded between two limits of Isentropic process or Isothermal process.

In this analysis, in order to obtain the essence of the process, some simplified assumptions are made. The assumptions can be relaxed or removed and the model will be more general. Of course, the payment is by far more complex model that sometime clutter the physics. First, a model based on Fanno flow model is constructed. Second, model is studied in which the flow in the tube is Isothermal. The flow in the tube in many cases is somewhere between the Fanno flow model to Isothermal flow model. This reality is an additional reason for the construction of two models in which they can be compared.

Effects such as chemical reactions (or condensation/evaporation) are neglected. There are two suggested itself possibilities to the connection between the tube to the tank (see the Figure 11.2): one) direct two) through a reduction. The direct connection is when the tube is connect straight to tank like in a case where pipe is welded into the tank. The reduction is typical when a ball is filled trough an one–way valve (filling a baseball ball, also in manufacturing processes). The second possibility leads itself to an additional parameter that is independent of the resistance. The first kind connection tied the resistance, $\frac{4fL}{D}$, with the tube area.

The simplest model for gas inside the chamber as a first approximation is the isotropic model. It is assumed that kinetic change in the chamber is negligible. Therefore, the pressure in the chamber is equal to the stagnation pressure, $P \approx P_0$ (see Figure 11.4). Thus, the stagnation pressure at the tube's entrance is the same as the pressure in the chamber.

The mass in the chamber and mass flow out are expressed in terms of the chamber variables (see Figure 11.3. The mass in the tank for perfect gas reads

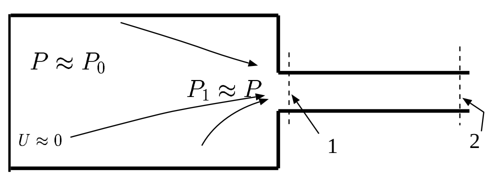


Fig. 11.4: The pressure assumptions in the chamber and tube entrance

$$\frac{dm}{dt} - \dot{m}_{out} = 0 \tag{11.1}$$

And for perfect gas the mass at any given time is

$$m = \frac{P(t)V(t)}{RT(t)} \tag{11.2}$$

The mass flow out is a function of the resistance in tube, $\frac{4fL}{D}$ and the pressure difference between the two sides of the tube $\dot{m}_{out}(\frac{4fL}{D}, P_1/P_2)$. The initial conditions in the chamber are $T(0)$, $P(0)$ and etc. If the mass occupied in the tube is neglected (only for filling process) the most general equation ideal gas (11.1) reads

$$\frac{d}{dt}\overbrace{\left(\frac{PV}{RT}\right)}^{m} \pm \rho_1 A \overbrace{\overbrace{c_1 M_1(\tfrac{4fL}{D}, \frac{P_2}{P_1})}^{U}}^{\dot{m}_{out}} = 0 \tag{11.3}$$

When the plus sign is for filling process and the negative sign is for evacuating process.

11.2 General Model and Non-dimensioned

It is convenient to non-dimensioned the properties in chamber by dividing them by their initial conditions. The dimensionless properties of chamber as

$$\bar{T} = \frac{T(t=\bar{t})}{T(t=0)}$$

$$\bar{V} = \frac{V(t=\bar{t})}{V(t=0)}$$

$$\bar{P} = \frac{P(t=\bar{t})}{P(t=0)} \tag{11.4a}$$

$$\bar{t} = \frac{t}{t_c} \tag{11.4b}$$

where t_c is the characteristic time of the system defined as followed

$$t_c = \frac{V(0)}{AM_{max}\sqrt{kRT(0)})} \tag{11.5}$$

The physical meaning of characteristic time, t_c is the time that will take to evacuate the chamber if the gas in the chamber was in its initial state, the flow rate was at its maximum (choking flow), and the gas was incompressible in the chamber.

Utilizing these definitions (11.4) and substituting into equation (11.3) yields

$$\frac{P(0)V(0)}{t_c RT(0)}\frac{d}{d\bar{t}}\left(\frac{\bar{P}\bar{V}}{\bar{T}}\right) \pm \overbrace{\frac{\bar{P}_1}{R\bar{T}_1}\frac{P(0)}{T(0)}}^{\rho} A \overbrace{\sqrt{kR\bar{T}_1 T(0)}}^{c(t)} M_{max}\bar{M}(\bar{t}) = 0 \tag{11.6}$$

where the following definition for the reduced Mach number is added as

$$\bar{M} = \frac{M_1(t)}{M_{max}} \tag{11.7}$$

After some rearranging equation (11.6) obtains the form

$$\frac{d}{d\bar{t}}\left(\frac{\bar{P}\bar{V}}{\bar{T}}\right) \pm \frac{t_c AM_{max}\sqrt{kRT(0)}}{V(0)}\frac{\bar{P}_1\bar{M}_1}{\sqrt{\bar{T}_1}}\bar{M} = 0 \tag{11.8}$$

and utilizing the definition of characteristic time, equation (11.5), and substituting into equation (11.8) yields

$$\frac{d}{d\bar{t}}\left(\frac{\bar{P}\bar{V}}{\bar{T}}\right) \pm \frac{\bar{P}_1\bar{M}}{\sqrt{\bar{T}_1}} = 0 \tag{11.9}$$

Note that equation (11.9) can be modified by introducing additional parameter which referred to as external time, t_{max}[3]. For cases, where the process time is important parameter equation (11.9) transformed to

$$\frac{d}{d\tilde{t}}\left(\frac{\bar{P}\bar{V}}{\bar{T}}\right) \pm \frac{t_{max}}{t_c}\frac{\bar{P}_1\bar{M}}{\sqrt{\bar{T}_1}} = 0 \tag{11.10}$$

when $\bar{P}, \bar{V}, \bar{T}$, and $\bar{M}$ are all are function of $\tilde{t}$ in this case. And where $\tilde{t} = t/t_{max}$.

It is more convenient to deal with the stagnation pressure then the actual pressure at the entrance to the tube. Utilizing the equations developed in Chapter 4 between the stagnation condition, denoted without subscript, and condition in a tube denoted with subscript 1. The ratio of $\frac{\bar{P}_1}{\sqrt{\bar{T}_1}}$ is substituted by

$$\frac{\bar{P}_1}{\sqrt{\bar{T}_1}} = \frac{\bar{P}}{\sqrt{\bar{T}}}\left[1 + \frac{k-1}{2}M^2\right]^{\frac{-(k+1)}{2(k-1)}} \tag{11.11}$$

It is convenient to denote

$$f[M] = \left[1 + \frac{k-1}{2}M^2\right]^{\frac{-(k+1)}{2(k-1)}} \tag{11.12}$$

Note that $f[M]$ is a function of the time. Utilizing the definitions (11.11) and substituting equation (11.12) into equation (11.9) to be transformed into

$$\frac{d}{d\bar{t}}\left(\frac{\bar{P}\bar{V}}{\bar{T}}\right) \pm \frac{\bar{P}\bar{M}(\bar{t})f[M]}{\sqrt{\bar{T}}} = 0 \tag{11.13}$$

Equation (11.13) is a first order nonlinear differential equation that can be solved for different initial conditions. At this stage, the author isn't aware that is a general solution for this equation[4]. Nevertheless, many numerical methods are available to solve this equation.

11.2.1 Isentropic process

The relationship between the pressure and the temperature in the chamber can be approximated as isotropic and therefore

$$\bar{T} = \frac{T(t)}{T(0)} = \left[\frac{P(t)}{P(0)}\right]^{\frac{k-1}{k}} = \bar{P}^{\frac{k-1}{k}} \tag{11.14}$$

[3]This notation is used in many industrial processes where time of process referred to sometime as the maximum time.

[4]To those mathematically included, fi nd the general solution for this equation.

The ratios can be expressed in term of the reduced pressure as followed:

$$\frac{\bar{P}}{\bar{T}} = \frac{\bar{P}}{\bar{P}^{\frac{k-1}{k}}} = \bar{P}^{\frac{1}{k}} \tag{11.15}$$

and

$$\frac{\bar{P}}{\sqrt{\bar{T}}} = \bar{P}^{\frac{k+1}{2k}} \tag{11.16}$$

So equation (11.13) is simplified into three different forms:

$$\begin{aligned}
\frac{d}{d\bar{t}}\left(\bar{V}\bar{P}^{\frac{1}{k}}\right) \pm \bar{P}^{\frac{k+1}{2k}}\bar{M}(\bar{t})f[M] = 0 \\
\frac{1}{k}\bar{P}^{\frac{1-k}{k}}\frac{d\bar{P}}{d\bar{t}}\bar{V} + \bar{P}^{\frac{1}{k}}\frac{d\bar{V}}{d\bar{t}} \pm \bar{P}^{\frac{k+1}{2k}}\bar{M}(\bar{t})f[M] = 0 \\
\bar{V}\frac{d\bar{P}}{d\bar{t}} + k\bar{P}\frac{d\bar{V}}{d\bar{t}} \pm k\bar{P}^{\frac{3k-1}{2k}}\bar{M}(\bar{t})f[M] = 0
\end{aligned} \tag{11.17}$$

Equation (11.17) is a general equation for evacuating or filling for isentropic process in the chamber. It should be point out that, in this stage, the model in the tube could be either Fanno flow or Isothermal flow. The situations where the chamber undergoes isentropic process but the the flow in the tube is Isothermal are limited. Nevertheless, the application of this model provide some kind of a limit where to expect when some heat transfer occurs. Note the temperature in the tube entrance can be above or below the surrounding temperature. Simplified calculations of the entrance Mach number are described in the advance topics section.

11.2.2 Isothermal Process in the Chamber

11.2.3 A Note on the entrance Mach number

The value of Mach number, M_1 is a function of the resistance, $\frac{4fL}{D}$ and the ratio of pressure in the tank to the back pressure, P_B/P_1. The exit pressure, P_2 is different from P_B in some situations. As it was shown before, once the flow became choked the Mach number, M_1 is only a function of the resistance, $\frac{4fL}{D}$. These statements are correct for both Fanno flow and the Isothermal flow models. The method outlined in Chapters 8 and 9 is appropriate for solving for entrance Mach number, M_1.

Two equations must be solved for the Mach numbers at the duct entrance and exit when the flow is in a chockless condition. These equations are combinations of the momentum and energy equations in terms of the Mach numbers. The characteristic equations for Fanno flow (9.50), are

$$\frac{4fL}{D} = \left[\frac{4fL_{max}}{D}\right]_1 - \left[\frac{4fL_{max}}{D}\right]_2 \tag{11.18}$$

and

$$\frac{P_2}{P_0(t)} = \left[1 + \frac{k-1}{2}{M_2}^2\right]^{\frac{k}{1-k}} \frac{M_1}{M_2}\sqrt{\left[\frac{1+\frac{k-1}{2}{M_2}^2}{1+\frac{k-1}{2}{M_1}^2}\right]^{\frac{k+1}{k-1}}} \tag{11.19}$$

where $\frac{4fL}{D}$ is defined by equation (9.49).

The solution of equations (11.18) and (11.19) for given $\frac{4fL}{D}$ and $\frac{P_{exit}}{P_0(t)}$ yields the entrance and exit Mach numbers. See advance topic about approximate solution for large resistance, $\frac{4fL}{D}$ or small entrance Mach number, M_1.

11.3 Rigid Tank with Nozzle

The most simplest possible combination is discussed here before going trough the more complex cases A chamber is filled or evacuated by a nozzle. The gas in the chamber assumed to go an isentropic processes and flow is bounded in nozzle between isentropic flow and isothermal flow[5]. Here, it also will be assumed that the flow in the nozzle is either adiabatic or isothermal.

11.3.1 Adiabatic Isentropic Nozzle Attached

The mass flow out is given by either by Fliegner's equation (4.47) or simply use $cM\rho A^*$ and equation (11.17) becomes

$$\frac{1}{k}\bar{P}^{\frac{1-k}{k}}\frac{d\bar{P}}{d\bar{t}} \pm \bar{P}^{\frac{k+1}{2k}}(\bar{t})f[M] = 0 \tag{11.20}$$

It was utilized that $\bar{V} = 1$ and $\bar{M}$ definition is simplified as $\bar{M} = 1$. It can be noticed that the characteristic time defined in equation (11.5) reduced into:

$$t_c = \frac{V(0)}{A\sqrt{kRT(0)})} \tag{11.21}$$

Also it can be noticed that equation (11.12) simplified into

$$f[M] = \left[1 + \frac{k-1}{2}1^2\right]^{\frac{-(k+1)}{2(k-1)}} = \left(\frac{k+1}{2}\right)^{\frac{-(k+1)}{2(k-1)}} \tag{11.22}$$

Equation (11.20) can be simplified as

$$\frac{1}{k}\left(P^{\frac{1-k}{2k}}\right)dP \pm f[m]d\bar{t} = 0 \tag{11.23}$$

[5]This work is suggested by Donald Katze the point out that this issue appeared in Shapiro's Book Vol 1, Chapter 4, p. 111 as a question 4.31.

Equation (11.23) can be integrated as

$$\int_{1}^{\bar{P}} P^{\frac{1-k}{2k}} dP \pm \int_{0}^{t} dt = 0 \tag{11.24}$$

The integration limits are obtained by simply using the definitions of reduced pressure, at $P(\bar{t}=0)=1$ and $P(\bar{t}=\bar{t})=\bar{P}$. After the integration, equation (11.24) and rearrangement becomes

$$\bar{P} = \left[1 \pm \left(\frac{k-1}{2}\right) f[M]\bar{t}\right]^{\frac{2k}{1-k}} \tag{11.25}$$

Example 11.1:
A chamber is connected to a main line with pressure line with a diaphragm and nozzle. The initial pressure at the chamber is $1.5[Bar]$ *and the volume is* $1.0[m^3]$*. Calculate time it requires that the pressure to reach 5[Bar] for two different nozzles throat area of 0.001, and 0.1* $[m^2]$ *when diaphragm is erupted. Assumed the stagnation temperature at the main line is the ambient of* $27[°\mathrm{C}]$*.*

<u>SOLUTION</u>
The characteristic time is

$$t_{max} = \frac{V}{A^{*}c} = \frac{V}{A^{*}c} = \frac{1.0}{0.1\sqrt{1.4\times 287\times 300}} = 0.028[sec] \tag{11.26}$$

And for smaller area

$$t_{max} = \frac{1.0}{0.001\sqrt{1.4\times 287\times 300}} = 2.8[sec]$$

$$\bar{P} = \frac{P(t)}{P(0)} = \frac{4.5}{1.5} = 3.0$$

The time is

$$t = t_{max}\left[\bar{P}^{\frac{1-k}{k}} - 1\right]\left(\frac{k+1}{2}\right)^{-()} \tag{11.27}$$

Substituting values into equation (11.27) results

$$t = 0.028\left[3^{\frac{1-1.4}{2.8}} - 1\right]\left(\frac{2.4}{2}\right)^{\frac{-2.4}{0.8}} = 0.013[sec] \tag{11.28}$$

11.3.1.1 Filling/evacuating the chamber under upchucked condition

The flow in the nozzle can became upchucked and it can be analytically solved. Owczarek [1964] found that analytical solution which described here.

11.3.2 Isothermal Nozzle Attached

In this case the process in nozzle is assumed to isothermal but the process in the chamber is isentropic. The temperature in the nozzle is changing because the temperature in the chamber is changing. Yet, the differential temperature change in the chamber is slower than the temperature change in nozzle. For rigid volume, $\bar{V} = 1$ and for isothermal nozzle $\bar{T} = 1$ Thus, equation (11.13) is reduced into

$$\frac{d\bar{P}}{d\bar{t}} = \pm f[M]\bar{P} = 0 \tag{11.29}$$

Separating the variables and rearranging equation (11.30) converted into

$$\int_1^{\bar{P}} \frac{d\bar{P}}{\bar{P}} \pm f[M] \int_0^{\bar{t}} d\bar{t} = 0 \tag{11.30}$$

Here, $f[M]$ is expressed by equation (11.22). After the integration, equation (11.30) transformed into

$$\begin{aligned} \ln \bar{P} &= \left(\frac{k+1}{2}\right)^{\frac{-(k+1)}{2(k-1)}} \bar{t} \\ \bar{P} &= e^{\left[\left(\frac{k+1}{2}\right)^{\frac{-(k+1)}{2(k-1)}} \bar{t}\right]} \end{aligned} \tag{11.31}$$

11.4 Rapid evacuating of a rigid tank

11.4.1 With Fanno Flow

The relative Volume, $\bar{V}(t) = 1$, is constant and equal one for a completely rigid tank. In such case, the general equation (11.17) “shrinks” and doesn't contain the relative volume term.

A reasonable model for the tank is isentropic (can be replaced polytropic relationship) and Fanno flow are assumed for the flow in the tube. Thus, the specific governing equation is

$$\frac{d\bar{P}}{d\bar{t}} - k\bar{M} f[M]\bar{P}^{\frac{3k-1}{2k}} = 0 \tag{11.32}$$

For a choked flow the entrance Mach number to the tube is at its maximum, M_{max} and therefore $\bar{M} = 1$. The solution of equation (11.37) is obtained by noticing that $\bar{M}$ is not a function of time and by variables separation results in

$$\int_0^{\bar{t}} d\bar{t} = \int_1^{\bar{P}} \frac{d\bar{P}}{k\bar{M} f[M]\bar{P}^{\frac{3k-1}{2k}}} = \frac{1}{k\bar{M} f[M]} \int_1^{\bar{P}} \bar{P}^{\frac{1-3k}{2k}} d\bar{P} \tag{11.33}$$

direct integration of equation (11.40) results in

$$\bar{t} = \frac{2}{(k-1)\bar{M}f[M]}\left[\bar{P}^{\frac{1-k}{2k}} - 1\right] \tag{11.34}$$

It has to be realized that this is “reversed” function i.e. $\bar{t}$ is a function of P and can be reversed for case. But for the chocked case it appears as

$$\bar{P} = \left[1 + \frac{(k-1)\bar{M}f[M]}{2}\bar{t}\right]^{\frac{2k}{1-k}} \tag{11.35}$$

The function is drawn as shown here in figure 11.6. The figure 11.6 shows

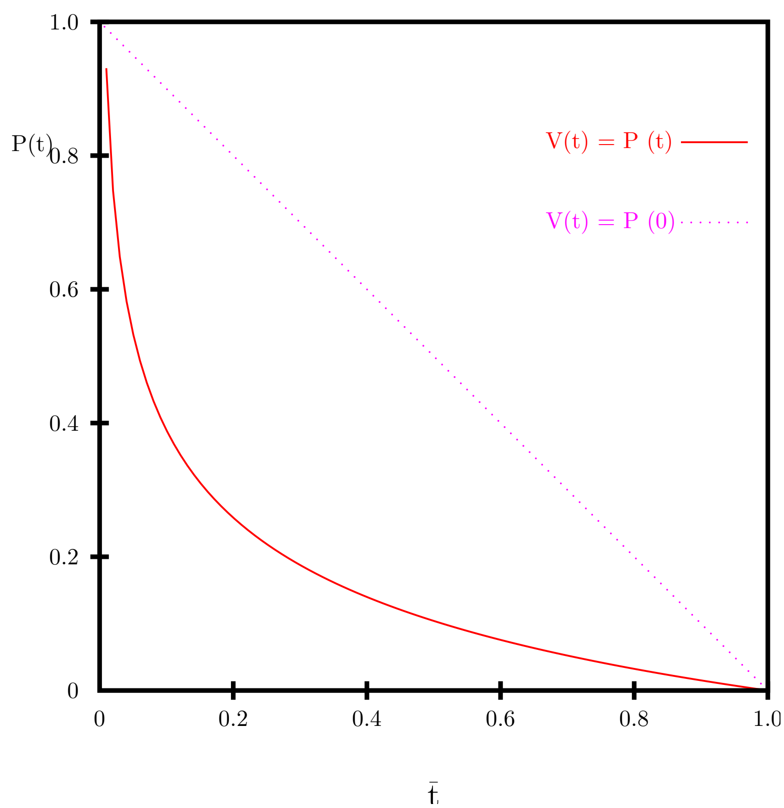


Fig. 11.5: The reduce time as a function of the modifi ed reduced pressure

that when the modified reduced pressure equal to one the reduced time is zero. The reduced time increases with decrease of the pressure in the tank.

The big struggle look for suggestion for better notation.

At certain point the flow becomes chockless flow (unless the back pressure is complete vacuum). The transition point is denoted here as chT. Thus, equation

(11.40) has to include the entrance Mach under the integration sign as

$$\bar{t} - \bar{t}_{chT} = \int_{P_{chT}}^{\bar{P}} \frac{1}{k\bar{M} f[M]} \bar{P}^{\frac{1-3k}{2k}} d\bar{P} \tag{11.36}$$

For practical purposes if the flow is choked for more than 30% of the charecteristic time the choking equation can be used for the whole range, unless extra long time or extra low pressure is calculated/needed. Further, when the flow became chockless the entrance Mach number does not change much from the choking condition.

Again, for the special cases where the choked equation is not applicable the integration has to be separated into zones: choked and chockless flow regions. And in the choke region the calculations can use the choking formula and numerical calculations for the rest.

