

Radu MIRON

Lagrangian and Hamiltonian  
geometries. Applications to  
Analytical Mechanics



## Preface

The aim of the present monograph is twofold: 1° to provide a Compendium of Lagrangian and Hamiltonian geometries and 2° to introduce and investigate new analytical Mechanics: Finslerian, Lagrangian and Hamiltonian.

One knows (R. Abraham, J. Klein, R. Miron et al.) that the geometrical theory of nonconservative mechanical systems can not be rigorously constructed without the use of the geometry of the tangent bundle of the configuration space.

The solution of this problem is based on the Lagrangian and Hamiltonian geometries. In fact, the construction of these geometries relies on the mechanical principles and on the notion of Legendre transformation.

The whole edifice has as support the sequence of inclusions:

$$\{\mathcal{R}^n\} \subset \{F^n\} \subset \{L^n\} \subset \{GL^n\}$$

formed by Riemannian, Finslerian, Lagrangian, and generalized Lagrangian spaces. The  $\mathcal{L}$ -duality transforms this sequence into a similar one formed by Hamiltonian spaces.

Of course, these sequences suggest the introduction of the correspondent Mechanics: Riemannian, Finslerian, Lagrangian, Hamiltonian etc.

The fundamental equations (or evolution equations) of these Mechanics are derived from the variational calculus applied to the integral of action and these can be studied by using the methods of Lagrangian or Hamiltonian geometries.

More general, the notions of higher order Lagrange or Hamilton spaces have been introduced by the present author [161], [162], [163] and developed is realized by means of two sequences of inclusions similarly with those of the geometry of order 1. The problems raised by the geometrical theory of Lagrange and Hamilton spaces of order  $k \geq 1$  have been investigated by Ch. Ehresmann, W. M. Tulczyjew, A. Kawaguchi, K. Yano, M. Crampin, Manuel de Léon, R. Miron,

M. Anastasiei, I. Bucătaru et al. [175]. The applications lead to the notions of Lagrangian or Hamiltonian Analytical Mechanics of order  $k$ .

For short, in this monograph we aim to solve some difficult problems:

- The problem of geometrization of classical non conservative mechanical systems;
- The foundations of geometrical theory of new mechanics: Finslerian, Lagrangian and Hamiltonian;
- To determine the evolution equations of the classical mechanical systems for whose external forces depend on the higher order accelerations.

This monograph is based on the terminology and important results taken from the books: Abraham, R., Marsden, J., *Foundation of Mechanics*, Benjamin, New York, 1978; Arnold, V.I., *Mathematical Methods in Classical Mechanics*, Graduate Texts in Math, Springer Verlag, 1989; Asanov, G.S., *Finsler Geometry Relativity and Gauge theories*, D. Reidel Publ. Co, Dordrecht, 1985; Bao, D., Chern, S.S., Shen, Z., *An Introduction to Riemann-Finsler Geometry*, Springer Verlag, Grad. Text in Math, 2000; Bucataru, I., Miron, R., *Finsler-Lagrange geometry. Applications to dynamical systems*, Ed. Academiei Romane, 2007; Miron, R., Anastasiei, M., *The geometry of Lagrange spaces. Theory and applications to Relativity*, Kluwer Acad. Publ. FTPH no. 59, 1994; Miron, R., *The Geometry of Higher-Order Lagrange Spaces. Applications to Mechanics and Physics*, Kluwer Acad. Publ. FTPH no. 82, 1997; Miron, R., *The Geometry of Higher-Order Finsler Spaces*, Hadronic Press Inc., SUA, 1998; Miron, R., Hrimiuc, D., Shimada, H., Sabau, S., *The geometry of Hamilton and Lagrange Spaces*, Kluwer Acad. Publ., FTPH no. 118, 2001.

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The purpose of the book is a short presentation of the geometrical theory of Lagrange and Hamilton spaces of order 1 or of order  $k \geq 1$ , as well as the definition and investigation of some new Analytical Mechanics of Lagrangian and Hamiltonian type of order  $k \geq 1$ .

In the last thirty five years, geometers, mechanicians and physicists from all over the world worked in the field of Lagrange or Hamilton geometries and their applications. We mention only some important names: P.L. Antonelli [21], M. Anastasiei [10], [11]. G. S. Asanov [27], A. Bejancu [38], I. Bucătaru [47], M. Crampin [64], R. S. Ingarden [114], S. Ikeda [116], M. de Leon [138], M. Matsumoto [144], R. Miron [164], [165], [166], H. Rund [218], H. Shimada [223], P. Stavrinou [237], L. Tamassy [247] and S. Vacaru [251].

The book is divided in three parts: I. Lagrange and Hamilton spaces; II. Lagrange and Hamilton spaces of higher order; III. Analytical Mechanics of Lagrangian and Hamiltonian mechanical systems.

The part I starts with the geometry of tangent bundle  $(TM, \pi, M)$  of a differentiable, real,  $n$ -dimensional manifold  $M$ . The main geometrical objects on  $TM$ , as Liouville vector field  $\mathbb{C}$ , tangent structure  $J$ , semispray  $S$ , nonlinear connection  $N$  determined by  $S, N$ -metrical structure  $D$  are pointed out. It is continued with the notion of Lagrange space, defined by a pair  $L^n = (M, L(x, y))$  with  $L : TM \rightarrow \mathbb{R}$  as a regular Lagrangian and  $g_{ij} = \frac{1}{2} \dot{\partial}_i \dot{\partial}_j L$  as fundamental tensor field. Of course,  $L(x, y)$  is a regular Lagrangian if the Hessian matrix  $\|g_{ij}(x, y)\|$  is nonsingular. In the definition of Lagrange space  $L^n$  we assume that the fundamental tensor  $g_{ij}(x, y)$  has a constant signature. The known Lagrangian from Electrodynamics assures the existence of Lagrange spaces.

The variational problem associated to the integral of action

$$I(c) = \int_0^1 L(x, \dot{x}) dt$$

allows us to determine the Euler-Lagrange equations, conservation law of energy,  $\mathcal{E}_L$ , as well as the canonical semispray  $S$  of  $L^n$ . But  $S$  determines the canonical nonlinear connection  $N$  and the metrical  $N$ -linear connection  $D$ , given by the generalized Christoffel symbols. The structure equations of  $D$  are derived. This theory is applied to the study of the electromagnetic and gravitational fields of the space  $L^n$ . An almost Kählerian model is constructed. This theory suggests to define the notion of generalized Lagrange space  $GL^n = (M, g(x, y))$ , where  $g(x, y)$  is a metric tensor on the manifold  $TM$ . The space  $GL^n$  is not reducible to a space  $L^n$ .

A particular case of Lagrange space  $L^n$  leads to the known concept of Finsler space  $F^n = (M, F(x, y))$ . It follows that the geometry

of Finsler space  $F^n$  can be constructed only by means of Analytical Mechanics principles.

Since a Riemann space  $\mathcal{R}^n = (M, g(x))$  is a particular Finsler space  $F^n = (M, F(x, y))$  we get the following sequence of inclusions:

$$(I) \quad \{\mathcal{R}^n\} \subset \{F^n\} \subset \{L^n\} \subset \{GL^n\}.$$

The Lagrangian geometry is the geometrical study of the sequence (I).

The geometrical theory of the Hamilton spaces can be constructed step by step following the theory of Lagrange spaces. The legitimacy of this theory is due to  $\mathcal{L}$ -duality (Legendre duality) between a Lagrange space  $L^n = (M, L(x, y))$  and a Hamilton space  $H^n = (M, H(x, p))$ .

Therefore, we begin with the geometrical theory of cotangent bundle  $(T^*M, \pi^*, M)$ , continue with the notion of Hamilton space  $H^n(M, H(x, p))$ , where  $H : T^*M \rightarrow R$  is a regular Hamiltonian, with the variational problem

$$I(c) = \int_0^1 \left[ p_i(t) \frac{dx^i}{dt} - \frac{1}{2} L(x(t), p(t)) \right] dt,$$

from which the Hamilton–Jacobi equations are derived, Hamiltonian vector field of  $H^n$ , nonlinear connection  $N$ ,  $N^*$ –linear connection etc. The Legendre transformation  $\mathcal{L} : L^n \rightarrow H^n$  transforms the fundamental geometrical object fields on  $L^n$  into the fundamental geometrical object fields on  $H^n$ . The restriction of  $\mathcal{L}$  to the Finsler spaces  $\{F^n\}$  has as image a new class of spaces  $\mathcal{C}^n = (M, K(x, p))$  called the Cartan spaces. They have the same beauty, symmetry and importance as the Finsler spaces. A pair  $GH^n = (M, g^*(x, p))$ , where  $g^*$  is a metric tensor on  $T^*M$  is named a generalized Hamilton space. Remarking that  $\mathcal{R}^{*n} = (M, g^*(x))$ , with  $g^{*ij}(x)$  the contravariant Cartan space, we obtain a dual sequence of the sequence (I):

$$(II) \quad \{\mathcal{R}^{*n}\} \subset \{\mathcal{C}^n\} \subset \{H^n\} \subset \{GH^n\}.$$

The Hamiltonian geometry is geometrical study of the sequence II.

The Lagrangian and Hamiltonian geometries are useful for applications in: Variational calculus, Mechanics, Physics, Biology etc.

The part II of the book is devoted to the notions of Lagrange and Hamilton spaces of higher order. We study the geometrical theory of the total space of  $k$ –tangent bundle  $T^kM$ ,  $k \geq 1$ , generalizing, step by step the theory from case  $k = 1$ . So, we introduce the Liouville vector fields, study the variational problem for a given integral of action on  $T^kM$ , continue with the notions of  $k$ –semispray, nonlinear con-

nection, the prolongation to  $T^kM$  of the Riemannian structure defined on the base manifold  $M$ . The notion of  $N$ -metrical connections is pointed out, too.

It follows the notion of Lagrange space of order  $k \geq 1$ . It is defined as a pair  $L^{(k)n} = (M, L(x, y^{(1)}, y^{(2)}, \dots, y^{(k)}))$ , where  $L$  is a Lagrangian depending on the (material point)  $x = (x^i)$  and on the accelerations

$$y^{(k)i} = \frac{1}{k!} \frac{d^k x^i}{dt^k}, \quad k = 1, 2, \dots, k; \quad i = 1, 2, \dots, n, \quad n = \dim M.$$

Some examples prove the existence of the spaces  $L^{(k)n}$ . The canonical nonlinear connection  $N$  and metrical  $N$ -linear connection are pointed out. The Riemannian almost contact model for  $L^{(k)n}$  as well as the Generalized Lagrange space of order  $k$  end this theory.

The methods used in the construction of the Lagrangian geometry of higher-order are the natural extensions of those used in the theory of Lagrangian geometries of order  $k = 1$ .

Next chapter treats the geometry of Finsler spaces of order  $k \geq 1$ ,  $F^{(k)1} = (M, F(x, y^{(1)}, \dots, y^{(k)}))$ ,  $F : T^kM \rightarrow R$  being positive  $k$ -homogeneous on the fibres of  $T^kM$  and fundamental tensor has a constant signature. Finally, one obtains the sequences

$$(III) \quad \{\mathcal{R}^{(k)n}\} \subset \{F^{(k)n}\} \subset \{L^{(k)1}\} \subset \{GL^{(k)n}\}.$$

The Lagrangian geometry of order  $k \geq 1$  is the geometrical theory of the sequences of inclusions (III).

The notion of Finsler spaces of order  $k$ , introduced by the present author, was investigated in the book *The Geometry of Higher-Order Finsler Spaces*, Hadronic Press, 1998. Here it is a natural extension to the manifold  $T^kM$  of the theory of Finsler spaces given in Section 3. It was developed by H. Shimada and S. Sabău [223].

The previous theory is continued with the geometry of  $k$  cotangent bundle  $T^{*k}M$ , defined as the fibered product:  $T^{k-1}M \times_M T^*M$ , with the notion of Hamilton space of order  $k \geq 1$  and with the particular case of Cartan spaces of order  $k$ . An extension to  $L^{(k)n}$  and  $H^{(k)n}$  of the Legendre transformation is pointed out. The Hamilton–Jacobi equations of  $H^{(k)n}$ , which are fundamental equations of these spaces, are presented, too.

For the Hamilton spaces of order  $k$ ,  $H^{(k)n} = (M, H(x, y)^{(k)}, \dots, y^{(k-1)}, p)$  a similar sequence of inclusions with (III) is introduced. These considerations allow to clearly define what means the Hamiltonian geometry of order  $k$  and what is the utility of this theory in applications.

Part III of this book is devoted to applications in Analytical Mechanics. One studies the geometrical theory of scleronomic noncon-

servative classical mechanical systems  $\Sigma_{\mathcal{R}} = (M, T, Fe)$ , it is introduced and investigated the notion of Finslerian mechanical systems  $\Sigma_F = (M, F, Fe)$  and is defined the concept of Lagrangian mechanical system  $\Sigma_L = (M, L, Fe)$ . In all these theories  $M$  is the configuration space,  $T$  is the kinetic energy of a Riemannian space  $\mathcal{R}^n = (M, g)$ ,  $F(x, y)$  is the fundamental function of a Finsler space  $F^n = (M, F(x, y))$ ,  $L(x, y)$  is a regular Lagrangian and  $Fe(x, y)$  are the external forces which depend on the material points  $x \in M$  and their

velocities  $y^i = \dot{x}^i = \frac{dx^i}{dt}$ .

In order to study these Mechanics we apply the methods from the Lagrangian geometries. The contents of these geometrical theory of mechanical systems  $\Sigma_{\mathcal{R}}$ ,  $\Sigma_F$  and  $\Sigma_L$  is based on the geometrical theory of the velocity space  $TM$ . The base of these investigations are the Lagrange equations. They determine a canonical semispray, which is fundamental for all constructions. For every case of  $\Sigma_{\mathcal{R}}$ ,  $\Sigma_F$  and  $\Sigma_L$  the law of conservation of energy is pointed out. We end with the corresponding almost Hermitian model on the velocity space  $TM$ .

The dual theory via Legendre transformation, leads to the geometrical study of the Hamiltonian mechanical systems  $\Sigma_{\mathcal{R}}^* = (M, T^*, Fe^*)$ ,  $\Sigma_{\mathcal{G}}^* = (M, K, Fe^*)$  and  $\Sigma_H^* = (M, H, Fe^*)$  where  $T^*$  is energy,  $K(x, p)$  is the fundamental function of a given Cartan space and  $H(x, p)$  is a regular Hamiltonian on the cotangent bundle  $T^*M$ . The fundamental equations in these Hamiltonian mechanical systems are the Hamilton equations

$$\frac{dx^i}{dt} = -\frac{\partial \mathcal{H}}{\partial p_i}; \quad \frac{dp_i}{dt} = \frac{\partial \mathcal{H}}{\partial x^i} + \frac{1}{2}F_i, \quad \left( \mathcal{H} = \frac{1}{2}H \right),$$

$F_i(x, p)$  being the covariant components of external forces. The used methods are those given by the sequence of inclusions  $\{\mathcal{R}^*\} \subset \{\mathcal{C}^n\} \subset \{H^n\} \subset \{GH^n\}$ .

More general, the notions of Lagrangian and Hamiltonian mechanical systems of order  $k \geq 1$  are introduced and studied. Therefore, to the sequences of inclusions III correspond the analytical mechanics of order  $k$  of the Riemannian, Finslerian, Lagrangian type. All these cases are the direct generalizations from case  $k = 1$ . This is the reason why we present here shortly the principal results. Especially, we paid attention to the following question:

*Considering a Riemannian (particularly Euclidian) mechanical system  $\Sigma_{\mathcal{R}} = (M, T, Fe)$ , with  $T = \frac{1}{2}g_{ij}(x)\dot{x}^i\dot{x}^j$  as energy, but with the external forces  $Fe$  depending on material point  $(x^i)$  and on higher*



order accelerations  $\left(\frac{dx^i}{dt}, \frac{d^2x^i}{dt^2}, \dots, \frac{d^kx^i}{dt^k}\right)$ . What are the evolution equations of the system  $\Sigma_{\mathcal{R}}$ ?

Clearly, the classical Lagrange equations are not valid. In the last part of this book we present the solution of this problem.

Finally, what is new in the present book?

- 1° A solution of the problem of geometrization of the classical non-conservative mechanical systems, whose external forces depend on velocities, based on the differential geometry of velocity space.
- 2° The introduction of the notion of Finslerian mechanical system.
- 3° The definition of Cartan mechanical system.
- 4° The study of theory of Lagrangian and Hamiltonian mechanical systems by means of the geometry of tangent and cotangent bundles.
- 5° The geometrization of the higher order Lagrangian and Hamiltonian mechanical systems.
- 6° The determination of the fundamental equations of the Riemannian mechanical systems whose external forces depend on the higher order accelerations.



**Part I**  
**Lagrange and Hamilton Spaces**

The purpose of the part I is a short presentation of the geometrical theory of Lagrange space and of Hamilton spaces. These spaces are basic for applications to Mechanics, Physics etc. and have been introduced by the present author in 1980 and 1987, respectively.

Therefore we present here the general framework of Lagrange and Hamilton geometries based on the books by R. Miron [161], [162], [163], R. Miron and M. Anastasiei [164], I. Bucătaru and R. Miron [49] and R. Miron, D. Hrimiuc, H. Shimada and S. Sabău [174]. Also, we use the papers R. Miron, M. Anastasiei and I. Bucătaru, *The geometry of Lagrange spaces*, Handbook of Finsler Geometry, P. L. Antonelli ed., Kluwer Academic; R. Miron, *Compendium on the Geometry of Lagrange spaces*, Handbook of Differential Geometry, vol. II, pp 438–512, Edited by F. J. E. Dillen and L. C. A. Verstraelen, 2006.

The geometry of Lagrange space starts here with the study of geometrical theory of tangent bundle  $(TM, \pi, M)$ . It is continued with the notion of Lagrange space  $L^n = (M, L(x, y))$  and with an important particular case the Finsler space  $F^n = (M, F)$ . The Lagrangian geometry is the study of the sequence of inclusions (I) from the Preface.

The geometry of Hamilton spaces follow the same pattern: the geometry of cotangent bundle  $(T^*M, \pi^*, M)$ , it continues with the notion of Hamilton space  $H^n = (M, H(x, p))$ , with the concept of Cartan space  $\mathcal{C} = (M, K(x, p))$  and ends with the sequence of inclusions (II) from Introduction.

The relation between the previous sequences are given by means of the Legendre transformation.

In the following, we assume that all the geometrical object fields and mappings are  $C^\infty$ -differentiable and we express this by words “differentiable” or “smooth”.

# Chapter 1

## The Geometry of tangent manifold

The total space  $TM$  of tangent bundle  $(TM, \pi, M)$  carries some natural object fields as Liouville vector field  $\mathbb{C}$ , tangent structure  $J$  and vertical distribution  $V$ . An important object field is the semispray  $S$  defined as a vector field  $S$  on the manifold  $TM$  with the property  $J(S) = \mathbb{C}$ . One can develop a consistent geometry of the pair  $(TM, S)$ .

### 1.1 The manifold $TM$

The differentiable structure on  $TM$  is induced by that of the base manifold  $M$  so that the natural projection  $\pi : TM \rightarrow M$  is a differentiable submersion and the triple  $(TM, \pi, M)$  is a differentiable vector bundle. Assuming that  $M$  is a real,  $n$ -dimensional differentiable manifold and  $(U, \varphi = (x^i))$  is a local chart at a point  $x \in M$ , then any curve  $\sigma : I \rightarrow M$ ,  $(\text{Im}\sigma \subset U)$  that passes through  $x$  at  $t = 0$  is analytically represented by  $x^i = x^i(t)$ ,  $t \in I$ ,  $\varphi(x) = (x^i(0))$ ,  $(i, j, \dots = 1, \dots, n)$ . The tangent vector  $[\sigma]_x$  is determined by the real numbers

$$x^i = x^i(0), y^i = \frac{dx^i}{dt}(0).$$

Then the pair  $(\pi^{-1}(U), \Phi)$ , with  $\Phi([\sigma]_x) = (x^i, y^i) \in \mathbb{R}^{2n}$ ,  $\forall [\sigma]_x \in \pi^{-1}(U)$  is a local chart on  $TM$ . It will be denoted by  $(\pi^{-1}(U), \phi = (x^i, y^i))$ . The set of these “induced” local charts determines a differentiable structure on  $TM$  such that  $\pi : TM \rightarrow M$  is a differentiable manifold of dimension  $2n$  and  $(TM, \pi, M)$  is a differentiable vector bundle.

A change of coordinate on  $M$ ,  $(U, \varphi = x^i) \rightarrow (V, \psi = \tilde{x}^i)$  given by  $\tilde{x}^i = \tilde{x}^i(x^j)$ , with  $\text{rank}\left(\frac{\partial \tilde{x}^i}{\partial x^j}\right) = n$  has the corresponding change of coordinates on  $TM : (\pi^{-1}(U), \Phi = (x^i, y^i)) \rightarrow (\pi^{-1}(V), \Psi = (\tilde{x}^i, \tilde{y}^i))$ ,  $(U \cap V \neq \emptyset)$ , given by:

$$\begin{cases} \tilde{x}^i = \tilde{x}^i(x^j), & \text{rank}\left(\frac{\partial \tilde{x}^i}{\partial x^j}\right) = n, \\ \tilde{y}^i = \frac{\partial \tilde{x}^i}{\partial x^j} y^j. \end{cases} \quad (1.1.1)$$

The Jacobian of  $\Psi \circ \Phi^{-1}$  is  $\det\left(\frac{\partial \tilde{x}^i}{\partial x^j}\right)^2 > 0$ . So the manifold  $TM$  is orientable.

The tangent space  $T_u TM$  at a point  $u \in TM$  to  $TM$  is a  $2n$ -dimensional vector space, having the natural basis  $\left\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^i}\right\}$  at  $u$ . A change of coordinates (1.1.1) on  $TM$  implies the change of natural basis, at point  $u$  as follows:

$$\begin{cases} \frac{\partial}{\partial x^i} = \frac{\partial \tilde{x}^j}{\partial x^i} \frac{\partial}{\partial \tilde{x}^j} + \frac{\partial \tilde{y}^j}{\partial x^i} \frac{\partial}{\partial \tilde{y}^j}, \\ \frac{\partial}{\partial y^i} = \frac{\partial \tilde{x}^j}{\partial x^i} \frac{\partial}{\partial \tilde{y}^j}. \end{cases} \quad (1.1.2)$$

A vector  $X_u \in T_u TM$  is given by  $X = X^i(u) \frac{\partial}{\partial x^i} + Y^i(u) \frac{\partial}{\partial y^i}$ . Then a vector field  $X$  on  $TM$  is section  $X : TM \rightarrow TTM$  of the projection  $\pi_* : TTM \rightarrow TM$ . This is  $\pi_*(x, y, X, Y) = (x, y)$ .

From the last formula (1.1.2) we can see that  $\left(\frac{\partial}{\partial y^i}\right)$  at point  $u \in TM$  span an  $n$ -dimensional vector subspace  $V(u)$  of  $T_u TM$ . The mapping  $V : u \in TM \rightarrow V(u) \subset T_u TM$  is an integrable distribution called the vertical distribution. Then  $VTM = \bigcup_{u \in TM} V(u)$  is a subbundle of the tangent bundle  $(TTM, \pi_*(u), TM)$  to  $TM$ . As  $\pi : TM \rightarrow M$  is a submersion it follows that  $\pi_{*,u} : T_u TM \rightarrow T_{\pi(u)} M$  is an epimorphism

of linear spaces. The kernel of  $\pi_{*,u}$  is exactly the vertical subspaces  $V(u)$ .

We denote by  $\mathcal{X}^v(TM)$  the set of all vertical vector field on  $TM$ . It is a real subalgebra of Lie algebra of vector fields on  $TM$ ,  $\mathcal{X}(TM)$ .

Consider the cotangent space  $T_u^*TM$ ,  $u \in TM$ . It is the dual of space  $T_uTM$  and  $(dx^i, dy^i)_u$  is the natural cobasis and with respect to (1.1.1) we have

$$d\tilde{x}^i = \frac{\partial \tilde{x}^i}{\partial x^j} dx^j, d\tilde{y}^i = \frac{\partial \tilde{x}^i}{\partial x^j} dy^j + \frac{\partial \tilde{y}^i}{\partial x^j} dx^j. \quad (1.1.3)$$

The almost tangent structure of tangent bundle is defined as

$$J = \frac{\partial}{\partial y^i} \otimes dx^i \quad (1.1.4)$$

By means of (1.1.2) and (1.1.3) we can prove that  $J$  is globally defined on  $TM$  and that we have

$$J\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial y^i}, J\left(\frac{\partial}{\partial y^i}\right) = 0. \quad (1.1.4')$$

It follows that the following formulae hold:

$$J^2 = J \circ J = 0, KerJ = ImJ = VTM.$$

The almost cotangent structure  $J^*$  is defined by

$$J^* = dx^i \otimes \frac{\partial}{\partial y^i}.$$

Therefore, we obtain

$$J^*(dx^i) = 0, J^*(dy^i) = dx^i.$$

The Liouville vector field on  $\widetilde{TM} = TM \setminus \{0\}$  is defined by

$$\mathbb{C} = y^i \frac{\partial}{\partial y^i}. \quad (1.1.5)$$

It is globally defined on  $\widetilde{TM}$  and  $\mathbb{C} \neq 0$ .

A smooth function  $f : TM \rightarrow \mathbb{R}$  is called  $r \in \mathbb{Z}$  homogeneous with respect to the variables  $y^i$  if,  $f(x, ay) = a^r f(x, y)$ ,  $\forall a \in \mathbb{R}^+$ . The Eu-

ler theorem holds: A function  $f \in \mathcal{F}(TM)$  differentiable on  $\widetilde{TM}$  is  $r$  homogeneous with respect to  $y^i$  if and only if

$$\mathcal{L}_{\mathbb{C}}f = \mathbb{C}f = y^i \frac{\partial f}{\partial y^i} = rf, \quad (1.1.6)$$

$\mathcal{L}_{\mathbb{C}}$  being the Lie derivation with respect to  $\mathbb{C}$ .

A vector field  $X \in \mathcal{X}(TM)$  is  $r$ -homogeneous with respect to  $y^i$  if  $\mathcal{L}_{\mathbb{C}}X = (r-1)X$ , where  $\mathcal{L}_{\mathbb{C}}X = [\mathbb{C}, X]$ .

Finally an 1-form  $\omega \in \mathcal{X}^*(TM)$  is  $r$ -homogeneous if  $\mathcal{L}_{\mathbb{C}}\omega = r\omega$ .

Evidently, the notion of homogeneity can be extended to a tensor field  $T$  of type  $(r, s)$  on the manifold  $TM$ .

## 1.2 Semisprays on the manifold $TM$

The notion of semispray on the total space  $TM$  of the tangent bundle is strongly related with the second order differential equations on the base manifold  $M$ :

$$\frac{d^2x^i}{dt^2} + 2G^i\left(x, \frac{dx}{dt}\right) = 0. \quad (1.2.1)$$

Writing the equation (1.2.1), on  $TM$ , in the equivalent form

$$\frac{dy^i}{dt^2} + 2G^i(x, y) = 0, y^i = \frac{dx^i}{dt} \quad (1.2.2)$$

we remark that with respect to the changing of coordinates (1.1.1) on  $TM$ , the functions  $G^i(x, y)$  transform according to:

$$2\tilde{G}^i = \frac{\partial \tilde{x}^i}{\partial x^j} 2G^j - \frac{\partial \tilde{y}^i}{\partial x^j} y^j. \quad (1.2.3)$$

But (1.2.2) are the integral curve of the vector field:

$$S = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y) \frac{\partial}{\partial y^i} \quad (1.2.4)$$



By means of (1.2.3) one proves:  $S$  is a vector field globally defined on  $TM$ . It is called a semispray on  $TM$  and  $G^i$  are the coefficients of  $S$ .

$S$  is homogeneous of degree 2 if and only if its coefficients  $G^i$  are homogeneous functions of degree 2. If  $S$  is 2-homogeneous then we say that  $S$  is a spray.

If the base manifold  $M$  is paracompact, then on  $TM$  there exist semisprays.

### 1.3 Nonlinear connections

As we have seen in the first section of this chapter, the vertical distribution  $V$  is a regular,  $n$ -dimensional, integrable distribution on  $TM$ . Then it is naturally to look for a complementary distribution of  $VTM$ . It will be called a horizontal distribution. Such distribution is equivalent with a nonlinear connection.

Consider the tangent bundle  $(TM, \pi, M)$  of the base manifold  $M$  and the tangent bundle  $(TTM, \pi_*, TTM)$  of the manifold  $TM$ . As we know the kernel of  $\pi_*$  is the vertical subbundle  $(VTM, \pi_V, TTM)$ . Its fibres are the vertical spaces  $V(u)$ ,  $u \in TTM$ .

For a vector field  $X \in \mathcal{X}(TTM)$ , in local natural basis, we can write:

$$X = X^i(x, y) \frac{\partial}{\partial x^i} + Y^i(x, y) \frac{\partial}{\partial y^i}.$$

Shorter  $X = (x^i, y^i, X^i, Y^i)$ .

The mapping  $\pi_* : TTM \rightarrow TTM$  has the local form  $\pi_*(x, y, X, Y) = (x, X)$ . The points of the submanifold  $VTM$  are of the form  $(x, y, 0, Y)$ .

Let us consider the pull-back bundle

$$\pi^*(TTM) = TTM \times_{\pi} TTM = \{(u, v) \in TTM \times TTM \mid \pi(u) = \pi(v)\}.$$

The fibres of  $\pi^*(TTM)$ , i.e.,  $\pi_u^*(TTM)$  are isomorphic to  $T_{\pi(u)}M$ . Then, we can define the following morphism  $\pi! : TTM \rightarrow \pi^*(TTM)$  by  $\pi!(X_u) = (u, \pi_{*,u}(X_u))$ . Therefore we have

$$\ker \pi! = \ker \pi_* = VTM.$$

We can prove, without difficulties that the following sequence of vector bundles over  $TTM$  is exact:

$$0 \longrightarrow VTM \xrightarrow{i} TTM \xrightarrow{\pi!} \pi^*(TTM) \longrightarrow 0 \quad (1.3.1)$$

Thus, we can give

**Definition 1.3.1.** A nonlinear connection on the tangent manifold  $TM$  is a left splitting of the exact sequence (1.3.1).

Consequently, a nonlinear connection on  $TM$  is a vector bundle morphism  $C : TTM \rightarrow VTM$ , with the property that  $C \circ i = 1_{VTM}$ .

The kernel of the morphism  $C$  is a vector subbundle of the tangent bundle  $(TTM, \pi_*, TM)$  denoted by  $(NTM, \pi_N, TM)$  and called the *horizontal* subbundle. Its fibres  $N(u)$  determine a regular  $n$ -dimensional distribution  $N : u \in TM \rightarrow N(u) \subset T_u TM$ , complementary to the vertical distribution  $V : u \in TM \rightarrow V(u) \subset T_u TM$  i.e.

$$T_u TM = N(u) \oplus V(u), \forall u \in TM. \quad (1.3.2)$$

Therefore, a nonlinear connection on  $TM$  induces the following Whitney sum:

$$TTM = NTM \oplus VTM. \quad (1.3.2')$$

The reciprocal of this property is true, too.

An adapted local basis to the direct decomposition (1.3.2) is of the form  $\left( \frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j} \right)_u$ , where

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N^j_i(x, y) \frac{\partial}{\partial y^j} \quad (1.3.3)$$

and  $\frac{\delta}{\delta x^i} \Big|_u$ ,  $(i = 1, \dots, n)$  are vector fields that belong to  $N(u)$ .

They are  $n$ -linear independent vector fields and are independent from the vector fields  $\left( \frac{\partial}{\partial y^i} \right)_u$   $(i = 1, \dots, n)$  which belong to  $V(u)$ .

The functions  $N^j_i(x, y)$  are called the coefficients of the nonlinear connection, denoted in the following by  $N$ .

Remarking that  $\pi_{*,u} : T_u TM \rightarrow T_{\pi(u)} M$  is an epimorphism the restriction of  $\pi_{*,u}$  to  $N(u)$  is an isomorphism from  $N(u)$  so  $T_{\pi(u)} M$ . So we can take the inverse map  $l_{h,u}$  the horizontal lift determined by the nonlinear connection  $N$ .

Consequently, the vector fields  $\frac{\delta}{\delta x^i} \Big|_u$  can be given in the form

$$\left( \frac{\delta}{\delta x^i} \right)_u = l_{h,u} \left( \frac{\partial}{\partial x^i} \right)_{\pi(u)}.$$

With respect to a change of local coordinates on the base manifold  $M$

we have  $\frac{\partial}{\partial x^i} = \frac{\partial \tilde{x}^j}{\partial x^i} \frac{\partial}{\partial \tilde{x}^j}$ .

Consequently  $\left(\frac{\delta}{\delta x^i}\right)_u$  are changed, with respect to (1.1.1), in the form

$$\frac{\delta}{\delta x^i} = \frac{\partial \tilde{x}^j}{\partial x^i} \frac{\delta}{\delta \tilde{x}^j}. \quad (1.3.4)$$

It follows, from (1.3.3), that the coefficients  $N_j^i(x, y)$  of the nonlinear connection  $N$ , with respect to a change of local coordinates on the manifold  $TM$ , (1.1.1) are transformed by the rule:

$$\frac{\partial \tilde{x}^j}{\partial x^k} N_i^k = \tilde{N}_k^j \frac{\partial \tilde{x}^k}{\partial x^i} + \frac{\partial \tilde{y}^j}{\partial x^i}. \quad (1.3.5)$$

The reciprocal property is true, too.

We can prove without difficulties that there exists a nonlinear connection on  $TM$  if  $M$  is a paracompact manifold. The validity of this sentence is assured by the following theorem:

**Theorem 1.3.1.** *If  $S$  is a semispray with the coefficients  $G^i(x, y)$  then the system of functions*

$$N_j^i(x, y) = \frac{\partial G^i}{\partial y^j} \quad (1.3.6)$$

*is the system of coefficients of a nonlinear connection  $N$ .*

Indeed, the formula (1.2.3) and  $\frac{\partial}{\partial y^i} = \frac{\partial \tilde{x}^j}{\partial x^i} \frac{\partial}{\partial \tilde{y}^j}$  give the rule of transformation (1.3.5) for  $N_j^i$  from (1.3.6).

The adapted dual basis  $\{dx^i, \delta y^i\}$  of the basis  $\left(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\right)$  has the 1-forms  $\delta y^i$  as follows:

$$\delta y^i = dy^i + N_j^i dx^j. \quad (1.3.7)$$

With respect to a change of coordinates, (1.1.1), we have

$$d\tilde{x}^i = \frac{\partial \tilde{x}^i}{\partial x^j} dx^j, \quad \delta \tilde{y}^i = \frac{\partial \tilde{x}^i}{\partial x^j} \delta y^j. \quad (1.3.7')$$

Now, we can consider the horizontal and vertical projectors  $h$  and  $v$  with respect to direct decomposition (1.3.2):

$$h = \frac{\delta}{\delta x^i} \otimes dx^i, \quad v = \frac{\partial}{\partial y^i} \otimes \delta y^i. \quad (1.3.8)$$

Some remarkable geometric structures, as the almost product  $\mathbb{P}$  and almost complex structure  $\mathbb{F}$ , are determined by the nonlinear connection  $N$ :

$$\mathbb{P} = \frac{\delta}{\delta x^i} \otimes dx^i - \frac{\partial}{\partial y^i} \otimes \delta y^i = h - v. \quad (1.3.9)$$

$$\mathbb{F} = \frac{\delta}{\delta x^i} \otimes \delta y^i - \frac{\partial}{\partial y^i} \otimes dx^i. \quad (1.3.9')$$

It is not difficult to see that  $\mathbb{P}$  and  $\mathbb{F}$  are globally defined on  $\widetilde{TM}$  and

$$\mathbb{P} \circ \mathbb{P} = Id, \quad \mathbb{F} \circ \mathbb{F} = -Id. \quad (1.3.10)$$

With respect to (1.3.2) a vector field  $X \in \mathcal{X}(TM)$  can be uniquely written in the form

$$X = hX + vX = X^H + X^V \quad (1.3.11)$$

with  $X^H = hX$  and  $X^V = vX$ .

An 1-form  $\omega \in \mathcal{X}^*(TM)$  has the similar form

$$\omega = h\omega + v\omega,$$

where  $h\omega(X) = \omega(X^H)$ ,  $v\omega(X) = \omega(X^V)$ .

A  $d$ -tensor field  $T$  on  $TM$  of type  $(r, s)$  is called a distinguished tensor field (shortly a  $d$ -tensor) if

$$T(\omega_1, \dots, \omega_r, X_1, \dots, X_s) = T(\varepsilon_1 \omega_1, \dots, \varepsilon_r \omega_r, \varepsilon_1 X_1, \dots, \varepsilon_s X_s)$$

where  $\varepsilon_1, \dots, \varepsilon_r, \dots$  are  $h$  or  $v$ .

Therefore  $hX = X^H$ ,  $vX = X^V$ ,  $h\omega = \omega^H$ ,  $v\omega = \omega^V$  are  $d$ -vectors or  $d$ -covectors. In the adapted basis  $\left( \frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i} \right)$  we have

$$X^H = X^i(x, y) \frac{\delta}{\delta x^i}, \quad X^V = \dot{X}^i \frac{\partial}{\partial y^i}$$

and

$$\omega^H = \omega_j(x,y)dx^j, \quad \omega^V = \dot{\omega}_j \delta y^j.$$

A change of local coordinates on  $TM : (x,y) \rightarrow (\tilde{x}, \tilde{y})$  leads to the change of coordinates of  $X^H, X^V, \omega^H, \omega^V$  by the classical rules of transformation:

$$\tilde{X}^i = \frac{\partial \tilde{x}^i}{\partial x^j} X^j, \quad \tilde{\omega}_j = \frac{\partial \tilde{x}^i}{\partial x^j} \omega_i \text{ etc.}$$

So, a  $d$ -tensor  $T$  of type  $(r,s)$  can be written as

$$T = T_{j_1 \dots j_s}^{i_1 \dots i_r}(x,y) \frac{\delta}{\delta x^{i_1}} \otimes \dots \otimes \frac{\partial}{\partial y^{i_r}} \otimes dx^{j_1} \otimes \dots \otimes \delta y^{j_s}. \quad (1.3.12)$$

A change of coordinates (1.1.1) implies the classical rule:

$$\tilde{T}_{j_1 \dots j_s}^{i_1 \dots i_r}(\tilde{x}, \tilde{y}) = \frac{\partial \tilde{x}^{i_1}}{\partial x^{h_1}} \dots \frac{\partial \tilde{x}^{i_r}}{\partial x^{h_r}} \frac{\partial x^{k_1}}{\partial \tilde{x}^{j_1}} \dots \frac{\partial x^{k_s}}{\partial \tilde{y}^{j_s}} = T_{k_1 \dots k_s}^{h_1 \dots h_r}. \quad (1.3.12')$$

Next, we shall study the integrability of the nonlinear connection  $N$  and of the structures  $\mathbb{P}$  and  $\mathbb{F}$ .

Since  $\left( \frac{\delta}{\delta x^i} \right) i = 1, \dots, n$  is an adapted basis to  $N$ , according to the Frobenius theorem it follows that  $N$  is integrable if and only if the Lie brackets  $\left[ \frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j} \right], i, j = 1, \dots, n$  are vector fields in the distribution  $N$ .

But we have

$$\left[ \frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j} \right] = R_{ij}^h \frac{\partial}{\partial y^h} \quad (1.3.13)$$

where

$$R_{ij}^h = \frac{\delta N_i^h}{\delta x^j} - \frac{\delta N_j^h}{\delta x^i}. \quad (1.3.13')$$

It is not difficult to prove that  $R_{ij}^h$  are the components of a  $d$ -tensor field of type (1,2). It is called the *curvature* tensor of the nonlinear connection  $N$ .

We deduce:

**Theorem 1.3.2.** *The nonlinear connection  $N$  is integrable if and only if its curvature tensor  $R_{ij}^h$  vanishes.*

The weak torsion  $t_{ij}^h$  of  $N$  is defined by

$$t_{ij}^h = \frac{\partial N_i^h}{\partial y^j} - \frac{\partial N_j^h}{\partial y^i}. \quad (1.3.14)$$

It is a d-tensor field of type (1,2), too. We say that  $N$  is a symmetric nonlinear connection if its weak torsion  $t_{ij}^h$  vanishes.

Now is not difficult to prove:

**Theorem 1.3.3.**  $1^\circ$  *The almost product structure  $\mathbb{P}$  is integrable if and only if the nonlinear connection  $N$  is integrable;*  
 $2^\circ$  *The almost complex structure  $\mathbb{F}$  is integrable if and only if the nonlinear connection  $N$  is symmetric and integrable.*

The proof is simple using the Nijenhuis tensors  $N_{\mathbb{P}}$  and  $N_{\mathbb{F}}$ . For  $N_{\mathbb{P}}$  we have the expression

$$N_{\mathbb{P}}(X, Y) = \mathbb{P}^2[X, Y] + [\mathbb{P}X, \mathbb{P}Y] - \mathbb{P}[\mathbb{P}X, Y] - \mathbb{P}[X, \mathbb{P}Y], \forall X, Y \in \chi(TM).$$

Also we can see that each structure  $\mathbb{P}$  or  $\mathbb{F}$  characterizes the nonlinear connection  $N$ .

**Autoparallel curves of a nonlinear connection** can be obtained considering the horizontal curves as follows.

A curve  $c : t \in I \subset \mathbb{R} \rightarrow (x^i(t), y^i(t)) \in TM$  has the tangent vector  $\dot{c}$  given by

$$\dot{c} = \dot{c}^H + \dot{c}^V = \frac{dx^i}{dt} \frac{\delta}{\delta x^i} + \frac{\delta y^i}{dt} \frac{\partial}{\partial y^i} \quad (1.3.15)$$

where

$$\frac{\delta y^i}{dt} = \frac{dy^i}{dt} + N_j^i(x, y) \frac{dx^j}{dt}. \quad (1.3.15')$$

The curve  $c$  is a *horizontal curve* if  $\dot{c}^V = 0$  or  $\frac{\delta y^i}{dt} = 0$ .

Evidently, if the functions  $x^i(t), t \in I$  are given and  $y^i(t)$  are the solutions of this system of differential equations, then we have an horizontal curve  $x^i = x^i(t), y^i = y^i(t)$  in  $TM$  with respect to  $N$ .

In the case  $y^i = \frac{dx^i}{dt}$ , the horizontal curves are called the autoparallel curves of the nonlinear connection  $N$ . They are characterized by the system of differential equations

$$\frac{dy^i}{dt} + N_j^i(x, y) \frac{dx^j}{dt} = 0, \quad y^i = \frac{dx^i}{dt}. \quad (1.3.16)$$

A theorem of existence and uniqueness can be easily formulated for the autoparallel curves of a nonlinear connection  $N$  given by its coefficients  $N_j^i(x, y)$ .

## 1.4 $N$ -linear connections

An  $N$ -linear connection on the manifold  $TM$  is a special linear connection  $D$  on  $TM$  that preserves by parallelism the horizontal distribution  $N$  and the vertical distribution  $V$ . We study such linear connections determining the curvature, torsion and structure equations.

Throughout this section  $N$  is an a priori given nonlinear connection with the coefficients  $N_j^i$ .

**Definition 1.4.1.** A linear connection  $D$  on the manifold  $TM$  is called a *distinguished connection* (a  $d$ -connection for short) if it preserves by parallelism the horizontal distribution  $N$ .

Thus, we have  $Dh = 0$ . It follows that we have also:  $Dv = 0$  and  $D\mathbb{F} = 0$ .

If  $Y = Y^H + Y^V$  we get

$$D_X Y = (D_X Y^H)^H + (D_X Y^V)^V, \quad \forall X, Y \in \chi(TM).$$

We can easily prove:

**Proposition 1.4.1.** For a  $d$ -connection the following conditions are equivalent:

- 1°  $DJ = 0$ ;
- 2°  $D\mathbb{F} = 0$ .

**Definition 1.4.2.** A  $d$ -connection  $D$  is called an  $N$ -linear connection if the structure  $J$  (or  $\mathbb{F}$ ) is absolute parallel with respect to  $D$ , i.e.  $DJ = 0$ .

In an adapted basis an  $N$ -linear connection has the form:

$$\begin{cases} D_{\frac{\delta}{\delta x^j}} \frac{\delta}{\delta x^i} = L_{ij}^h \frac{\delta}{\delta x^h}; & D_{\frac{\delta}{\delta x^j}} \frac{\partial}{\partial y^i} = L_{ij}^h \frac{\partial}{\partial y^h}, \\ D_{\frac{\partial}{\partial y^j}} \frac{\delta}{\delta x^i} = C_{ij}^h \frac{\delta}{\delta x^h}; & D_{\frac{\partial}{\partial y^j}} \frac{\partial}{\partial y^i} = C_{ij}^h \frac{\partial}{\partial y^h}. \end{cases} \quad (1.4.1)$$

The set of functions  $D\Gamma = (N_j^i(x, y), L_{ij}^h(x, y), C_{ij}^h(x, y))$  are called the local coefficients of the  $N$ -linear connection  $D$ . Since  $N$  is fixed

we denote sometimes by  $D\Gamma(N) = (L_{ij}^h(x, y), C_{ij}^h(x, y))$  the coefficients of  $D$ .

For instance the  $B\Gamma(N) = \left( \frac{\partial N_i^h}{\partial y^j}, 0 \right)$  are the coefficient of a special  $N$ -linear connection, derived only from the nonlinear connection  $N$ . It is called the **Berwald connection** of the nonlinear connection  $N$ .