Example 11.2:
A chamber with volume of 0.1[m^3] is filled with air at pressure of 10[Bar]. The chamber is connected with a rubber tube with $f = 0.025$, $d = 0.01[m]$ and length of $L = 5.0[m]$

SOLUTION
The first parameter that calculated is $\frac{4fL}{D}$ $\frac{4fL}{D} = 5$

11.4.2 Filling process

The governing equation is

$$\frac{d\bar{P}}{d\bar{t}} - k\bar{M} f[M] \bar{P}^{\frac{3k-1}{2k}} = 0 \tag{11.37}$$

For a choked flow the entrance Mach number to the tube is at its maximum, M_{max} and therefore $\bar{M} = 1$. The solution of equation (11.37) is obtained by noticing that $\bar{M}$ is not a function of time and by variable separation results in

$$\int_0^{\bar{t}} d\bar{t} = \int_1^{\bar{P}} \frac{d\bar{P}}{k\bar{M} f[M] \bar{P}^{\frac{3k-1}{2k}}} = \frac{1}{k\bar{M} f[M]} \int_1^{\bar{P}} \bar{P}^{\frac{1-3k}{2k}} d\bar{P} \tag{11.38}$$

direct integration of equation (11.40) results in

$$\bar{t} = \frac{2}{(k-1)\bar{M} f[M]} \left[\bar{P}^{\frac{1-k}{2k}} - 1 \right] \tag{11.39}$$

It has to be realized that this is a reversed function. Nevertheless, with today computer this should not be a problem and easily can be drawn as shown here in Figure 11.6. The Figure shows that when the modified reduced pressure

Fig. 11.6: The reduce time as a function of the modifi ed reduced pressure

equal to one the reduced time is zero. The reduced time increases with decrease of the pressure in the tank.

At some point the flow became chockless flow (unless the back pressure is a complete vacuum). The transition point is denoted here as chT. Thus, equation (11.40) has to include the entrance Mach under the integration sign as

$$\bar{t} - \bar{t}_{chT} = \int_{P_{chT}}^{\bar{P}} \frac{1}{k\bar{M}f[M]} \bar{P}^{\frac{1-3k}{2k}} d\bar{P} \tag{11.40}$$

11.4.3 The Isothermal Process

For Isothermal process, the relative temperature, $\bar{T} = 1$. The combination of the isentropic tank and Isothermal flow in the tube is different from Fanno flow in that the chocking condition occurs at $1/\sqrt{k}$. This model is reasonably appropriated when the chamber is insulated and not flat while the tube is relatively long and the process is relatively long.

It has to be remembered that the chamber can undergo isothermal process. For the double isothermal (chamber and tube) the equation (11.6) reduced into

$$\frac{P(0)V(0)}{t_c RT(0)} \frac{d(\bar{P}\bar{V})}{d\bar{t}} \pm \overbrace{\frac{\bar{P}_1}{R}\frac{P(0)}{T(0)}}^{\rho} A \overbrace{\sqrt{kRT(0)}}^{c(0)} M_{max} \bar{M}(\bar{t}) = 0 \tag{11.41}$$

11.4.4 Simple Semi Rigid Chamber

A simple relation of semi rigid chamber when the volume of the chamber is linearly related to the pressure as

$$V(t) = aP(t) \tag{11.42}$$

where a is a constant that represent the physics. This situation occurs at least in small ranges for airbag balloon etc. The physical explanation when it occurs beyond the scope of this book. Nevertheless, a general solution is easily can be obtained similarly to rigid tank. Substituting equation (11.42) into yields

$$\frac{d}{d\bar{t}}\left(\bar{P}^{\frac{1+k}{k}}\right) - \bar{P}^{\frac{k+1}{2k}} \bar{M} f[M] = 0 \tag{11.43}$$

Carrying differentiation result in

$$\frac{1+k}{k} \bar{P}^{\frac{1}{k}} \frac{d\bar{P}}{d\bar{t}} - \bar{P}^{\frac{k+1}{2k}} \bar{M} f[M] = 0 \tag{11.44}$$

Similarly as before, the variables are separated as

$$\int_0^{\bar{t}} dt = \frac{k}{1+k} \int_1^{\bar{P}} \frac{\bar{P}^{\frac{k-1}{2k}} d\bar{P}}{\bar{M} f[M]} \tag{11.45}$$

The equation (11.50) integration obtains the form

$$\bar{t} = \frac{2k^2}{\bar{M} f[M] (3k-1)(1+k)} \left[1 - \bar{P}^{\frac{3k-1}{2k}}\right] \tag{11.46}$$

The physical meaning that the pressure remains larger thorough evacuating process, as results in faster reduction of the gas from the chamber.

11.4.5 The “Simple” General Case

The relationship between the pressure and the volume from the physical point of view must be monotonous. Further, the relation must be also positive, increase of the pressure results in increase of the volume (as results of Hook’s law. After all, in the known situations to this author pressure increase results in volume decrease (at least for ideal gas.).

In this analysis and previous analysis the initial effect of the chamber container inertia is neglected. The analysis is based only on the mass conservation and if unsteady effects are required more terms (physical quantities) have taken into account. Further, it is assumed the ideal gas applied to the gas and this assumption isn’t relaxed here.

Any continuous positive monotonic function can be expressed into a polynomial function. However, as first approximation and simplified approach can be done by a single term with a different power as

$$V(t) = aP^n \tag{11.47}$$

When n can be any positive value including zero, 0. The physical meaning of $n = 0$ is that the tank is rigid. In reality the value of n lays between zero to one. When n is approaching to zero the chamber is approaches to a rigid tank and vis versa when the $n \to 1$ the chamber is flexible like a balloon.

There isn’t a real critical value to n. Yet, it is convenient for engineers to further study the point where the relationship between the reduced time and the reduced pressure are linear[6] Value of n above it will Convex and and below it concave.

[6]Some suggested this border point as infinite evocation to infinite time for evacuation etc. This undersigned is not aware situation where this indeed play important role. Therefore, is waiting to find such conditions before calling it as critical condition.

$$\frac{d}{d\bar{t}}\left(\bar{P}^{\frac{1+nk-k}{k}}\right) - \bar{P}^{\frac{k+1}{2k}}\bar{M}f[M] = 0 \tag{11.48}$$

Notice that when $n = 1$ equation (11.49) reduced to equation (11.43).
After carrying–out differentiation results

$$\frac{1+nk-k}{k}\bar{P}^{\frac{1+nk-2k}{k}}\frac{d\bar{P}}{d\bar{t}} - \bar{P}^{\frac{k+1}{2k}}\bar{M}f[M] = 0 \tag{11.49}$$

Again, similarly as before, variables are separated and integrated as follows

$$\int_0^{\bar{t}} dt = \frac{1+nk-k}{k}\int_1^{\bar{P}} \frac{\bar{P}^{\frac{1+2nk-5k}{2k}}d\bar{P}}{\bar{M}f[M]} \tag{11.50}$$

Carrying–out the integration for the initial part if exit results in

$$\bar{t} = \frac{2k^2}{\bar{M}f[M](3k-2nk-1)(1+k)}\left[1-\bar{P}^{\frac{3k-2nk-1}{2k}}\right] \tag{11.51}$$

The linear condition are obtain when

$$3k-2nk-1 = 1 \longrightarrow n = \frac{3k-2}{2k} \tag{11.52}$$

That is just bellow 1 ($n = 0.785714286$) for $k = 1.4$.

11.5 Advance Topics

The term $\frac{4fL}{D}$ is very large for small values of the entrance Mach number which requires keeping many digits in the calculation. For small values of the Mach numbers, equation (**??**) can be approximated as

$$\frac{4fL}{D} = \frac{1}{k}\frac{{M_{exit}}^2 - {M_{in}}^2}{{M_{exit}}^2{M_{in}}^2} \tag{11.53}$$

and equation (**??**) as

$$\frac{P_{exit}}{P_0(t)} = \frac{M_{in}}{M_{exit}}. \tag{11.54}$$

The solution of the last two equations yields

$$M_{in} = \sqrt{\frac{1-\left[\frac{P_{exit}}{P_0(t)}\right]^2}{k\frac{4fL}{D}}}. \tag{11.55}$$

This solution should used only for $M_{in} < 0.00286$; otherwise equations (**??**) and (**??**) must be solved numerically.

The solution of equation (**??**) and (**??**) is described in “Pressure die casting: a model of vacuum pumping” Bar-Meir, G; Eckert, E R G; Goldstein, R J Journal of Manufacturing Science and Engineering (USA). Vol. 118, no. 2, pp. 259-265. May 1996.

CHAPTER 12

Evacuating/Filing Chambers under External Volume Control

This chapter is the second on the section dealing with filling and evacuating chambers. Here the model deals with the case where the volume is controlled by external forces. This kind of model is applicable to many manufacturing processes such as die casting, extraction etc. In general the process of the displacing the gas (in many cases air) with a liquid is a very common process. For example, in die casting process liquid metal is injected to a cavity and after the cooling/solidification period a part is obtained in near the final shape. One can also view the exhaust systems of internal combustion engine in the same manner. In these processes, sometime is vital to obtain a proper evacuation of the gas (air) from the cavity.

12.1 Model

In this analysis, in order to obtain the essence of the process, some simplified assumptions are made. It simplest model of such process is when a piston is displacing the gas though a long tube. It assumed that no chemical reaction (or condensation/evaporation) occur in the piston or the tube [1]. It is further assumed that the process is relatively fast. The last assumption is a appropriate assumption in process such as die casting.

Two extreme possibilities again suggest themselves: rapid and slow processes. The two different connections, direct and through reduced area are combined in this analysis.

again to add the dimensional analysis what is rapid and what is slow.

[1]such reaction are possible and expected to be part of process but the complicates the analysis and do not contribute to understand to the compressibility effects.

12.1.1 Rapid Process

Clearly under the assumption of rapid process the heat transfer can be neglected and Fanno flow can be assumed for the tube. The first approximation isotropic process describe the process inside the cylinder (see figure 12.1.

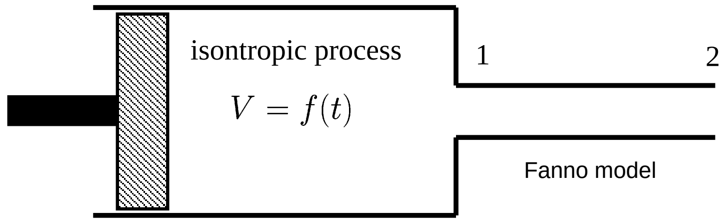


Fig. 12.1: The control volume of the “Cylinder”

Before introducing the steps of the analysis, it is noteworthy to think about the process in qualitative terms. The replacing incompressible liquid enter in the same amount as replaced incompressible liquid. But in a compressible substance the situation can be totally different, it is possible to obtain a situation where that most of the liquid entered the chamber and yet most of the replaced gas can be still be in the chamber. Obtaining conditions where the volume of displacing liquid is equal to the displaced liquid are called the critical conditions. These critical conditions are very significant that they provide guidelines for the design of processes.

Obviously, the best ventilation is achieved with a large tube or area. In manufacture processes to minimize cost and the secondary machining such as trimming and other issues the exit area or tube has to be narrow as possible. In the exhaust system cost of large exhaust valve increase with the size and in addition reduces the strength with the size of valve[2]. For these reasons the optimum size is desired. The conflicting requirements suggest an optimum area, which is also indicated by experimental studies and utilized by practiced engineers.

The purpose of this analysis to yields a formula for critical/optimum vent area in a simple form is one of the objectives of this section. The second objective is to provide a tool to “combine” the actual tube with the resistance in the tube, thus, eliminating the need for calculations of the gas flow in the tube to minimize the numerical calculations.

A linear function is the simplest model that decibels changes the volume. In reality, in some situations like die casting this description is appropriate. Nevertheless, this model can be extended numerical in cases where more complex function is applied.

$$V(t) = V(0)\left[1 - \frac{t}{t_{max}}\right] \tag{12.1}$$

[2]After certain sizes, the possibility of crack increases.

Equation (12.2) can be non–dimensionlassed as

$$\bar{V}(\bar{t}) = 1 - \bar{t} \tag{12.2}$$

The governing equation (11.10) that was developed in the previous chapter (11) obtained the form as

$$\left[\bar{P}\right]^{\frac{1}{k}} \left\{ \frac{1}{k}\frac{\bar{V}}{\bar{P}}\frac{d\bar{P}}{dt} + \frac{d\bar{V}}{d\bar{t}} \right\} + \frac{t_{max}\bar{M}f(M)}{t_c}\left[\bar{P}\right]^{\frac{k+1}{2k}} = 0 \tag{12.3}$$

where $\bar{t} = t/t_{max}$. Notice that in this case that there are two different characteristic times: the "characteristic" time, t_c and the "maximum" time, t_{max}. The first characteristic time, t_c is associated with the ratio of the volume and the tube characteristics (see equation (11.5)). The second characteristic time, t_{max} is associated with the imposed time on the system (in this case the elapsed time of the piston stroke).

Equation (12.3) is an nonlinear first order differential equation and can be rearranged as follows

$$\frac{d\bar{P}}{k\left(1 - \frac{t_{max}}{t_c}\bar{M}f[M]\bar{P}^{\frac{k-1}{2k}}\right)\bar{P}} = \frac{d\bar{t}}{1-\bar{t}} \quad ; \quad \bar{P}(0) = 1. \tag{12.4}$$

Equation (12.4) is can be solved only when the flow is chocked In which case $f[m]$ isn't function of the time.

The solution of equation (12.4) can be obtained by transforming and by introducing a new variable $\xi = \bar{P}^{\frac{k-1}{2k}}$ and therefore $\bar{P} = [\xi]^{\frac{2k}{k-1}}$. The reduced Pressure derivative, $d\bar{P} = \frac{2k}{k-1}[\xi]^{\left(\frac{2k}{k-1}\right)-1} d\xi$ Utilizing this definition and there implication reduce equation (12.4)

$$\frac{2\,[\xi]^{\left(\frac{2k}{k-1}\right)-1} d\xi}{(k-1)\,(1-B\xi)\,[\xi]^{\frac{2k}{k-1}}} = \frac{d\bar{t}}{1-\bar{t}} \tag{12.5}$$

where $B = \frac{t_{max}}{t_c}\bar{M}f[M]$ And equation (12.5) can be further simplified as

$$\frac{2d\xi}{(k-1)\,(1-B\xi)\,\xi} = \frac{d\bar{t}}{1-\bar{t}} \tag{12.6}$$

Equation (12.6) can be integrated to obtain

$$\frac{2}{(k-1)B}\ln\left|\frac{1-B\xi}{\xi}\right| = -\ln\bar{t} \tag{12.7}$$

or in a different form

$$\left|\frac{1-B\xi}{\xi}\right|^{\frac{2}{(1-k)B}} = \bar{t} \tag{12.8}$$

Now substituting to the "preferred" variable

$$\left[\frac{1-\frac{t_{max}}{t_c}\bar{M}f[M]\bar{P}^{\frac{k-1}{2k}}}{\bar{P}^{\frac{k-1}{2k}}}\right]^{\frac{2}{(1-k)\frac{t_{max}}{t_c}\bar{M}f[M]}}\Bigg|_{\bar{P}}^{1} = \bar{t} \tag{12.9}$$

The analytical solution is applicable only in the case which the flow is choked thorough all the process. The solution is applicable to indirect connection. This happen when vacuum is applied outside the tube (a technique used in die casting and injection molding to improve quality by reducing porosity.). In case when the flow chockless a numerical integration needed to be performed. In the literature, to create a direct function equation (12.4) is transformed into

$$\frac{d\bar{P}}{d\bar{t}} = \frac{k\left(1-\frac{t_{max}}{t_c}\bar{M}f[M]\bar{P}^{\frac{k-1}{2k}}\right)}{1-\bar{t}} \tag{12.10}$$

with the initial condition of

$$P(0) = 1 \tag{12.11}$$

The analytical solution also can be approximated by a simpler equation as

$$\bar{P} = [1-t]^{\frac{t_{max}}{t_c}} \tag{12.12}$$

The results for numerical evaluation in the case when cylinder is initially at an atmospheric pressure and outside tube is also at atmospheric pressure are presented in figure 12.2. In this case only some part of the flow is choked (the later part). The results of a choked case are presented in figure 12.3 in which outside tube condition is in vacuum. These figures 12.2 and 12.3 demonstrate the importance of the ratio of $\frac{t_{max}}{t_c}$. When $\frac{t_{max}}{t_c} > 1$ the pressure increases significantly and verse versa.

Thus, the question remains how the time ratio can be transfered to parameters that can the engineer can design in the system.

Denoting the area that creates the ratio $\frac{t_{max}}{t_c} = 1$ as the critical area, A_c provides the needed tool. Thus the exit area, A can be expressed as

$$A = \frac{A}{A_c}A_c \tag{12.13}$$

The actual times ratio $\left.\frac{t_{max}}{t_c}\right|_{@A}$ can be expressed as

$$\left.\frac{t_{max}}{t_c}\right|_{@A} = \left.\frac{t_{max}}{t_c}\right|_{@A} \overbrace{\left.\frac{t_{max}}{t_c}\right|_{@A_c}}^{1} \tag{12.14}$$

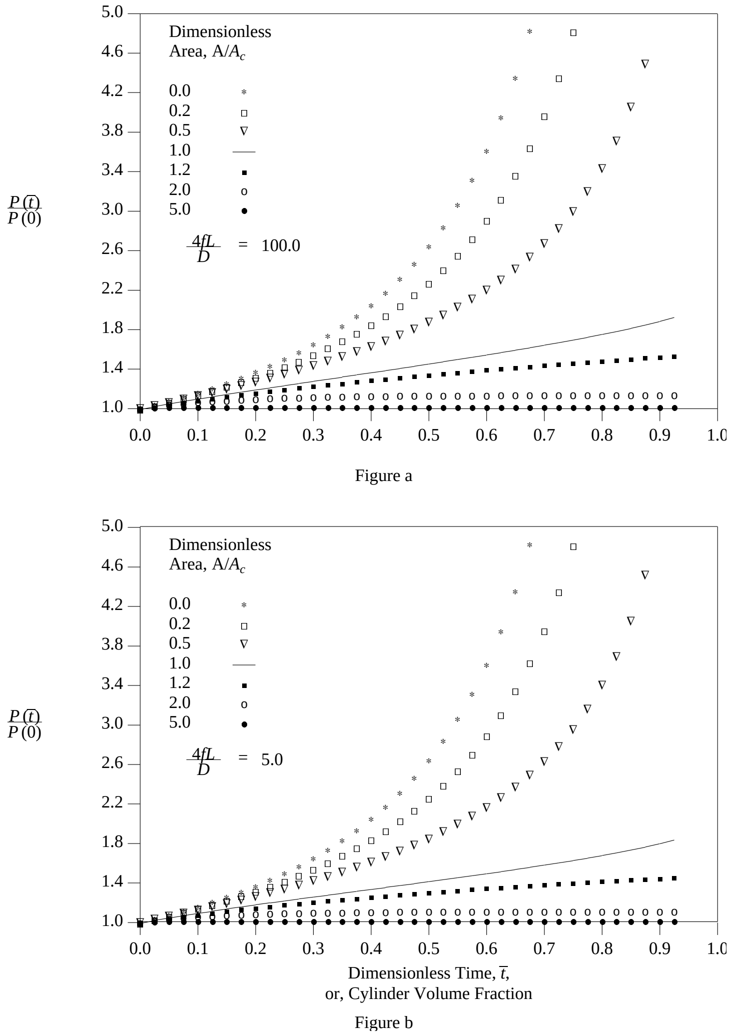


Fig. 12.2: The pressure ratio as a function of the dimensionless time for chockless condition

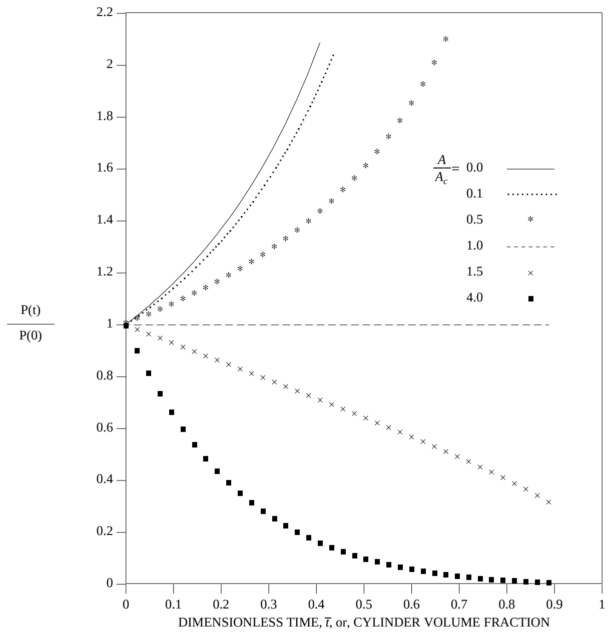


Fig. 12.3: The pressure ratio as a function of the dimensionless time for choked condition

According to equation (11.5) t_c is inversely proportional to to area, $t_c \propto 1/A$. Thus, equation (12.14) the t_{max} is canceled and reduced into

$$\left.\frac{t_{max}}{t_c}\right|_{@A} = \frac{A}{A_c} \tag{12.15}$$

Parameters influencing the process are the area ratio, $\frac{A}{A_c}$, and the friction parameter, $\frac{4fL}{D}$. From other detailed calculations [**?**] it was found that the influence of the parameter $\frac{4fL}{D}$ on the pressure development in the cylinder is quite small. The influence is small on the residual air mass in the cylinder but larger on the Mach number, M_{exit}. The effects of the area ratio, $\frac{A}{A_c}$, are studied here since it is the dominant parameter.