We can prove this showing that the system of functions  $\left( \frac{\partial N_i^h}{\partial y^j} \right)$  has the same rule of transformation, with respect to (1.1.1), as the coefficients  $L_{ij}^h$ . Indeed, under a change of coordinates (1.1.1) on  $TM$  the coefficients  $(L_{ij}^h, C_{ij}^h)$  are transformed by the rules:

$$\begin{cases} \tilde{L}_{ij}^h = \frac{\partial \tilde{x}^h}{\partial x^s} L_{pq}^s \frac{\partial x^p}{\partial \tilde{x}^i} \frac{\partial x^q}{\partial \tilde{x}^j} - \frac{\partial^2 \tilde{x}^h}{\partial x^p \partial x^q} \frac{\partial x^p}{\partial \tilde{x}^i} \frac{\partial x^q}{\partial \tilde{x}^j}, \\ \tilde{C}_{ij}^h = \frac{\partial \tilde{x}^h}{\partial x^s} C_{pq}^s \frac{\partial x^p}{\partial \tilde{x}^i} \frac{\partial x^q}{\partial \tilde{x}^j}. \end{cases} \quad (1.4.2)$$

So, the horizontal coefficients  $L_{ij}^h$  of  $D$  have the same rule of transformation of the local coefficients of a linear connection on the base manifold  $M$ . The vertical coefficients  $C_{ij}^h$  are the components of a (1,2)-type d-tensor field.

But, conversely, if the set of functions  $(L_{jk}^i(x, y), C_{jk}^i(x, y))$  are given, having the property (1.4.2), then the equalities (1.4.1) determine an  $N$ -linear connection  $D$  on  $TM$ .

For an  $N$ -linear connection  $D$  on  $TM$  we shall associate two operators of  $h$ - and  $v$ - covariant derivation on the algebra of  $d$ -tensor fields. For each  $X \in \chi(TM)$  we set:

$$D_X^H Y = D_{X^H} Y, \quad D_X^H f = X^H f, \quad \forall Y \in \chi(TM), \forall f \in \mathcal{F}(TM). \quad (1.4.3)$$

If  $\omega \in \chi^*(TM)$ , we obtain

$$(D_X^H \omega)(Y) = X^H(\omega(Y)) - \omega(D_X^H Y). \quad (1.4.3')$$

So we may extend the action of the operator  $D_X^H$  to any d-tensor field by asking that  $D_X^H$  preserves the type of d-tensor fields, is  $\mathbb{R}$ -linear, satisfies the Leibniz rule with respect to tensor product and commutes with the operation of contraction.  $D_X^H$  will be called the *h-covariant derivation operator*.

In a similar way, we set:



$$D_X^V Y = D_{X^V} Y, D_X^V f = X^V(f), \forall Y \in \mathcal{X}(TM), \forall f \in \mathcal{F}(TM). \quad (1.4.4)$$

and

$$D_X^V \omega = X^V(\omega(Y)) - \omega(D_X^V Y), \forall \omega \in \mathcal{X}(TM).$$

Also, we extend the action of the operator  $D_X^V$  to any d-tensor field in a similar way as we did for  $D_X^H$ .  $D_X^V$  is called the  $\nu$ -covariant derivation operator.

Consider now a d-tensor  $T$  given by (1.3.12). According to (1.4.1) its  $h$ -covariant derivation is given by

$$D_X^H T = X^k T_{j_1 \dots j_s | k}^{i_1 \dots i_r} \frac{\delta}{\delta x^{i_1}} \otimes \dots \otimes \frac{\partial}{\partial y^{i_r}} \otimes dx^{j_1} \otimes \dots \otimes dx^{j_s} \quad (1.4.5)$$

where

$$\begin{aligned} T_{j_1 \dots j_s | k}^{i_1 \dots i_r} &= \frac{\delta T_{j_1 \dots j_s}^{i_1 \dots i_r}}{\delta x^k} + L_{pk}^{i_1} T_{j_1 \dots j_s}^{pi_2 \dots i_r} + \dots + L_{pk}^{i_r} T_{j_1 \dots j_s}^{i_1 \dots i_{r-1} p} - \\ &\quad - L_{j_1 k}^p T_{pj_2 \dots j_s}^{i_1 \dots i_r} - \dots - L_{j_s k}^p T_{j_1 \dots j_{s-1} p}^{i_1 \dots i_r}. \end{aligned} \quad (1.4.5')$$

The  $\nu$ -covariant derivative  $D_X^V T$  is as follows:

$$D_X^V T = X^k T_{j_1 \dots j_s | k}^{i_1 \dots i_r} \frac{\delta}{\delta x^{i_1}} \otimes \dots \otimes \frac{\partial}{\partial y^{i_r}} \otimes dx^{j_1} \otimes \dots \otimes dx^{j_s}, \quad (1.4.6)$$

where

$$\begin{aligned} T_{j_1 \dots j_s | k}^{i_1 \dots i_r} &= \frac{\partial T_{j_1 \dots j_s}^{i_1 \dots i_r}}{\partial y^k} + C_{pk}^{i_1} T_{j_1 \dots j_s}^{pi_2 \dots i_r} + \dots + C_{pk}^{i_r} T_{j_1 \dots j_s}^{i_1 \dots i_{r-1} p} - \\ &\quad - C_{j_1 k}^p T_{pj_2 \dots j_s}^{i_1 \dots i_r} - \dots - C_{j_r k}^p T_{j_1 \dots j_{s-1} p}^{i_1 \dots i_r}. \end{aligned} \quad (1.4.6')$$

For instance, if  $g$  is a d-tensor of type (0,2) having the components  $g_{ij}(x, y)$ , we have

$$\begin{aligned} g_{ij|k} &= \frac{\delta g_{ij}}{\delta x^k} - L_{ik}^p g_{pj} - L_{jk}^p g_{ip}, \\ g_{ij|k} &= \frac{\partial g_{ij}}{\partial y^k} - C_{ik}^p g_{pj} - C_{jk}^p g_{ip}. \end{aligned} \quad (1.4.7)$$

For the operators “|” and “|” we preserve the same denomination of  $h$ - and  $v$ -covariant derivations.

The torsion  $T$  of an  $N$ -linear is given by:

$$T(X, Y) = D_X Y - D_Y X - [X, Y], \quad \forall X, Y \in \chi(TM). \quad (1.4.8)$$

The horizontal part  $hT(X, Y)$  and the vertical one  $vT(X, Y)$ , for  $X \in \{X^H, X^V\}$  and  $Y \in \{Y^H, Y^V\}$  determine five d-tensor fields of torsion  $T$ :

$$\begin{cases} hT(X^H, Y^H) = D_X^H Y^H - D_Y^H X^H - [X^H, Y^H]^H, \\ vT(X^H, Y^H) = -v[X^H, Y^H]^V, \\ hT(X^H, Y^V) = -D_Y^V X^H - [X^H, Y^V]^V, \\ vT(X^H, Y^V) = D_X^H Y^V - [X^H, Y^V]^V, \\ vT(X^V, Y^V) = D_X^V Y^V - D_Y^V X^V - [X^V, Y^V]^V. \end{cases} \quad (1.4.9)$$

With respect to the adapted basis, the components of torsion are as follows:

$$\begin{aligned} hT\left(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right) &= T_{ji}^k \frac{\delta}{\delta x^k}; \quad vT\left(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right) = R_{ji}^k \frac{\partial}{\partial y^k}; \\ hT\left(\frac{\partial}{\partial y^i}, \frac{\delta}{\delta x^j}\right) &= C_{ji}^k \frac{\delta}{\delta x^k}; \\ vT\left(\frac{\partial}{\partial y^i}, \frac{\delta}{\delta x^j}\right) &= P_{ji}^k \frac{\partial}{\partial y^k}; \\ vT\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right) &= S_{ji}^k \frac{\partial}{\partial y^k}, \end{aligned} \quad (1.4.9')$$

where  $C_{jk}^i$  are the  $v$ -coefficients of  $D$ ,  $R_{jk}^i$  is the curvature tensor of the nonlinear connection  $N$  and

$$T_{jk}^i = L_{jk}^i - L_{kj}^i, \quad S_{jk}^i = C_{jk}^i - C_{kj}^i, \quad P_{jk}^i = \frac{\partial N_j^i}{\partial y^k} - L_{kj}^i. \quad (1.4.10)$$

The  $N$ -linear connection  $D$  is said to be symmetric if  $T_{jk}^i = S_{jk}^i = 0$ .

Next, we study the curvature of an  $N$ -linear connection  $D$ :

$$R(X, Y)Z = D_X D_Y Z - D_Y D_X Z - D_{[X, Y]}Z, \quad \forall X, Y, Z \in \chi(TM). \quad (1.4.11)$$

As  $D$  preserves by parallelism the distribution  $N$  and  $V$  it follows that the operator  $R(X, Y) = D_X D_Y - D_Y D_X - D_{[X, Y]}$  carries the horizontal vector fields  $Z^H$  into horizontal vector fields and vertical vector fields into verticals. Consequently we have the formula:

$$R(X, Y)Z = hR(X, Y)Z^H + vR(X, Y)Z^V, \quad \forall X, Y, Z \in \chi(TM).$$

Since  $R(X, Y) = -R(Y, X)$ , we obtain:

**Theorem 1.4.1.** *The curvature  $R$  of a  $N$ -linear connection  $D$  on the tangent manifold  $TM$  is completely determined by the following six  $d$ -tensor fields:*

$$\left\{ \begin{array}{l} R(X^H, Y^H)Z^H = D_X^H D_Y^H Z^H - D_Y^H D_X^H Z^H - D_{[X^H, Y^H]}Z^H, \\ R(X^H, Y^H)Z^V = D_X^H D_Y^H Z^V - D_Y^H D_X^H Z^V - D_{[X^H, Y^H]}Z^V, \\ R(X^V, Y^H)Z^H = D_X^V D_Y^H Z^H - D_Y^H D_X^V Z^H - D_{[X^V, Y^H]}Z^H, \\ R(X^V, Y^H)Z^V = D_X^V D_Y^H Z^V - D_Y^H D_X^V Z^V - D_{[X^V, Y^H]}Z^V, \\ R(X^V, Y^V)Z^H = D_X^V D_Y^V Z^H - D_Y^V D_X^V Z^H - D_{[X^V, Y^V]}Z^H, \\ R(X^V, Y^V)Z^V = D_X^V D_Y^V Z^V - D_Y^V D_X^V Z^V - D_{[X^V, Y^V]}Z^V. \end{array} \right. \quad (1.4.12)$$

As the tangent structure  $J$  is absolute parallel with respect to  $D$  we have that

$$JR(X, Y)Z = R(X, Y)JZ.$$

Then  $R(X, Y)Z$  has only three components

$$R(X^H, Y^H)Z^H, R(X^V, Y^H)Z^H, R(X^V, Y^V)Z^H.$$

In the adapted basis these are:

$$\begin{aligned}
R\left(\frac{\delta}{\delta x^k}, \frac{\delta}{\delta x^j}\right) \frac{\delta}{\delta x^h} &= R_{h\ kj}^i \frac{\delta}{\delta x^i}; \\
R\left(\frac{\partial}{\partial y^k}, \frac{\delta}{\delta x^j}\right) \frac{\delta}{\delta x^h} &= P_{h\ kj}^i \frac{\delta}{\delta x^i}; \\
R\left(\frac{\partial}{\partial y^k}, \frac{\partial}{\partial y^j}\right) \frac{\delta}{\delta x^h} &= S_{h\ kj}^i \frac{\delta}{\delta x^i}.
\end{aligned} \tag{1.4.13}$$

The other three components are obtained by applying the operator  $J$  to the previous ones. So, we have  $R\left(\frac{\delta}{\delta x^h}, \frac{\delta}{\delta x^j}\right) \frac{\partial}{\partial y^h} = R_{h\ jk}^i \frac{\partial}{\partial y^h}$  etc. Therefore, the curvature of an  $N$ -linear connection  $D\Gamma = (N_j^i, L_{jk}^i, C_{jk}^i)$  has only three local components  $R_{h\ jk}^i$ ,  $P_{h\ jk}^i$  and  $S_{h\ jk}^i$ . They are given by

$$\begin{aligned}
R_{h\ jk}^i &= \frac{\delta L_{hj}^i}{\delta x^k} - \frac{\delta L_{hk}^i}{\delta x^j} + L_{hj}^s L_{sk}^i - L_{hk}^s L_{sj}^i + C_{hs}^i R_{jk}^s; \\
P_{h\ jk}^i &= \frac{\partial L_{hj}^i}{\partial y^k} - C_{hk|j}^i + C_{hs}^i P_{jk}^s; \\
S_{h\ jk}^i &= \frac{\partial C_{hj}^i}{\partial y^k} - \frac{\partial C_{hk}^i}{\partial y^j} + C_{hj}^s C_{sk}^i - C_{hk}^s C_{sj}^i.
\end{aligned} \tag{1.4.14}$$

If  $X^i(x, y)$  are the components of a  $d$ -vector field on  $TM$  then from (1.4.12) we may derive the Ricci identities for  $X^i$  with respect to an  $N$ -linear connection  $D$ . They are:

$$\begin{aligned}
X_{|j|k}^i - X_{|k|j}^i &= X^s R_{s\ jk}^i - X_{|s}^i T_{jk}^s - X^i |_{s} R_{jk}^s, \\
X_{|j}^i |_{k} - X^i |_{k} |_{j} &= X^s P_{s\ jk}^i - X_{|s}^i C_{jk}^s - X^i |_{s} P_{jk}^s, \\
X^i |_{j} |_{k} - X^i |_{k} |_{j} &= X^s S_{s\ jk}^i - X^i |_{s} S_{jk}^s.
\end{aligned} \tag{1.4.15}$$

We deduce some fundamental identities for the  $N$ -linear connection  $D$ , applying the Ricci identities to the Liouville vector field  $\mathbb{C} = y^i \frac{\partial}{\partial y^i}$ . Considering the d-tensors

$$D_j^i = y^i |_{j}, \quad d_j^i = y^i |_{j} \tag{1.4.16}$$

called  $h$ - and  $v$ -deflection tensors of  $D$  we obtain:

**Theorem 1.4.2.** *For any  $N$ -linear connection  $D$  the following identities hold:*

$$\begin{aligned} D^i_k|_h - D^i_h|_k &= y^s R^i_{s kh} - D^i_s T^s_{kh} - d^i_s R^s_{kh}, \\ D^i_k|_h - d^i_h|_k &= y^s P^i_{s kh} - D^i_s C^s_{kh} - d^i_s P^s_{kh}, \\ d^i_k|_h - d^i_h|_k &= y^s S^i_{s kh} - d^i_s S^s_{kh}. \end{aligned} \quad (1.4.17)$$

Others fundamental identities are the Bianchi identities, which are obtained by writing in the adapted basis the following Bianchi identities:

$$\begin{aligned} \Sigma [D_X T(Y, Z) - R(X, Y)Z + T(T(X, Y), Z)] &= 0, \\ \Sigma [(D_X R)(U, Y, Z) + R(T(X, Y), Z)U] &= 0, \end{aligned} \quad (1.4.18)$$

where  $\Sigma$  means cyclic summation over  $X, Y, Z$ .

## 1.5 Parallelism. Structure equations

Let  $D\Gamma(N) = (L^i_{jk}, C^i_{jk})$  be an  $N$ -linear connection and the adapted basis  $\left(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\right)$ ,  $i = \overline{1, n}$ .

As we know, a curve  $c : t \in I \rightarrow (x^i(t), y^i(t)) \in TM$ , has the tangent vector  $\dot{c} = \dot{c}^H + \dot{c}^V$  given by (3.15), i.e.  $\dot{c} = \frac{dx^i}{dt} \frac{\delta}{\delta x^i} + \frac{\delta y^i}{dt} \frac{\partial}{\partial y^i}$ . The curve  $c$  is horizontal if  $\frac{\delta y^i}{dt} = 0$ , and it is an autoparallel curve with respect to the nonlinear connection  $N$  if  $\frac{\delta y^i}{dt} = 0$ ,  $y^i = \frac{dx^i}{dt}$ .

We set

$$\frac{DX}{dt} = D_{\dot{c}}X, \quad DX = \frac{DX}{dt} dt, \quad \forall X \in \chi(TM). \quad (1.5.1)$$

Here  $\frac{DX}{dt}$  is the covariant differential along the curve  $c$  with respect to the the  $N$ -linear connection  $D$ .

Setting  $X = X^H + X^V$ ,  $X^H = X^i \frac{\delta}{\delta x^i}$ ,  $X^V = \dot{X}^i \frac{\partial}{\partial y^i}$  we have

$$\begin{aligned} \frac{DX}{dt} &= \frac{DX^H}{dt} + \frac{DX^V}{dt} = \\ &= \left\{ X^i \Big|_k \frac{dx^k}{dt} + X^i \Big|_k \frac{\delta y^k}{dt} \right\} \frac{\delta}{\delta x^i} + \left\{ \dot{X}^i \Big|_k \frac{dx^k}{dt} + \dot{X}^i \Big|_k \frac{\delta y^k}{dt} \right\}. \end{aligned} \quad (1.5.2)$$

Consider the *connection 1-forms* of  $D$ :

$$\omega^i_j = L^i_{jk} dx^k + C^i_{jk} \delta y^k. \quad (1.5.3)$$

Then  $\frac{DX}{dt}$  takes the form

$$\frac{DX}{dt} = \left\{ \frac{dX^i}{dt} + X^s \frac{\omega^i_s}{dt} \right\} \frac{\delta}{\delta x^i} + \left\{ \frac{d\dot{X}^i}{dt} + \dot{X}^s \frac{\omega^i_s}{dt} \right\} \frac{\partial}{\partial y^i}. \quad (1.5.4)$$

The vector field  $X$  on  $TM$  is said to be parallel along the curve  $c$ , with respect to the  $N$ -linear connection  $D(N)$  if  $\frac{DX}{dt} = 0$ . Using (1.5.2), the equation  $\frac{DX}{dt} = 0$  is equivalent to  $\frac{DX^H}{dt} = 0$ ,  $\frac{DX^V}{dt} = 0$ . According to (1.5.4) we obtain:

**Proposition 1.5.1.** *The vector field  $X = X^i \frac{\delta}{\delta x^i} + \dot{X}^i \frac{\partial}{\partial y^i}$  from  $\chi(TM)$  is parallel along the parametrized curve  $c$  in  $TM$ , with respect to the  $N$ -linear connection  $D\Gamma(N) = (L^i_{jk}, C^i_{jk})$  if and only if its coefficients  $X^i(x(t), y(t))$  and  $\dot{X}^i(x(t), y(t))$  are solutions of the linear system of differential equations*

$$\frac{dZ^i}{dt} + Z^s(x(t), y(t)) \frac{\omega^i_s(x(t), y(t))}{dt} = 0.$$

A theorem of existence and uniqueness for the parallel vector fields along a curve  $c$  on the manifold  $TM$  can be formulated on the classical way.

The *horizontal geodesic* of  $D$  are the horizontal curves  $c : I \rightarrow TM$  with the property  $D_c \dot{c} = 0$ . Taking  $X^i = \frac{dx^i}{dt}$ ,  $\dot{X}^i = \frac{\delta y^i}{dt} = 0$  we get:

**Theorem 1.5.1.** *The horizontal geodesics of an  $N$ -linear connection are characterized by the system of differential equations:*

$$\frac{d^2x^i}{dt^2} + L_{jk}^i(x, y) \frac{dx^j}{dt} \frac{dx^k}{dt} = 0, \quad \frac{dy^i}{dt} + N^i_j(x, y) \frac{dx^j}{dt} = 0. \quad (1.5.5)$$

Now we can consider a curve  $c_{x_0}^V$  on the fibre  $T_{x_0}M = \pi^{-1}(x_0)$ . It is represented by

$$x^i = x_0^i, \quad y^i = y^i(t), \quad t \in I,$$

$c_{x_0}^V$  is called a *vertical curve* of  $TM$  at the point  $x_0 \in M$ .

The *vertical geodesic* of  $D$  are the vertical curves  $c_{x_0}^V$  with the property  $D_{\dot{c}_{x_0}^V} \dot{c}_{x_0}^V = 0$ .

**Theorem 1.5.2.** *The vertical geodesics at the point  $x_0 \in M$ , of the  $N$ -linear connection  $D\Gamma(N) = (L_{jk}^i, C_{jk}^i)$  are characterized by the following system of differential equations*

$$x^i = x_0^i, \quad \frac{d^2y^i}{dt^2} + C_{jk}^i(x_0, y) \frac{dy^j}{dt} \frac{dy^k}{dt} = 0. \quad (1.5.6)$$

Obviously, the local existence and uniqueness of horizontal or vertical geodesics are assured if initial conditions are given.

Now, we determine the structure equations of an  $N$ -linear connection  $D$ , considering the connection 1-forms  $\omega_j^i$ , (1.5.3).

First of all, we have:

**Lemma 1.5.1.** *The exterior differential of 1-forms  $\delta y^i = dy^i + N^i_j dx^j$  are given by:*

$$d(\delta y^i) = \frac{1}{2} R^i_{j_s} dx^s \wedge dx^j + B^i_{j_s} \delta y^s \wedge dx^j \quad (1.5.7)$$

where

$$B^i_{jk} = \frac{\partial N^i_j}{\partial y^k}. \quad (1.5.7')$$

*Remark 1.5.1.*  $B^i_{jk}$  are the coefficients of the Berwald connection.

**Lemma 1.5.2.** *With respect to a change of local coordinate on the manifold  $TM$ , the following 2-forms*

$$d(dx^i) - dx^s \wedge \omega_s^i; \quad d(\delta y^i) - \delta y^s \wedge \omega_s^i$$

transform as the components of a  $d$ -vector field.

The 2-forms

$$d\omega^i_j - \omega^s_j \wedge \omega^i_s$$

transform as the components of a  $d$ -tensor field of type  $(1, 1)$ .

**Theorem 1.5.3.** *The structure equations of an  $N$ -linear connection  $D\Gamma(N) = (L^i_{jk}, C^i_{jk})$  on the manifold  $TM$  are given by*

$$\begin{aligned} d(dx^i) - dx^s \wedge \omega^i_s &= -\Omega^{(0)i} \\ d(\delta y^i) - \delta y^s \wedge \omega^i_s &= -\Omega^{(1)i} \\ d\omega^i_j - \omega^s_j \wedge \omega^i_s &= -\Omega^i_j \end{aligned} \quad (1.5.8)$$

where  $\Omega^{(0)i}$  and  $\Omega^{(1)i}$  are the 2-forms of torsion:

$$\begin{aligned} \Omega^{(0)i} &= \frac{1}{2} T^i_{jk} dx^j \wedge dx^k + C^i_{jk} dx^j \wedge \delta y^k \\ \Omega^{(1)i} &= \frac{1}{2} R^i_{jk} dx^j \wedge dx^k + P^i_{jk} dx^j \wedge \delta y^k + \frac{1}{2} S^i_{jk} \delta y^j \wedge \delta y^k \end{aligned} \quad (1.5.9)$$

and the 2-forms of curvature  $\Omega^i_j$  are given by

$$\Omega^i_j = \frac{1}{2} R^i_{jkh} dx^k \wedge dx^h + P^i_{jkh} dx^k \wedge \delta y^h + \frac{1}{2} S^i_{jkh} \delta y^j \wedge \delta y^h. \quad (1.5.10)$$

*Proof.* By means of Lemma (1.5.2), the general structure equations of a linear connection on  $TM$  are particularized for an  $N$ -linear connection  $D$  in the form (1.5.8). Using the connection 1-forms  $\omega^i_j$  (1.5.3)

and the formula (1.5.7), we can calculate the forms  $\Omega^{(0)i}$ ,  $\Omega^{(1)i}$  and  $\Omega^i_j$ .