It is important to point out the significance of the $\frac{t_{max}}{t_c}$. This parameter represents the ratio between the filling time and the evacuating time, the time which would be required to evacuate the cylinder for constant mass flow rate at the maximum Mach number when the gas temperature and pressure remain in their initial values. This parameter also represents the dimensionless area, $\frac{A}{A_c}$, according to the following equation

Figure 12.4 describes the pressure as a function of the dimensionless time for various values of $\frac{A}{A_c}$. The line that represents $\frac{A}{A_c} = 1$ is almost straight. For

Fig. 12.4: The pressure ratio as a function of the dimensionless time

large values of $\frac{A}{A_c}$ the pressure increases the volume flow rate of the air until a quasi steady state is reached. This quasi steady state is achieved when the volumetric air flow rate out is equal to the volume pushed by the piston. The pressure and the mass flow rate are maintained constant after this state is reached. The pressure in this quasi steady state is a function of $\frac{A}{A_c}$. For small values of $\frac{A}{A_c}$ there is no steady state stage. When $\frac{A}{A_c}$ is greater than one the pressure is concave upward and when $\frac{A}{A_c}$ is less than one the pressure is concave downward as shown in Figures 12.4, which was obtained by an integration of equation (12.9).

12.1.2 Examples

Example 12.1:
Calculate the minimum required vent area for die casting process when the die volume is $0.001[m^3]$ and $\frac{4fL}{D} = 20$. The required solidification time, $t_{max} = 0.03[sec]$.

SOLUTION

12.1.3 Direct Connection

In the above analysis is applicable to indirect connection. It should be noted that critical area, A_c, is not function of the time. The direct connection posts more mathematical difficulty because the critical area is not constant and time dependent.

To continue

12.2 Summary

The analysis indicates there is a critical vent area below which the ventilation is poor and above which the resistance to air flow is minimal. This critical area depends on the geometry and the filling time. The critical area also provides a mean to "combine" the actual vent area with the vent resistance for numerical simulations of the cavity filling, taking into account the compressibility of the gas flow.

CHAPTER 13

Oblique-Shock

This author would like to acknowledge the contribution and help of Ralph Menikoff who was the first to provide real peer review on this chapter. Of course, all the mistakes and omissions that are still in this chapter are the sole responsibility of this Author. The new addition Maximum Deflection Mach number's equation was developed by Menikoff but if you find any mistakes than you still can blame this author.

13.1 Preface to Oblique Shock

In Chapter 5 a discussion on a normal shock was presented. The normal shock is a special case of shock and other situations exist for example the oblique shock. Commonly in literature the oblique shock, normal shock and Prandtl–Meyer function are presented as three and separate and different issues. However, one can view all these cases as three different regions of flow over plate with deflection section. Clearly, variation of the deflection angle from a zero ($\delta = 0$) to positive values results in the oblique shock. Further, changing of the deflection angle to a negative value results in expansion waves. The common presentation is done by avoiding to show the boundaries of these models. Here, it is attempted to show the boundaries and the limits or connections of

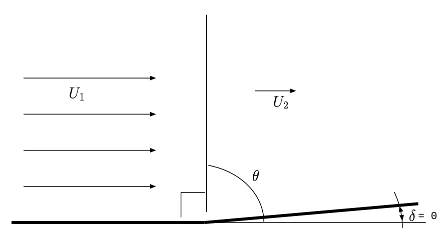


Fig. 13.1: A view of a straight normal shock as limited case for the oblique shock

these models[1].

13.2 Introduction

13.2.1 Introduction to Oblique Shock

The normal shock occurs when there is a disturbance downstream which imposed on the flow and in which the fluid/gas can react only by a sharp change to the flow. As it might be recalled, the normal shock occurs when a wall is straight/flat ($\delta = 0$) as shown in Figure 13.1 which occurs when somewhere downstream a disturbance[2] appears. When the deflection angle is increased the shock must match the boundary conditions. This matching can occur only when there is a discontinuity in the flow field. Thus, the direction of the flow is changed by a shock wave with an angle. This shock communally is referred to as the oblique shock. Alternatively, as discussed in Chapter 1[3] the flow behaves as it in hyperbolic field. In such case, flow fluid is governed by hyperbolic equation which is where the information (like boundary conditions) reaches from downstream only if they are within the range of influence. For information such as the disturbance (boundary condition) reaches deep into flow from the side requires time. During this time, the flow moves ahead downstream which creates an angle.

13.2.2 Introduction to Prandtl–Meyer Function

Decreasing the defection angle results in the same results as before: the boundary conditions must match the geometry. Yet, for a negative (in this section notation) defection angle, the flow must be continuous. The analysis shows that velocity of the flow must increased to achieve this requirement. This velocity increase is referred to as the expansion waves. As it will be shown in the next Chapter, as oppose to oblique shock analysis,

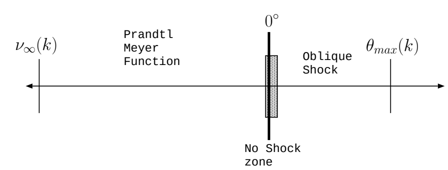


Fig. 13.2: The regions where the oblique shock or Prandtl–Meyer function exist. Notice that both a maximum point and “no solution” zone around zero. However, Prandtls-Meyer Function approaches to closer to zero.

[1] In this chapter, even the whole book, a very limited discussion about reflection shocks and collisions of weak shock, Von Neumann paradox, triple shock intersection, etc is presented. This author believes that these issues are not relevant to most engineering students and practices. Furthermore, these issues should not be introduced in introductory textbook of compressible flow. Those who would like to obtain more information, should refer to J.B. Keller, “Rays, waves and asymptotics,” Bull. Am. Math. Soc. 84, 727 (1978), and E.G. Tabak and R.R. Rosales, “Focusing of weak shock waves and the Von Neuman paradox of oblique shock reflection,” Phys. Fluids 6, 1874 (1994).

[2] Zero velocity, pressure boundary condition are example of forcing shock. The zero velocity can be found in a jet flowing into a still medium of gas.

[3] This section is under construction and does not appear in the book yet.

the upstream Mach number increases
and determined the downstream Mach number and the “negative” deflection angle.

One has to point that both oblique shock and Prandtl–Meyer Function have maximum point for $M_1 \rightarrow \infty$. However, the maximum point for Prandtl–Meyer Function is match larger than the Oblique shock by a factor of more than two. The reason for the larger maximum point is because of the effective turning (less entropy) which will be explained in the next chapter (see Figure (13.2)).

13.2.3 Introduction to zero inclination

What happened when the inclination angle is zero? Which model is correct to use? Can these two conflicting models co-exist? Or perhaps a different model better describes the physics. In some books and in the famous NACA report 1135 it was assumed that Mach wave and oblique shock co–occur in the same zone. Previously (see Chapter 5), it also assumed that normal shock occurs in the same time. In this chapter, the stability issue will be examined in some details.

13.3 Oblique Shock

The shock occurs in reality in situations where the shock has three–dimensional effects. The three–dimensional effects of the shock make it appears as a curved plan. However, for a chosen arbitrary accuracy requires a specific small area, a one dimensional shock can be considered. In such a case the change of the orientation makes the shock considerations a two dimensional. Alternately, using an infinite (or two dimensional) object produces a two dimensional shock. The two dimensional effects occur when the flow is affected from the “side” i.e. change in the flow direction[4].

To match the boundary conditions, the flow turns after the shock to be parallel to the inclination angle. In Figure 13.3 exhibits the schematic of the oblique shock. The deflection angle, δ, is the direction of the flow after the shock (parallel to the wall). The normal shock analysis dictates that after the shock, the flow is always subsonic. The total flow after oblique shock can be also supersonic which depends boundary layer.

Only the oblique shock’s normal component undergoes the “shock.” The tangent component doesn’t change because it doesn’t “moves” across the shock line. Hence, the mass balance reads

$$\rho_1 {U_1}_n = \rho_2 {U_2}_n \tag{13.1}$$

The momentum equation reads

$$P_1 + \rho_1 {U_1}_n{}^2 = P_2 + \rho_2 {U_2}_n{}^2 \tag{13.2}$$

[4]This author beg for forgiveness from those who view this description offensive (There was unpleasant eMail to this author accusing him to revolt against the holy of the holy.). If you do not like this description, please just ignore it. You can use the tradition explanation, you do not need this author permission.

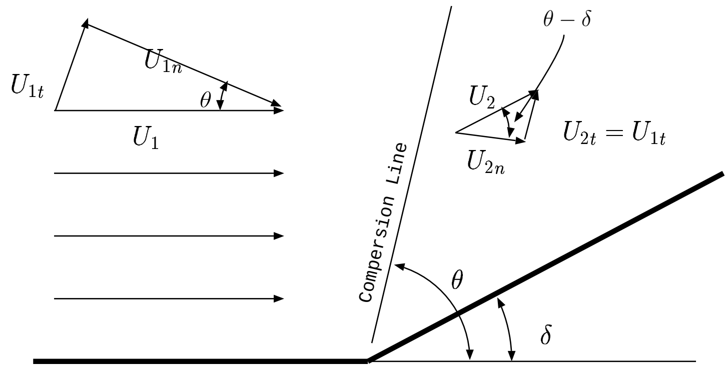


Fig. 13.3: A typical oblique shock schematic

The momentum equation in the tangential direction yields

$$U_{1t} = U_{2t} \tag{13.3}$$

The energy balance reads

$$C_pT_1 + \frac{{U_{1n}}^2}{2} = C_pT_2 + \frac{{U_{2n}}^2}{2} \tag{13.4}$$

Equations (13.1), (13.2) and (13.4) are the same equations as the equations for normal shock with the exception that the total velocity is replaced by the perpendicular components. Yet, the new issue of relationship between the upstream Mach number and the deflection angle, δ and the Mach angel, θ has to be solved. From the geometry it can be observed that

$$\tan\theta = \frac{U_{1n}}{U_{1t}} \tag{13.5}$$

and

$$\tan(\theta - \delta) = \frac{U_{2n}}{U_{2t}} \tag{13.6}$$

Not as in the normal shock, here there are three possible pair[5] of solutions to these equations one is referred to as the weak shock, two the strong shock, and three the impossible solution (thermodynamically)[6]. Experiments and experience

[5]This issue is due to R. Menikoff, which raise to completeness of the solution. He pointed out the full explanation to what happened to the negative solution.

[6]This solution requires to solve the entropy conservation equation. The author is not aware of “simple” proof and a call to fi nd a simple proof is needed.

show that the common solution is the weaker shock, in which the flow turn to lesser extent[7].

$$\frac{\tan\theta}{\tan(\theta-\delta)} = \frac{U_{1n}}{U_{2n}} \tag{13.7}$$

The above velocity–geometry equations also can be expressed in term of Mach number as

$$\sin\theta = \frac{M_{1n}}{M_1} \tag{13.8}$$

$$\sin(\theta-\delta) = \frac{M_{2n}}{M_2} \tag{13.9}$$

$$\cos\theta = \frac{M_{1t}}{M_1} \tag{13.10}$$

$$\cos(\theta-\delta) = \frac{M_{2t}}{M_2} \tag{13.11}$$

The total energy across oblique shock wave is constant, and it follows that the **total** speed of sound is constant across the (oblique) shock. It should be noted that although $U_{1t} = U_{2t}$ the Mach number $M_{1t} \neq M_{2t}$ because the temperatures on both sides of the shock are different, $T_1 \neq T_2$.

As opposed to the normal shock, here angles (the second dimension) have to be solved. The solution of this set of four equations (13.8) through (13.11) are function of four unknowns of M_1, M_2, θ, and δ. Rearranging this set set with utilizing the the geometrical identity such as $\sin\alpha = 2\sin\alpha\cos\alpha$ results in

$$\tan\delta = 2\cot\theta\left[\frac{{M_1}^2\sin^2\theta - 1}{{M_1}^2\left(k+\cos 2\theta\right)+2}\right] \tag{13.12}$$

The relationship between the properties can be found by substituting $M_1\sin\theta$ instead of M_1 into the normal shock relationship and results in

$$\frac{P_2}{P_1} = \frac{2k{M_1}^2\sin^2\theta - (k-1)}{k+1} \tag{13.13}$$

[7]Actually this term is used from historical reason. The lesser extent angle is the unstable and the weak angle is the middle solution. But because the literature referred to only two roots the term lesser extent is used.

The density and normal velocities ratio can be found from the following equation

$$\frac{\rho_2}{\rho_1} = \frac{U_{1n}}{U_{2n}} = \frac{(k+1){M_1}^2 \sin^2\theta}{(k-1){M_1}^2 \sin^2\theta + 2} \tag{13.14}$$

The temperature ratio is expressed as

$$\frac{T_2}{T_1} = \frac{2k{M_1}^2 \sin^2\theta - (k-1)\left[(k-1){M_1}^2 + 2\right]}{(k+1)^2 M_1} \tag{13.15}$$

Prandtl's relation for oblique shock is

$$U_{n_1} U_{n_2} = c^2 - \frac{k-1}{k+1}{U_t}^2 \tag{13.16}$$

The Rankine–Hugoniot relations are the same as the relationship for the normal shock

$$\frac{P_2 - P_1}{\rho_2 - \rho_1} = k\frac{P_2 - P_1}{\rho_2 - \rho_1} \tag{13.17}$$

13.4 Solution of Mach Angle

The oblique shock orientated in coordinate perpendicular and parallel shock plan is like a normal shock. Thus, the properties relationship can be founded by using the normal components or utilizing the normal shock table developed earlier. One has to be careful to use the normal components of the Mach numbers. The stagnation temperature contains the total velocity.

Again, as it may be recalled, the normal shock is one dimensional problem, thus, only one parameter was required (to solve the problem). The oblique shock is a two dimensional problem and two properties must be provided so a solution can be found. Probably, the most useful properties, are upstream Mach number, M_1 and the deflection angle which create somewhat complicated mathematical procedure and it will be discussed momentarily. Other properties combinations provide a relatively simple mathematical treatment and the solutions of selected pairs and selected relationships are presented.

13.4.1 Upstream Mach number, M_1, and deflection angle, δ

Again, this set of parameters is, perhaps, the most common and natural to examine. Thompson (1950) has shown that the relationship of shock angle is obtained from the following cubic equation:

$$x^3 + a_1 x^2 + a_2 x + a_3 = 0 \tag{13.18}$$

Where

$$x = \sin^2\theta \tag{13.19}$$

And

$$a_1 = -\frac{{M_1}^2 + 2}{{M_1}^2} - k\sin^2\delta \tag{13.20}$$

$$a_2 = -\frac{2{M_1}^2 + 1}{{M_1}^4} + \left[\frac{(k+1)^2}{4} + \frac{k-1}{{M_1}^2}\right]\sin^2\delta \tag{13.21}$$

$$a_3 = -\frac{\cos^2\delta}{{M_1}^4} \tag{13.22}$$

Equation (13.19) requires that x has to be a real and positive number to obtain real deflection angle[8]. Clearly, $\sin\theta$ must be possible the and the negative sign is refers to the mirror image of the solution. Thus, the negative root of $\sin\theta$ must be disregarded

Solution of a cubic equation like (13.18) provides three roots[9]. These roots can be expressed as

$$x_1 = -\frac{1}{3}a_1 + (S+T) \tag{13.23}$$

$$x_2 = -\frac{1}{3}a_1 - \frac{1}{2}(S+T) + \frac{1}{2}i\sqrt{3}(S-T) \tag{13.24}$$

and

$$x_3 = -\frac{1}{3}a_1 - \frac{1}{2}(S+T) - \frac{1}{2}i\sqrt{3}(S-T) \tag{13.25}$$

Where

$$S = \sqrt[3]{R + \sqrt{D}}, \tag{13.26}$$

$$T = \sqrt[3]{R - \sqrt{D}} \tag{13.27}$$

and where the definitions of the D is

$$D = Q^3 + R^2 \tag{13.28}$$

[8]This point was pointed by R. Menikoff. He also suggested that θ is bounded by $\sin^{-1} 1/M_1$ and 1.

[9]The highest power of the equation (only with integer numbers) is the number of the roots. For example, in a quadratic equation there are two roots.

and where the definitions of Q and R are

$$Q = \frac{3a_2 - {a_1}^2}{9} \tag{13.29}$$

and

$$R = \frac{9a_1a_2 - 27a_3 - 2{a_1}^3}{54} \tag{13.30}$$

Only three roots can exist for Mach angle, θ. From mathematical point of view, if $D > 0$ one root is real and two roots are complex. For the case $D = 0$ all the roots are real and at least two are identical. In the last case where $D < 0$ all the roots are real and unequal.

The physical meaning of the above analysis demonstrates that in the range where $D > 0$ no solution exist because no imaginary solution can exist[10]. $D > 0$ occurs when no shock angle can be found so that the shock normal component is reduced to be subsonic and yet be parallel to inclination angle.

Furthermore, only in some cases when $D = 0$ the solution has physical meaning. Hence, the solution in the case of $D = 0$ has to be examined in the light of other issues to determine the validity of the solution.

When $D < 0$ the three unique roots are reduced to two roots at least for steady state because the thermodynamics dictation[11]. Physically, it can be shown that the first solution(13.23), referred sometimes as thermodynamically unstable root which also related to decrease in entropy, is "unrealistic." Therefore, the first solution dose not occur in reality, at least, in steady state situations. This root has only a mathematical meaning for steady state analysis[12].

These two roots represents two different situations. One, for the second root, the shock wave keeps the flow almost all the time as supersonic flow and it refereed to as the weak solution (there is a small branch (section) that the flow is subsonic). Two, the third root always turns the flow into subsonic and it refereed to a strong solution. It should be noted that this case is where entropy increases in the largest amount.

In summary, if there was hand which moves the shock angle starting with the deflection angle, reach the first angle that satisfies the boundary condition, however, this situation is unstable and shock angle will jump to the second angle (root). If additional "push" is given, for example by additional boundary conditions,

[10] A call for suggestions, should explanation about complex numbers and imaginary numbers should be included. Maybe insert example where imaginary solution results in no physical solution.

[11] This situation is somewhat similar to a cubical body rotation. The cubical body has three symmetrical axises which the body can rotate around. However, the body will freely rotate only around two axis with small and large moments of inertia. The body rotation is unstable around the middle axes. The reader simply can try it.

[12] There is no experimental evidence, that this author found, showing that it is totally impossible. Though, those who are dealing with rapid transient situations should be aware that this angle of the oblique shock can exist. The shock initially for a very brief time will transient in it and will jump from this angle to the thermodynamically stable angles.

the shock angle will jump to the third root[13]. These two angles of the strong and weak shock are stable for two-dimensional wedge (See for the appendix of this Chapter for a limit discussion on the stability[14]).

13.4.2 In What Situations No Oblique Shock Exist or When $D > 0$

13.4.2.1 Large deflection angle for given, M_1

The first range is when the deflection angle reaches above the maximum point. For given upstream Mach number, M_1, a change in the inclination angle requires a larger energy to change the flow direction. Once, the inclination angle reaches "maximum potential energy" to change the flow direction and no change of flow direction is possible. Alternative view, the fluid "sees" the disturbance (here, in this case, the wedge) in–front of it. Only the fluid away the object "sees" the object as object with a different inclination angle. This different inclination angle sometimes referred to as imaginary angle.

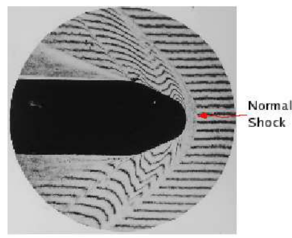


Fig. 13.4: Flow around spherically blunted $30°$ cone-cylinder with Mach number 2.0. It can be noticed that a normal shock, strong shock, and weak shock co-exist.

13.4.2.1.1 The simple procedure For example, in Figure 13.4 and 13.5, the imaginary angle is shown. The flow far away from the object does not "see' the object. For example, for $M_1 \longrightarrow \infty$ the maximum defection angle is calculated when $D = Q^3 + R^2 = 0$. This can be done by evaluating the terms a_1, a_2, and a_3 for $M_1 = \infty$.

$$a_1 = -1 - k\sin^2\delta$$
$$a_2 = \frac{(k+1)^2 \sin^2\delta}{4}$$
$$a_3 = 0$$

With these values the coefficients, R and Q are

$$R = \frac{9(-)(1+k\sin^2\delta)\left(\frac{(k+1)^2\sin^2\delta}{4}\right) - (2)(-)(1+k\sin^2\delta)^2}{54}$$

[13] See for historical discussion on the stability. There are those who view this question not as a stability equation but rather as under what conditions a strong or a weak shock will prevail.

[14] This material is extra and not recommended for standard undergraduate students.

and

$$Q = \frac{(1 + k\sin^2\delta)^2}{9}$$

Solving equation (13.28) after substituting these values of Q and R provides series

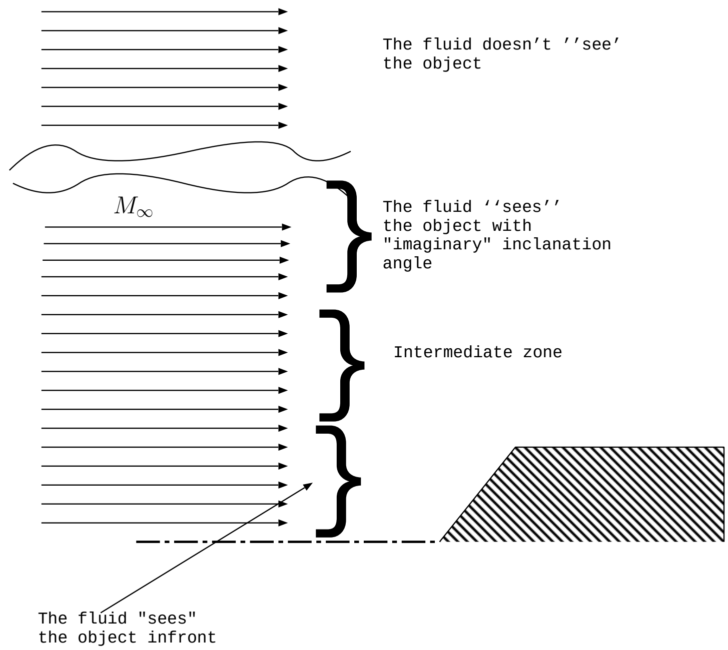


Fig. 13.5: The view of large inclination angle from different points in the fluid field

of roots from which only one root is possible. This root, in the case $k = 1.4$, is just above $\delta_{max} \sim \frac{\pi}{4}$ (note that the maximum is also a function of heat ratio, k).

13.4.2.1.2 The Procedure for Calculating The Maximum (Deflection point)

The maximum is obtained when $D = 0$. When the right terms defined in (13.20)-(13.21), (13.29), and (13.30) are substitute into this equation and utilizing the trigonometrical $\sin^2\delta + \cos^2\delta = 1$ and other trigonometrical identities results in to Maximum Deflection Mach number's equation in which is

$$M_1{}^2\,(k+1)\,({M_{1n}}^2 + 1) = 2(k{M_{1n}}^4 + 2{M_{1n}}^2 - 1) \qquad (13.31)$$

This equation and it twin equation can be obtained by alternative procedure. C. J.Chapman, English mathematician[15] suggest another way to approached this

[15]Mathematician have a different way in looking at things. At time, their approach seems simpler to explain.

issue. He noticed that equation (13.12) the deflection angle is a function of Mach angle and upstream Mach number, M_1. Thus, one can conclude that the maximum Mach angle is only a function of the upstream Much number, M_1. This can be shown mathematical by the argument that differentiating equation (13.12) and equating the results to zero crates relationship between the Mach number, M_1 and the maximum Mach angle, θ. Since in that equation there appears only the heat ratio, k, and Mach number, M_1, θ_{max} is a function of of only these parameters. The differentiation of the equation (13.12) yields

$$\frac{d\tan\delta}{d\theta} = \frac{k{M_1}^4\sin^4\theta + \left(2 - \frac{(k+1)}{2}{M_1}^2\right){M_1}^2\sin^2\theta - \left(1 + \frac{(k+1)}{2}{M_1}^2\right)}{k{M_1}^4\sin^4\theta - \left[(k-1) + \frac{(k+1)^2{M_1}^2}{4}\right]{M_1}^2\sin^2\theta - 1} \tag{13.32}$$

Because $\tan$ is a monotonous function the maximum appears when θ has its maximum. The numerator of equation (13.32) is zero at different values of denominator. Thus it is sufficient to equate the numerator to zero to find the maximum. The nominator produce quadratic equation for $\sin^2\theta$ and only the positive value for $\sin^2\theta$ is applied here. Thus, the $\sin^2\theta$ is

$$\sin^2\theta_{max} = \frac{-1 + \frac{k+1}{4}{M_1}^2 + \sqrt{(k+1)\left[1 + \frac{k-1}{2}{M_1}^2 + \left(\frac{k+1}{2}M_1\right)^4\right]}}{k{M_1}^2} \tag{13.33}$$

Equation (13.33) should be referred as Chapman's equation. It should be noted that both Maximum Mach Deflection equation and Chapman's equation lead to the same conclusion that maximum M_{1n} is only a function of of upstream Mach number and the heat ratio, k. It can be noticed that this Maximum Deflection Mach Number's equation is also quadratic equation for ${M_{1n}}^2$. Once, M_{1n} is found than the Mach angle can be easily calculated by equation (13.8). To compare these two equations the simple case of Maximum for infinite Mach number is examined. A simplified case of Maximum Deflection Mach Number's equation for Large Mach number becomes

$$M_{1n} = \sqrt{\frac{k+1}{2k}}M_1 \quad \text{for} \quad M_1 >> 1 \tag{13.34}$$

Hence, for large Mach number the Mach angle is $\sin\theta = \sqrt{\frac{k+1}{2k}}$ which make $\theta = 1.18$ or $\theta = 67.79°$.

With the value of θ utilizing equation (13.12) the maximum deflection angle can be computed. Note this procedure does not require that approximation of M_{1n} has to be made. The general solution of equation (13.31) is

$$M_{1n} = \frac{\sqrt{(k+1){M_1}^2 + 1 + \sqrt{({M_1}^2\left[{M_1}^2(k+1)^2 + 8(k^2-1)\right] + 16(1+k)}}}{2\sqrt{k}} \tag{13.35}$$

Note that Maximum Deflection Mach Number's equation can be extend to more complicated equation of state (aside the perfect gas model).

This typical example for these who like mathematics.

Example 13.1:
Derive perturbation of Maximum Deflection Mach Number's equation for the case of a very small upstream Mach number number of the form of $M_1 = 1 + \epsilon$. Hint, Start with equation (13.31) *and neglect all the terms that are relatively small.*

SOLUTION
under construction

13.4.2.2 The case of $D \geq 0$ of $0 \geq \delta$

The second range in which $D > 0$ is when $\delta < 0$. Thus, first the transition line in–which $D = 0$ has to be determined. This can be achieved by the standard mathematical procedure by equating $D = 0$. The analysis shows regardless to the value of upstream Mach number $D = 0$ when $\delta = 0$. This can be partially demonstrated by evaluating the terms a_1, a_2, and a_3 for specific value of M_1 as following

$$\begin{aligned}
a_1 &= \frac{{M_1}^2 + 2}{{M_1}^2} \\
a_2 &= -\frac{2{M_1}^2 + 1}{{M_1}^4} \\
a_3 &= -\frac{1}{{M_1}^4}
\end{aligned} \tag{13.36}$$

With values presented in equations (13.36) for R and Q becomes

$$\begin{aligned}
R &= \frac{9\left(\frac{{M_1}^2+2}{{M_1}^2}\right)\left(\frac{2{M_1}^2+1}{{M_1}^4}\right) - 27\left(\frac{-1}{{M_1}^4}\right) - 2\left(\frac{{M_1}^2+2}{{M_1}^2}\right)^2}{54} \\
&= \frac{9\left({M_1}^2+2\right)\left(2{M_1}^2+1\right) + 27{M_1}^2 - 2{M_1}^2\left({M_1}^2+2\right)^2}{54{M_1}^6}
\end{aligned} \tag{13.37}$$

and

$$Q = \frac{3\left(\frac{2{M_1}^2+1}{{M_1}^4}\right) - \left(\frac{{M_1}^2+2}{{M_1}^2}\right)^3}{9} \tag{13.38}$$

Substituting the values of Q and R equations (13.37)(13.38) into equation (13.28) provides the equation to be solved for δ.

$$\left[\frac{3\left(\frac{2{M_1}^2+1}{{M_1}^4}\right)-\left(\frac{{M_1}^2+2}{{M_1}^2}\right)^3}{9}\right]^3 + \left[\frac{9\left({M_1}^2+2\right)\left(2{M_1}^2+1\right)+27{M_1}^2-2{M_1}^2\left({M_1}^2+2\right)^2}{54{M_1}^6}\right]^2 = 0 \tag{13.39}$$

This author is not aware of any analytical demonstration in the literature which showing that the solution is identity zero for $\delta = 0$[16]. Nevertheless, this identity can be demonstrated by checking several points for example, $M_1 = 1., 2.0, \infty$. Table (13.6) is provided for the following demonstration. Substitution of all the above values into (13.28) results in $D = 0$.

Utilizing the symmetry and antisymmetry of the qualities of the $\cos$ and $\sin$ for $\delta < 0$ demonstrates that $D > 0$ regardless to Mach number. Hence, the physical interpretation of this fact that either that no shock can exist and the flow is without any discontinuity or a normal shock exist[17]. Note, in the previous case, positive large deflection angle, there was transition from one kind of discontinuity to another.

coefficients / M_1	a_1	a_2	a_3
1.0	-3	-1	-$\frac{3}{2}$
2.0	3	0	$\frac{9}{16}$
∞	-1	0	-$\frac{1}{16}$

Fig. 13.6: The various coeffi cients of three different Mach number to demonstrate that D is zero

In the range where $\delta \leq 0$, the question whether it is possible for the oblique shock to exist? The answer according to this analysis and stability analysis is not. And according to this analysis no Mach wave can be generated from the wall with **zero deflection**. In other words, the wall doesn't emit any signal to the flow (assuming zero viscosity) which contradicts the common approach. Nevertheless, in the literature, there are several papers suggesting zero strength Mach wave, other suggest singular point[18]. The question of singular point or zero Mach wave strength are only of mathematical interest.

[16]A mathematical challenge for those who like to work it out.

[17]There are several papers that attempted to prove this point in the past. Once this analytical solution was published, this proof became trivial. But for non ideal gas (real gas) this solution is only indication.

[18]See for example, paper by Rosles, Tabak, "Caustics of weak shock waves," 206 Phys. Fluids 10 (1) , January 1998.

Suppose that there is a Mach wave at the wall at zero inclination (see Figure 13.7). Obviously, another Mach wave occurs after a small distance. But because the velocity after a Mach wave (even for a extremely weak shock wave) is reduced, thus, the Mach angle will be larger ($\mu_2 > \mu_1$). If the situation is keeping on occurring over a finite distance there will be a point where the Mach number will be one and a normal shock will occur according the common explanation. However, the reality is that no continues Mach wave can occur because the viscosity (boundary layer).

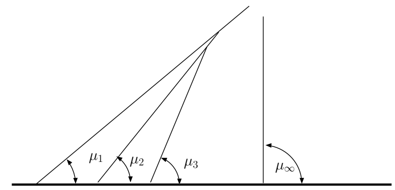


Fig. 13.7: The Mach waves that supposed to be generated at zero inclination

In reality, there are imperfections in the wall and in the flow and there is the question of boundary layer. It is well known, in engineering world, that there no such thing as a perfect wall. The imperfections of the wall can be, for simplicity sake, assumed to be as a sinusoidal shape. For such wall the zero inclination changes from small positive value to a negative value. If the Mach number is large enough and wall is rough enough there will be points where a weak[19] weak will be created. On the other hand, the boundary layer covers or smooths the bumps. With these conflicting mechanisms, and yet both not allowing situation of zero inclination with emission of Mach wave. At the very extreme case, only in several points (depends on the bumps) at the leading edge a very weak shock occurs. Therefor, for the purpose of introductory class, no Mach wave at zero inclination should be assumed.

Furthermore, if it was assumed that no boundary layer exist and wall is perfect, any deviations from the zero inclination angle creates a jump between a positive angle (Mach wave) to a negative angle (expansion wave). This theoretical jump occurs because in Mach wave the velocity decreases while in expansion waves the velocity increases. Further, the increase and the decrease depend on the upstream Mach number but in different direction. This jump has to be in reality either smoothed or has physical meaning of jump (for example, detach normal shock). The analysis started by looking at normal shock which occurs when there is a zero inclination. After analysis of the oblique shock, the same conclusion must be found, i.e. that normal shock can occur at zero inclination. The analysis of the oblique shock impose that the inclination angle is not the source (boundary condition) that creates the shock. There must be another boundary condition(s) that forces a shock. In the light of this discussion, at least for a simple engineering analysis, the zone in the proximity of zero inclination (small positive and negative inclination angle) should be viewed as zone without any change unless the boundary conditions force a normal shock.

Nevertheless, emission of Mach wave can occur in other situations. The

[19]It is not a mistake, there two "weaks." These words mean two different things. The first "weak" means more of compression "line" while the other means the weak shock.

approximation of weak weak wave with non zero strength has engineering applicability in limited cases especially in acoustic engineering but for most cases it should be ignored.

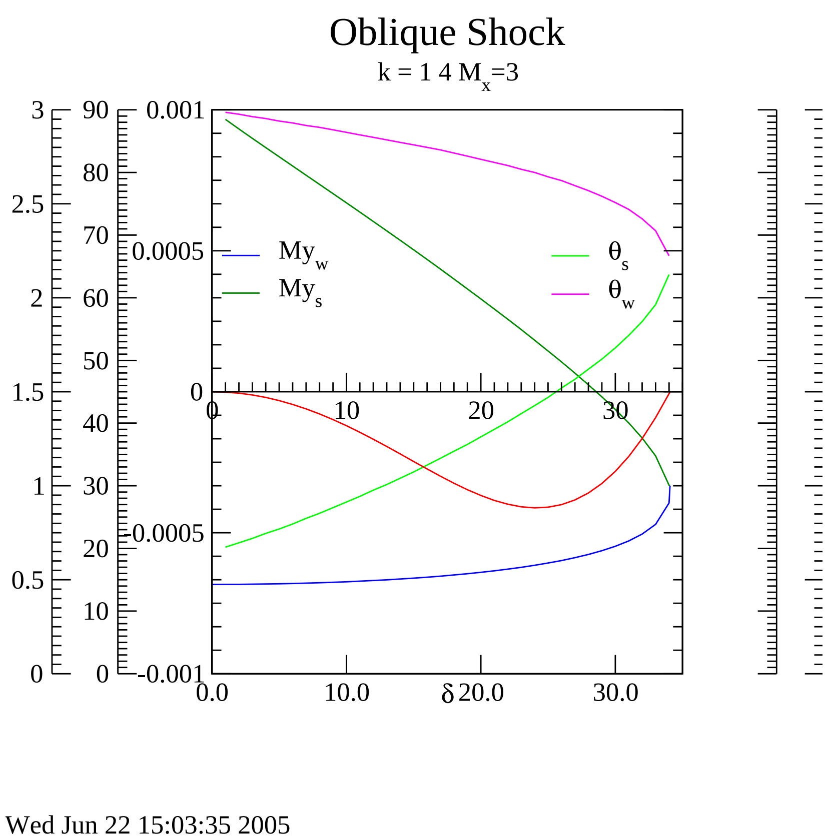


Fig. 13.8: The calculation of D (possible error), shock angle and exit Mach number for $M_1 = 3$

13.4.3 Upstream Mach Number, M_1, and Shock Angle, θ

The solution for upstream Mach number, M_1, and shock angle, θ, are far more simpler and an unique solution exist. The deflection angle can be expressed as a function of these variable as

$$\cot\delta = \tan\theta \left[\frac{(k+1){M_1}^2}{2({M_1}^2\sin^2\theta - 1)} - 1 \right] \tag{13.40}$$

or

$$\tan\delta = \frac{2\cot\theta({M_1}^2\sin^2\theta - 1)}{2 + {M_1}^2(k + 1 - 2\sin^2\theta)} \tag{13.41}$$

The pressure ratio can be expressed as

$$\frac{P_2}{P_1} = \frac{2k{M_1}^2\sin^2\theta - (k-1)}{k+1} \tag{13.42}$$

The density ratio can be expressed as

$$\frac{\rho_2}{\rho_1} = \frac{U_{1n}}{U_{2n}} = \frac{(k+1){M_1}^2\sin^2\theta}{(k-1){M_1}^2\sin^2\theta + 2} \tag{13.43}$$

The temperature ratio expressed as

$$\frac{T_2}{T_1} = \frac{{c_2}^2}{{c_1}^2} = \frac{\left(2k{M_1}^2\sin^2\theta - (k-1)\right)\left((k-1){M_1}^2\sin^2\theta + 2\right)}{(k+1){M_1}^2\sin^2\theta} \tag{13.44}$$

The Mach number after the shock is

$${M_2}^2\sin(\theta-\delta) = \frac{(k-1){M_1}^2\sin^2\theta + 2}{2k{M_1}^2\sin^2\theta - (k-1)} \tag{13.45}$$

or explicitly

$${M_2}^2 = \frac{(k+1)^2{M_1}^4\sin^2\theta - 4({M_1}^2\sin^2\theta - 1)(k{M_1}^2\sin^2\theta + 1)}{\left(2k{M_1}^2\sin^2\theta - (k-1)\right)\left((k-1){M_1}^2\sin^2\theta + 2\right)} \tag{13.46}$$

The ratio of the total pressure can be expressed as

$$\frac{P_{0_2}}{P_{0_1}} = \left[\frac{(k+1){M_1}^2\sin^2\theta}{(k-1){M_1}^2\sin^2\theta + 2}\right]^{\frac{k}{k-1}}\left[\frac{k+1}{2k{M_1}^2\sin^2\theta - (k-1)}\right]^{\frac{1}{k-1}} \tag{13.47}$$

Eventhough that the solution for these variables, M_1 and θ, is unique the possible range deflection angle, δ, is limited. Examining equation (13.40) shows that shock angel, θ , has to be in the range of $\sin^{-1}(1/M_1) \geq \theta \geq (\pi/2)$ (see Figure 13.9). The range of given θ, upstream Mach number M_1, is limited between ∞ and $\sqrt{1/\sin^2\theta}$.

13.4.4 For Given Two Angles, δ and θ

It is sometimes useful to obtains relationship where the two angles are known. The first upstream Mach number, M_1 is

$${M_1}^2 = \frac{2(\cot\theta + \tan\delta)}{\sin 2\theta - (\tan\delta)(k + \cos 2\theta)} \tag{13.48}$$

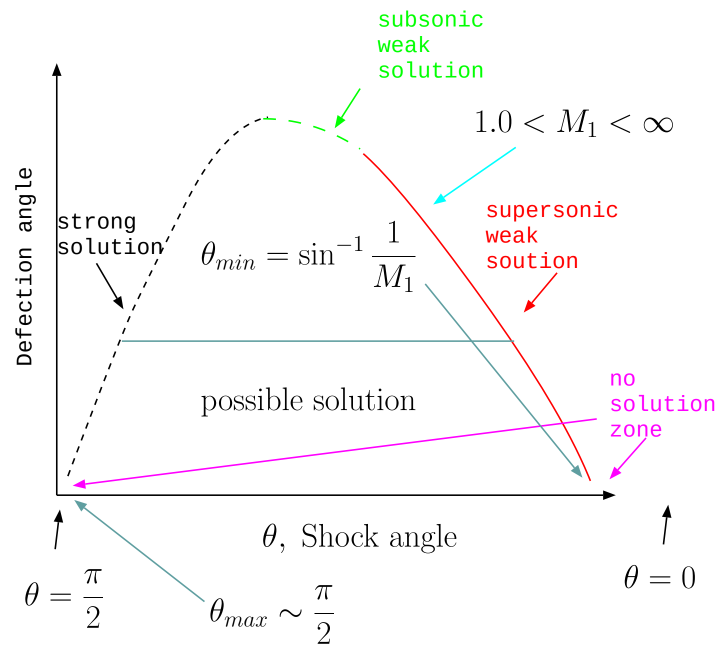


Fig. 13.9: The possible range of solution for different parameters for given upstream Mach number

The reduced pressure difference is

$$\frac{2(P_2 - P_1)}{\rho U^2} = \frac{2\sin\theta\sin\delta}{\cos(\theta - \delta)} \tag{13.49}$$

The reduced density is

$$\frac{\rho_2 - \rho_1}{\rho_2} = \frac{\sin\delta}{\sin\theta\cos(\theta - \delta)} \tag{13.50}$$

For large upstream Mach number M_1 and small shock angle (yet non approaching zero), θ, the deflection angle, δ must be small as well. Equation (13.40) can be simplified into

$$\theta \cong \frac{k+1}{2}\delta \tag{13.51}$$

The results are consistent with the initial assumption shows that it was appropriate assumption.

13.4.5 Flow in a Semi–2D Shape

The discussion so far was about the straight infinite long wedge[20] which is a “pure” 2–D configuration. Clearly, for any finite length of the wedge, the analysis needs to account for edge effects. The end of the wedge must have a different configuration (see Figure 13.10). Yet, the analysis for middle section produces close results to reality (because symmetry). The section where the current analysis is close to reality can be estimated from a dimensional analysis for the required accuracy or by a numerical method. The dimensional analysis shows that only doted area to be area where current solution can be assumed as correct[21]. In spite of the small area were the current solution can be assumed, this solution is also act as “reality check” to any numerical analysis. The analysis also provides additional value of the expected range.

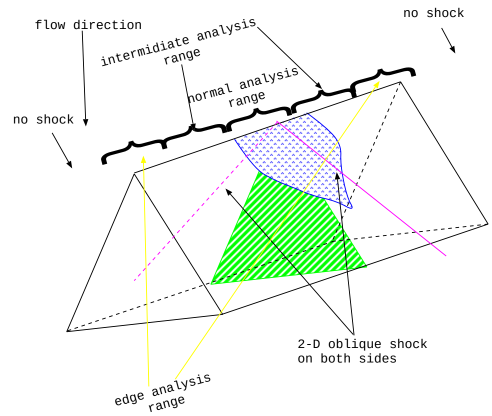


Fig. 13.10: Schematic of finite wedge with zero angle of attack

Another geometry that can be considered as two dimensional is the cone (some referred to as Taylor–Maccoll flow). Eventhough, the cone is a three dimensional problem, the symmetrical nature of the cone creates a semi–2D problem. In this case there are no edge effects and the geometry dictates slightly different results. The mathematics is much more complicated but there are three solutions. As before, the first solution is thermodynamical unstable. Experimental and analytical work shows that the weak solution is the stable solution and a discussion is provided in the appendix of this chapter. As oppose to the weak shock, the strong shock is unstable, at least, for steady state and no know experiments showing that it exist can be found the literature. All the literature, known to this author, reports that only a weak shock is possible.

13.4.6 Small δ “Weak Oblique shock”

This topic has interest mostly from academic point of view. It is recommend to skip this issue and devote the time to other issues. This author, is not aware of a

[20]Even finite wedge with limiting wall can be considered as example for this discussion if the B.L. is neglected.