Then it is very easy to determine the structure equations (1.5.9).

*Remark 1.5.2.* The Bianchi identities of an  $N$ -linear connection  $D$  can be obtained from (1.5.8) by calculating the exterior differential of (1.5.8), modulo the same system (1.5.8) and using the exterior differential of  $\Omega^{(0)i}$ ,  $\Omega^{(1)i}$  and  $\Omega^i_j$ .



## Chapter 2

### Lagrange spaces

The notion of Lagrange spaces was introduced and studied by the present author. The term “Lagrange geometry” is due to J. Kern, [121]. We study the geometry of Lagrange spaces as a subgeometry of the geometry of tangent bundle  $(TM, \pi, M)$  of a manifold  $M$ , using the principles of Analytical Mechanics given by variational problem on the integral of action of a regular Lagrangian, the law of conservation, Nöther theorem etc. Remarking that the Euler - Lagrange equations determine a canonical semispray  $S$  on the manifold  $TM$  we study the geometry of a Lagrange space using this canonical semi-spray  $S$  and following the methods given in the first chapter.

Beginning with the year 1987 there were published by author, alone or in collaborations, some books on the Lagrange spaces and the Hamilton spaces [174], and on the higher-order Lagrange and Hamilton spaces [161], as well.

#### 2.1 The notion of Lagrange space

First we shall define the notion of differentiable Lagrangian over the tangent manifold  $TM$  and  $\widetilde{TM} = TM \setminus \{0\}$ ,  $M$  being a real  $n$ -dimensional manifold.

**Definition 2.1.1.** A differentiable Lagrangian is a mapping  $L : (x, y) \in \widetilde{TM} \rightarrow L(x, y) \in \mathbb{R}$ , of class  $C^\infty$  on  $\widetilde{TM}$  and continuous on the null section  $0 : M \rightarrow TM$  of the projection  $\pi : TM \rightarrow M$ .

The Hessian of a differentiable Lagrangian  $L$ , with respect to  $y^i$ , has the elements:

$$g_{ij} = \frac{1}{2} \frac{\partial^2 L}{\partial y^i \partial y^j}. \quad (2.1.1)$$

Evidently, the set of functions  $g_{ij}(x, y)$  are the components of a  $d$ -tensor field, symmetric and covariant of order 2.

**Definition 2.1.2.** A differentiable Lagrangian  $L$  is called *regular* if:

$$\text{rank}(g_{ij}(x, y)) = n, \text{ on } \widetilde{TM}. \quad (2.1.2)$$

Now we can give the definition of a Lagrange space:

**Definition 2.1.3.** A Lagrange space is a pair  $L^n = (M, L(x, y))$  formed by a smooth, real  $n$ -dimensional manifold  $M$  and a regular Lagrangian  $L(x, y)$  for which the  $d$ -tensor  $g_{ij}$  has a constant signature over the manifold  $\widetilde{TM}$ .

For the Lagrange space  $L^n = (M, L(x, y))$  we say that  $L(x, y)$  is the *fundamental function* and  $g_{ij}(x, y)$  is the *fundamental* (or metric) tensor.

### Examples.

1° Every Riemannian manifold  $(M, g_{ij}(x))$  determines a Lagrange space  $L^n = (M, L(x, y))$ , where

$$L(x, y) = g_{ij}(x)y^i y^j. \quad (2.1.3)$$

This example allows to say:

If the manifold  $M$  is paracompact, then there exist Lagrangians  $L(x, y)$  such that  $L^n = (M, L(x, y))$  is a Lagrange space.

2° The following Lagrangian from electrodynamics

$$L(x, y) = mc\gamma_{ij}(x)y^i y^j + \frac{2e}{m}A_i(x)y^i + \mathcal{U}(x), \quad (2.1.4)$$

where  $\gamma_{ij}(x)$  is a pseudo-Riemannian metric,  $A_i(x)$  a covector field and  $\mathcal{U}(x)$  a smooth function,  $m, c, e$  are physical constants, is a Lagrange space  $L^n$ . This is called the Lagrange space of electrodynamics.

We already have seen that  $g_{ij}(x, y)$  from (2.1.1) is a  $d$ -tensor field, i.e. with respect to (1.1.1), we have

$$\tilde{g}_{ij}(\tilde{x}, \tilde{y}) = \frac{\partial x^h}{\partial \tilde{x}^i} \frac{\partial x^k}{\partial \tilde{x}^j} g_{hk}(x, y).$$

Now we can prove without difficulties:

**Theorem 2.1.1.** For a Lagrange space  $L^n$  the following properties hold:

1° *The system of functions*

$$p_i = \frac{1}{2} \frac{\partial L}{\partial y^i}$$

*determines a d-covector field.*

2° *The functions*

$$C_{ijk} = \frac{1}{4} \frac{\partial^3 L}{\partial y^i \partial y^j \partial y^k} = \frac{1}{2} \frac{\partial g_{ij}}{\partial y^k}$$

*are the components of a symmetric d-tensor field of type (0, 3).*

3° *The 1-form*

$$\omega = p_i dx^i = \frac{1}{2} \frac{\partial L}{\partial y^i} dx^i \quad (2.1.5)$$

*depend on the Lagrangian L only and are globally defined on the manifold  $\widetilde{TM}$*

4° *The 2-form*

$$\theta = d\omega = dp_i \wedge dx^i \quad (2.1.6)$$

*is globally defined on  $\widetilde{TM}$  and defines a symplectic structure on  $\widetilde{TM}$ .*

## 2.2 Variational problem. Euler-Lagrange equations

The variational problem can be formulated for differentiable Lagrangians and can be solved in the case when the integral of action is defined on the parametrized curves.

Let  $L : TM \rightarrow R$  be a differentiable Lagrangian and  $c : t \in [0, 1] \rightarrow (x^i(t)) \in U \subset M$  be a smooth curve, with a fixed parametrization, having  $\text{Im}c \subset U$ , where  $U$  is a domain of a local chart on the manifold  $M$ . The curve  $c$  can be extended to  $\pi^{-1}(U) \subset \widetilde{TM}$  by

$$\tilde{c} : t \in [0, 1] \rightarrow (x^i(t), \frac{dx^i}{dt}(t)) \in \pi^{-1}(U).$$

So,  $\text{Im}\tilde{c} \subset \pi^{-1}(U)$ .

The integral of action of the Lagrangian  $L$  on the curve  $c$  is given by the functional:

$$I(c) = \int_0^1 L\left(x, \frac{dx}{dt}\right) dt. \quad (2.2.1)$$

Consider the curves

$$c_\varepsilon : t \in [0, 1] \rightarrow (x^i(t) + \varepsilon V^i(t)) \in M \quad (2.2.2)$$

which have the same end points  $x^i(0)$  and  $x^i(1)$  as the curve  $c$ ,  $V^i(t) = V^i(x^i(t))$  being a regular vector field on the curve  $c$ , with the property  $V^i(0) = V^i(1) = 0$  and  $\varepsilon$  is a real number, sufficiently small in absolute value, so that  $\text{Im}c_\varepsilon \subset U$ .

The extension of a curve  $c_\varepsilon$  to  $\widetilde{TM}$  is given by

$$\widetilde{c}_\varepsilon : t \in [0, 1] \mapsto \left( x^i(t) + \varepsilon V^i(t), \frac{dx^i}{dt} + \varepsilon \frac{dV^i}{dt} \right) \in \pi^{-1}(U).$$

The integral of action of the Lagrangian  $L$  on the curve  $c_\varepsilon$  is

$$I(c_\varepsilon) = \int_0^1 L\left(x + \varepsilon V, \frac{dx}{dt} + \varepsilon \frac{dV}{dt}\right) dt. \quad (2.2.1')$$

A necessary condition for  $I(c)$  to be an extremal value of  $I(c_\varepsilon)$  is

$$\left. \frac{dI(c_\varepsilon)}{d\varepsilon} \right|_{\varepsilon=0} = 0. \quad (2.2.3)$$

Under our condition of differentiability, the operator  $\frac{d}{d\varepsilon}$  is permuting with the operator of integration.  
From (2.2.1') we obtain

$$\frac{dI(c_\varepsilon)}{d\varepsilon} = \int_0^1 \frac{d}{d\varepsilon} L\left(x + \varepsilon V, \frac{dx}{dt} + \varepsilon \frac{dV}{dt}\right) dt. \quad (2.2.4)$$

But we have

$$\begin{aligned} \frac{d}{d\varepsilon} L\left(x + \varepsilon V, \frac{dx}{dt} + \varepsilon \frac{dV}{dt}\right) \Big|_{\varepsilon=0} &= \frac{\partial L}{\partial x^i} V^i + \frac{\partial L}{\partial y^i} \frac{dV^i}{dt} = \\ &= \left\{ \frac{\partial L}{\partial x^i} - \frac{d}{dt} \frac{\partial L}{\partial y^i} \right\} V^i + \frac{d}{dt} \left\{ \frac{\partial L}{\partial y^i} V^i \right\}, \quad y^i = \frac{dx^i}{dt}. \end{aligned}$$

Substituting in (2.2.4) and taking into account the fact that  $V^i(x(t))$  is arbitrary, we obtain the following theorem.

**Theorem 2.2.1.** *In order for the functional  $I(c)$  to be an extremal value of  $I(c_\varepsilon)$  it is necessary for the curve  $c(t) = (x^i(t))$  to satisfy the Euler-Lagrange equations:*

$$E_i(L) := \frac{\partial L}{\partial x^i} - \frac{d}{dt} \frac{\partial L}{\partial y^i} = 0, \quad y^i = \frac{dx^i}{dt}. \quad (2.2.5)$$

For the Euler-Lagrange operator  $E_i = \frac{\partial}{\partial x^i} - \frac{d}{dt} \frac{\partial}{\partial y^i}$  we have:

**Theorem 2.2.2.** *The following properties hold true:*

1°  $E_i(L)$  is a  $d$ -covector field.

2°  $E_i(L + L') = E_i(L) + E_i(L')$ .

3°  $E_i(aL) = aE_i(L), a \in \mathbb{R}$ .

4°  $E_i\left(\frac{dF}{dt}\right) = 0, \forall F \in \mathcal{F}(TM)$  with  $\frac{\partial F}{\partial y^i} = 0$ .

The notion of *energy of a Lagrangian*  $L$  can be introduced as in the Theoretical Mechanics [17], [164], by

$$E_L = y^i \frac{\partial L}{\partial y^i} - L. \quad (2.2.6)$$

We obtain, without difficulties:

**Theorem 2.2.3.** *For every smooth curve  $c$  on the base manifold  $M$  the following formula holds:*

$$\frac{dE_L}{dt} = -\frac{dx^i}{dt} E_i(L), \quad y^i = \frac{dx^i}{dt}. \quad (2.2.7)$$

Consequently:

**Theorem 2.2.4.** *For any differentiable Lagrangian  $L(x, y)$  the energy  $E_L$  is conserved along every solution curve  $c$  of the Euler-Lagrange equations*

$$E_i(L) = 0, \quad \frac{dx^i}{dt} = y^i.$$

A Noether theorem can be proved:

**Theorem 2.2.5.** *For any infinitesimal symmetry on  $M \times \mathbb{R}$  of the Lagrangian  $L(x, y)$  and for any smooth function  $\phi(x)$  the following function:*

$$\mathcal{F}(L, \phi) = V^i \frac{\partial L}{\partial y^i} - \tau E_L - \phi(x)$$

*is conserved on every solution curve  $c$  of the Euler-Lagrange equations  $E_i(L) = 0$ ,  $y^i = \frac{dx^i}{dt}$ .*

**Remark.** An infinitesimal symmetry on  $M \times \mathbb{R}$  is given by  $x^i = x^i + \varepsilon V^i(x, t)$ ,  $t' = t + \varepsilon \tau(x, t)$ .

### 2.3 Canonical semispray. Nonlinear connection

Now we can apply the previous theory in order to study the Lagrange space  $L^n = (M, L(x, y))$ . As we shall see that  $L^n$  determines a canonical semispray  $S$  and  $S$  gives a canonical nonlinear connection on the manifold  $\widetilde{TM}$ .

As we know, the fundamental tensor  $g_{ij}$  of the space  $L^n$  is nondegenerate, and  $E_i(L)$  is a  $d$ -covector field, so the equations  $g^{ij}E_j(L) = 0$  have a geometrical meaning.

**Theorem 2.3.1.** *If  $L^n = (M, L)$  is a Lagrange space, then the system of differential equations*

$$g^{ij}E_j(L) = 0, y^i = \frac{dx^i}{dt} \quad (2.3.1)$$

*can be written in the form:*

$$\frac{d^2x^i}{dt^2} + 2G^i \left( x, \frac{dx}{dt} \right) = 0, y^i = \frac{dx^i}{dt} \quad (2.3.1')$$

*where*

$$2G^i(x, y) = \frac{1}{2}g^{ij} \left\{ \frac{\partial^2 L}{\partial y^j \partial x^k} y^k - \frac{\partial L}{\partial x^j} \right\}. \quad (2.3.2)$$

*Proof.* We have

$$E_i(L) = \frac{\partial L}{\partial x^i} - \left\{ \frac{\partial^2 L}{\partial y^j \partial x^k} + 2g_{ij} \frac{dy^j}{dt} \right\}, y^i = \frac{dx^i}{dt}.$$

So, (2.3.1) implies (2.3.1'), (2.3.2).

The previous theorem tells us that the Euler Lagrange equations for a Lagrange space are given by a system of  $n$  second order ordinary differential equations. According with theory from Section 1.2, Ch. 1, it follows that the equations (2.3.1) determine a semispray with the coefficients  $G^i(x, y)$ :

$$S = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y) \frac{\partial}{\partial y^i}. \quad (2.3.3)$$

$S$  is called the canonical semispray of the Lagrange space  $L^n$ .

By means of Theorem 1.3.1, it follows:

**Theorem 2.3.2.** *Every Lagrange space  $L^n = (M, L)$  has a canonical nonlinear connection  $N$  which depends only on the fundamental function  $L$ . The local coefficients of  $N$  are given by*

$$N^i_j = \frac{\partial G^i}{\partial y^j} = \frac{1}{4} \frac{\partial}{\partial y^j} \left\{ g^{ik} \left( \frac{\partial^2 L}{\partial y^k \partial x^h} y^h - \frac{\partial L}{\partial x^k} \right) \right\}. \quad (2.3.4)$$

Evidently:

**Proposition 2.3.1.** *The canonical nonlinear connection  $N$  is symmetric, i.e.  $t^i_{jk} = \frac{\partial N^i_j}{\partial y^k} - \frac{\partial N^i_k}{\partial y^j} = 0$ .*

**Proposition 2.3.2.** *The canonical nonlinear connection  $N$  is invariant with respect to the Carathéodory transformation*

$$L'(x, y) = L(x, y) + \frac{\partial \varphi(x)}{\partial x^i} y^i. \quad (2.3.5)$$

Indeed, we have

$$E_i(L') = E_i \left( L(x, y) + \frac{d\varphi}{dt} \right) = E_i(L). \square$$

So,  $E_i(L'(x, y)) = 0$  determines the same canonical semispray as the one determined by  $E_i(L(x, y)) = 0$ . Thus, the Carathéodory transformation (2.3.5) preserves the nonlinear connection  $N$ .

**Example.** The Lagrange space of electrodynamics,  $L^n = (M, L(x, y))$ , where  $L(x, y)$  is given by (2.1.4) with  $U(x) = 0$  has the canonical semispray with the coefficients:

$$G^i(x, y) = \frac{1}{2} \gamma^i_{jk}(x) y^j y^k - g^{ij}(x) F_{jk}(x) y^k, \quad (2.3.6)$$

where  $\gamma^i_{jk}(x)$  are the Christoffel symbols of the metric tensor  $g_{ij}(x) = mc\gamma_{ij}(x)$  of the space  $L^n$  and  $F_{jk}$  is the electromagnetic tensor

$$F_{jk}(x, y) = \frac{e}{2m} \left( \frac{\partial A_k}{\partial x^j} - \frac{\partial A_j}{\partial x^k} \right). \quad (2.3.7)$$

Therefore, the integral curves of the Euler-Lagrange equation are given by the solution curves of the *Lorentz equations*:

$$\frac{d^2 x^i}{dt^2} + \gamma^i_{jk}(x) \frac{dx^j}{dt} \frac{dx^k}{dt} = g^{ij}(x) F_{jk}(x) \frac{dx^k}{dt}. \quad (2.3.8)$$

According to (2.3.4), the canonical nonlinear connection of the Lagrange space of electrodynamics  $L^n$  has the local coefficients given by

$$N^i_j(x, y) = \gamma^i_{jk}(x) y^k - g^{ik}(x) F_{kj}(x). \quad (2.3.9)$$

It is remarkable that the coefficients  $N^i_j$  from (2.3.9) are linear with respect to  $y^i$ .

**Proposition 2.3.3.** *The Berwald connection of the canonical nonlinear connection  $N$  has the coefficients  $B\Gamma(N) = (\gamma^i_{jk}(x), 0)$ .*

**Proposition 2.3.4.** *The solution curves of the Euler-Lagrange equations and the autoparallel curves of the canonical nonlinear connection  $N$  are given by the Lorentz equations (2.3.8).*

In the last part of this section, we underline the following theorem:

**Theorem 2.3.3.** *The autoparallel curves of the canonical nonlinear connection  $N$  are given by the following system:*

$$\frac{d^2 x^i}{dt^2} + N^i_j \left( x, \frac{dx}{dt} \right) \frac{dx^j}{dt} = 0,$$

where  $N^i_j$  are given by (2.3.4).



## 2.4 Hamilton-Jacobi equations

Consider a Lagrange space  $L^n = (M, L(x, y))$  and  $N(N^i_j)$  its canonical nonlinear connection. The adapted basis  $\left(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\right)$  to the horizontal distribution  $N$  and the vertical distribution  $V$  has the horizontal vector fields:

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N^j_i \frac{\partial}{\partial y^j}. \quad (2.4.1)$$

Its dual is  $(dx^i, \delta y^i)$ , with

$$\delta y^i = dy^i + N^i_j dx^j. \quad (2.4.2)$$

Theorem 1.1.1 give us the momenta

$$p_i = \frac{1}{2} \frac{\partial L}{\partial y^i}, \quad (2.4.3)$$

the 1-form

$$\omega = p_i dx^i \quad (2.4.4)$$

and the 2-form

$$\theta = d\omega = dp_i \wedge dx^i. \quad (2.4.5)$$

These geometrical object fields are globally defined on  $\widetilde{TM}$ .  $\theta$  is a *symplectic structure* on the manifold  $\widetilde{TM}$ .

**Proposition 2.4.5.** *In the adapted basis the 2-form  $\theta$  is given by*

$$\theta = g_{ij} \delta y^i \wedge dx^j. \quad (2.4.6)$$

Indeed,  $\theta = dp_i \wedge dx^i = \frac{1}{2} \left( \frac{\delta}{\delta x^s} \frac{\partial L}{\partial y^i} dx^s + \frac{\partial}{\partial y^s} \frac{\partial L}{\partial y^i} \delta y^s \right) \wedge dx^i =$   
 $\frac{1}{4} \left( \frac{\delta}{\delta x^s} \frac{\partial L}{\partial y^i} - \frac{\delta}{\delta x^i} \frac{\partial L}{\partial y^s} \right) dx^s \wedge dx^i + g_{is} \delta y^s \wedge dx^i.$

But is not difficult to see that the coefficient of  $dx^s \wedge dx^i$  vanishes.

The triple  $(\widetilde{TM}, \theta, L)$  is called a Lagrangian system.

The energy  $E_L$  of the space  $L^n$  is given by (2.2.6). Denoting  $\mathcal{H} = \frac{1}{2} E_L$ ,  $\mathcal{L} = \frac{1}{2} L$ , then (2.2.6) can be written as:

$$\mathcal{H} = p_i y^i - \mathcal{L}(x, y). \quad (2.4.7)$$

But, along the integral curve of the Euler-Lagrange equations (2.2.5) we have

$$\frac{\partial \mathcal{H}}{\partial x^i} = -\frac{\partial \mathcal{L}}{\partial x^i} = -\frac{dp_i}{dt}.$$

And from (2.4.7), we get

$$\frac{\partial \mathcal{H}}{\partial p_i} = y^i = \frac{dx^i}{dt}.$$

So, we obtain:

**Theorem 2.4.1.** *Along to integral curves of the Euler-Lagrange equations the Hamilton-Jacobi equations:*

$$\frac{dx^i}{dt} = \frac{\partial \mathcal{H}}{\partial p_i}; \quad \frac{dp_i}{dt} = -\frac{\partial \mathcal{H}}{\partial x^i}, \quad (2.4.8)$$

where  $\mathcal{H}$  is given by (2.4.7) and  $p_i = \frac{1}{2} \frac{\partial L}{\partial y^i}$ , are satisfied.

These equations are important in applications.

**Example.** For the Lagrange space of Electrodynamics with the fundamental function  $L(x, y)$  from (2.1.4) and  $U(x) = 0$  we obtain

$$\mathcal{H} = \frac{1}{2mc} \gamma^{ij}(x) p_i p_j - \frac{e}{mc^2} A^i(x) p_i + \frac{e^2}{2mc^3} A^i(x) A_i(x)$$

( $A^i = \gamma^{ij} A_j$ ).

Then, the Hamilton - Jacobi equations can be written without difficulties.

Now we remark that  $\theta$  being a symplectic structure on  $\widetilde{TM}$ , exterior differential  $d\theta$  vanishes. But in adapted basis

$$d\theta = dg_{ij} \wedge \delta y^i \wedge dx^j + g_{ij} d\delta y^i \wedge dx^j = 0$$

reduces to:

$$\begin{aligned} & \frac{1}{2} \left( \frac{\delta g_{ij}}{\delta x^k} - \frac{\delta g_{ik}}{\delta x^j} \right) \delta y^i \wedge dx^j \wedge dx^k + \frac{1}{2} \left( \frac{\partial g_{ij}}{\partial x^k} - \frac{\partial g_{kj}}{\partial y^i} \right) \delta y^k \wedge \delta y^i \wedge dx^j + \\ & + g_{ij} \left( \frac{1}{2} R^i_{rs} dx^s \wedge dx^r + B^i_{rs} \delta y^s \wedge dx^r \right) \wedge dx^j = 0. \end{aligned}$$

We obtain

**Theorem 2.4.2.** *For any Lagrange space  $L^n$  the following identities hold*

$$g_{ij|k} - g_{ik|j} = 0, \quad g_{ij|k} - g_{ik|j} = 0. \quad (2.4.9)$$

Indeed, taking into account the  $h$ - and  $v$ -covariant derivations of the metric  $g_{ij}$  with respect to Berwald connection  $B\Gamma(N) = \left( \frac{\partial N_j^i}{\partial y^k}, 0 \right)$ ,

i.e.

$$g_{ij|k} = \frac{\delta g_{ij}}{\delta x^k} - B_{ik}^r g_{kj} - B_{jk}^r g_{ir}$$

and  $g_{ij|k} = \frac{\partial g_{ij}}{\partial y^k}$ , according with the properties  $\frac{\partial g_{ij}}{\partial y^k} = 2C_{ijk}$ ,  $B_{jk}^i = B_{kj}^i$ , we obtain (2.4.9).