[21]At this stage dimensional analysis is not competed. This author is not aware of any such analysis in literature. The common approach is to carry numerical analysis. In spite recent trends, for most engineering application, simple tool are sufficient for limit accuracy. In additionally, the numerical works require many times a “reality check.”

single case that this topic used in a real world calculations. In fact, after expressed analytical solution is provided, devoted time, seems to come on the count of many important topics. However, this author admits that as long there are instructors who examine their students on this issue, it should be covered in this book.

For small deflection angle, δ, and small normal upstream Mach number, $M_1 \sim 1 + \epsilon$,

$$\tan\theta = \frac{1}{\sqrt{{M_1}^2 - 1}} \tag{13.52}$$

... under construction.

13.4.7 Close and Far Views of The Oblique Shock

In many cases, the close proximity view provides continuous turning of the deflection angle, δ. Yet, the far view shows a sharp transition. The traditional approach to reconcile these two views, is by suggesting that the far view shock is a collection of many small weak shocks (see Figure 13.11). At the local view close to wall the oblique shock is a weak "weak oblique" shock. From the far view the oblique shock is accumulations of many small (or again weak) "weak shocks." However, there small "shocks" are built or accumulate into a large and abrupt change (shock). In this theory, the Boundary Layer (B.L.) doesn't enter into the calculation. In reality, the B.L. increases the zone where continuous flow exist. The B.L. reduces the upstream flow velocity and therefore the shock doesn't exist close proximity to the wall. In larger distance form the the wall, the shock starts to be possible.

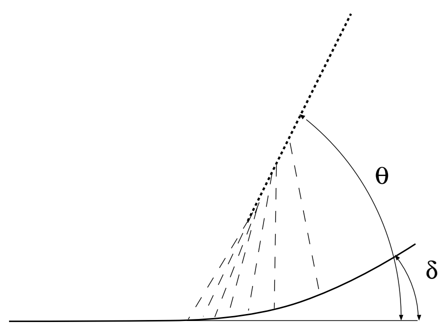


Fig. 13.11: Two different views from local and far on the oblique shock

13.4.8 Maximum value of of Oblique shock

The maximum values are summarized in the following Table .

Table 13.1: Table of Maximum values of the obliques Shock

$\mathbf{M_x}$	$\mathbf{M_y}$	$\delta_{\mathbf{max}}$	$\theta_{\mathbf{max}}$
1.1000	0.97131	1.5152	76.2762
1.2000	0.95049	3.9442	71.9555

Table 13.1: Maximum values of oblique shock (continue)

$\mathbf{M_x}$	$\mathbf{M_y}$	$\delta_{\mathbf{max}}$	$\theta_{\mathbf{max}}$
1.3000	0.93629	6.6621	69.3645
1.4000	0.92683	9.4272	67.7023
1.5000	0.92165	12.1127	66.5676
1.6000	0.91941	14.6515	65.7972
1.7000	0.91871	17.0119	65.3066
1.8000	0.91997	19.1833	64.9668
1.9000	0.92224	21.1675	64.7532
2.0000	0.92478	22.9735	64.6465
2.2000	0.93083	26.1028	64.6074
2.4000	0.93747	28.6814	64.6934
2.6000	0.94387	30.8137	64.8443
2.8000	0.94925	32.5875	65.0399
3.0000	0.95435	34.0734	65.2309
3.2000	0.95897	35.3275	65.4144
3.4000	0.96335	36.3934	65.5787
3.6000	0.96630	37.3059	65.7593
3.8000	0.96942	38.0922	65.9087
4.0000	0.97214	38.7739	66.0464
5.0000	0.98183	41.1177	66.5671
6.0000	0.98714	42.4398	66.9020
7.0000	0.99047	43.2546	67.1196
8.0000	0.99337	43.7908	67.2503
9.0000	0.99440	44.1619	67.3673
10.0000	0.99559	44.4290	67.4419

It must be noted that the calculations are for ideal gas model. In some cases this assumption might not be sufficient and different analysis is needed. Henderson and Menikoff[22] suggested a procedure to calculate the maximum deflection angle for arbitrary equation of state[23].

13.4.9 Detached shock

When the mathematical quantity D becomes positive, for large deflection angle, there isn't a physical solution to oblique shock. Since the flow "sees" the obstacle, the only possible reaction is by a normal shock which occurs at some distance from the body. This shock referred to as the detach shock. The detach shock

[22]Henderson and Menikoff "Triple Shock Entropy Theorem" Journal of Fluid Mechanics 366 (1998) pp. 179–210.

[23]The effect of the equation of state on the maximum and other parameters at this state is unknown at this moment and there are more works underway.

distance from the body is a complex analysis and should be left to graduate class and for researchers in the area. Nevertheless, a graph and general explanation to engineers is provided. Even though there are very limited applications to this topic some might be raised in certain situations, which this author isn't aware of.

Analysis of the detached shock can be carried out by looking at a body with a round section moving in a supper sonic flow (the absolute velocity isn't important for this discussion). Figure 13.12 exhibits a bullet with a round tip which the shock is detach. The distance of the detachment determined to a large degree the resistance to the body. The zone A is zone where the flow must be subsonic because at the body the velocity must be zero (the no slip condition). In such case, the gas must go through a shock. While at at zone C the flow must be supersonic (The weak oblique shock is predicted for flow around cone.). The flow in zone A has to go thorough some acceleration to became supersonic flow. The explanation to such phenomenon is above the level of this book (where is the "throat" area question[24]. Yet, it can be explained as the subsonic is "sucked" into gas in zone C. Regardless the explanation, these calculations can be summarized in the flowing equation

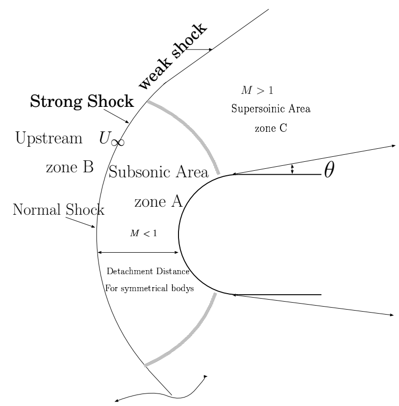


Fig. 13.12: The schematic for round tip bullet in a supersonic flow

$$\frac{\text{detachment distance}}{\text{body thickness}} = \text{constant} \times (\theta - f(M_\infty)) \tag{13.53}$$

where $f(M_\infty)$ is a function of the upstream Mach number which tabulated in the literature.

The constant and the function are different for different geometries. As general rule the increase in the upstream Mach results in decrease of the detachment. Larger shock results in smaller detachment distance, or alternatively the flow becomes "blinder" to obstacles. Thus, this phenomenon has a larger impact for relatively smaller supersonic flow.

To insert the table for the constants and functions

13.4.10 Issues related to the Maximum Deflection Angle

The issue of maximum deflection has practical application aside to the obvious configuration shown as a typical simple example. In the typical example a wedge or cone moves into a still medium or gas flow into it. If the deflection angle exceeds

[24] See example 13.5.

the maximum possible a detached shock is occurs. However, the configuration that detached shock occurs in many design configurations and the engineers need to take it into considerations. Such configurations seem sometimes at the first glance not related to the detached shock issue. Consider, for example, a symmetrical suction section in which the deflection angle is just between the maximum deflection angle and above the half of the maximum deflection angle. In this situation, at least two oblique shocks occur and after their interaction is shown in Figure (13.13). No detached shock issues are raised when only the first oblique shock is considered. However, the second oblique shock complicates the situation and the second oblique shock can cause detached shock. This situation known in the scientific literature as the Mach reflection.

It can be observed that the maximum of the oblique shock for Ideal gas model depends only on upstream Mach number i.e. for every upstream Mach number there is only one maximum deflection angle.

$$\delta_{max} = f(M_1) \tag{13.54}$$

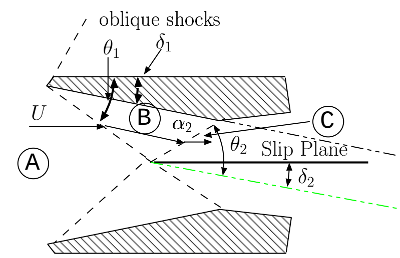


Fig. 13.13: The schematic for symmetrical suction section with Mach reflection

Additionally, it can be observed that for non maximum oblique shock that for a constant deflection angle decrease of Mach number results in increase of Mach angle (weak shock only) $M_1 > M_2 \Longrightarrow \theta_1 < \theta_2$. The Mach number decreases after every shock. Therefore, the maximum deflection angle decreases with decrease of the Mach number. Additionally due to the symmetry a slip plane angle can be guessed to be parallel to original flow, hence $\delta_1 = \delta_2$. Thus, this situation causes the detach shock to appear in the second oblique shock. This detached shock manifested itself in form of curved shock (see Figure 13.14).

The analysis of this situation logically is very simple yet the mathematics is somewhat complicated. The maximum deflection angle in this case is, as before, only function of the upstream Mach number. The calculations of such case can be carried by several approaches. It seems to this author that most straight way is by the following procedure:

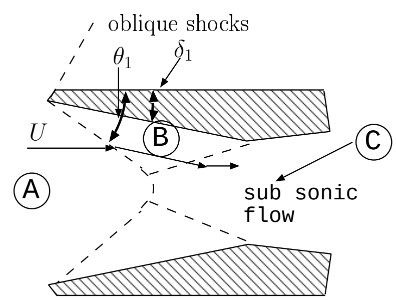


Fig. 13.14: The "detached" shock in complicated configuration some times referred as Mach reflection

(a) Let calculation carried for M_{1B};

(b) Calculate the maximum deflection angle, θ_2 utilizing (13.31) equation.

(c) Calculate the deflection angle, δ_2 utilizing equation (13.12)

(d) Use the deflection angle, $\delta_2 = \delta_1$ and Mach number M_{1B} to calculate M_{1B}. Note that no maximum angle is achieved in this shock. POTTO–GDC can be used to calculate this ratio.

This procedure can be extended to calculated the maximum incoming Mach number, M_1 by check the relationship between the intermediate Mach number to M_1.

In discussing these issues one must be aware that there zone of dual solution in which sharp shock line co–exist with curved line. In general this zone is larger with Mach number, for example at Mach 5 the this zone is 8.5°. For engineering purpose when seldom Mach number reaching this value it can be ignored.

13.4.11 Examples

Example 13.2:
Air flows at Mach number (M_1) or $M_x = 4$ is approaching a wedge. What is the maximum wedge angle which the oblique shock can occur? If the wedge angle is 20° calculated the weak and the strong Mach numbers and what are the respective shock angle.

SOLUTION
The find maximum wedge angle for ($M_x = 4$) D has to be equal to zero. The wedge angel that satisfy this requirement by equation (13.28) is the solution (a side to the case proximity of $\delta = 0$). The maximum values are:

$\mathbf{M_x}$	$\mathbf{M_y}$	$\delta_{\mathbf{max}}$	$\theta_{\mathbf{max}}$
4.0000	0.97234	38.7738	66.0407

To obtain the results of the weak and the strong solutions either utilize the equation (13.28) or the GDC which yields the following results

$\mathbf{M_x}$	$\mathbf{M_{y_s}}$	$\mathbf{M_{y_w}}$	$\theta_{\mathbf{s}}$	$\theta_{\mathbf{w}}$	δ
4.0000	0.48523	2.5686	1.4635	0.56660	0.34907

Example 13.3:
A cone shown in the Figure 13.15 exposed to supersonic flow and create an oblique shock. Is the shock shown in the photo is weak or strong shock? explain. Using the geometry provided in the photo predict at which of the Mach number the photo was taken based on the assumption that the cone is a wedge.

SOLUTION
The measurement shows that cone angle is 14.43° and shock angle is 30.099°. With given two angle the solution can be obtained utilizing equation (13.48) or the Potto-GDC.

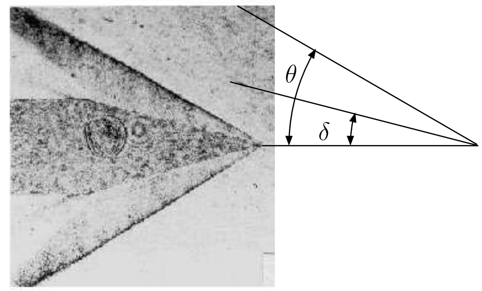


Fig. 13.15: Oblique shock occurs around a cone. This photo is courtesy of Dr. Grigory Toker a Research Professor at Cuernavaco University at Mexico. According to his measurement the cone half angle is $15°$ and the Mach number is 2.2.

$\mathbf{M_1}$	$\mathbf{M_{y_s}}$	$\mathbf{M_{y_w}}$	$\theta_\mathbf{s}$	$\theta_\mathbf{w}$	δ	$\frac{\mathbf{P_{0_y}}}{\mathbf{P_{0_x}}}$
3.2318	0.56543	2.4522	71.0143	30.0990	14.4300	0.88737

Because the flow is around Cone it must be a weak shock. Even if the cone was a wedge, the shock will be weak because the maximum (transition to a strong shock) occurs at about $60°$. Note that Mach number is larger than the predicted by the wedge.

13.4.12 Application of oblique shock

One of the practical application of the oblique shock is the design of inlet suction for supersonic flow. It is suggested that series of weak shocks should replace one normal shock to increase the efficiency (see Figure 13.17)[25].

Clearly with a proper design, the flow can be brought to a subsonic flow just below $M = 1$. In such case there is less entropy production (less pressure loss.). To illustrate the design significance of the oblique shock the following example is provided.

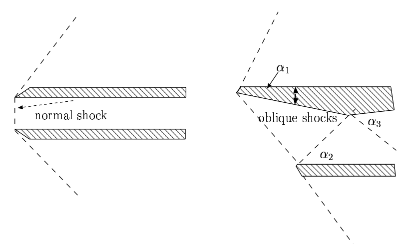


Fig. 13.17: Two variations of inlet suction for supersonic flow.

[25] In fact, there is a general proof that regardless to the equation of state (any kind of gas) the entropy is be minimized through a series of oblique shocks rather than a single normal shock. See for details in Henderson and Menikoff "Triple Shock Entropy Theorem," Journal of Fluid Mechanics 366 (1998) pp. 179–210.

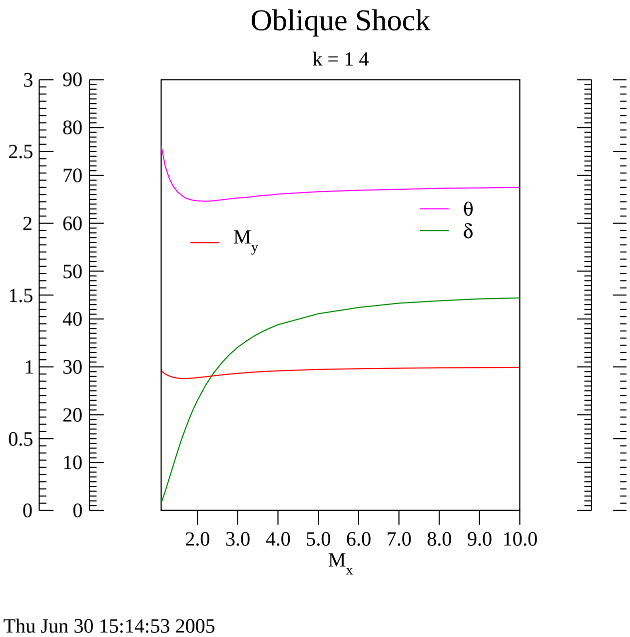


Fig. 13.16: Maximum values of the properties in oblique shock

Example 13.4:
The Section described in Figure 13.18 air is flowing into a suction section at $M = 2.0$, $P = 1.0[bar]$, *and* $T = 17°\mathrm{C}$. *Compare the different conditions in the two different configurations. Assume that only weak shock occurs.*

SOLUTION
The first configuration is of a normal shock. For which the results[26] are

$\mathbf{M_x}$	$\mathbf{M_y}$	$\frac{\mathbf{T_y}}{\mathbf{T_x}}$	$\frac{\rho_\mathbf{y}}{\rho_\mathbf{x}}$	$\frac{\mathbf{P_y}}{\mathbf{P_x}}$	$\frac{\mathbf{P_{0y}}}{\mathbf{P_{0x}}}$
2.0000	0.57735	1.6875	2.6667	4.5000	0.72087

In the oblique shock the first angle shown is

$\mathbf{M_x}$	$\mathbf{M_y}_\mathbf{s}$	$\mathbf{M_y}_\mathbf{w}$	$\theta_\mathbf{s}$	$\theta_\mathbf{w}$	δ	$\frac{\mathbf{P_{0y}}}{\mathbf{P_{0x}}}$
2.0000	0.58974	1.7498	85.7021	36.2098	7.0000	0.99445

and the additional information by the minimal info in Potto-GDC

[26]The results in this example are obtained using the graphical interface of POTTO–GDC thus, no input explanation is given. In the past the input fi le was given but the graphical interface it is no longer needed.

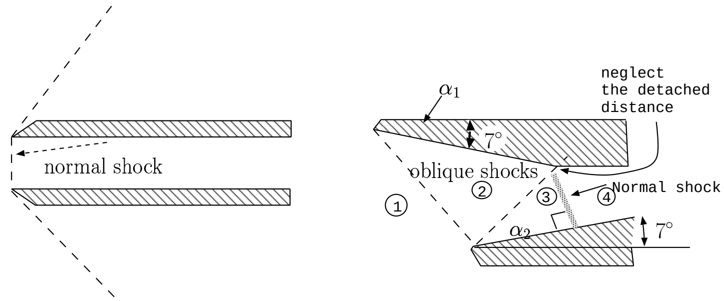


Fig. 13.18: Schematic for example 13.4

$\mathbf{M_x}$	$\mathbf{M_y}_\mathbf{w}$	$\theta_\mathbf{w}$	δ	$\frac{\mathbf{P_y}}{\mathbf{P_x}}$	$\frac{\mathbf{T_y}}{\mathbf{T_x}}$	$\frac{\mathbf{P_{0y}}}{\mathbf{P_{0x}}}$
2.0000	1.7498	36.2098	7.0000	1.2485	1.1931	0.99445

In the new region new angle is $7° + 7°$ with new upstream Mach number of $M_x = 1.7498$ results in

$\mathbf{M_x}$	$\mathbf{M_y}_\mathbf{s}$	$\mathbf{M_y}_\mathbf{w}$	$\theta_\mathbf{s}$	$\theta_\mathbf{w}$	δ	$\frac{\mathbf{P_{0y}}}{\mathbf{P_{0x}}}$
1.7498	0.71761	1.2346	76.9831	51.5549	14.0000	0.96524

And the additional information is

$\mathbf{M_x}$	$\mathbf{M_y}_\mathbf{w}$	$\theta_\mathbf{w}$	δ	$\frac{\mathbf{P_y}}{\mathbf{P_x}}$	$\frac{\mathbf{T_y}}{\mathbf{T_x}}$	$\frac{\mathbf{P_{0y}}}{\mathbf{P_{0x}}}$
1.7498	1.5088	41.8770	7.0000	1.2626	1.1853	0.99549

A oblique shock is not possible and normal shock occurs. In such case, the results are:

$\mathbf{M_x}$	$\mathbf{M_y}$	$\frac{\mathbf{T_y}}{\mathbf{T_x}}$	$\frac{\rho_\mathbf{y}}{\rho_\mathbf{x}}$	$\frac{\mathbf{P_y}}{\mathbf{P_x}}$	$\frac{\mathbf{P_{0y}}}{\mathbf{P_{0x}}}$
1.2346	0.82141	1.1497	1.4018	1.6116	0.98903

With two weak shock waves and a normal shock the total pressure loss is

$$\frac{P_{0\,4}}{P_{0\,1}} = \frac{P_{0\,4}}{P_{0\,3}}\frac{P_{0\,3}}{P_{0\,2}}\frac{P_{0\,2}}{P_{0\,1}} = 0.98903 \times 0.96524 \times 0.99445 = 0.9496$$

The static pressure ratio for the second case is

$$\frac{P_4}{P_1} = \frac{P_4}{P_3}\frac{P_3}{P_2}\frac{P_2}{P_1} = 1.6116 \times 1.2626 \times 1.285 = 2.6147$$

The loss in this case is much less than in direct normal shock. In fact, the loss in the normal shock is 31% larger for the total pressure.

Example 13.5:

A supersonic flow approaching a very long two dimensional bland wedge body and creates a detached shock at Mach 3.5 (see Figure 13.19). The half wedge angle is $10°$. What is the requited "throat" area ratio to achieve acceleration from subsonic region to supersonic region assuming one-dimensional flow.

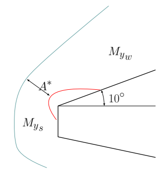


Fig. 13.19: Schematic for example 13.5

SOLUTION

The detach shock is a normal shock and the results are

$\mathbf{M_x}$	$\mathbf{M_y}$	$\frac{\mathbf{T_y}}{\mathbf{T_x}}$	$\frac{\rho_\mathbf{y}}{\rho_\mathbf{x}}$	$\frac{\mathbf{P_y}}{\mathbf{P_x}}$	$\frac{\mathbf{P_{0y}}}{\mathbf{P_{0x}}}$
3.5000	0.45115	3.3151	4.2609	14.1250	0.21295

Now utilizing the isentropic relationship for $k = 1.4$ yields

$\mathbf{M}$	$\frac{\mathbf{T}}{\mathbf{T_0}}$	$\frac{\rho}{\rho_0}$	$\frac{\mathbf{A}}{\mathbf{A^\star}}$	$\frac{\mathbf{P}}{\mathbf{P_0}}$	$\frac{\mathbf{A\times P}}{\mathbf{A^*\times P_0}}$
0.45115	0.96089	0.90506	1.4458	0.86966	1.2574

Thus the area ratio has to be 1.4458. Note that the pressure after the weak shock is irrelevant to area ratio between the normal shock to the "throat" according to the standard nozzle analysis.

Example 13.6:

The effects of double wedge were explained in government web site as shown in figure 13.20. Adopted this description and assumed that turn of $6°$ is made out of two equal angles of $3°$ (see Figure 13.20). Assume that there are no boundary layers and all the shocks are weak and straight. Carry the calculation for $M_1 = 3.0$. Find the required angle of shock BE. Then, explain why this description has internal conflict.

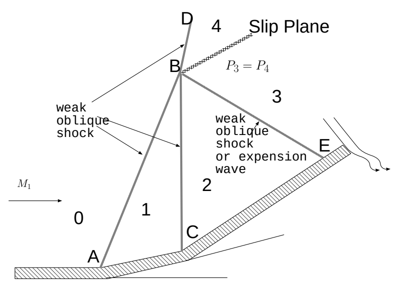


Fig. 13.20: Schematic of two angles turn with two weak shocks

SOLUTION

The shock BD is an oblique shock which response to total turn of $6°$. The condition for this shock are:

$\mathbf{M_x}$	$\mathbf{M_{y_s}}$	$\mathbf{M_{y_w}}$	$\theta_{\mathbf{s}}$	$\theta_{\mathbf{w}}$	δ	$\frac{\mathbf{P_{0_y}}}{\mathbf{P_{0_x}}}$
3.0000	0.48013	2.7008	87.8807	23.9356	6.0000	0.99105

The transition for shock AB is

$\mathbf{M_x}$	$\mathbf{M_{y_s}}$	$\mathbf{M_{y_w}}$	$\theta_{\mathbf{s}}$	$\theta_{\mathbf{w}}$	δ	$\frac{\mathbf{P_{0_y}}}{\mathbf{P_{0_x}}}$
3.0000	0.47641	2.8482	88.9476	21.5990	3.0000	0.99879

For the Shock BC the results are:

$\mathbf{M_x}$	$\mathbf{M_{y_s}}$	$\mathbf{M_{y_w}}$	$\theta_{\mathbf{s}}$	$\theta_{\mathbf{w}}$	δ	$\frac{\mathbf{P_{0_y}}}{\mathbf{P_{0_x}}}$
2.8482	0.48610	2.7049	88.8912	22.7080	3.0000	0.99894

And the isentropic relationship for $M = 2.7049,\ 2.7008$ are

$\mathbf{M}$	$\frac{\mathbf{T}}{\mathbf{T_0}}$	$\frac{\rho}{\rho_0}$	$\frac{\mathbf{A}}{\mathbf{A^\star}}$	$\frac{\mathbf{P}}{\mathbf{P_0}}$	$\frac{\mathbf{A\times P}}{\mathbf{A^*\times P_0}}$
2.7049	0.40596	0.10500	3.1978	0.04263	0.13632
2.7008	0.40669	0.10548	3.1854	0.04290	0.13665

The combined shocks AB and BC provides the base to calculation of the total pressure ratio at zone 3. The total pressure ratio at zone 2 is

$$\frac{P_{0\,2}}{P_{0\,0}} = \frac{P_{0\,2}}{P_{0\,1}}\frac{P_{0\,1}}{P_{0\,0}} = 0.99894 \times 0.99879 = 0.997731283 \tag{13.55}$$

On the other hand, the pressure at 4 has to be

$$\frac{P_4}{P_{0\,1}} = \frac{P_4}{P_{0\,4}}\frac{P_{0\,4}}{P_{0\,1}} = 0.04290 \times 0.99105 = 0.042516045 \tag{13.56}$$

The static pressure at zone 4 and zone 3 have to match according to the government suggestion hence, the angle for BE shock which cause this pressure ratio needed to be found. To that, check whether the pressure at 2 is above or below or above the pressure (ratio) in zone 4.

$$\frac{P_2}{P_{0\,2}} = \frac{P_{0\,2}}{P_{0\,0}}\frac{P_2}{P_{0\,2}} = 0.997731283 \times 0.04263 = 0.042436789$$

Since $\frac{P_2}{P_{0\,2}} < \frac{P_4}{P_{0\,1}}$ a weak shock must occur to increase the static pressure (see Figure 5.4). The increase has to be

$$P_3/P_2 = 0.042516045/0.042436789 = 1.001867743$$

To achieve this kind of pressure ratio perpendicular component has to be

$\mathbf{M_x}$	$\mathbf{M_y}$	$\frac{\mathbf{T_y}}{\mathbf{T_x}}$	$\frac{\rho_\mathbf{y}}{\rho_\mathbf{x}}$	$\frac{\mathbf{P_y}}{\mathbf{P_x}}$	$\frac{\mathbf{P_{0_y}}}{\mathbf{P_{0_x}}}$
1.0008	0.99920	1.0005	1.0013	1.0019	1.00000

The shock angle, θ can be calculated from

$$\theta = \sin^{-1} 1.0008/2.7049 = 21.715320879°$$

The deflection angle for such shock angle with Mach number is

$\mathbf{M_x}$	$\mathbf{M_{y_s}}$	$\mathbf{M_{y_w}}$	$\theta_\mathbf{s}$	$\theta_\mathbf{w}$	δ	$\frac{\mathbf{P_{0_y}}}{\mathbf{P_{0_x}}}$
2.7049	0.49525	2.7037	0.0	21.72	0.026233	1.00000

For the last calculation is clear that the government proposed schematic of the double wedge is conflict with boundary condition. The flow in zone 3 will flow into the wall in about $2.7°$. In reality the flow of double wedge produce curved shock surfaces with several zones. Only far away for the double wedge the flow behaves as only angle exist of $6°$.

Example 13.7:
Calculate the flow deflection angle and other parameters downstream when Mach angle of $34°$ *and* $P_1 = 3[bar]$, $T_1 = 27°\mathrm{C}$ $U_1 = 1000 m/sec$. *Assume* $k = 1.4$ *and* $R = 287 J/KgK$

<u>SOLUTION</u>
The angle of Mach angle of $34°$ while below maximum deflection means that the it is a weak shock. Yet, the Upstream Mach number, M_1, has to be determined

$$M_1 = \frac{U_1}{\sqrt{kRT}} = \frac{1000}{1.4 \times 287 \times 300} = 2.88$$

With this Mach number and the Mach deflection either using the Table or the figure or POTTO-GDC results in

$\mathbf{M_x}$	$\mathbf{M_{y_s}}$	$\mathbf{M_{y_w}}$	$\theta_\mathbf{s}$	$\theta_\mathbf{w}$	δ	$\frac{\mathbf{P_{0_y}}}{\mathbf{P_{0_x}}}$
2.8800	0.48269	2.1280	0.0	34.00	15.78	0.89127

The relationship for the temperature and pressure can be obtained by using equation (13.15) and (13.13) or simply converting the M_1 to perpendicular component.

$$M_{1n} = M_1 * \sin\theta = 2.88 \sin(34.0) = 1.61$$

From the Table (5.1) or GDC the following can be obtained.

$\mathbf{M_x}$	$\mathbf{M_y}$	$\frac{\mathbf{T_y}}{\mathbf{T_x}}$	$\frac{\rho_\mathbf{y}}{\rho_\mathbf{x}}$	$\frac{\mathbf{P_y}}{\mathbf{P_x}}$	$\frac{\mathbf{P_{0_y}}}{\mathbf{P_{0_x}}}$
1.6100	0.66545	1.3949	2.0485	2.8575	0.89145

The temperature ratio combined upstream temperature yield

$$T_2 = 1.3949 \times 300 \sim 418.5K$$

and the same for the pressure

$$P_2 = 2.8575 \times 3 = 8.57[bar]$$

And the velocity

$$U_{n2} = M_{y_w}\sqrt{kRT} = 2.128\sqrt{1.4 \times 287 \times 418.5} = 872.6[m/sec]$$

Example 13.8:
For Mach number 2.5 and wedge with total angle of $22°$ calculate the ratio of the stagnation pressure.

SOLUTION
Utilizing GDC for Mach number 2.5 and angle of $11°$ results in

$\mathbf{M_x}$	$\mathbf{M_{y_s}}$	$\mathbf{M_{y_w}}$	$\theta_\mathbf{s}$	$\theta_\mathbf{w}$	δ	$\frac{\mathbf{P_{0_y}}}{\mathbf{P_{0_x}}}$
2.5000	0.53431	2.0443	85.0995	32.8124	11.0000	0.96873

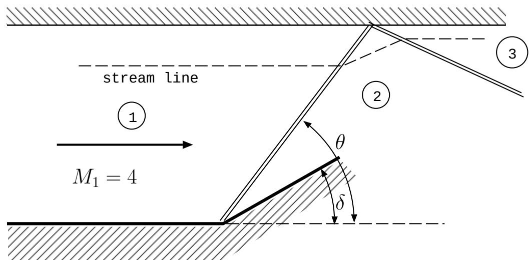
Example 13.9:
What the maximum pressure ratio that can be obtained on wedge when the gas is flowing in 2.5 Mach without any close boundaries. Would it make any difference if the wedge was flowing into the air? if so what is the difference.

SOLUTION
It has to be recognized that without any other boundary condition the shock is weak shock. For weak shock the maximum pressure ratio is obtained when at the deflection point because it is the closest to normal shock. The obtain the maximum point for 2.5 Mach number either use Maximum Deflection Mach number's equation or POTTO–GDC

$\mathbf{M_x}$	$\mathbf{M_{y_{max}}}$	$\theta_\mathbf{max}$	δ	$\frac{\mathbf{P_y}}{\mathbf{P_x}}$	$\frac{\mathbf{T_y}}{\mathbf{T_x}}$	$\frac{\mathbf{P_{0_y}}}{\mathbf{P_{0_x}}}$
2.5000	0.94021	64.7822	29.7974	4.3573	2.6854	0.60027

In these calculation Maximum Deflection Mach's equation was used to calculate Normal component of the upstream than Mach angle was calculated using the geometrical relationship of $\theta = \sin^{-1} M_{1n}/M_1$. With these two quantities utilizing equation (13.12) the deflection angle, δ is obtained.

Example 13.10:
Consider the schematic shown in the following Figure.



Assume that upstream Mach number is 4 and the deflection angle is $\delta = 15°$. Compute the pressure ratio and temperature ratio after the second shock (sometimes it referred as the reflective shock while the first shock is called as the incidental shock).

SOLUTION

This kind problem is essentially two wedges placed in a certain geometry. It is clear that the flow must be parallel to the wall. For the first shock, the upstream Mach number is known with deflection angle. Utilizing the table or POTTO–GDC, the following can be obtained.

$\mathbf{M_x}$	$\mathbf{M_{y_s}}$	$\mathbf{M_{y_w}}$	$\theta_\mathbf{s}$	$\theta_\mathbf{w}$	δ	$\frac{\mathbf{P_{0_y}}}{\mathbf{P_{0_x}}}$
4.0000	0.46152	2.9290	85.5851	27.0629	15.0000	0.80382

And the additional information is by using minimal information ratio button in POTTO–GDC

$\mathbf{M_x}$	$\mathbf{M_{y_w}}$	$\theta_\mathbf{w}$	δ	$\frac{\mathbf{P_y}}{\mathbf{P_x}}$	$\frac{\mathbf{T_y}}{\mathbf{T_x}}$	$\frac{\mathbf{P_{0_y}}}{\mathbf{P_{0_x}}}$
4.0000	2.9290	27.0629	15.0000	1.7985	1.7344	0.80382

With Mach number of $M = 2.929$ the second deflection angle is also $15°$. with these values the following can be obtained.

$\mathbf{M_x}$	$\mathbf{M_{y_s}}$	$\mathbf{M_{y_w}}$	$\theta_\mathbf{s}$	$\theta_\mathbf{w}$	δ	$\frac{\mathbf{P_{0_y}}}{\mathbf{P_{0_x}}}$
2.9290	0.51367	2.2028	84.2808	32.7822	15.0000	0.90041

and the additional information is

$\mathbf{M_x}$	$\mathbf{M_{y_w}}$	$\theta_\mathbf{w}$	δ	$\frac{\mathbf{P_y}}{\mathbf{P_x}}$	$\frac{\mathbf{T_y}}{\mathbf{T_x}}$	$\frac{\mathbf{P_{0_y}}}{\mathbf{P_{0_x}}}$
2.9290	2.2028	32.7822	15.0000	1.6695	1.5764	0.90041

With the combined tables the ratios can be easily calculated. Note that hand calculations requires endless time looking up graphical representation of the solution. Utilizing the POTTO–GDC provides a solution in just a few clicks.

$$\frac{P_1}{P_3} = \frac{P_1}{P_2}\frac{P_2}{P_3} = 1.7985 \times 1.6695 = 3.0026$$

$$\frac{T_1}{T_3} = \frac{T_1}{T_2}\frac{T_2}{T_3} = 1.7344 \times 1.5764 = 2.632$$

Example 13.11:
Similar example as before but here Mach angel is $29°$ *and Mach number is 2.85. Again calculate the ratios downstream after the second shock and the deflection angle.*

SOLUTION
Here the Mach number and the Mach angle are given. With these pieces of information utilizing the GDC provides the following:

$\mathbf{M_x}$	$\mathbf{M_y}_\mathbf{s}$	$\mathbf{M_y}_\mathbf{w}$	$\theta_\mathbf{s}$	$\theta_\mathbf{w}$	δ	$\frac{\mathbf{P_0}_\mathbf{y}}{\mathbf{P_0}_\mathbf{x}}$
2.8500	0.48469	2.3575	0.0	29.00	10.51	0.96263

and the additional information by utilizing the minimal info button in GDC provides

$\mathbf{M_x}$	$\mathbf{M_y}_\mathbf{w}$	$\theta_\mathbf{w}$	δ	$\frac{\mathbf{P_y}}{\mathbf{P_x}}$	$\frac{\mathbf{T_y}}{\mathbf{T_x}}$	$\frac{\mathbf{P_0}_\mathbf{y}}{\mathbf{P_0}_\mathbf{x}}$
2.8500	2.3575	29.0000	10.5131	1.4089	1.3582	0.96263

With the deflection Angle of $\delta = 10.51$ the so called reflective shock provide the following information

$\mathbf{M_x}$	$\mathbf{M_y}_\mathbf{s}$	$\mathbf{M_y}_\mathbf{w}$	$\theta_\mathbf{s}$	$\theta_\mathbf{w}$	δ	$\frac{\mathbf{P_0}_\mathbf{y}}{\mathbf{P_0}_\mathbf{x}}$
2.3575	0.54894	1.9419	84.9398	34.0590	10.5100	0.97569

and the additional information of

$\mathbf{M_x}$	$\mathbf{M_y}_\mathbf{w}$	$\theta_\mathbf{w}$	δ	$\frac{\mathbf{P_y}}{\mathbf{P_x}}$	$\frac{\mathbf{T_y}}{\mathbf{T_x}}$	$\frac{\mathbf{P_0}_\mathbf{y}}{\mathbf{P_0}_\mathbf{x}}$
2.3575	1.9419	34.0590	10.5100	1.3984	1.3268	0.97569

$$\frac{P_1}{P_3} = \frac{P_1}{P_2}\frac{P_2}{P_3} = 1.4089 \times 1.3984 \sim 1.97$$

$$\frac{T_1}{T_3} = \frac{T_1}{T_2}\frac{T_2}{T_3} = 1.3582 \times 1.3268 \sim 1.8021$$

Example 13.12:
Compare a direct normal shock to oblique shock with a normal shock. Where will be total pressure loss (entropy) larger? Assume that upstream Mach number is 5 and the first oblique shock has Mach angle of $30°$. What is the deflection angle in this case?

SOLUTION
For the normal shock the results are

$\mathbf{M_x}$	$\mathbf{M_y}$	$\frac{\mathbf{T_y}}{\mathbf{T_x}}$	$\frac{\rho_\mathbf{y}}{\rho_\mathbf{x}}$	$\frac{\mathbf{P_y}}{\mathbf{P_x}}$	$\frac{\mathbf{P_{0_y}}}{\mathbf{P_{0_x}}}$
5.0000	0.41523	5.8000	5.0000	29.0000	0.06172

While the results for the oblique shock are

$\mathbf{M_x}$	$\mathbf{M_{y_s}}$	$\mathbf{M_{y_w}}$	$\theta_\mathbf{s}$	$\theta_\mathbf{w}$	δ	$\frac{\mathbf{P_{0_y}}}{\mathbf{P_{0_x}}}$
5.0000	0.41523	3.0058	0.0	30.00	20.17	0.49901

And the additional information is

$\mathbf{M_x}$	$\mathbf{M_{y_w}}$	$\theta_\mathbf{w}$	δ	$\frac{\mathbf{P_y}}{\mathbf{P_x}}$	$\frac{\mathbf{T_y}}{\mathbf{T_x}}$	$\frac{\mathbf{P_{0_y}}}{\mathbf{P_{0_x}}}$
5.0000	3.0058	30.0000	20.1736	2.6375	2.5141	0.49901

The normal shock that follows this oblique is

$\mathbf{M_x}$	$\mathbf{M_y}$	$\frac{\mathbf{T_y}}{\mathbf{T_x}}$	$\frac{\rho_\mathbf{y}}{\rho_\mathbf{x}}$	$\frac{\mathbf{P_y}}{\mathbf{P_x}}$	$\frac{\mathbf{P_{0_y}}}{\mathbf{P_{0_x}}}$
3.0058	0.47485	2.6858	3.8625	10.3740	0.32671

The pressure ratios of the oblique shock with normal shock is the total shock in the second case.

$$\frac{P_1}{P_3} = \frac{P_1}{P_2}\frac{P_2}{P_3} = 2.6375 \times 10.374 \sim 27.36$$

$$\frac{T_1}{T_3} = \frac{T_1}{T_2}\frac{T_2}{T_3} = 2.5141 \times 2.6858 \sim 6.75$$

Note the static pressure raised less the combination shocks compare the the normal shock but the total pressure has the opposite results.

Example 13.13:
A flow in tunnel end up with two deflection angles from both sides (see the following Figure).

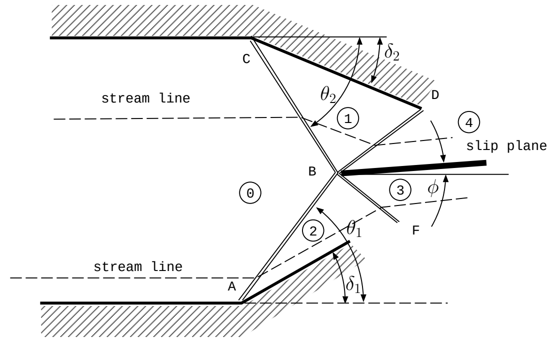


Illustration for example 13.13

For upstream Mach number of 5 and deflection angle of $12°$ and $15°$, calculate the the pressure at zones 3 and 4 based on the assumption that the slip plan is half of difference between the two deflection angles. Based on these calculations, explain whether the slip angle is larger or smaller the the difference of the deflection angle.

SOLUTION

The first two zones immediately after are computed using the same techniques that were developed and discussed earlier.

For the first direction is for $15°$ and Mach number =5.

$\mathbf{M_x}$	$\mathbf{M_y}_\mathbf{s}$	$\mathbf{M_y}_\mathbf{w}$	$\theta_\mathbf{s}$	$\theta_\mathbf{w}$	δ	$\frac{\mathbf{P_0}_\mathbf{y}}{\mathbf{P_0}_\mathbf{x}}$
5.0000	0.43914	3.5040	86.0739	24.3217	15.0000	0.69317

And the addition conditions are

$\mathbf{M_x}$	$\mathbf{M_y}_\mathbf{w}$	$\theta_\mathbf{w}$	δ	$\frac{\mathbf{P_y}}{\mathbf{P_x}}$	$\frac{\mathbf{T_y}}{\mathbf{T_x}}$	$\frac{\mathbf{P_0}_\mathbf{y}}{\mathbf{P_0}_\mathbf{x}}$
5.0000	3.5040	24.3217	15.0000	1.9791	1.9238	0.69317

For the second direction is for $12°$ and Mach number =5.

$\mathbf{M_x}$	$\mathbf{M_y}_\mathbf{s}$	$\mathbf{M_y}_\mathbf{w}$	$\theta_\mathbf{s}$	$\theta_\mathbf{w}$	δ	$\frac{\mathbf{P_0}_\mathbf{y}}{\mathbf{P_0}_\mathbf{x}}$
5.0000	0.43016	3.8006	86.9122	21.2845	12.0000	0.80600

And the additional conditions are

$\mathbf{M_x}$	$\mathbf{M_y}_\mathbf{w}$	$\theta_\mathbf{w}$	δ	$\frac{\mathbf{P_y}}{\mathbf{P_x}}$	$\frac{\mathbf{T_y}}{\mathbf{T_x}}$	$\frac{\mathbf{P_0}_\mathbf{y}}{\mathbf{P_0}_\mathbf{x}}$
5.0000	3.8006	21.2845	12.0000	1.6963	1.6625	0.80600

The conditions in zone 4 and zone 3 have to have two things that are equal, and they are the pressure and the velocity direction. It has to be noticed that the velocity magnitudes in zone 3 and 4 do not have to be equal. This non continuous velocity profile can occurs in our model because it is assumed that fluid is non–viscous.