## 2.5 Metrical $N$ -linear connections

Let  $N(N_j^i)$  be the canonical nonlinear connection of the Lagrange space  $L^n = (M, L)$  and  $D$  an  $N$ -linear connection with the coefficients  $D\Gamma(N) = (L_{jk}^i, C_{jk}^i)$ . Then, the  $h$ - and  $v$ -covariant derivations of the fundamental tensor  $g_{ij}$ ,  $g_{ij|k}$  and  $g_{ij|k}$  are given by (1.4.7).

Applying the theory of  $N$ -linear connection from Chapter 1, one proves without difficulties, the following theorem:

**Theorem 2.5.1.**  $1^\circ$  *On the manifold  $\widetilde{TM}$  there exist only one  $N$ -linear connection  $D$  which verifies the following axioms:*

- $A_1$   $N$  is canonical nonlinear connection of the space  $L^n$ .
- $A_2$   $g_{ij|k} = 0$  ( $D$  is  $h$ -metrical);
- $A_3$   $g_{ij|k} = 0$  ( $D$  is  $v$ -metrical);
- $A_4$   $T_{jk}^i = 0$  ( $D$  is  $h$ -torsion free);
- $A_5$   $S_{jk}^i = 0$  ( $D$  is  $v$ -torsion free).

$2^\circ$  *The coefficients  $D\Gamma(N) = (L_{jk}^i, C_{jk}^i)$  of  $D$  are expressed by the following generalized Christoffel symbols:*

$$\begin{aligned}
L_{jk}^i &= \frac{1}{2}g^{ir} \left( \frac{\delta g_{rk}}{\delta x^j} + \frac{\delta g_{rj}}{\delta x^k} - \frac{\delta g_{jk}}{\delta x^r} \right) \\
C_{jk}^i &= \frac{1}{2}g^{ir} \left( \frac{\partial g_{rk}}{\partial y^j} + \frac{\partial g_{rj}}{\partial y^k} - \frac{\partial g_{jk}}{\partial y^r} \right)
\end{aligned} \tag{2.5.1}$$

3° This connection depends only on the fundamental function  $L(x, y)$  of the Lagrange space  $L^n$ .

The  $N$ -linear connection  $D$  given by the previous theorem is called the *canonical metric connection* and denoted by  $CG(N) = (L_{jk}^i, C_{jk}^i)$ .

By means of §1.5, Ch. 1, the connection 1-forms  $\omega^i_j$  of the  $CG(N)$  are

$$\omega^i_j = L_{jk}^i dx^k + C_{jk}^i \delta y^k, \tag{2.5.2}$$

**Theorem 2.5.2.** *The canonical metrical connection  $CG(N)$  satisfies the following structure equations:*

$$\begin{aligned}
d(dx^i) - dx^k \wedge \omega^i_k &= -\Omega^{(0)i}, \\
d(\delta y^i) - \delta y^k \wedge \omega^i_k &= -\Omega^{(1)i}, \\
d\omega^i_j - \omega_j^k \wedge \omega^i_k &= -\Omega^i_j
\end{aligned} \tag{2.5.3}$$

where 2-forms of torsion  $\Omega^{(0)i}$  and  $\Omega^{(1)i}$  are as follows

$$\begin{aligned}
\Omega^{(0)i} &= C_{jk}^i dx^j \wedge \delta y^k, \\
\Omega^{(1)i} &= \frac{1}{2}R^i_{jk} dx^j \wedge dx^k + P^i_{jk} dx^j \wedge \delta y^k
\end{aligned} \tag{2.5.4}$$

and the 2-forms of curvature  $\Omega^i_j$  are

$$\Omega^i_j = \frac{1}{2}R^i_{jkh} dx^k \wedge dx^h + P^i_{jk} dx^k \wedge \delta y^h + \frac{1}{2}S^i_{jkh} \delta y^k \wedge \delta y^h. \tag{2.5.5}$$

The  $d$ -tensors of torsion  $R^i_{jk}$ ,  $P^i_{jk}$  are given by (1.3.13') and (1.4.10), and the  $d$ -tensors of curvature  $R^i_{jkh}$ ,  $P^i_{jkh}$ ,  $S^i_{jkh}$  have the expressions (1.4.14).

Starting from the canonical metrical connection  $CG(N) = (L_{jk}^i, C_{jk}^i)$  we can derive other  $N$ -linear connections depend only on the space  $L^n$ :

Berwald connection  $B\Gamma(N) = \left( \frac{\partial N_j^i}{\partial y^k}, 0 \right)$ ; Chern-Rund connection  $R\Gamma(N) = (L_{jk}^i, 0)$  and Hashiguchi connection  $H\Gamma(N) = \left( \frac{\partial N_j^i}{\partial y^k}, C_{jk}^i \right)$ .

For special transformations of these connections, the following commutative diagram holds:

$$\begin{array}{ccccc} & & R\Gamma(N) & & \\ & \nearrow & & \searrow & \\ CG(N) & & \longrightarrow & & B\Gamma(N) \\ & \searrow & & \nearrow & \\ & & H\Gamma(N) & & \end{array}$$

Some properties of the canonical metrical connection  $CG(N)$  are given by:

**Proposition 2.5.6.** *We have:*

- 1°  $\sum_{(ijk)} R_{ijk} = 0$ ,  $(R_{ijk} = g_{ih}R^h{}_{jk})$ .
- 2°  $P_{ijk} = g_{ih}P^h{}_{jk}$  is totally symmetric.
- 3° The covariant curvature  $d$ -tensors  $R_{ijkh} = g_{jr}R_i{}^r{}_{kh}$ ,  $P_{ijkh} = g_{jr}P_i{}^r{}_{kh}$  and  $S_{ijkh} = g_{ir}S_i{}^r{}_{kh}$  are skew-symmetric with respect to the first two indices.
- 4°  $S_{ijkh} = C_{iks}C^s{}_{jh} - C_{ih}sC^s{}_{jk}$ .
- 5°  $C_{ikh} = g_{is}C^s{}_{jh}$ .

These properties can be proved using the property  $d\theta = 0$ , with  $\theta = g_{ij}\delta y^i \wedge dx^j$ , the Ricci identities applied to the fundamental tensor  $g_{ij}$  and the equations  $g_{ij|k} = 0$ ,  $g_{ij|k} = 0$ .

By the same method we can study the metrical connections with a priori given  $h$ - and  $v$ - torsions.

**Theorem 2.5.3.** 1° *There exists only one  $N$ -linear connection  $\bar{D}\Gamma(N) = (\bar{L}_{jk}^i, \bar{C}_{jk}^i)$  which satisfies the following axioms:*

- $A'_1$   $N$  is canonical nonlinear connection of the space  $L^n$ ,
- $A'_2$   $g_{ij|k} = 0$  ( $\bar{D}$  is  $h$ -metrical),
- $A'_3$   $g_{ij|k} = 0$  ( $\bar{D}$  is  $v$ -metrical),
- $A'_4$  The  $h$ -tensor of torsion  $\bar{T}_{jk}^i$  is a priori given.
- $A'_5$  The  $v$ -tensor of torsion  $\bar{S}_{jk}^i$  is a priori given.

2° The coefficients  $(\bar{L}_{jk}^i, \bar{C}_{jk}^i)$  of  $\bar{D}$  are given by

$$\begin{aligned}\bar{L}_{jk}^i &= L_{jk}^i + \frac{1}{2}g^{ih}(g_{jr}\bar{T}^r_{kh} + g_{kr}\bar{T}^r_{jh} - g_{hr}\bar{T}^r_{kj}), \\ \bar{C}_{jk}^i &= C_{jk}^i + \frac{1}{2}g^{ih}(g_{jr}\bar{S}^r_{kh} + g_{kr}\bar{S}^r_{jh} - g_{hr}\bar{S}^r_{kj})\end{aligned}\quad (2.5.6)$$

where  $(L_{jk}^i, C_{jk}^i)$  are the coefficients of the canonical metrical connection.

From now on  $\bar{T}_{jk}^i, \bar{S}_{jk}^i$  will be denoted by  $T_{jk}^i, S_{jk}^i$  and the  $N$ -linear connection given by the previous theorem will be called *metrical  $N$ -connection* of the Lagrange space  $L^n$ .

Some particular cases can be studied using the expressions of the coefficients  $\bar{L}_{jk}^i$  and  $\bar{C}_{jk}^i$ . For instance the semi-symmetric case will be obtained taking  $T_{jk}^i = \delta_j^i \sigma_k - \delta_k^i \sigma_j$ ,  $S_{jk}^i = \delta_j^i \tau_k - \delta_k^i \tau_j$ .

**Proposition 2.5.7.** *The Ricci identities of the metrical  $N$ -linear connection  $D\Gamma(N)$  are given by:*

$$\begin{aligned}X^i|_j|_k - X^i|_k|_j &= X^r R_r^i{}_{jk} - X^i|_r T^r{}_{jk} - X^i|_r R^r{}_{jk}, \\ X^i|_j|_k - X^i|_k|_j &= X^r P_r^i{}_{jk} - X^i|_r C^r{}_{jk} - X^i|_r P^r{}_{jk}, \\ X^i|_j|_k - X^i|_k|_j &= X^r S_r^i{}_{jk} - X^i|_r S^r{}_{jk}.\end{aligned}\quad (2.5.7)$$

Of course these identities can be extended to a  $d$ -tensor field of type  $(r, s)$ .

Denoting

$$D^i{}_j = y^i|_j, \quad d^i{}_j = y^i|_j. \quad (2.5.8)$$

we have the  $h$ - and  $v$ - deflection tensors. They have the known expressions:

$$D^i{}_j = y^s L^i{}_{sj} - N^i{}_j; \quad d^i{}_j = \delta^i{}_j + y^s C^i{}_{sj}. \quad (2.5.7')$$

According to Ricci identities (2.5.7) we obtain:

**Theorem 2.5.4.** *For any metrical  $N$ -linear connection the following identities hold:*

$$\begin{aligned}D^i{}_j|_k - D^i{}_k|_j &= y^s R_s^i{}_{jk} - D^i{}_s T^s{}_{jk} - d^i{}_s R^s{}_{jk}, \\ D^i{}_j|_k - d^i{}_k|_j &= y^s P_s^i{}_{jk} - D^i{}_s C^s{}_{jk} - d^i{}_s P^s{}_{jk}, \\ d^i{}_j|_k - d^i{}_k|_j &= y^s S_s^i{}_{jk} - d^i{}_s S^s{}_{jk}.\end{aligned}\quad (2.5.9)$$

We will apply this theory in a next section taking into account the canonical metrical connection  $C\Gamma(N)$  and taking  $T^i_{jk} = 0, S^i_{jk} = 0$ .

Of course the theory of parallelism of vector fields and the  $h$ -geodesics or  $v$ -geodesics for the metrical connection  $N$ -linear connections can be obtained as a consequence of the corresponding theory from Ch. 1.

## 2.6 The electromagnetic and gravitational fields

Let us consider a Lagrange spaces  $L^n = (M, L)$  endowed with the canonical nonlinear connection  $N$  and with the canonical metrical  $N$ -connection  $C\Gamma(N) = (L^i_{jk}, C^i_{jk})$ .

The covariant deflection tensors  $D_{ji}$  and  $d_{ji}$  are given by  $D_{ij} = g_{is}D^s_j, d_{ij} = g_{is}d^s_j$ . We have:

$$D_{ij|k} = g_{is}D^s_{j|k}, d_{ij|k} = g_{is}d^s_{j|k}$$

etc. So, we have

**Proposition 2.6.8.** *The covariant deflection tensors  $D_{ij}$  and  $d_{ij}$  of the canonical metrical  $N$ -connection  $C\Gamma(N)$  satisfy the identities:*

$$\begin{aligned} D_{ij|k} - D_{ik|j} &= y^s R_{sijk} - d_{is} R^s_{jk}, \\ D_{ij|k} - d_{ik|j} &= y^s P_{sijk} - D_{is} C^s_{jk} - d_{is} P^s_{jk}, \\ d_{ij|k} - d_{ik|j} &= y^s S_{sijk}. \end{aligned} \quad (2.6.1)$$

The Lagrangian theory of electrodynamics lead us to introduce [182], [184], [183], [175]:

**Definition 2.6.1.** The d-tensor fields:

$$F_{ij} = \frac{1}{2}(D_{ij} - D_{ji}), f_{ij} = \frac{1}{2}(d_{ij} - d_{ji}) \quad (2.6.2)$$

are the  $h$ - and  $v$ -electromagnetic tensor of the Lagrange space  $L^n = (M, L)$ .

The Bianchi identities for  $C\Gamma(N)$  and the identities (2.6.1) lead to the following important result:

**Theorem 2.6.1.** *The following generalized Maxwell equations hold:*

$$F_{ij|k} + F_{jk|i} + F_{ki|j} = - \sum_{(ijk)} C_{ios} R^s{}_{jk}, \quad (2.6.3)$$

$$F_{ij|k} + F_{jk|i} + F_{ki|j} = 0,$$

where  $C_{ios} = C_{ijs}y^j$ , and  $\sum_{(ijk)}$  means cyclic sum.

**Corollary 2.6.1.** *If the canonical nonlinear connection  $N$  of the space  $L^n$  is integrable then the equations (2.6.3) reduce to:*

$$\sum_{(ijk)} F_{ij|k} = 0, \quad \sum_{(ijk)} F_{ij|k} = 0. \quad (2.6.3')$$

If we put

$$F^{ij} = g^{is} g^{jr} F_{sr} \quad (2.6.4)$$

and

$$hJ^i = F^{ij}|_j, \quad vJ^i = F^{ij}|_j, \quad (2.6.5)$$

then one can prove:

**Theorem 2.6.2.** *The following laws of conservation hold:*

$$hJ^i|_i = \frac{1}{2} \{ F^{ij} (R_{ij} - R_{ji}) + F^{ij}|_r R^r{}_{ij} \}, \quad (2.6.6)$$

$$vJ^i|_i = 0,$$

where  $R_{ij}$  is the Ricci tensor  $R_i{}^h{}_{jh}$ .

*Remark 2.6.1.* In the Lagrange space of electrodynamics the tensor  $F_{jk}$  is given by (2.3.6). The previous theory one reduces to the classical theory. Namely  $F_{ij}(x)$  satisfy the Maxwell equations  $\sum_{(ijk)} F_{ij|k} = 0$  and

$$F_{ij|k} = 0, \quad hJ^i|_i = 0, \quad vJ^i = 0.$$

Now, considering the lift to  $\widetilde{TM}$  of the fundamental tensor  $g_{ij}(x, y)$  of the space  $L^n$ , given by

$$\mathbb{G} = g_{ij} dx^i \otimes dx^j + g_{ij} \delta y^i \otimes \delta y^j$$

we can obtain the Einstein equations of the canonical metric connection  $CT(N)$ . The curvature Ricci and scalar curvatures:



$$\begin{aligned} R_{ij} &= R_i^h{}_{jh}, S_{ij} = S_i^h{}_{jh}, {}'P_{ij} = P_i^h{}_{jh}, {}''P_{ij} = P_i^h{}_{hj} \\ R &= g^{ij}R_{ij}, S = g^{ij}S_{ij}. \end{aligned} \quad (2.6.7)$$

Let us denote by  $\overset{H}{T}_{ij}$ ,  $\overset{V}{T}_{ij}$ ,  $\overset{1}{T}_{ij}$  and  $\overset{2}{T}_{ij}$  the components in adapted basis  $\left(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\right)$  of the energy momentum tensor on the manifold  $\widetilde{TM}$ .

Thus we obtain:

**Theorem 2.6.3.**  $1^\circ$  The Einstein equations of the Lagrange space  $L^n = (M, L(x, y))$  with respect to the canonical metrical connection  $CG(N) = (L_{jk}^i, C_{jk}^i)$  are as follows:

$$\begin{aligned} R_{ij} - \frac{1}{2}Rg_{ij} &= \kappa \overset{H}{T}_{ij}, {}'P_{ij} = \kappa \overset{1}{T}_{ij} \\ S_{ij} - \frac{1}{2}Sg_{ij} &= \kappa \overset{V}{T}_{(i)(j)}, {}''P_{ij} = \kappa \overset{2}{T}_{ij}, \end{aligned} \quad (2.6.8)$$

where  $\kappa$  is a real constant.

$2^\circ$  The energy momentum tensors  $\overset{H}{T}_{ij}$  and  $\overset{V}{T}_{ij}$  satisfy the following laws of conservation

$$\kappa \overset{H}{T}_{ij} = -\frac{1}{2}(P_{js}^{ih}R_{hi}^s + 2R_{ij}^s P_s^i), \kappa \overset{V}{T}_{ij} |_{i} = 0. \quad (2.6.9)$$

The physical background of the previous theory is discussed by Satoshi Ikeda in the last chapter of the book [116].

The previous theory is very simple in the particular Lagrange spaces  $L^n$  having  $P_i^h{}_{jk} = 0$ .

We have:

**Corollary 2.6.2.**  $1^\circ$  If the canonical metrical connection  $CG(N)$  has the property  $P_j^i{}_{kh} = 0$ , then the Einstein equations are

$$R_{ij} - \frac{1}{2}Rg_{ij} = \kappa \overset{H}{T}_{ij}, S_{ij} - \frac{1}{2}Sg_{ij} = \kappa \overset{V}{T}_{(i)(j)} \quad (2.6.10)$$

$2^\circ$  The following laws of conservation hold:

$$\begin{aligned} H^i \\ T_{j|i} = 0, \quad V^i \\ T_j |i = 0. \end{aligned}$$

*Remark 2.6.2.* The Lagrange space of Electrodynamics,  $L^n$ , has  $C\Gamma(N) = (\gamma_{jk}^i(x), 0)$ ,  $P_{jkh}^i = 0$ ,  $S_{jkh}^i = 0$ . The Einstein equations (2.6.10) reduce to the classical Einstein equations of the space  $L^n$ .

## 2.7 The almost Kählerian model of a Lagrange space $L^n$

A Lagrange space  $L^n = (M, L)$  can be thought as an almost Kähler space on the manifold  $\widetilde{TM} = TM \setminus \{0\}$ , called the geometrical model of the space  $L^n$ .

As we know from section 3, Ch. 1 the canonical nonlinear connection  $N$  determines an almost complex structure  $\mathbb{F}(\widetilde{TM})$ , expressed in (1.3.9'). This is

$$\mathbb{F} = \frac{\delta}{\delta x^i} \otimes \delta y^i - \frac{\partial}{\partial y^i} \otimes dx^i. \quad (2.7.1)$$

$\mathbb{F}$  is integrable if and only if  $R_{jk}^i = 0$ .

$\mathbb{F}$  is globally defined on  $\widetilde{TM}$  and it can be considered as a  $\mathcal{F}(\widetilde{TM})$ -linear mapping from  $\chi(\widetilde{TM})$  to  $\chi(\widetilde{TM})$ :

$$\mathbb{F} \left( \frac{\delta}{\delta x^i} \right) = -\frac{\partial}{\partial y^i}, \quad \mathbb{F} \left( \frac{\partial}{\partial y^i} \right) = \frac{\delta}{\delta x^i}, \quad (i = 1, \dots, n). \quad (2.7.1')$$

The lift of the fundamental tensor  $g_{ij}$  of the space  $L^n$  with respect to  $N$  is defined by

$$\mathbb{G} = g_{ij} dx^i \otimes dx^j + g_{ij} \delta y^i \otimes \delta y^j. \quad (2.7.2)$$

Evidently  $\mathbb{G}$  is a (pseudo-)Riemannian metric on the manifold  $\widetilde{TM}$ . The following result can be proved without difficulties:

**Theorem 2.7.1.** 1° The pair  $(\mathbb{G}, \mathbb{F})$  is an almost Hermitian structure on  $\widetilde{TM}$ , determined only by the fundamental function  $L(x, y)$  of  $L^n$ .  
2° The almost symplectic structure associated to the structure  $(\mathbb{G}, \mathbb{F})$  is given by

$$\theta = g_{ij} \delta y^i \wedge dx^j. \quad (2.7.3)$$

3° The space  $(\widetilde{TM}, \mathbb{G}, \mathbb{F})$  is almost Kählerian.

Indeed:

1°  $N, \mathbb{G}, \mathbb{F}$  are determined only by  $L(x, y)$ .

We have  $\mathbb{G}(\mathbb{F}X, \mathbb{F}Y) = \mathbb{G}(X, Y), \forall X, Y \in \chi(\widetilde{TM})$ .

2° In the adapted basis  $\theta(X, Y) = \mathbb{G}(\mathbb{F}X, Y)$  is (2.7.3).

3° Taking into account (2.1.1), it follows that  $\theta$  is a symplectic structure (i.e.  $d\theta = 0$ ).

The space  $H^{2n} = (\widetilde{TM}, \mathbb{G}, \mathbb{F})$  is called *the almost Kählerian model* of the Lagrange space  $L^n$ . It has a remarkable property:

**Theorem 2.7.2.** *The canonical metrical connection  $D$  with coefficients  $C\Gamma(N) = (L_{jk}^i, C_{jk}^i)$  of the Lagrange space  $L^n$  is an almost Kählerian connection, i.e.*

$$D\mathbb{G} = 0, \quad D\mathbb{F} = 0. \quad (2.7.4)$$

Indeed, (2.7.1), (2.7.2), in the adapted basis imply (2.7.4).

We can use this geometrical model to study the geometry of Lagrange space  $L^n$ . For instance, the Einstein equations of the (pseudo) Riemannian space  $(\widetilde{TM}, \mathbb{G})$  equipped with the metrical canonical connection  $C\Gamma(N)$  are the Einstein equations of the Lagrange space studied in the previous section of this chapter.

G.S. Asanov showed [27] that the metric  $\mathbb{G}$  given by the lift (2.7.2) does not satisfies the principle of the Post-Newtonian calculus because the two terms of  $\mathbb{G}$  have not the same physical dimensions. This is the reason to introduce a new lift which can be used in a gauge theory of physical fields.

Let us consider the scalar field:

$$\mathcal{E} = ||y||^2 = g_{ij}(x, y)y^i y^j. \quad (2.7.5)$$

$\mathcal{E}$  is called the absolute energy of the Lagrange space  $L^n$ .

We assume  $||y||^2 > 0$  and consider the following lift of the fundamental tensor  $g_{ij}$ :

$${}^0\mathbb{G} = g_{ij}dx^i \otimes dx^j + \frac{a^2}{||y||^2} g_{ij} \delta y^i \otimes \delta y^j \quad (2.7.6)$$

where  $a > 0$  is a constant.

Let us consider also the tensor field on  $\widetilde{TM}$ :

$${}^0\mathbb{F} = -\frac{||y||}{a} \frac{\partial}{\partial y^i} \otimes dx^j + \frac{a}{||y||} \frac{\delta}{\delta x^i} \otimes \delta y^j, \quad (2.7.7)$$

and 2-form

$$\overset{0}{\theta} = \frac{a}{\|y\|} \theta, \quad (2.7.8)$$

where  $\theta$  is from (2.7.3).

We can prove:

**Theorem 2.7.3.**  $1^\circ$  The pair  $(\overset{0}{\mathbb{G}}, \overset{0}{\mathbb{F}})$  is an almost Hermitian structure on the manifold  $\widetilde{TM}$ , depending only on the the fundamental function  $L(x, y)$  of the space  $L^n$ .

$2^\circ$  The almost symplectic structure  $\overset{0}{\theta}$  associated to the structure  $(\overset{0}{\mathbb{G}}, \overset{0}{\mathbb{F}})$  is given by (2.7.8).

$3^\circ$   $\overset{0}{\theta}$  being conformal to symplectic structure  $\theta$ , the pair  $(\overset{0}{\mathbb{G}}, \overset{0}{\mathbb{F}})$  is conformal to the almost Kählerian structure  $(\mathbb{G}, \mathbb{F})$ .

## 2.8 Generalized Lagrange spaces

A first natural generalization of the notion of Lagrange space is provided by a notion which we call a generalized Lagrange space. This notion was introduced by author in the paper [183].

**Definition 2.8.1.** A generalized Lagrange space is a pair  $GL^n = (M, g_{ij}(x, y))$ , where  $g_{ij}(x, y)$  is a d-tensor field on the manifold  $\widetilde{TM}$ , of type  $(0, 2)$ , symmetric, of rank  $n$  and having a constant signature on  $\widetilde{TM}$ .

We continue to call  $g_{ij}(x, y)$  the *fundamental tensor* on  $GL^n$ .

One easily sees that any Lagrange space  $L^n = (M, L(x, y))$  is a generalized Lagrange space with the fundamental tensor

$$g_{ij}(x, y) = \frac{1}{2} \frac{\partial^2 L(x, y)}{\partial y^i \partial y^j}. \quad (2.8.1)$$

But not any space  $GL^n$  is a Lagrange space  $L^n$ .

Indeed, if  $g_{ij}(x, y)$  is given, it may happen that the system of partial differential equations (2.8.1) does not admits solutions in  $L(x, y)$ .

**Proposition 2.8.9.**  $1^\circ$  A necessary condition in order that the system

(2.8.1) admit a solution  $L(x, y)$  is that the d-tensor field  $\frac{\partial g_{ij}}{\partial y^k} =$

$2C_{ijk}$  be completely symmetric.

2° If the condition 1° is verified and the functions  $g_{ij}(x, y)$  are 0-homogeneous with respect to  $y^i$ ; then the function

$$L(x, y) = g_{ij}(x, y)y^i y^j + A_i(x)y^i + U(x) \quad (2.8.2)$$

is a solution of the system of partial differential equations (2.8.1) for any arbitrary d-covector field  $A_i(x)$  and any arbitrary function  $U(x)$ , on the base manifold  $M$ .

The proof of previous statement is not complicated.

In the case when the system (2.8.1) does not admit solutions in the functions  $L(x, y)$  we say that the generalized Lagrange space  $GL^2 = (M, g_{ij}(x, y))$  is not reducible to a Lagrange space.

*Remark 2.8.3.* The Lagrange spaces  $L^n$  with the fundamental function (2.8.2) give us an important class of Lagrange spaces which includes the Lagrange space of electrodynamics.

### Examples.

1° The pair  $GL^n = (M, g_{ij})$  with the fundamental tensor field

$$g_{ij}(x, y) = e^{2\sigma(x, y)} \gamma_{ij}(x) \quad (2.8.3)$$

where  $\sigma$  is a function on  $(\widetilde{TM})$  and  $\gamma_{ij}(x)$  is a pseudo-Riemannian metric on the manifold  $M$  is a generalized Lagrange space if the d-covector field  $\frac{\partial \sigma}{\partial y^i}$  no vanishes.

It is not reducible to a Lagrange space. R. Miron and R. Tavakol [183] proved that  $GL^n = (M, g_{ij}(x, y))$  defined by (2.8.3) satisfies the Ehlers - Pirani - Schild's axioms of General Relativity.

2° The pair  $GL^n = (M, g_{ij}(x, y))$ , with

$$g_{ij}(x, y) = \gamma_{ij}(x) + \left(1 - \frac{1}{n^2(x, y)}\right) y_i y_j, \quad y_i = \gamma_{ij}(x) y^j \quad (2.8.4)$$

where  $\gamma_{ij}(x)$  is a pseudo-Riemannian metric and  $n(x, y) > 1$  is a smooth function ( $n$  is a refractive index) give us a generalized Lagrange space  $GL^n$  which is not reducible to a Lagrange space. This metric has been called by R. G. Beil, [40] the Miron's metric from Relativistic Optics.

The restriction of the fundamental tensor  $g_{ij}(x, y)$  (2.8.4) to a section  $S_V : x^i = x^i, y^i = V^i(x)$ , ( $V^i$  being a vector field) of the projection

$\pi : TM \rightarrow M$ , is given by  $g_{ij}(x, V(x))$ . It gives us the known Synge's metric tensor of the Relativistic Optics [243].

For a generalized Lagrange space  $GL^n = (M, g_{ij}(x, y))$  an important problem is to determine a nonlinear connection obtained from the fundamental tensor  $g_{ij}(x, y)$ . In the particular cases, given by the previous two examples this is possible. But, generally no.

We point out a method of determining a nonlinear connection  $N$ , strongly connected to the fundamental tensor  $g_{ij}$  of the space  $GL^n$ , if such kind of nonlinear connection exists.

Consider the *absolute energy*  $\mathcal{E}(x, y)$  of space  $GL^n$ :

$$\varepsilon(x, y) = g_{ij}(x, y)y^i y^j \quad (2.8.5)$$

$\varepsilon(x, y)$  is a Lagrangian.

The Euler - Lagrange equations of  $\varepsilon(x, y)$  are

$$\frac{\partial \varepsilon}{\partial x^i} - \frac{d}{dt} \frac{\partial \varepsilon}{\partial y^i} = 0, \quad y^i = \frac{dx^i}{dt}. \quad (2.8.6)$$

Of course, according the general theory, the energy  $E_\varepsilon$  of the Lagrangian  $\mathcal{E}(x, y)$ , is  $E_\varepsilon = y^i \frac{\partial \varepsilon}{\partial y^i} - \varepsilon$  and it is preserved along the integral curves of the differential equations (2.8.6).

If  $\mathcal{E}(x, y)$  is a regular Lagrangian - we say that the space  $GL^n$  is weakly regular - it follows that the Euler-Lagrange equations determine a semispray with the coefficients

$$2G^i(x, y) = \overset{\vee}{g}{}^{is} \left( \frac{\partial^2 \varepsilon}{\partial y^s \partial x^j} y^j - \frac{\partial \varepsilon}{\partial x^s} \right), \quad \left( \overset{\vee}{g}{}_{ij} = \frac{1}{2} \frac{\partial^2 \varepsilon}{\partial y^i \partial y^j} \right). \quad (2.8.7)$$

Consequently, the nonlinear connection  $N$  with the coefficients  $N_j^i = \frac{\partial G^i}{\partial y^j}$  is determined only by the fundamental tensor  $g_{ij}(x, y)$  of the space  $GL^n$ .

In the case when we can not derive a nonlinear connection from the fundamental tensor  $g_{ij}$ , we give a priori a nonlinear connection  $N$  and study the geometry of pair  $(GL^n, N)$  by the methods of the geometry of Lagrange space  $L^n$ .

For instance, using the adapted basis  $\left( \frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i} \right)$  to the distributions  $N$  and  $V$ , respectively and its dual  $(dx^i, \delta y^i)$  we can lift  $g_{ij}(x, y)$

to  $\widetilde{TM}$ :

$$G(x,y) = g_{ij}(x,y)dx^i \otimes dx^j + \frac{a}{\|y\|^2}g_{ij}(x,y)\delta y^i \otimes \delta y^j$$

and can consider the almost complex structure

$$\mathbb{F} = -\frac{\|y\|}{a} \frac{\partial}{\partial y^i} \otimes dx^i + \frac{a}{\|y\|} \frac{\delta}{\delta x^i} \otimes \delta y^i$$

with  $\varepsilon(x,y) > 0$  and  $\|y\| = \varepsilon^{1/2}(x,y)$ .

The space  $(\widetilde{TM}, \mathbb{F})$  is an almost Hermitian space geometrical associated to the pair  $(GL^n, N)$ .

J. Silagy, [241], make an exhaustive and interesting study of this difficult problem.





## Chapter 3

# Finsler Spaces

An important class of Lagrange Spaces is provided by the so-called Finsler spaces.

The notion of Finsler space was introduced by Paul Finsler in 1918 and was developed by remarkable mathematicians as L. Berwald [42], E. Cartan [56], H. Buseman [52], H. Rund [218], S.S. Chern [60], M. Matsumoto [144], and many others.

This notion is a generalization of Riemann space, which gives an important geometrical framework in Physics, especially in the geometrical theory of physical fields, [18], [116], [129], [246], [252].

In the last 40 years, some remarkable books on Finsler geometry and its applications were published by H. Rund, M. Matsumoto, R. Miron and M. Anastasiei, A. Bejancu, Abate-Patrizio, D. Bao, S.S. Chern and Z. Shen, P. Antonelli, R. Ingarden and M. Matsumoto, R. Miron, D. Hrimiuc, H. Shimada and S. Sabău, G.S. Asanov, M. Crampin, P.L. Antonelli, S. Vacaru, S. Ikeda.

In the present chapter we will study the Finsler spaces considered as Lagrange spaces and applying the mechanical principles. This method simplifies the theory of Finsler spaces. So, we will treat: Finsler metric, Cartan nonlinear connection derived from the canonical spray, Cartan metrical connection and its structure equations. Some examples: Randers spaces, Kropina spaces and some new classes of spaces more general as Finsler spaces: the almost Finsler Lagrange spaces and the Ingarden spaces will close this chapter.

### 3.1 Finsler metrics

**Definition 3.1.1.** A Finsler space is a pair  $F^n = (M, F(x, y))$  where  $M$  is a real  $n$ -dimensional differentiable manifold and  $F : TM \rightarrow R$  is a scalar function which satisfies the following axioms:

- 1°  $F$  is a differentiable function on  $\widetilde{TM}$  and  $F$  is continuous on the null section of the projection  $\pi : TM \rightarrow M$ .  
 2°  $F$  is a positive function.  
 3°  $F$  is positively 1-homogeneous with respect to the variables  $y^i$   
 4° The Hessian of  $F^2$  with the elements:

$$g_{ij}(x, y) = \frac{1}{2} \frac{\partial F^2}{\partial y^i \partial y^j} \quad (3.1.1)$$

is positively defined on the manifold  $\widetilde{TM}$ .

Of course, the axiom 4° is equivalent with the following:

- 4° The pair  $(M, F^2(x, y)) = L_F^n$  is a Lagrange space with positively defined fundamental tensor,  $g_{ij}$ .  $L_F^n$  will be called the Lagrange space associated to the Finsler space  $F^n$ . It follows that all properties of the Finsler space  $F^n$  derived from the fundamental function  $F^2$  and the fundamental tensor  $g_{ij}$  of the associated Lagrange space  $L_F^n$ .

#### Remarks

- 1° Sometimes we will ask for  $g_{ij}$  to be of constant signature and  $\text{rank}(g_{ij}(x, y)) = n$  on  $\widetilde{TM}$ .  
 2° Any Finsler space  $F^n = (M, F(x, y))$  is a Lagrange space  $L_F^n = (M, F^2(x, y))$ , but not conversely.

#### Examples

- 1° A Riemannian manifold  $(M, \gamma_{ij}(x))$  determines a Finsler space  $F^n = (M, F(x, y))$ , where

$$F(x, y) = \sqrt{\gamma_{ij}(x)y^i y^j}. \quad (3.1.2)$$

The fundamental tensor is  $g_{ij}(x, y) = \gamma_{ij}(x)$ .

- 2° Let us consider, in a preferential local system of coordinates, the following function:

$$F(x, y) = \sqrt[4]{(y^1)^4 + \dots + (y^n)^4}. \quad (3.1.3)$$

Then  $F$  satisfy the axioms 1°-4°.

*Remark 3.1.1.* This example was given by B. Riemann.

3° Antonelli-Shimada's ecological metric is given, in a preferential local system of coordinates, on  $\widetilde{TM}$ , by

$$F(x, y) = e^\phi L, \quad \phi = \alpha_i x^i, \quad (\alpha_i \text{ are positive constants}),$$

and where

$$L = \{(y^1)^m + (y^2)^m + \dots + (y^n)^m\}^{1/m}, \quad m \geq 3, \quad (3.1.4)$$

$m$  being even.

4° Randers metric is defined by

$$F(x, y) = \alpha(x, y) + \beta(x, y), \quad (3.1.5)$$

where

$$\alpha^2(x, y) := a_{ij}(x) y^i y^j,$$

$(M, a_{ij}(x))$  being a Riemannian manifold and

$$\beta(x, y) := b_i(x) y^i.$$

The fundamental tensor  $g_{ij}$  is expressed by:

$$g_{ij} = \frac{\alpha + \beta}{\alpha} h_{ij} + d_i d_j, \quad h_{ij} := a_{ij} - l_i l_j, \quad (3.1.5')$$

$$d_i = b_i + l_i, \quad l_i := \frac{\partial \alpha}{\partial y^i}.$$

One can prove that  $g_{ij}$  is positively defined under the condition  $b^2 = a^{ij} b_i b_j < 1$ . In this case the pair  $F^n = (M, \alpha + \beta)$  is a Finsler space.

The first example motivates the following theorem:

**Theorem 3.1.1.** *If the base manifold  $M$  is paracompact, then there exist functions  $F : TM \rightarrow \mathbb{R}$  such that the pair  $(M, F)$  is a Finsler spaces.*

Regarding the axioms 1° – 4° we can see without difficulties:

**Theorem 3.1.2.** *The system of axioms of a Finsler space is minimal.*

Some properties of Finsler space  $F^n$ :

1° The components of the fundamental tensor  $g_{ij}(x, y)$  are 0-homogeneous with respect to  $y^i$ .

2° The components of 1-form

$$p_i = \frac{1}{2} \frac{\partial F^2}{\partial y^i} \quad (3.1.6)$$

are 1-homogeneous with respect to  $y^i$ .

3° The components of the Cartan tensor

$$C_{ijk} = \frac{1}{4} \frac{\partial^3 F^2}{\partial y^i \partial y^j \partial y^k} = \frac{1}{2} \frac{\partial g_{ij}}{\partial y^k} \quad (3.1.7)$$

are  $-1$ -homogeneous with respect to  $y^i$ .

Consequently we have

$$C_{oij} = y^s C_{sij} = 0. \quad (3.1.8)$$

If  $X^i$  and  $Y^i$  are d-vector fields, then

$\|X\|^2 := g_{ij}(x,y)X^iY^i$  is a scalar field.

$\langle X, Y \rangle := g_{ij}(x,y)X^iY^j$  is a scalar field.

Assuming  $\|X\|_u \neq 0$ ,  $\|Y\|_u \neq 0$  the angle  $\varphi = \angle(X, Y)$  at a point  $u \in \widetilde{TM}$  is given by

$$\cos \varphi = \frac{\langle X, Y \rangle (u)}{\|X\|_u \cdot \|Y\|_u}.$$

The vectors  $X_u, Y_u$  are orthogonal if  $\langle X, Y \rangle (u) = 0$ .

**Proposition 3.1.1.** *In a Finsler space  $F^n$  the following identities hold:*

$$1^\circ F^2(x, y) = g_{ij}(x, y)y^i y^j$$

$$2^\circ p_i y^i = F^2.$$

**Proposition 3.1.2.** *1° The 1-form*

$$\omega = p_i dx^i \quad (3.1.9)$$

*is globally defined on  $\widetilde{TM}$ .*

*2° The 2-form*

$$\theta = d\omega = dp_i \wedge dx^i \quad (3.1.10)$$

*is globally defined on  $\widetilde{TM}$ .*

*3°  $\theta$  is a symplectic structure on  $\widetilde{TM}$ .*

**Definition 3.1.2.** A Finsler space  $F^n = (M, F)$  is called reducible to a Riemann space if its fundamental tensor  $g_{ij}(x, y)$  does not depend on the variable  $y^i$ .

**Proposition 3.1.3.** A Finsler space  $F^n$  is reducible to a Riemann space if and only if the tensor  $C_{ijk}$  is vanishing on  $\widetilde{TM}$ .

### 3.2 Geodesics

In a Finsler space  $F^n = (M, F(x, y))$  one can define the notion of arc length of a smooth curve.

Let  $c$  be a parametrized curve in the manifold  $M$ :

$$c : t \in [0, 1] \rightarrow (x^i(t)) \in U \subset M \quad (3.2.1)$$

$U$  being a domain of a local chart in  $M$ .

The extension  $\tilde{c}$  of  $c$  to  $\widetilde{TM}$  has the equations

$$x^i = x^i(t), \quad y^i = \frac{dx^i}{dt}(t), \quad t \in [0, 1]. \quad (3.2.1')$$

Thus the restriction of the fundamental function  $F(x, y)$  to  $\tilde{c}$  is  $F(x(t), \frac{dx}{dt}(t))$ ,  $t \in [0, 1]$ .

We define the "length" of curve  $c$  with extremities  $c(0), c(1)$  by the number

$$\ell(c) = \int_0^1 F(x(t), \frac{dx}{dt}(t)) dt. \quad (3.2.2)$$

The number  $\ell(c)$  does not depend on a changing of coordinates on  $\widetilde{TM}$  and, by means of 1-homogeneity of function  $F$ ,  $\ell(c)$  does not depend on the parametrization of the curve  $c$ .  $\ell(c)$  depends on the curve  $c$ , only.

We can choose a canonical parameter on  $c$ , considering the following function  $s = s(t)$ ,  $t \in [0, 1]$ :

$$s(t) = \int_0^t F(x(\tau), \frac{dx}{dt}(\tau)) d\tau.$$

This function is derivable and its derivative is

$$\frac{ds}{dt} = F(x(t), \frac{dx}{dt}(t)) > 0, t \in (0, 1).$$

So the function  $s = s(t)$ ,  $t \in [0, 1]$ , is invertible. Let  $t = t(s)$ ,  $s \in [s_0, s_1]$  be its inverse. The change of parameter  $t \rightarrow s$  has the property

$$F\left(x(s), \frac{dx}{ds}(s)\right) = 1. \quad (3.2.3)$$

Variational problem on the functional  $\ell$  will give the curves on  $\widetilde{TM}$  which extremize the arc length. These curves are geodesics of the Finsler space  $F^n$ .

So, the solution curves of the Euler-Lagrange equations:

$$\frac{d}{dt} \left( \frac{\partial F}{\partial y^i} \right) - \frac{\partial F}{\partial x^i} = 0, y^i = \frac{dx^i}{dt} \quad (3.2.4)$$

are the geodesics of the space  $F^n$ .

**Definition 3.2.1.** The curves  $c = (x^i(t))$ ,  $t \in [0, 1]$ , solutions of the Euler-Lagrange equations (3.2.4) are called the *geodesics* of the Finsler space  $F^n$ .

The system of differential equations (3.2.4) is equivalent to the following system

$$\frac{d}{dt} \frac{\partial F^2}{\partial y^i} - \frac{\partial F^2}{\partial x^i} = 2 \frac{dF}{dt} \frac{\partial F}{\partial y^i}, y^i = \frac{dx^i}{dt}.$$

In the canonical parametrization, according to (3.2.3) we have:

**Theorem 3.2.1.** *In the canonical parametrization the geodesics of the Finsler space  $F^n$  are given by the system of differential equations*

$$E_i(F^2) := \frac{d}{ds} \frac{\partial F^2}{\partial y^i} - \frac{\partial F^2}{\partial x^i} = 0, y^i = \frac{dx^i}{ds}. \quad (3.2.5)$$

Now, remarking that  $F^2 = g_{ij}y^iy^j$ , the previous equations can be written in the form:

$$\frac{d^2x^i}{ds^2} + \gamma_{jk}^i \left( x, \frac{dx}{ds} \right) \frac{dx^j}{ds} \frac{dx^k}{ds} = 0, y^i = \frac{dx^i}{ds}, \quad (3.2.6)$$

where  $\gamma_{jk}^i$  are the Christoffel symbols of the fundamental tensor  $g_{ij}$ :

$$\gamma^j{}_{jk} = \frac{1}{2}g^{ir} \left( \frac{\partial g_{rk}}{\partial x^j} + \frac{\partial g_{jr}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^r} \right). \quad (3.2.7)$$

A theorem of existence and uniqueness of the solution of differential equations (3.2.6) can be formulated in the classical manner.

### 3.3 Cartan nonlinear connection

Considering the Lagrange space  $L_F^n = (M, F^2)$  associated to the Finsler space  $F^n = (M, F)$  we can obtain some main geometrical object field of  $F^n$ .

So, Theorem 2.3.1, affirms:

**Theorem 3.3.1.** *For the Finsler space  $F^n$  the equations*

$$g^{ij}E_j(F^2) := g^{ij} \left( \frac{d}{dt} \frac{\partial F^2}{\partial y^j} \right) - \frac{\partial F^2}{\partial x^i} = 0, y^j = \frac{dx^j}{dt}$$

can be written in the form

$$\frac{d^2x^i}{dt^2} + 2G^i \left( x, \frac{dx}{dt} \right) = 0, y^j = \frac{dx^j}{dt} \quad (3.3.1)$$

where

$$2G^i(x, y) = \gamma^i{}_{jk}(x, y)y^jy^k \quad (3.3.1')$$

Consequently the equations (3.3.1) give the integral curves of the semispray:

$$S = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y) \frac{\partial}{\partial y^i}. \quad (3.3.2)$$

Since  $G^i$  are 2-homogeneous functions with respect to  $y^i$  it follows that  $S$  is a *spray*.

$S$  determine a canonical nonlinear connection  $N$  with the coefficients

$$N^i{}_j = \frac{\partial G^i}{\partial y^j} = \frac{1}{2} \frac{\partial}{\partial y_j} \{ \gamma^i{}_{rs}(x, y)y^r y^s \}. \quad (3.3.3)$$

$N$  is called the Cartan nonlinear connection of the space  $F^n$ .