If the two sides were equal because symmetry the slip angle was zero. It is to say, for the analysis, that only one deflection angle exist. For the two different deflection angles, the slip angle has two extreme cases. The first case is where match lower deflection angle and second to match the higher deflection angle. In this case, it is assumed that the slip angle moves half of the angle to satisfy both of the deflection angles (first approximation). Under this assumption the continuous in zone 3 are solved by looking at deflection angle of $12^\circ + 1.5^\circ = 13.5^\circ$ which results in

$\mathbf{M_x}$	$\mathbf{M_{y_s}}$	$\mathbf{M_{y_w}}$	$\theta_\mathbf{s}$	$\theta_\mathbf{w}$	δ	$\frac{\mathbf{P_{0_y}}}{\mathbf{P_{0_x}}}$
3.5040	0.47413	2.6986	85.6819	27.6668	13.5000	0.88496

with the additional information

$\mathbf{M_x}$	$\mathbf{M_{y_w}}$	$\theta_\mathbf{w}$	δ	$\frac{\mathbf{P_y}}{\mathbf{P_x}}$	$\frac{\mathbf{T_y}}{\mathbf{T_x}}$	$\frac{\mathbf{P_{0_y}}}{\mathbf{P_{0_x}}}$
3.5040	2.6986	27.6668	13.5000	1.6247	1.5656	0.88496

And in zone 4 the conditions are due to deflection angle of 13.5° and Mach 3.8006

$\mathbf{M_x}$	$\mathbf{M_{y_s}}$	$\mathbf{M_{y_w}}$	$\theta_\mathbf{s}$	$\theta_\mathbf{w}$	δ	$\frac{\mathbf{P_{0_y}}}{\mathbf{P_{0_x}}}$
3.8006	0.46259	2.9035	85.9316	26.3226	13.5000	0.86179

with the additional information

$\mathbf{M_x}$	$\mathbf{M_{y_w}}$	$\theta_\mathbf{w}$	δ	$\frac{\mathbf{P_y}}{\mathbf{P_x}}$	$\frac{\mathbf{T_y}}{\mathbf{T_x}}$	$\frac{\mathbf{P_{0_y}}}{\mathbf{P_{0_x}}}$
3.8006	2.9035	26.3226	13.5000	1.6577	1.6038	0.86179

From these tables the pressure ratio at zone 3 and 4 can be calculated

$$\frac{P_3}{P_4} = \frac{P_3}{P_2}\frac{P_2}{P_0}\frac{P_0}{P_1}\frac{P_1}{P_4} = 1.6247 \times 1.9791 \frac{1}{1.6963}\frac{1}{1.6038} \sim 1.18192$$

To reduce the pressure ratio the deflection angle has to be reduced (remember that at weak weak shock almost no pressure change). Thus, the pressure at zone 3 has to be reduced. To reduce the pressure the angle of slip plane has to increase from $1.5\circ$ to a larger number.

Example 13.14:
The previous example give raise to another question the order of the deflection angles. Consider the same values as previous analysis, if oblique shock with first with angle of $15°$ and $12°$ or opposite order make a difference ($M = 5$)? If not what order make bigger entropy production or pressure loss? (No general proof is needed).

SOLUTION
Waiting for the solution

13.4.13 Optimization of Suction Section Design

Under heavy construction please ignore

The question raises what is the optimum design for inlet suction unit. The are several considerations that have to be taken into account aside to supersonic flow which include for example the material strength consideration and operation factors.

The optimum deflection angle is a function of the Mach number range in with suction section is operated in. The are researchers that suggest that the numerical work with possibility to work the abrupt solution.

is presentation of the experimental works is useful here? or present the numerical works? Perhaps to present the simplified model.

13.5 Summary

As normal shock, the oblique shock the upstream Mach number, M_1 is always greater than 1. However, not as the normal shock downstream Mach number, M_2 could be larger or smaller then 1. The perpendicular component of the downstream Mach number, M_{1n} is always smaller than 1. For given M_1 and deflection angle, δ there could be three solutions: the first one is the "impossible" solution in case where D is negative two the weak shock, and three the strong shock. When D is positive there no physical solution and only normal shock exist. When D is equal to zero, a spacial case is created for the weak and strong solution are equal (for large deflection angle). When $D > 0$, for large deflection angle, there is possibility of no two-dimensional solution resulting in a detached shock case.

13.6 Appendix: Oblique Shock Stability Analysis

The stability analysis is an analysis which answer the question what happen if for some reasons, the situation moves away from the expected solution. If the answer turned out to be that situation will return to its

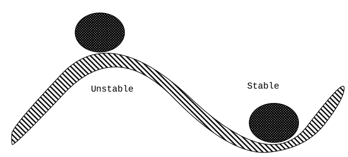


Fig. 13.21: Typical examples of of unstable and stable situations

original state then it referred to as the stable situation. On the other hand, if the answer is negative, then the situation is referred to as unstable. An example to this situation, is a ball shown in the Figure 13.21. Instinctively, the stable and unstable can be recognized. There is also the situation where the ball is between the stable and unstable situations when the ball is on plan field which referred as the neutrally stable. In the same manner, the analysis for the oblique shock wave is carried out. The only difference is that here, there are more than one parameter that can changed, for example, the shock angle, deflection angle, upstream Mach number. In this example only the weak solution is explained. The similar analysis can be applied to strong shock. Yet, in that analysis it has to remember that when the flow became subsonic the equation change from hyperbolic to elliptic equation. This change complicates the explanation and omitted in this section. Of course, in the analysis the strong shock results in elliptic solution (or region) as oppose to hyperbolic in weak shock. As results, the discussion is more complicated but similar analysis can be applied to the strong shock.

The change in the inclination angel results in a different upstream Mach number and a different pressure. On the other hand, to maintain same direction stream lines the virtual change in the deflection angle has to be opposite direction of the change of shock angle. The change is determined from the solution provided before or from the approximation (13.51).

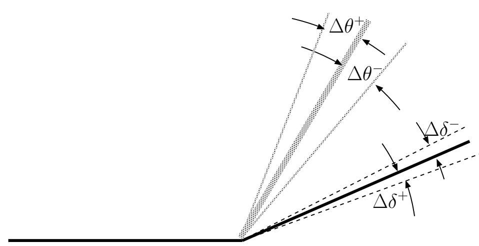


Fig. 13.22: The schematic of stability analysis for oblique shock

$$\Delta\theta = \frac{k+1}{2}\Delta\delta \tag{13.57}$$

Equation (13.57) can be applied either to positive, $\Delta\theta^+$ or negative $\Delta\theta^-$ values. The pressure difference at the wall becomes negative increment which tends to pull the shock angle to opposite direction. The opposite when the deflection increment became negative the deflection angle becomes positive which increase the pressure at the wall. Thus, the weak shock is stable.

Please note this analysis doesn't applied to the case in the close proximity of the $\delta = 0$. In fact, the shock wave is unstable according to this analysis to one direction but stable to the other direction. Yet, it must be point out that doesn't mean that flow is unstable but rather that the model are incorrect. There isn't known experimental evidence showing that flow is unstable for $\delta = 0$.

CHAPTER 14

Prandtl-Meyer Function

14.1 Introduction

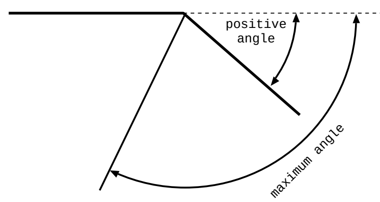


Fig. 14.1: The defi nition of the angle for Prandtl–Meyer function here

As it was discussed in Chapter (13) when the deflection turns to the opposite direction of the flow and accelerated the flow to match the boundary condition. The transition as opposite to the oblique shock is smooth without any jump in properties. Here because the tradition, the deflection angle is denoted as a positive when the it appears away from the flow (see the Figure (14.8)). In somewhat similar concept to oblique shock there exist a "detachment" point above which this model breaks and another model have to be implemented. Yet, when this model breaks, the flow becomes complicate and flow separation occurs and no known simple model describes the situation. As oppose to the oblique shock, there is no limitation of the Prandtl-Meyer function to approach zero. Yet, for very small angles, because imperfections of the wall have to be assumed insignificant.

Supersonic expansion and isentropic compression (Prandtl-Meyer function), is extension of the Mach Line concept. Reviewing the Mach line shows that a disturbance in a field of supersonic flow moves in an angle of μ, which is defined as (see Figure (14.2))

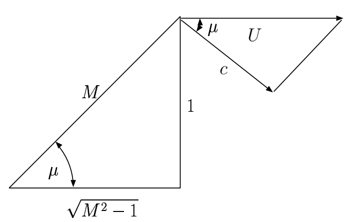


Fig. 14.2: The angles of the Mach line triangle

$$\mu = \sin^{-1}\left(\frac{1}{M}\right) \tag{14.1}$$

or

$$\mu = \tan^{-1} \frac{1}{\sqrt{M^1 - 1}} \tag{14.2}$$

A Mach line results of a small disturbance of the wall contour is discussed here. This Mach line is assumed to be results of positive angle. The reasons that "negative" angle is not applicable is because coalescing of small Mach wave results in a shock wave. However, no shock is created for many small positive angles.

The reason that Mach line is the chief line in the analysis because this line is the line on which the information of the shape of contour of the wall propagates. Once, the contour is changed the direction of the flow changes to fit the wall. This change results in a change of the flow properties and is assumed here to be isotropic for a positive angle. This assumption turned out to be not far way from realty. In this chapter a discussion on the relationship between the flow properties and the flow direction is presented.

14.2 Geometrical Explanation

The change in the flow direction is results of the change in the tangential component. Hence, the total Mach number increases. Therefore, the Mach angle results is increase and a change in the direction of the flow appears. The velocity component at direction of the Mach line assumed to be constant to satisfy the assumption that the change is results of the contour only. Later, this assumption will be examined. This change results in the change in the direction of the flow. The typical simplifications for geometrical functions are used

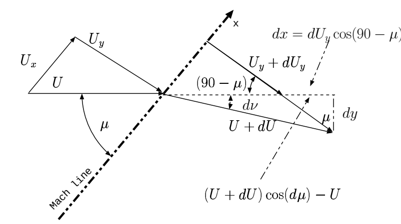


Fig. 14.3: The schematic of the turning fbw

$$\begin{aligned} d\nu &\sim \sin(d\nu); \\ \cos(d\nu) &\sim 1 \end{aligned} \tag{14.3}$$

These simplifications are the core why the change occurs only in the perpendicular direction ($d\nu << 1$). The change of the velocity in the flow direction, dx is

$$dx = (U + dU)\cos\nu - U = dU \tag{14.4}$$

Also in the same manner the velocity in perpendicular to the flow, dy, is

$$dy = (U + dU)\sin(d\nu) = U d\nu \tag{14.5}$$

The $\tan\mu$ is the ratio of dy/dx (see Figure (14.3))

$$\tan\mu = \frac{dx}{dy} = \frac{dU}{U d\nu} \tag{14.6}$$

The ratio dU/U was shown to be

$$\frac{dU}{U} = \frac{dM^2}{2M^2\left(1 + \frac{k-1}{2}M^2\right)} \tag{14.7}$$

Combining equation (14.6) and (14.7) transform to

$$d\nu = -\frac{\sqrt{M^2-1}dM^2}{2M^2\left(1 + \frac{k-1}{2}M^2\right)} \tag{14.8}$$

After integration of the equation (14.8) results in

$$\nu(M) = -\sqrt{\frac{k+1}{k-1}}\tan^{-1}\sqrt{\frac{k-1}{k+1}\left(M^2-1\right)} + \tan^{-1}\sqrt{\left(M^2-1\right)} + constant \tag{14.9}$$

The constant can be chosen in a such a way that $\nu = 0$ at $M = 1$.

14.2.1 Alternative Approach to Governing equations

In the previous section, a simplified version was derived based on geometrical arguments. In this section more rigorous explanation is provided. It must be recognized that here the cylindrical coordinates are advantageous because the flow turned around a single point.

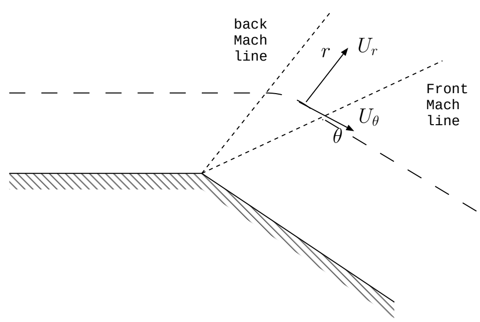


Fig. 14.4: The schematic of the coordinate for the mathematical description

For this coordinate system, the mass conservation can be written as

$$\frac{\partial\left(\rho r U_r\right)}{\partial r} + \frac{\partial\left(\rho U_\theta\right)}{\partial\theta} = 0 \tag{14.10}$$

The momentum equations are expressed as

$$U_r\frac{\partial U_r}{\partial r} + \frac{U_\theta}{r}\frac{\partial U_r}{\partial\theta} - \frac{{U_\theta}^2}{r} = -\frac{1}{\rho}\frac{\partial P}{\partial r} = -\frac{c^2}{\rho}\frac{\partial\rho}{\partial r} \tag{14.11}$$

$$U_r\frac{\partial U_\theta}{\partial r} + \frac{U_\theta}{r}\frac{\partial U_\theta}{\partial\theta} - \frac{U_\theta U_r}{r} = -\frac{1}{r\rho}\frac{\partial P}{\partial\theta} = -\frac{c^2}{r\rho}\frac{\partial\rho}{\partial\theta} \tag{14.12}$$

If it is assumed that the flow isn't a function of the radios, r, then all the derivatives with respect to radios vanish. One has to remember that when r enter to the function, like the first term in mass equation, the derivative isn't zero. Hence, the mass equation reduced to

$$\rho U_r + \frac{\partial \left(\rho U_\theta\right)}{\partial \theta} = 0 \tag{14.13}$$

After rearrangement equation (14.13) transforms into

$$-\frac{1}{U_\theta}\left(U_r + \frac{\partial U_\theta}{\partial \theta}\right) = \frac{1}{\rho}\frac{\partial \rho}{\partial \theta} \tag{14.14}$$

The momentum equations are obtained the form of

$$\begin{aligned} \frac{U_\theta}{r}\frac{\partial U_r}{\partial \theta} - \frac{{U_\theta}^2}{r} &= 0 \\ U_\theta\left(\frac{\partial U_r}{\partial \theta} - U_\theta\right) &= 0 \end{aligned} \tag{14.15}$$

$$\begin{aligned} \frac{U_\theta}{r}\frac{\partial U_\theta}{\partial \theta} - \frac{U_\theta U_r}{r} &= -\frac{c^2}{r\rho}\frac{\partial \rho}{\partial \theta} \\ U_\theta\left(\frac{\partial U_\theta}{\partial \theta} - U_r\right) &= -\frac{c^2}{\rho}\frac{\partial \rho}{\partial \theta} \end{aligned} \tag{14.16}$$

Substituting the term $\frac{1}{\rho}\frac{\partial \rho}{\partial \theta}$ from equation (14.14) into equation (14.16) results in

$$U_\theta\left(\frac{\partial U_\theta}{\partial \theta} - U_r\right) = \frac{c^2}{U_\theta}\left(U_r + \frac{\partial U_\theta}{\partial \theta}\right) \tag{14.17}$$

or

$${U_\theta}^2\left(U_r + \frac{\partial U_\theta}{\partial \theta}\right) = c^2\left(U_r + \frac{\partial U_\theta}{\partial \theta}\right) \tag{14.18}$$

And additional rearrangement results in

$$\left(c^2 - {U_\theta}^2\right)\left(U_r + \frac{\partial U_\theta}{\partial \theta}\right) = 0 \tag{14.19}$$

From equation (14.19) it follows that

$$U_\theta = c \tag{14.20}$$

It is remarkable that tangential velocity at every turn is the speed of sound! It must be point out that the total velocity isn't at the speed of sound but only the tangential component. In fact, based on definition of the Mach angle, the component shown in Figure (14.3) under U_y is equal to speed of sound, $M = 1$.

After some additional rearrangement equation (14.15) becomes

$$\frac{U_\theta}{r}\left(\frac{\partial U_r}{\partial \theta} - U_\theta\right) = 0 \tag{14.21}$$

If r isn't approaching infinity, ∞ and since $U_\theta \neq 0$ leads to

$$\frac{\partial U_r}{\partial \theta} = U_\theta \tag{14.22}$$

In the literature, these results associated with line of characteristic line[1]. This analysis can be also applied to the same equation when they normalized by Mach number. However, the dimensionlization can be applied at this stage as well.

The energy equation for any point on stream line is

$$h(\theta) + \frac{{U_\theta}^2 + {U_r}^2}{2} = h_0 \tag{14.23}$$

For enthalpy in ideal gas with a constant specific heat, k, is

$$h(\theta) = C_p T = C_p \frac{R}{R} T = \frac{1}{(k-1)} \overbrace{\overbrace{\frac{C_p}{C_v}}^{k} RT}^{c(\theta)^2} = \frac{c^2}{k-1} \tag{14.24}$$

and substituting this equality, (equation (14.24)) into equation (14.23) results

$$\frac{c^2}{k-1} + \frac{{U_\theta}^2 + {U_r}^2}{2} = h_0 \tag{14.25}$$

Utilizing equation (14.20) for the speed of sound and substituting the radial velocity equation (14.22) transformed equation (14.25) into

$$\frac{\left(\frac{\partial U_r}{\partial \theta}\right)^2}{k-1} + \frac{\left(\frac{\partial U_r}{\partial \theta}\right)^2 + {U_r}^2}{2} = h_0 \tag{14.26}$$

After some rearrangement equation (14.27) becomes

$$\frac{k+1}{k-1}\left(\frac{\partial U_r}{\partial \theta}\right)^2 + {U_r}^2 = 2h_0 \tag{14.27}$$

Note, U_r must be positive. The solution of the differential equation (14.27) incorporating the constant into it becomes

$$U_r = \sqrt{2h_0}\sin\left(\theta\sqrt{\frac{k-1}{k+1}}\right) \tag{14.28}$$

[1]This topic is under construction.

which satisfied equation (14.27) (because $\sin^2\theta + \cos^2\theta = 1$). The arbitrary constant in equation (14.28) is chosen such that $U_r(\theta = 0) = 0$. The tangential velocity obtains the form

$$U_\theta = c = \frac{\partial U_r}{\partial \theta} = \sqrt{\frac{k-1}{k+1}}\sqrt{2\,h_0}\;\;\cos\left(\theta\sqrt{\frac{k-1}{k+1}}\right) \tag{14.29}$$

The Mach number in the turning area is

$$M^2 = \frac{{U_\theta}^2 + {U_r}^2}{c^2} = \frac{{U_\theta}^2 + {U_r}^2}{{U_\theta}^2} = 1 + \left(\frac{U_r}{U_\theta}\right)^2 \tag{14.30}$$

Now utilizing the expression that were obtained for U_r and U_θ equations (14.29) and (14.28) results for the Mach number

$$M^2 = 1 + \frac{k+1}{k-1}\tan^2\left(\theta\sqrt{\frac{k-1}{k+1}}\right) \tag{14.31}$$

or the reverse function for θ is

$$\theta = \sqrt{\frac{k+1}{k-1}}\tan^{-1}\left(\sqrt{\frac{k-1}{k+1}}\left(M^2-1\right)\right) \tag{14.32}$$

What happened when the upstream Mach number is not 1? That is when initial condition for the turning angle doesn't start with $M = 1$ but at already at different angle. The upstream Mach number denoted in this segment as, $M_{starting}$. For this upstream Mach number (see Figure (14.2))

$$\tan\nu = \sqrt{{M_{starting}}^2 - 1} \tag{14.33}$$

The deflection angle ν, has to match to definition of the angle that chosen here ($\theta = 0$ when $M = 1$) so

$$\nu(M) = \theta(M) - \theta(M_{starting}) \tag{14.34}$$

$$= \sqrt{\frac{k+1}{k-1}}\tan^{-1}\left(\sqrt{\frac{k-1}{k+1}}\sqrt{M^2-1}\right) - \tan^{-1}\sqrt{M^2-1} \tag{14.35}$$

These relationship are plotted in Figure (14.6).

14.2.2 Comparison Between The Two Approaches, And Limitations

The two models produce the exact the same results but the assumptions that construction of the models are different. In the geometrical model the assumption was that the velocity in the radial direction is zero. While the rigorous model the assumption was that radial velocity is only function of θ. Whence, the statement for the construction of the geometrical can be improved by assuming that the frame of reference moving in a constant velocity radially.

Regardless, to the assumption that were used in the construction of these models, the fact remains that that there is a radial velocity at $U_r(r=0) = constant$. At this point ($r = 0$) these models falls to satisfy the boundary conditions and something else happen there. On top the complication of the turning point, the question of boundary layer arises. For example, how the gas is accelerated to above the speed of sound where there is no nozzle (where is the nozzle?)? These questions have engineering interest but are beyond the scope of this book (at least at this stage). Normally, this author recommend to use this function every ever beyond 2-4 the thickness of the boundary layer based on the upstream length.

In fact, analysis of design commonly used in the industry and even questions posted for students shows that many assumed that the turning point can be sharp. At small Mach number, $(1+\epsilon)$ the radial velocity is small ϵ. but increase of the Mach number can result in a very significant radial velocity. The radial velocity is “fed” through the reduction of the density. Aside to close proximity to turning point, mass balance maintained by reduction of the density. Thus, some researchers recommend that in many instances, the sharp point should be replaced by a smother transition.

14.3 The Maximum Turning Angle

The maximum turning angle is obtained when the starting Mach number is one and end Mach number approach infinity. In this case, Prandtl–Meyer function became

$$\nu_\infty = \frac{\pi}{2}\left[\sqrt{\frac{k+1}{k-1}} - 1\right] \tag{14.36}$$

The maximum of the deflection point and and maximum turning point are only function of the specific heat ratios. However, the maximum turning angle is match larger than the maximum deflection point because the process is isentropic.