The tangent bundle  $T(TM)$ , the horizontal distribution  $N$  and the vertical distribution  $V$  give us the direct decomposition of vectorial spaces

$$T_u(\widetilde{TM}) = N(u) \oplus V(u), \quad \forall u \in \widetilde{TM}. \quad (3.3.4)$$

The adapted basis to  $N$  and  $V$  is  $\left(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\right)$  and the dual adapted basis is  $(dx, \delta y^i)$ , where

$$\begin{cases} \frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N_i^j(x, y) \frac{\partial}{\partial y^j} \\ \delta y^i = dy^i + N_j^i(x, y) dx^j. \end{cases} \quad (3.3.4')$$

One obtains:

**Theorem 3.3.2.** 1° *The horizontal curves in  $F^n$  are given by*

$$x^i = x^i(t), \quad \frac{\delta y^i}{dt} = 0.$$

2° *The autoparallel curves of the Cartan nonlinear connection  $N$  coincide to the integral curves of the spray  $S$ , (3.3.2).*

### 3.4 Cartan metrical connection

Let  $N(N_j^i)$  be the Cartan nonlinear connection of the Finsler space  $F^n$ . According to section 5 of chapter 2 one introduces the canonical metrical  $N$ -linear connection of the space  $F^n$ .

But, for these spaces the system of axioms, from Theorem 2.5.1 can be given in the Matsumoto's form [144], [145].

**Theorem 3.4.1.** 1° *On the manifold  $\widetilde{TM}$ , for any Finsler space  $F^n = (M, F)$  there exists only one linear connection  $D$ , with the coefficients  $C\Gamma = (N_j^i, F_{jk}^i, C_{jk}^i)$  which verifies the following axioms:*

A<sub>1</sub>. *The deflection tensor field  $D_j^i = y_{|j}^i$  vanishes.*

A<sub>2</sub>.  $g_{ij|k} = 0$ , *( $D$  is  $h$ -metrical).*

A<sub>3</sub>.  $g_{ij|k} = 0$ , *( $D$  is  $v$ -metrical).*

A<sub>4</sub>.  $T_{jk}^i = 0$ , *( $D$  is  $h$ -torsion free).*

A<sub>5</sub>.  $S_{jk}^i = 0$ , *( $D$  is  $v$ -torsion free).*

2° *The coefficients  $(N_j^i, F_{jk}^i, C_{jk}^i)$  are as follows:*

a.  $N_j^i$  *are the coefficients (3.3.3) of the Cartan nonlinear connection.*



b.  $F_{jk}^i, C_{jk}^i$  are expressed by the generalized Christoffel symbols:

$$\begin{aligned} F_{jk}^i &= \frac{1}{2} g^{is} \left( \frac{\delta g_{sk}}{\delta x^j} + \frac{\delta g_{js}}{\delta x^k} - \frac{\delta g_{jk}}{\delta x^s} \right) \\ C_{jk}^i &= \frac{1}{2} g^{is} \left( \frac{\partial g_{sk}}{\partial y^j} + \frac{\partial g_{js}}{\partial y^k} - \frac{\partial g_{jk}}{\partial y^s} \right) \end{aligned} \quad (3.4.1)$$

3° This connection depends only on the fundamental function  $F$ .

A proof can be found in the books [21], [164].

The previous connection is named the *Cartan metrical connection* of the Finsler space  $F^n$ .

Now we can develop the geometry of Finsler spaces, exactly as the geometry of the associated Lagrange space  $L_F^n = (M, F^2)$ .

Also, in the case of Finsler space the geometrical model  $H^{2n} = (\widehat{TM}, \mathbb{G}, \mathbb{F})$  is an almost Kählerian space.

A very good example is provided by the Randers space, introduced by R.S. Ingarden.

A Randers space is the Finsler space  $F^n = (M, \alpha + \beta)$  equipped with Cartan nonlinear connection  $N$ .

It is denoted by  $RF^n = (M, \alpha + \beta, N)$ .

The geometry of these spaces was much studied by many geometers. A good monograph in this respect is the D. Bao, S.S. Chern and Z. Shen's book [35].

The Randers spaces  $RF^n$  can be generalized considering the Finsler spaces  $F^n = (M, \alpha + \beta)$ , where  $\alpha(x, y)$  is the fundamental function of a Finsler space  $F^m = (M, \alpha)$ . The Finsler space  $F^n = (M, \alpha + \beta)$  equipped with the Cartan nonlinear connection  $N$  of the space  $F^m = (M, \alpha)$  is a generalized Randers space [152], [35]. Evidently, this notion has some advantages, since we can take some remarkable Finsler spaces  $F^m$ , (M. Anastasiei  $\beta$ -transformation of a Finsler space [14]).

As an application of the previous notions, we define the notion of Ingarden space  $IF^n$ , [115], [49]. This is the Finsler space  $F^n = (M, \alpha + \beta)$  equipped with the nonlinear connection  $N = \gamma_{jk}^i(x)y^k - F_j^i(x)$ ,  $\gamma_{jk}^i(x)$  being the Christoffel symbols of the Riemannian metric  $a_{ij}(x)$ , which defines  $\alpha^2 = a_{ij}(x)y^i y^j$  and the electromagnetic tensor  $F_j^i(x)$  determined by  $\beta = b_i(x)y^i$ . While the spaces  $RF^n$  have not

the electromagnetic field  $\mathcal{F} = \frac{1}{2}(D_{ij} - D_{ji})$ , the Ingarden spaces have such kind of tensor fields and they give us the remarkable Maxwell equations, [49]. Also, the autoparallel curve of the nonlinear connection  $N$  are given by the known Lorentz equations.

An example of a special Lagrange space derived from a Finsler one, [164] is as follows:

Let us consider the Lagrange space  $L^n = (M, L(x, y))$  with the fundamental function

$$L(x, y) = F^2(x, y) + \beta,$$

where  $F$  is the fundamental function of a priori given Finsler space  $F^n = (M, F)$  and  $\beta = b_i(x)y^i$ .

These spaces have been called *the almost Finsler Lagrange Spaces* (shortly AFL-spaces), [164], [49]. They generalize the Lagrange space from Electrodynamics.

Indeed, the Euler - Lagrange equations of AFL-spaces are exactly the Lorentz equations

$$\frac{d^2x^i}{dt^2} + \gamma_{jk}^i(x, y) \frac{dx^j}{dt} \frac{dx^k}{dt} = \frac{1}{2} F_j^i(x) \frac{dx^j}{dt}.$$

To the end of these three chapter we can do a general remark:

The class of Riemann spaces  $\{\mathcal{R}^n\}$  is a subclass of the class of Finsler spaces  $\{F^n\}$ , the class  $\{F^n\}$  is a subclass of class of Lagrange spaces  $\{L^n\}$  and this is a subclass of class of generalized Lagrange spaces  $\{GL^n\}$ . So, we have the following sequence of inclusions:

$$(I) \quad \{\mathcal{R}^n\} \subset \{F^n\} \subset \{L^n\} \subset \{GL^n\}.$$

Therefore, we can say: The Lagrange geometry is the geometric study of the terms of the sequence of inclusions (I).

## Chapter 4

# The Geometry of Cotangent Manifold

The geometrical theory of cotangent bundle  $(T^*M, \pi^*, M)$  on a real, finite dimensional manifold  $M$  is important in differential geometry. Correlated with that of tangent bundle  $(TM, \pi, M)$ , introduced in Ch. 1, one obtains a framework for construction of Lagrangian and Hamiltonian mechanical systems, as well as, for the duality between them, via Legendre transformation.

We study here the fundamental geometric objects on  $T^*M$ , as Liouville vector field  $C^*$ , Liouville 1-form  $\omega$ , symplectic structure  $\theta = d\omega$ , Poisson structure,  $N$ -linear connection etc.

### 4.1 Cotangent bundle

Let  $M$  be a real  $n$ -dimensional differentiable manifold. Its cotangent bundle  $(T^*M, \pi^*, M)$  can be constructed as the dual of the tangent bundle  $(TM, \pi, M)$ , [154]. If  $(x^i)$  is a local coordinate system on a domain  $U$  of a chart on  $M$ , the induced system of coordinates on  $\pi^{*-1}(U)$  is  $(x^i, p_i)$ ,  $(i, j, k, .. = 1, 2, \dots, n)$ ,  $p_1, \dots, p_n$  are called “momentum variables”. We denote  $(x^i, p_i) = (x, p) = u$ .

A change of coordinates on  $T^*M$  is given by

$$\begin{cases} \tilde{x}^i = \tilde{x}^i(x^1, \dots, x^n), \text{rank} \left( \frac{\partial \tilde{x}^i}{\partial x^j} \right) = n \\ \tilde{p}_i = \frac{\partial x^j}{\partial \tilde{x}^i} p_j. \end{cases} \quad (4.1.1)$$

The natural frame  $\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial p_i}\right) = (\partial_i, \dot{\partial}^i)$  is transformed by (4.1.1) in the form

$$\partial_i = \frac{\partial \tilde{x}^j}{\partial x^i} \tilde{\partial}_j + \frac{\partial \tilde{p}_j}{\partial x^i} \dot{\partial}^j, \quad \dot{\partial}^i = \frac{\partial \tilde{x}^i}{\partial x^j} \dot{\partial}^j. \quad (4.1.2)$$

The natural coframe  $(dx^i, dp_i)$  is changed by the rule

$$d\tilde{x}^i = \frac{\partial \tilde{x}^i}{\partial x^j} dx^j; \quad d\tilde{p}_i = \frac{\partial x^j}{\partial \tilde{x}^i} dp_j + \frac{\partial^2 x^j}{\partial \tilde{x}^i \partial \tilde{x}^k} p_j d\tilde{x}^k. \quad (4.1.2')$$

The Jacobian matrix of change of coordinates (4.1.1) is

$$J(u) = \begin{pmatrix} \frac{\partial \tilde{x}^i}{\partial x^j} & 0 \\ \frac{\partial \tilde{p}_j}{\partial x^i} & \frac{\partial x^j}{\partial \tilde{x}^i} \end{pmatrix}_u.$$

It follows

$$\det J(u) = 1 \text{ for every } u \in T^*M.$$

So we get:

**Theorem 4.1.1.** *The differentiable manifold  $T^*M$  is orientable.*

One can prove that if  $M$  is a paracompact manifold, then  $T^*M$  is paracompact, too.

The kernel of differential  $d\pi^* : TT^*M \rightarrow T^*M$  is the vertical subbundle  $VT^*M$  of the tangent bundle  $TT^*M$ . The fibres  $V_u$  of  $VT^*M$ ,  $\forall u \in T^*M$  determine a distribution  $V$  on  $T^*M$ , called the *vertical* distribution. It is locally generated by the tangent vector fields  $(\dot{\partial}^1, \dots, \dot{\partial}^n)$ . Consequently,  $V$  is an integrable distribution of local dimension  $n$ .

Noticing the formulae (4.1.2), (4.1.2') one can introduce the following geometrical object fields:

$$C^* = p_i \dot{\partial}^i, \quad (4.1.3)$$

called the Liouville–Hamilton vector field on  $T^*M$ ,

$$\omega = p_i dx^i, \quad (4.1.4)$$

called the Liouville 1-form on  $T^*M$ ,

$$\theta = d\omega = dp_i \wedge dx^i, \quad (4.1.5)$$

$\theta$  is a symplectic structure on  $T^*M$ . All these geometrical object fields do not depend on the change of coordinates (4.1.1).

The Poisson brackets  $\{, \}$  on the manifold  $T^*M$  are defined by

$$\{f, g\} = \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial x^i} - \frac{\partial g}{\partial p_i} \frac{\partial f}{\partial x^i}, \quad \forall f, g \in \mathcal{F}(T^*M). \quad (4.1.6)$$

Of course,  $\{f, g\} \in \mathcal{F}(T^*M)$  and  $\{f, g\}$  does not depend on the change of coordinates (4.1.1).

Also, the following properties hold:

- 1°  $\{f, g\} = -\{g, f\}$ ,
- 2°  $\{f, g\}$  is  $R$ -linear in every argument,
- 3°  $\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0$  (Jacobi identity),
- 4°  $\{\cdot, gh\} = \{\cdot, g\}h + \{\cdot, h\}g$ .

The pair  $\{\mathcal{F}(T^*M), \{, \}\}$  is a Lie algebra, called the Poisson–Lie algebra.

The relation between the structures  $\theta$  and  $\{, \}$  can be given by means of the notion of Hamiltonian system.

**Definition 4.1.1.** A differentiable Hamiltonian is a real function  $H : T^*M \rightarrow R$  which is of  $C^\infty$  class on  $TM^* = T^*M \setminus \{0\}$  and continuous on  $T^*M$ .

An example: Let  $g_{ij}(x)$  be a  $C^\infty$ -Riemannian structure on  $M$ . Then  $H = \sqrt{g^{ij}(x)p_i p_j}$  is a differentiable Hamiltonian on  $T^*M$ .

**Definition 4.1.2.** A Hamiltonian system is a triple  $(T^*M, \theta, H)$ .

Let us consider the  $\mathcal{F}(T^*M)$ -modules  $\chi(T^*M)$  (tangent vector fields on  $T^*M$ ),  $\chi^*(T^*M)$  (cotangent vector fields on  $T^*M$ ).

The following  $\mathcal{F}(T^*M)$ -linear mapping

$$S_\theta = \chi(T^*M) \rightarrow \chi^*(TM)$$

can be defined by

$$S_\theta(X) = i_X \theta, \quad \forall X \in \chi(T^*M). \quad (4.1.7)$$

One proves, without difficulties:

**Proposition 4.1.1.**  $S_\theta$  is an isomorphism. We have:

$$S_\theta \left( \frac{\partial}{\partial x^i} \right) = -dp_i, \quad S_\theta \left( \frac{\partial}{\partial p_i} \right) = dx^i \quad (4.1.6')$$

$$S_\theta(C^*) = \omega. \quad (4.1.7')$$

**Theorem 4.1.2.** *The following properties of the Hamiltonian system  $(T^*M, \theta, H)$  hold:*

1° *There exists a unique vector field  $X_H \in \mathcal{X}(T^*M)$  for which:*

$$i_H\theta = -dH. \quad (4.1.8)$$

2° *The integral curves of the vector field  $X_H$  are given by the Hamilton–Jacobi equations:*

$$\frac{dx^i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial x^i}. \quad (4.1.9)$$

*Proof.* 1° The existence and uniqueness of the vector field  $X_H$  is assured by the isomorphism  $S_\theta$ :

$$X_H = S_\theta^{-1}(-dH). \quad (4.1.10)$$

$X_H$  is called the *Hamiltonian vector field*.

2° The local expression of  $X_H$  is given (by (4.1.6)):

$$X_H = \frac{\partial H}{\partial p_i} \frac{\partial}{\partial x^i} - \frac{\partial H}{\partial x^i} \frac{\partial}{\partial p_i}. \quad (4.1.11)$$

Consequently: the integral curves of the vector field  $X_H$  are given by the Hamilton–Jacobi equations (4.1.8).

Along the integral curves of  $X_H$ , we have

$$\frac{dH}{dt} = \{H, H\} = 0.$$

Thus: *The differentiable Hamiltonian  $H(x, p)$  is constant along the integral curves of the Hamilton vector field  $X_H$ .*

The structures  $\theta$  and  $\{, \}$  have a fundamental property:

**Theorem 4.1.3.** *The following formula holds:*

$$\{f, g\} = \theta(X_f, X_g), \quad \forall f, g \in \mathcal{X}^*(T^*M), \forall X \in \mathcal{X}(T^*M). \quad (4.1.12)$$

*Proof.* From (4.1.10):

$$\{f, g\} = X_f g = -X_g f = -df(X_g) = (i_{X_f}\theta)(X_g) = \theta(X_f, X_g).$$

As a consequence, we obtain:

$$\frac{dx^i}{dt} = \{H, x^i\}, \quad \frac{dp_i}{dt} = \{H, p_i\}. \quad (4.1.13)$$

It is remarkable the Jacobi method for integration of Hamilton–Jacobi equations (4.1.9). Namely, we look for a solution curves  $\gamma(t)$  in  $T^*M$  of the form

$$x^i = x^i(t), \quad p_i = \frac{\partial S}{\partial x^i}(x(t)) \quad (4.1.14)$$

where  $S \in \mathcal{F}(M)$ . Substituting in (4.1.9), we have

$$\frac{dx^i}{dt} = \frac{\partial H}{\partial p_i}(x(t)) \frac{\partial S}{\partial x^i}(x(t)); \quad \frac{dp_i}{dt} = \frac{\partial^2 S}{\partial x^i \partial x^j} \frac{\partial H}{\partial p_j} = -\frac{\partial H}{\partial x^i}. \quad (4.1.15)$$

It follows

$$dH \left( x, \frac{\partial S}{\partial x} \right) = \left( \frac{\partial H}{\partial x^i} \frac{\partial H}{\partial p_i} - \frac{\partial H}{\partial p_i} \frac{\partial H}{\partial x^i} \right) dt = 0.$$

Consequently,  $H \left( x, \frac{\partial S}{\partial x} \right) = \text{const}$ , which is called the Hamilton–Jacobi equation of Mechanics. This equation determines the function  $S$  and the first equation (4.1.14) gives us the curves  $\gamma(t)$ .

The Jacobi method suggests to obtain the Hamilton–Jacobi equation by the variational principle.

## 4.2 Variational problem. Hamilton–Jacobi equations

The variational problem for the Hamiltonian systems  $(T^*M, \theta, H)$  is defined as follows:

Let us consider a smooth curve  $c$  defined on a local chart  $\pi^{*-1}(U)$  of the cotangent manifold  $T^*M$  by:

$$c : t \in [0, 1] \rightarrow (x(t); p(t)) \in \pi^{*-1}(U)$$

analytically expressed by

$$x^i = x^i(t), p_i = p_i(t), t \in [0, 1]. \quad (4.2.1)$$

Consider a vector field  $V^i(t)$  and a covector field  $\eta_i(t)$  on the domain  $U$  of the local chart  $(U, \varphi)$  on  $M$ , and assume that we have

$$\begin{aligned} V^i(0) = V^i(1) = 0, \eta_i(0) = \eta_i(1) = 0 \\ \frac{dV^i}{dt}(0) = \frac{dV^i}{dt}(1) = 0. \end{aligned} \quad (4.2.2)$$

The variations of the curve  $c$  determined by the pair  $(V^i(t), \eta_i(t))$  are defined by the curves  $\bar{c}(\varepsilon_1, \varepsilon_2)$ :

$$\begin{aligned} \bar{x}^i = x^i(t) + \varepsilon_1 V^i(t), \\ \bar{p}_i = p_i(t) + \varepsilon_2 \eta_i(t), t \in [0, 1], \end{aligned} \quad (4.2.3)$$

when  $\varepsilon_1$  and  $\varepsilon_2$  are constants, small in absolute value such that the image of any curve  $\bar{c}$  belongs to the open set  $\pi^{*-1}(U)$  in  $T^*M$ .

The integral of action of Hamiltonian  $H(x, p)$  along the curve  $c$  is defined by

$$I(c) = \int_0^1 \left[ p_i(t) \frac{dx^i}{dt} - H(x(t), p(t)) \right] dt. \quad (4.2.4)$$

The integral of action  $I(\bar{c}(\varepsilon_1, \varepsilon_2))$  is:

$$\begin{aligned} I(\bar{c}(\varepsilon_1, \varepsilon_2)) = \int_0^1 \left[ (p_i + \varepsilon_2 \eta_i(t)) \left( \frac{dx^i}{dt} + \varepsilon_1 \frac{dV^i}{dt} \right) - \right. \\ \left. - H(x + \varepsilon_1 V, p + \varepsilon_2 \eta) \right] dt. \end{aligned} \quad (4.2.5)$$

The necessary conditions in order that  $I(c)$  is an extremal value of  $I(\bar{c}(\varepsilon_1, \varepsilon_2))$  are

$$\left. \frac{\partial I(\bar{c}(\varepsilon_1, \varepsilon_2))}{\partial \varepsilon_1} \right|_{\varepsilon_1 = \varepsilon_2 = 0} = 0, \quad \left. \frac{\partial I(\bar{c}(\varepsilon_1, \varepsilon_2))}{\partial \varepsilon_2} \right|_{\varepsilon_1 = \varepsilon_2 = 0} = 0. \quad (4.2.6)$$



Under our conditions of differentiability, the operators  $\frac{\partial}{\partial \varepsilon_1}$ ,  $\frac{\partial}{\partial \varepsilon_2}$  and the operator of integration commute. Therefore, from (4.2.5) we deduce:

$$\int_0^1 \left[ p_i(t) \frac{dV^i}{dt}(t) - \frac{\partial H}{\partial x^i} V^i \right] dt = 0 \quad (4.2.7)$$

$$\int_0^1 \left( \frac{dx^i}{dt} - \frac{\partial H}{\partial p_i} \right) \eta_i dt = 0.$$

Denoting:

$$\overset{\circ}{E}_i(H) = \frac{dp_i}{dt} + \frac{\partial H}{\partial x^i} \quad (4.2.8)$$

and noticing the conditions (4.2.2) one obtain that the equations (4.2.7) are equivalent to:

$$\int_0^1 \overset{\circ}{E}_i(H) V^i dt = 0; \quad \int_0^1 \left[ \frac{dx^i}{dt} - \frac{\partial H}{\partial p_i} \right] \eta_i dt = 0. \quad (4.2.9)$$

But  $(V^i, \eta_i)$  are arbitrary. Thus, (4.2.9) lead to the following result:

**Theorem 4.2.1.** *In order to the integral of action  $I(c)$  to be an extremal value for the functionals  $I(\bar{c})$  is necessary that the curve  $c$  to satisfy the Hamilton–Jacobi equations:*

$$\frac{dx^i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial x^i}. \quad (4.2.10)$$

The operator  $\overset{\circ}{E}_i(H)$  has a geometrical meaning:

**Theorem 4.2.2.**  *$\overset{\circ}{E}_i(H)$  is a covector field.*

*Proof.* With respect to a change of local coordinates (4.1.1) on the manifold  $T^*M$ , we have

$$\begin{aligned} & \int_0^1 \overset{\circ}{E}_i(\tilde{H}) \tilde{V}^i dt - \int_0^1 \overset{\circ}{E}_i(H) V^i dt = \\ & = \int_0^1 \left[ \overset{\circ}{E}_i(\tilde{H}) \frac{\partial \tilde{x}^i}{\partial x^j} - \overset{\circ}{E}_j(H) \right] V^j dt = 0. \end{aligned}$$

Since the vector  $V^i$  is arbitrary, we obtain:

$$\overset{\circ}{\tilde{E}}_i(\tilde{H}) \frac{\partial \tilde{x}^i}{\partial x^j} = \overset{\circ}{E}_j(H). \square$$

A consequence of the last theorem: The Hamilton–Jacobi equation have a geometrical meaning on the cotangent manifold  $T^*M$ .

The notion of homogeneity for the Hamiltonian systems is defined in the classical manner [174].