What happen when the deflection angel exceeds the maximum angle? The flow in this case behaves as if there almost maximum angle and in that region beyond will became vortex street see Figure (14.5) i

14.4 The Working Equations For Prandtl-Meyer Function

The change in deflection angle is calculated by

$$\nu_2 - \nu_1 = \nu(M_2) - \nu(M_1) \tag{14.37}$$

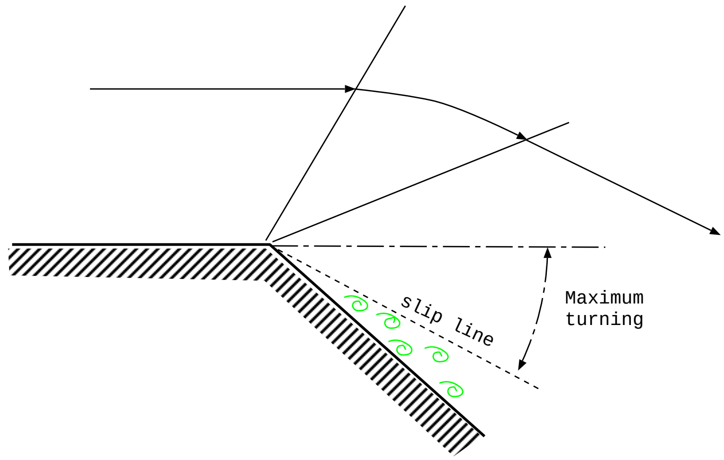


Fig. 14.5: Expansion of Prandtl-Meyer function when it exceeds the maximum angle

14.5 d'Alembert's Paradox

In ideal inviscid incompressible flow, movement of body doesn't encoder any resistance. This results is known as d'Alembert's Paradox and this paradox is examined here.

Supposed that a two dimensional diamond shape body is stationed in a supersonic flow as shown in Figure (14.7). Again it is assumed that the fluid is inviscid. The net force in flow direction, the drag, is

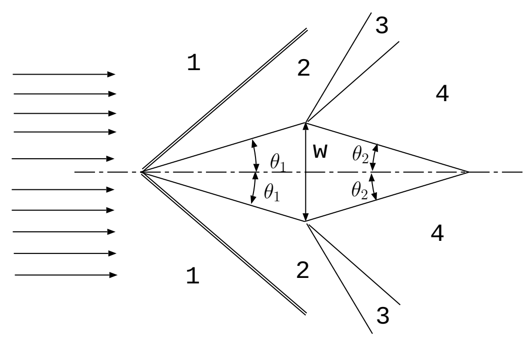


Fig. 14.7: A simplifi ed Diamond Shape to illustrate the Supersonic d'Alembert's Paradox

$$D = 2\left(\frac{w}{2}(P_2 - P_4)\right) = w(P_2 - P_4) \tag{14.38}$$

It can be noticed that only the area "seems" by the flow was used in expressing equation (14.38). The relation between P_2 and P_4 is such that it depends on the upstream Mach number, M_1 and the specific heat, k. Regardless, to equation of state of the gas, the pressure at zone 2 P_2 is larger than the pressure at zone 4, P_4. Thus, there is always drag when the flow is supersonic which depends on the upstream Mach number, M_1, specific heat, k and the "visible" area of the object. This drag known in the literature as (shock) wave drag.

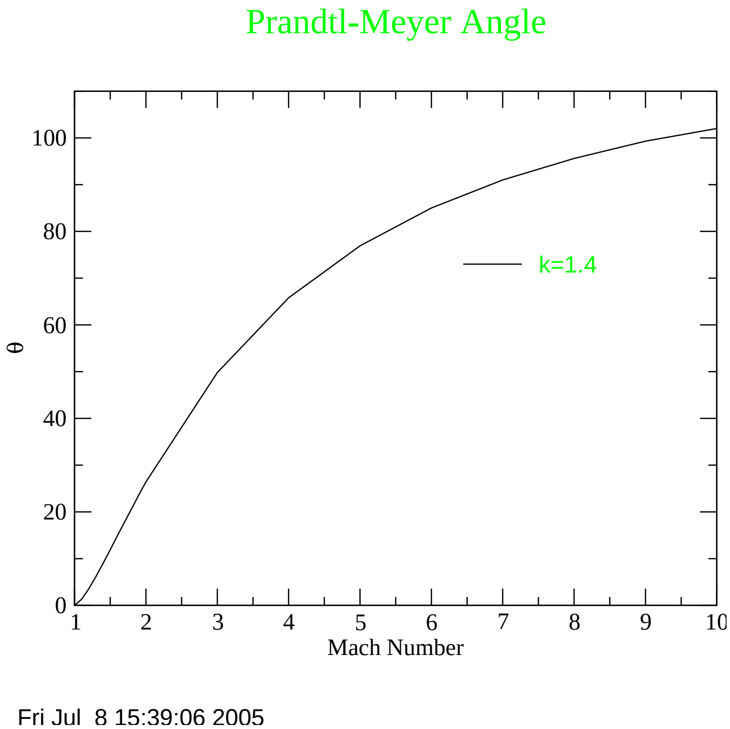


Fig. 14.6: Mach number as a function of

14.6 Flat Body with angle of Attack

Previously the thickness of a body was shown to have drag. Now, A body with zero thickness but with angle of attack will be examine. As oppose the thickness of the body, in addition to the drag, the body also obtains lift. Again, the slip condition is such that pressure in region 5 and 7 is the same in additional the direction of the velocity must be the same. As before the magnitude of the velocity will be different between the two regions.

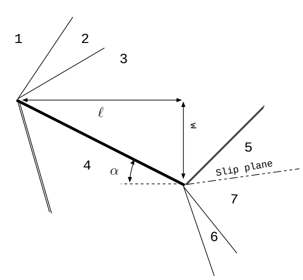


Fig. 14.8: The definition of the angle for Prandtl–Meyer function here

14.7 Examples

Example 14.1:
A wall is include with $20.0°$ inclination. A flow of air with temperature of $20°\mathrm{C}$ and speed of $U = 450m/sec$ flows (see Figure 14.9). Calculate the pressure reduction ratio, and Mach number after the bending point. If the air flows in a imaginary 2-dimensional tunnel with width of 0.1$[m]$ what will the width of this imaginary tunnel after the bend? Calculate the "fan" angle. Assume the specific heat ratio is $k = 1.4$.

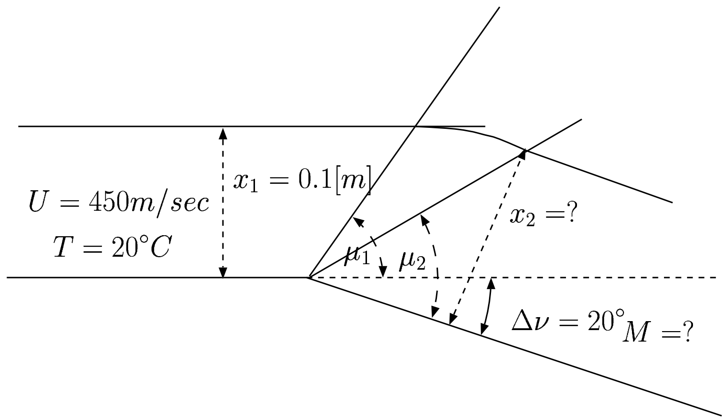


Fig. 14.9: The schematic of the Example 14.1

SOLUTION
First the initial Mach number has to calculated (the initial speed of sound).

$$a = \sqrt{kRT} = \sqrt{1.4 * 287 * 293} = 343.1m/sec$$

The Mach number is then

$$M = \frac{450}{343.1} = 1.31$$

This Mach number associated with

$\mathbf{M}$	ν	$\frac{\mathbf{P}}{\mathbf{P_0}}$	$\frac{\mathbf{T}}{\mathbf{T_0}}$	$\frac{\rho}{\rho_0}$	μ
1.3100	6.4449	0.35603	0.74448	0.47822	52.6434

The "new" angle should be

$$\nu_2 = 6.4449 + 20 = 26.4449°$$

and results in

$\mathbf{M}$	ν	$\frac{\mathbf{P}}{\mathbf{P_0}}$	$\frac{\mathbf{T}}{\mathbf{T_0}}$	$\frac{\rho}{\rho_0}$	μ
2.0024	26.4449	0.12734	0.55497	0.22944	63.4620

Note that $P_{01} = P_{02}$

$$\frac{P_2}{P_1} = \frac{P_{01}}{P_1}\frac{P_2}{P_{02}} = \frac{0.12734}{0.35603} = 0.35766$$

The "new" width can be calculated from the mass conservation equation.

$$\rho_1 x_1 M_1 c_1 = \rho_2 x_2 M_2 c_2 \Longrightarrow x_2 = x_1 \frac{\rho_1}{\rho_2}\frac{M_1}{M_2}\sqrt{\frac{T_1}{T_2}}$$

$$x_2 = 0.1 \times \frac{0.47822}{0.22944} \times \frac{1.31}{2.0024}\sqrt{\frac{0.74448}{0.55497}} = 0.1579[m]$$

Note that the compression "fan" stream lines are note and their function can be obtain either by numerical method of going over small angle increments. The other alternative is using the exact solution[2]. The expansion "fan" angle change in the Mach angle between the two sides of the bend

$$\text{fan angle} = 63.4 + 20.0 - 52.6 = 30.8^\circ$$

Reverse example, this time the pressure is given on both sides and angle is needed to be found[3].

Example 14.2:
Gas with $k = 1.67$ flows over bend (see Figure 14.2) . Compute the Mach number after the bend, and the bend angle.

SOLUTION
The Mach number is determined by satisfying the condition that the pressure down steam are Mach the given one. The relative pressure downstream can be calculated by the relationship

$$\frac{P_2}{P_{02}} = \frac{P_2}{P_1}\frac{P_1}{P_{01}} = \frac{1}{1.2} \times 0.31424 = 0.2619$$

$\mathbf{M}$	ν	$\frac{\mathbf{P}}{\mathbf{P_0}}$	$\frac{\mathbf{T}}{\mathbf{T_0}}$	$\frac{\rho}{\rho_0}$	μ
1.4000	7.7720	0.28418	0.60365	0.47077	54.4623

[2]Not really different from this explanation but shown in more mathematical form, due to Landau and friends. It will be presented in the future version. It isn't present now because the low priority to this issue present for a text book on this subject.

[3]This example is for academic understanding. There is very little with practical problems.

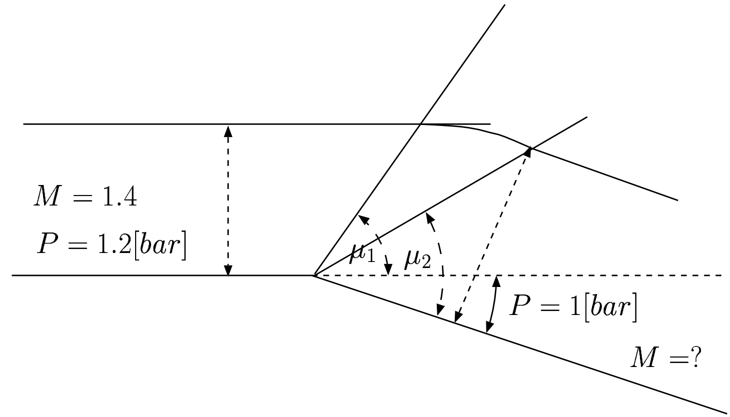


Fig. 14.10: The reversed example schematic 14.2

With this pressure ratio $\bar{P} = 0.2619$ require either locking in the table or using the enclosed program.

$\mathbf{M}$	ν	$\frac{\mathbf{P}}{\mathbf{P_0}}$	$\frac{\mathbf{T}}{\mathbf{T_0}}$	$\frac{\rho}{\rho_0}$	μ
1.4576	9.1719	0.26190	0.58419	0.44831	55.5479

For the rest of the calculation the initial condition are used. The Mach number after the bend is $M = 1.4576$. It should be noted that specific heat isn't $k = 1.4$ but $k = 1.67$. The bend angle is

$$\Delta\nu = 9.1719 - 7.7720 \sim 1.4°$$

$$\Delta\mu = 55.5479 - 54.4623 = 1.0°$$

14.8 Combination of The Oblique Shock and Isentropic Expansion

Example 14.3:
Consider two dimensional flat thin plate at angle of attack of $4°$ and Mach number of 3.3. Assume that specific heat ratio at stage is $k = 1.3$, calculate the drag coefficient and lift coefficient.

SOLUTION
For $M = 3.3$ the following table can be obtained

$\mathbf{M}$	ν	$\frac{\mathbf{P}}{\mathbf{P_0}}$	$\frac{\mathbf{T}}{\mathbf{T_0}}$	$\frac{\rho}{\rho_0}$	μ
3.3000	62.3113	0.01506	0.37972	0.03965	73.1416

With the angle of attack the region 3 will at $\nu \sim 62.31 + 4$ for which the following table can be obtain (Potto-GDC)

$\mathbf{M}$	ν	$\mathbf{\frac{P}{P_0}}$	$\mathbf{\frac{T}{T_0}}$	$\frac{\rho}{\rho_0}$	μ
3.4996	66.3100	0.01090	0.35248	0.03093	74.0528

On the other side the oblique shock (assuming weak shock) results in

$\mathbf{M_x}$	$\mathbf{M_{y_s}}$	$\mathbf{M_{y_w}}$	$\theta_\mathbf{s}$	$\theta_\mathbf{w}$	δ	$\mathbf{\frac{P_{0_y}}{P_{0_x}}}$
3.3000	0.43534	3.1115	88.9313	20.3467	4.0000	0.99676

And the additional information by clicking on the minimal button provides

$\mathbf{M_x}$	$\mathbf{M_{y_w}}$	$\theta_\mathbf{w}$	δ	$\mathbf{\frac{P_y}{P_x}}$	$\mathbf{\frac{T_y}{T_x}}$	$\mathbf{\frac{P_{0_y}}{P_{0_x}}}$
3.3000	3.1115	20.3467	4.0000	1.1157	1.1066	0.99676

The pressure ratio at point 3 is

$$\frac{P_3}{P_1} = \frac{P_3}{P_{03}}\frac{P_{03}}{P_{01}}\frac{P_{01}}{P_1} = 0.0109 \times 1 \times \frac{1}{0.01506} \sim 0.7238$$

The pressure ratio at point 4

$$\frac{P_3}{P_1} = 1.1157$$

$$d_L = \frac{2}{kP_1{M_1}^2}(P_4 - P_3)\cos\alpha = \frac{2}{k{M_1}^2}\left(\frac{P_4}{P_1} - \frac{P_3}{P_1}\right)\cos\alpha = \frac{2}{1.33.3^2}(1.1157 - 0.7238)\cos 4^\circ \sim .054$$

$$d_d = \frac{2}{k{M_1}^2}\left(\frac{P_4}{P_1} - \frac{P_3}{P_1}\right)\sin\alpha = \frac{2}{1.33.3^2}(1.1157 - 0.7238)\sin 4^\circ \sim .0039$$

This shows that on expense of small drag large lift can be obtained. Question of optimum design what is left for the next versions.

CHAPTER 15

Topics in Steady state Two Dimensional flow

shock–expansion theory, linearized potential flow: thin airfoil theory, 2D method of characteristics

APPENDIX A

Computer Program

A.1 About the Program

The program is written in a C++ language. This program was used to generate all the data in this book. Some parts of the code are in FORTRAN (old code especially for chapters 11 and 12 and not included here.[1]. The program has the base class of basic fluid mechanics and utilities functions to calculate certain properties given data. The derived class are Fanno, isothermal, shock and others.

At this stage only the source code of the program is available no binary available. This program is complied under gnu g++ in /Gnu/Linux system. As much support as possible will be provided if it is in Linux systems. NO Support whatsoever will be provided for any Microsoft system. In fact even PLEASE do not even try to use this program under any Microsoft window system.

A.2 Usage

To use the program some information has to be provided. The necessary input parameter(s), the kind of the information needed, where it has to be in a LaTeX format or not, and in many case where it is a range of parameter(s).

machV The Mach number and it is used in stagnation class

fldV The $\frac{4fL}{D}$ and it is used in Fanno class isothermal class

p2p1V The pressure ratio of the two sides of the tubes

M1V Entrance Mach M1 to the tube Fanno and isothermal classes

[1]when will be written in C++ will be add to this program.

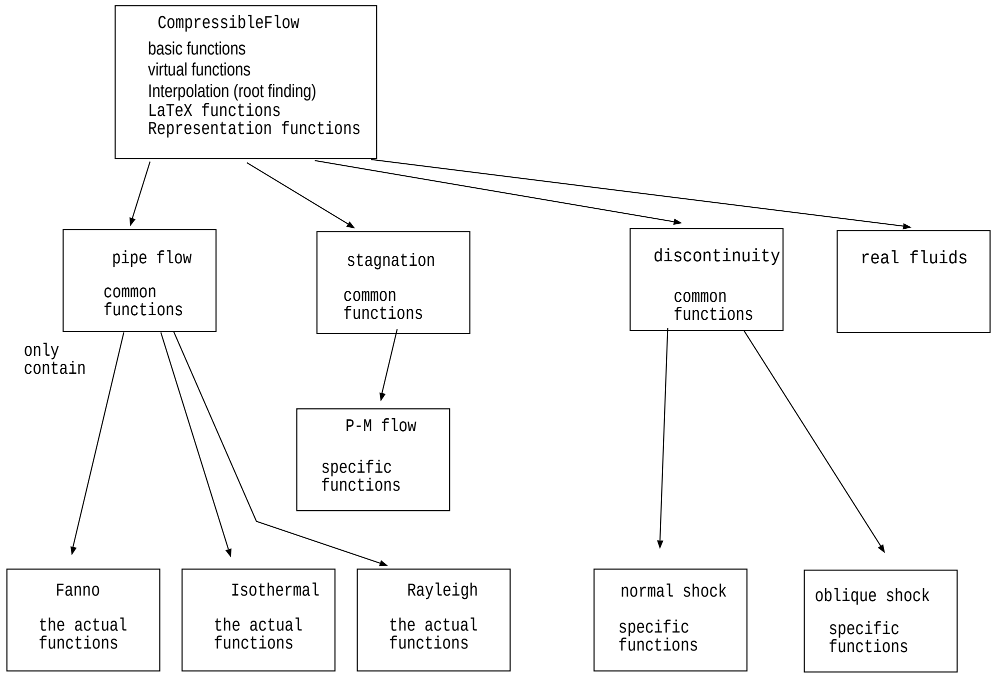


Fig. A.1: Schematic diagram that explains the structure of the program

M1ShockV Entrance Mach M1 when expected shock to the tube Fanno and isothermal classes

FLDShockV FLD with shock in the in Fanno class

M1fldV both M_1 and $\frac{4fL}{D}$ are given

M1fldP2P1V three part info $\frac{P_1}{P_2}$, M_1 and $\frac{4fL}{D}$ are given

MxV M_x or M_y

infoStagnation print standard (stagnation) info

infoStandard standard info for (Fanno, shock etc)

infoTube print tube side info for (Fanno, etc) including

infoShock print shock sides info

infoTubeShock print tube info shock main info

infoTubeProfile the Mach number and pressure ratio profiles

infoTubeShockLimits print tube limits with shock

To get the shock results in LaTeX of M_x The following lines have to be inserted in the end of the main function.

```
int isTex = yes;
int isRange = no;
whatInfo = infoStandard ;
variableName =  MxV;
Mx = 2.0 ;
s.makeTable(whatInfo, isRange, isTex,  variableName, variableValue);
```

```
*******************************************
The following stuff is the same as above/below
if you use showResults with showHeads but the
information is setup for the latex text processing.
You can just can cut and paste it in your latex file.
You must use longtable style file and dcolumn
style files.


*******************************************
\setlongtables
\begin{longtable}
{|D..{1.4}|D..{1.4}|D..{1.4}|D..{1.4}|D..{1.4}|D..{1.4}|D..{1.4}|}
\caption{ ??  \label{?:tab:?}}\\
```

```
\hline
\multicolumn{1}{|c|} {$\rule[-0.1in]{0.pt}{0.3 in}\mathbf{M} $}  &
\multicolumn{1}{|c|} {$\mathbf{4fL \over D} $}  &
\multicolumn{1}{|c|} {$\mathbf{P \over P^{*}} $}  &
\multicolumn{1}{|c|} {$\mathbf{P_0 \over {P_0}^{*}} $}  &
\multicolumn{1}{|c|} {$\mathbf{\rho \over \rho^{*}} $}  &
\multicolumn{1}{|c|} {$\mathbf{U \over {U}^{*}} $}  &
\multicolumn{1}{|c|} {$\mathbf{T \over T^{*}} $}

\\\hline

\endfirsthead
\caption{ ?? (continue)} \\\hline
\multicolumn{1}{|c|} {$\rule[-0.1in]{0.pt}{0.3 in}\mathbf{M} $}  &
\multicolumn{1}{|c|} {$\mathbf{4fL \over D} $}  &
\multicolumn{1}{|c|} {$\mathbf{P \over P^{*}} $}  &
\multicolumn{1}{|c|} {$\mathbf{P_0 \over {P_0}^{*}} $}  &
\multicolumn{1}{|c|} {$\mathbf{\rho \over \rho^{*}} $}  &
\multicolumn{1}{|c|} {$\mathbf{U \over {U}^{*}} $}  &
\multicolumn{1}{|c|} {$\mathbf{T \over T^{*}} $}

\\\hline
\endhead
    2.176&    2.152&   0.3608&    1.000&   0.5854&    3.773&   0.6164 \\
\hline\end{longtable}
```

A.3 Program listings

Can be download from www.potto.org.