A smooth function  $f : T^*M \rightarrow R$  is called  $r$ –homogeneous  $r \in \mathbb{Z}$  with respect to the momenta variables  $p_i$  if we have:

$$\mathcal{L}_{\mathbb{C}^*} f = \mathbb{C}^* f = p : \dot{\partial}^i f = r f. \quad (4.2.11)$$

A vector field  $X \in \chi(T^*M)$  is  $r$ –homogeneous if

$$\mathcal{L}_{\mathbb{C}^*} X = (r - 1)X, \quad (4.2.12)$$

where  $\mathcal{L}_{\mathbb{C}^*} X = [\mathbb{C}^*; X]$ . So, we have

1°  $\frac{\partial}{\partial x^i}, \frac{\partial}{\partial p_i} = \dot{\partial}^i$  are 1- and 0-homogeneous.

2° If  $f \in \mathcal{F}(T^*M)$  is  $s$ –homogeneous and  $X \in \chi(T^*M)$  is  $r$ –homogeneous then  $fX$  is  $r + s$ –homogeneous.

3°  $\mathbb{C}^* = p_i \dot{\partial}^i$  is 1-homogeneous.

4° If  $X$  is  $r$ –homogeneous and  $f$  is  $s$ –homogeneous then  $Xf$  is  $r + s - 1$ –homogeneous.

5° If  $f$  is  $s$ –homogeneous, then  $\dot{\partial}^i \dot{\partial}^j f$  is  $s - 1$ –homogeneous.

Analogously, for  $q$ –forms  $\omega \in \Lambda^q(T^*M)$ . The  $q$ –form  $\omega$  is  $r$ –homogeneous if

$$\mathcal{L}_{\mathbb{C}^*} \omega = r \omega. \quad (4.2.13)$$

It follows:

1' If  $\omega, \omega'$  are  $s$ – respectively  $s'$ –homogeneous, then  $\omega \wedge \omega'$  is  $s + s'$ –homogeneous.

2'  $dx^i, dp_i$  are 0– respectively 1–homogeneous.

3' The Liouville 1–form  $\omega = p_i dx^i$  is 1-homogeneous.

4' The canonical symplectic structure  $\theta$  is 1-homogeneous.

### 4.3 Nonlinear connections

On the manifold  $T^*M$  there exists a remarkable distribution. It is the vertical distribution  $V$ . As we know,  $V$  is integrable, having  $(\dot{\partial}^1, \dot{\partial}^2, \dots, \dot{\partial}^n)$  as a local adapted basis and  $\dim V = n = \dim M$ .

**Definition 4.3.1.** A nonlinear connection  $N$  on the manifold  $T^*M$  is a differentiable distribution  $N$  on  $T^*M$  supplementary to the vertical distribution  $V$ :

$$T_u T^*M = N_u \oplus V_u, \quad \forall u \in T^*M. \quad (4.3.1)$$

$N$  is called the *horizontal* distribution on  $T^*M$ .

It follows that  $\dim N = n$ .

When a nonlinear connection  $N$  is given we can consider an *adapted* basis  $\left(\frac{\delta}{\delta x^1}, \dots, \frac{\delta}{\delta x^n}\right)$  expressed in the form

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} + N_{ij} \frac{\partial}{\partial p_j} \quad (i = 1, 2, \dots, n). \quad (4.3.2)$$

The system of function  $N_{ij}(x, p)$  is well determined by (4.3.1). It is called system of coefficients of the nonlinear connection  $N$ . This system defines a geometrical object field on  $T^*M$ .

The set of vector fields  $\left(\frac{\delta}{\delta x^i}, \dot{\partial}^i\right)$  give us an adapted basis to the direct decomposition (4.3.1). Its dual adapted basis is  $(dx^i, \delta p_i)$ ,  $(i = 1, \dots, n)$ , where

$$\delta p_i = dp_i - N_{ji} dx^j. \quad (4.3.3)$$

It is not difficult to prove that if  $M$  is a paracompact manifold, then on the cotangent manifold  $T^*M$  there exists a nonlinear connection.

Let  $N$  be a nonlinear connection with the coefficients  $N_{ij}(x, p)$  and define the set of function  $\tau_{ij}(x, y)$  by

$$\tau_{ij} = \frac{1}{2}(N_{ij} - N_{ji}). \quad (4.3.4)$$

It is not difficult to see that  $\tau_{ij}$  is transformed, with respect to (4.1.1) as a covariant tensor field on the base manifold  $M$ . So, it is a distinguished tensor field, shortly a  $d$ -tensor of *torsion* of the nonlinear connection.  $\tau_{ij} = 0$  has a geometrical meaning. In this case  $N$  is a symmetric nonlinear connection.

With respect to a symmetric nonlinear connection  $N$  the symplectic structure  $\theta$  can be written in an invariant form:

$$\theta = \delta p_i \wedge dx^i, \quad (4.3.5)$$

and the Poisson structure  $\{, \}$  can be expressed in the invariant form:

$$\{f, g\} = \frac{\partial f}{\partial p_i} \frac{\delta g}{\delta x^i} - \frac{\partial g}{\partial p_i} \frac{\delta f}{\delta x^i}. \quad (4.3.6)$$

Of course, we can consider the curvature tensor of  $N$  as the  $d$ -tensor field

$$R_{ijh} = \frac{\delta N_{ji}}{\delta x^h} - \frac{\delta N_{hi}}{\delta x^j}. \quad (4.3.7)$$

It is given by

$$\left[ \frac{\delta}{\delta x^j}, \frac{\delta}{\delta x^h} \right] = R_{ijh} \dot{\partial}^i. \quad (4.3.8)$$

Therefore  $R_{ijh}(x, p)$  is the *integrability* tensor of the horizontal distribution  $N$ . Thus  $N$  is an integrable distribution if and only if the curvature tensor  $R_{ijh}(x, p)$  vanishes.

A curve  $c : I \subset \mathbb{R} \rightarrow c(t) \in T^*M$ .  $\text{Im}c \subset \pi^*(U)$  expressed by  $(x^i = x^i(t), p_i = p_i(t), t \in I)$ . The tangent vector  $\frac{dc}{dt}$  can be written in the adapted basis as:

$$\frac{dc}{dt} = \frac{dx^i}{dt} \frac{\delta}{\delta x^i} + \frac{\delta p_i}{dt} \frac{\partial}{\partial p_i},$$

where

$$\frac{\delta p_i}{dt} = \frac{dp_i}{dt} - N_{ji}(x(t), p(t)) \frac{dx^j}{dt}.$$

The curve  $c$  is horizontal if  $\frac{\delta p_i}{dt} = 0$ . We obtain the system of differential equations which characterize the horizontal curves:

$$x^i = x^i(t), \quad \frac{dp_i}{dt} - N_{ji}(x, p) \frac{dx^j}{dt} = 0. \quad (4.3.9)$$

Let  $g_{ij}(x, y)$  be a  $d$ -tensor by ( $d$ -means distinguished) with the properties  $g_{ij} = g_{ji}$  and  $\det(g_{ij}) \neq 0$  on  $T^*M$ . Its contravariant  $g^{ij}(x, y)$  can be considered,  $g_{ij}g^{ts} = \delta_i^s$ . As usually, we put  $\frac{\delta}{\delta x^i} = \delta_i$ ,  $\frac{\partial}{\partial p_i} = \dot{\partial}^i$

and  $\dot{\partial}_i = g_{ij}\dot{\partial}^j$ . So, one can consider the following  $\mathcal{F}(T^*M)$ –linear mapping  $\check{\mathbb{F}} : \mathcal{X}(T^*M) \rightarrow \mathcal{X}(T^*M)$ :

$$\check{\mathbb{F}}(\delta_i) = -\dot{\partial}_i, \check{\mathbb{F}}(\dot{\partial}_i) = \delta_i, (i = 1, \dots, n). \quad (4.3.10)$$

It is not difficult to prove, [174]:

**Theorem 4.3.1.** 1° The mapping  $\check{\mathbb{F}}$  is globally defined on  $T^*M$ .

2°  $\check{\mathbb{F}}$  is a tensor field of type  $(1, 1)$  on  $T^*M$ .

3° The local expression of  $\check{\mathbb{F}}$  in the adapted basis  $(\delta_i, \dot{\partial}^i)$  is

$$\check{\mathbb{F}} = -g_{ij}\dot{\partial}^i \otimes dx^j + g^{ij}\delta_i \otimes \delta p_j. \quad (4.3.11)$$

4°  $\check{\mathbb{F}}$  is an almost complex structure on  $T^*M$  determined by  $N$  and by  $g_{ij}(x, p)$ , i.e.

$$\check{\mathbb{F}} \circ \check{\mathbb{F}} = -I. \quad (4.3.12)$$

Also, nonlinear symmetric connection  $N$  being considered, we can define the tensor:

$$\mathbb{G}(x, p) = g_{ij}(x, p)dx^i \otimes dx^j + g^{ij}\delta p_i \otimes \delta p_j. \quad (4.3.13)$$

If  $d$ –tensor  $g_{ij}(x, p)$  has a constant signature on  $T^*M$  (for instance it is positively defined), then it follows:

**Theorem 4.3.2.** 1°  $\mathbb{G}$  is a Riemannian structure on  $T^*M$  determined by  $N$  and  $g_{ij}$ .

2° The distributions  $N$  and  $V$  are orthogonal with respect to  $\mathbb{G}$ .

3° The pair  $(\mathbb{G}, \check{\mathbb{F}})$  is an almost Hermitian structure on  $T^*M$ .

4° The associated almost symplectic structure to  $(\mathbb{G}, \check{\mathbb{F}})$  is the canonical symplectic structure  $\theta = \delta p_i \wedge dx^i$ .

Remarking that the vector fields  $(\delta_i, \dot{\partial}^i)$  are transformed by (4.1.1) in the form

$$\tilde{\delta}_i = \frac{\partial x^j}{\partial \tilde{x}^i} \delta_j, \tilde{\partial}^i = \frac{\partial \tilde{x}^i}{\partial x^j} \dot{\partial}^j, \quad (4.3.14)$$

and the 1-forms  $(dx^i, \delta p_i)$  are transformed by

$$d\tilde{x}^i = \frac{\partial \tilde{x}^i}{\partial x^j} dx^j, \delta \tilde{p}_i = \frac{\partial x^j}{\partial \tilde{x}^i} \delta p_j, \quad (4.3.13')$$

we can consider the horizontal and vertical projectors with respect to the direct decomposition (4.3.1):

$$h = \frac{\delta}{\delta x^i} \otimes dx^i, \quad v = \dot{\partial}^i \otimes \delta p_i. \quad (4.3.15)$$

They have the properties:

$$h + v = I, \quad h^2 = h, \quad v^2 = v, \quad h \circ v + v \circ h = 0.$$

We set

$$hX = X^H, \quad vX = X^V, \quad \forall X \in \mathcal{X}(T^*M)$$

$$\omega^H = \omega \circ h, \quad \omega^V = \omega \circ v, \quad \forall \omega \in \mathcal{X}^*(T^*M).$$

Therefore, for every vector field  $X$  on  $T^*M$ , represented in adapted basis in the form

$$X = X^i \delta_i + \dot{X}_i \dot{\partial}^i$$

we have

$$X^H = hX = X^i \frac{\delta}{\delta x^i}, \quad X^V = vX = \dot{X}_i \dot{\partial}^i$$

where the coefficients  $X^i(x, p)$  and  $\dot{X}_i(x, p)$  are transformed by (4.1.1):

$$\tilde{X}^i = \frac{\partial \tilde{x}^i}{\partial x^j} X^j, \quad \tilde{\dot{X}}_i = \frac{\partial x^j}{\partial \tilde{x}^i} \dot{X}_j.$$

For this reason,  $X^i(x, p)$  are the coefficients of a distinguished vector field and  $\dot{X}_i(x, p)$  are the coefficients of a distinguished covector field, shortly denoted by  $d$ -vector (covector) fields.

Analogously, for the 1-form  $\omega$ :

$$\omega = \omega_i dx^i + \dot{\omega}^i \delta p_i.$$

Therefore,

$$\omega^H = \omega_i dx^i, \quad \omega^V = \dot{\omega}^i \delta p_i.$$

Then  $\omega_i(x, p)$  are component of an one  $d$ -form on  $T^*M$  and  $\dot{\omega}^i(x, p)$  are the components of a  $d$ -vector field on  $T^*M$ .

On the same way, we can define a distinguished ( $d$ -) tensor field.

A  $d$ -tensor field can be represented in adapted basis in the form

$$T = T_{j_1 \dots j_s}^{i_1 \dots i_r}(x, p) \delta_{i_1} \otimes \dots \otimes \dot{\partial}^{j_s} \otimes dx^{j_1} \otimes \dots \otimes \delta p_r. \quad (4.3.16)$$

Its coefficients are transformed by (4.1.1) in the form:

$$\tilde{T}_{j_1 \dots j_s}^{i_1 \dots i_r}(\tilde{x}, \tilde{p}) = \frac{\partial \tilde{x}^{i_1}}{\partial x^{h_1}} \dots \frac{\partial \tilde{x}^{i_r}}{\partial x^{h_r}} \frac{\partial x^{k_1}}{\partial \tilde{x}^{j_1}} \dots \frac{\partial x^{k_s}}{\partial \tilde{x}^{j_s}} T_{k_1 \dots k_s}^{h_1 \dots h_r}(x, p). \quad (4.3.15')$$

So, a  $d$ -tensor field  $T$  can be given by the coefficients  $T_{j_1 \dots j_s}^{i_1 \dots i_r}(x, p)$  whose rule of transformation, with respect to (4.1.1) is the same with that of a tensor of the same type on the base manifold  $M$  of cotangent bundle  $(T^*M, M, \pi^*)$ .

One can speak of the  $d$ -tensor algebra on  $T^*M$ , which is not difficult to be defined.

#### 4.4 $N$ -linear connections

As we know, a nonlinear connection  $N$  determines a direct decomposition (4.3.1) in respect to which we have

$$X = X^H + X^V, \quad \omega = \omega^H + \omega^V, \quad \forall X \in \mathcal{X}(T^*M), \quad \forall \omega \in \mathcal{X}^*(T^*M). \quad (4.4.1)$$

Assuming that  $N$  is a symmetric nonlinear connection we can give:

**Definition 4.4.1.** A linear connection  $D$  on the manifold  $T^*M$  is called an  $N$ -linear connection if:

- 1°  $D$  preserves by parallelism the distributions  $N$  and  $V$ .
- 2° The canonical symplectic structure  $\theta = \delta p_i \wedge dx^i$  has the associate tensor  $\bar{\theta} = \delta p_i \otimes dx^i$  parallel with respect to  $D$ :

$$D\bar{\theta} = 0. \quad (4.4.2)$$

It follows that:

$$\begin{aligned} D_X h &= D_X v = 0 \\ D_X Y &= D_{X^H} Y + D_{X^V} Y. \end{aligned} \quad (4.4.3)$$

We obtain new operators of derivations in the algebras of  $d$ -tensors, defined by

$$D_X^H = D_{X^H}, \quad D_X^V = D_{X^V}, \quad \forall X \in \mathcal{X}(T^*M). \quad (4.4.4)$$

We have

$$\begin{aligned}
D_X &= D_X^H + D_X^V; \quad D_X^H f = X^H f, \quad D_X^V f = X^V f \\
D_X^H(fY) &= X^H fY + fD_X^H Y; \quad D_X^V(fY) = X^V fY + fD_X^V Y \\
D_{fX}^H &= fD_X^H, \quad D_{fX}^V = fD_X^V \\
D_X^H \theta &= 0, \quad D_X^V \theta = 0.
\end{aligned} \tag{4.4.5}$$

The operators  $D^H, D^V$  have the property of localization. They have the similarly properties with covariant derivative but they are not covariant derivative.  $D^H$  is called  $h$ -covariant derivative and  $D^V$  is called  $v$ -covariant derivative. They act on the 1-forms  $\omega$  on  $T^*M$  by the rules:

$$\begin{aligned}
(D_X^H \omega)(Y) &= X^H \omega(Y) - \omega(D_X^H Y) \\
(D_X^V \omega)(Y) &= X^V \omega(Y) - \omega(D_X^V Y).
\end{aligned} \tag{4.4.6}$$

The extension of these operators to the  $d$ -tensor fields is immediate. The torsion  $\mathbb{T}$  of an  $N$ -linear connection has the form

$$\mathbb{T}(X, Y) = D_X Y - D_Y X - [X, Y]. \tag{4.4.7}$$

It is characterized by the following vector fields:

$$\mathbb{T}(X^H, Y^H), \mathbb{T}(X^H, Y^V), \mathbb{T}(X^V, X^V).$$

Taking the  $h$ - and  $v$ -components we obtain the  $d$ -tensor of torsion

$$T(X^H, Y^H) = h\mathbb{T}(X^H, Y^H) + v\mathbb{T}(X^H, Y^H), \text{ etc.}$$

**Proposition 4.4.2.** *The  $d$ -tensors of torsion of an  $N$ -linear connection  $D$  are:*

$$\begin{aligned}
h\mathbb{T}(X^H, Y^H) &= D_X^H Y^H - D_Y^H X^H - [X^H, Y^H]^H \\
h\mathbb{T}(X^H, Y^V) &= -D_Y^V X^H + [X^H, Y^V]^H \\
v\mathbb{T}(X^H, Y^H) &= -[X^H, Y^H]^V \\
v\mathbb{T}(X^H, Y^V) &= D_X^H Y^V - [X^H, Y^V]^V \\
v\mathbb{T}(X^V, Y^V) &= D_X^V Y^V - D_Y^V X^V - [X^V, Y^V]^V = 0.
\end{aligned} \tag{4.4.8}$$

The curvature  $\mathbb{R}$  of an  $N$ -linear connection  $D$  is given by



$$\mathbb{R}(X, Y)Z = (D_X D_Y - D_Y D_X)Z - D_{[X, Y]}Z. \quad (4.4.9)$$

Remarking that the vector field  $\mathbb{R}(X, Y)Z^H$  is horizontal one and  $\mathbb{R}(X, Y)Z^V$  is a vertical one, we have

$$v(\mathbb{R}(X, Y)Z^H) = 0, \quad h(\mathbb{R}(X, Y)Z^V) = 0. \quad (4.4.10)$$

We will see that the  $d$ -tensors of curvature of  $D$  are:

$$\mathbb{R}(X^H, Y^H)Z^H, \quad \mathbb{R}(X^H, Y^V)Z^H, \quad \mathbb{R}(X^V, Y^V)Z^H. \quad (4.4.11)$$

By means of (4.4.9) one obtains:

**Proposition 4.4.3.** *1° The Ricci identities of  $D$  are*

$$[D_X, D_Y]Z = \mathbb{R}(X, Y)Z - D_{[X, Y]}Z. \quad (4.4.12)$$

*2° The Bianchi identities are given by*

$$\sum_{(XYZ)} \{(D_X \mathbb{T})(Y, Z) - \mathbb{R}(X, Y)Z + \mathbb{T}(\mathbb{T}(X, Y), Z)\} = 0, \quad (4.4.13)$$

$$\sum_{(X, Y, Z)} \{(D_X \mathbb{R})(U, Y, Z) - \mathbb{R}(\mathbb{T}(X, Y), Z)U\} = 0,$$

where  $\sum_{(XYZ)}$  means the cyclic sum.

The previous formulae can be expressed in local coordinates, taking  $\delta_i, \dot{\partial}^i$ , as the vectors  $X, Y, Z, U$ . But, first of all, we must introduce the local coefficients of an  $N$ -linear connection  $D$ . In the adapted basis  $(\delta_i, \dot{\partial}^i)$  we take into account the properties:

$$D_{\delta_j} = D_{\delta_j}^H, \quad D_{\dot{\partial}^j} = D_{\dot{\partial}^j}^V. \quad (4.4.14)$$

Then, the following theorem holds:

**Theorem 4.4.1.** *1° An  $N$ -linear connection  $D$  on  $T^*M$  can be uniquely represented in adapted basis  $(\delta_i, \dot{\partial}^i)$  in the form:*

$$\begin{aligned} D_{\delta_j} \delta_i &= H_{ij}^h \delta_h, \quad D_{\delta_j} \dot{\partial}^i = -H_{hj}^i \dot{\partial}^h, \\ D_{\dot{\partial}^j} \delta_i &= C_i^{hj} \delta_h, \quad D_{\dot{\partial}^j} \dot{\partial}^i = -C_h^{ij} \dot{\partial}^h. \end{aligned} \quad (4.4.15)$$

2° With respect to (4.4.1) on  $T^*M$  the coefficients  $H_{jk}^i(x, p)$  transform by the rule

$$\tilde{H}_{rs}^i \frac{\partial x^r}{\partial \tilde{x}^j} \frac{\partial x^s}{\partial \tilde{x}^k} = \frac{\partial \tilde{x}^i}{\partial x^r} H_{jk}^r - \frac{\partial^2 \tilde{x}^i}{\partial x^j \partial x^k} \quad (4.4.16)$$

and  $C_i^{jk}(x, p)$  is a  $d$ -tensor of type  $(2, 1)$ .

The proof of this theorem is not difficult.

Conversely, if  $N$  is an a priori given nonlinear connection and a set of functions  $H_{jk}^i(x, p), C_i^{jk}(x, p)$  on  $T^*M$ , verifying 2° is given, then there exists an unique  $N$ -linear connection  $D$  with the property (4.4.16).

The action of  $D$  on the adapted cobasis  $(dx^i, \delta p_i)$  is given by

$$D_{\delta_j} dx^i = -H_{kj}^i dx^k, \quad D_{\delta_j} \delta p_i = H_{ij}^k \delta p_k, \quad (4.4.17)$$

$$D_{\partial_j} dx^i = -C_k^{ij} dx^k, \quad D_{\partial_j} \delta p_i = C_i^{kj} \delta p_k.$$

The pair  $D\Gamma(N) = (H_{jk}^i, C_i^{jk})$  is called the system of coefficients of  $D$ .

Let us consider a  $d$ -tensor field  $T$  with the local coefficients  $T_{j_1 \dots j_s}^{i_1 \dots i_r}(x, p)$ , (see (4.3.15)) and a horizontal vector field  $X = X^H = X^i \delta_i$ .

By means of previous theorem we obtain for  $h$ -covariant derivation  $D_X^H$  of tensor  $T$ :

$$D_X^H T = X^k T_{j_1 \dots j_s | k}^{i_1 \dots i_r} \delta_{i_1} \otimes \dots \otimes \partial^{j_s} \otimes dx^{j_1} \otimes \dots \otimes \delta p_{i_r}, \quad (4.4.18)$$

where

$$\begin{aligned} T_{j_1 \dots j_s | k}^{i_1 \dots i_r} &= \delta_k T_{j_1 \dots j_s}^{i_1 \dots i_r} + T_{j_1 \dots j_s}^{mi_2 \dots i_r} H_{mk}^{i_1} + \dots + T_{j_1 \dots j_s}^{ir_1 \dots m} H_{mk}^{i_r} - \\ &- T_{m \dots j_s}^{i_1 \dots i_r} H_{j_1 k}^m - \dots - T_{j_1 \dots m}^{i_1 \dots i_r} H_{j_s k}^m. \end{aligned} \quad (4.4.16')$$

The operator “|” is called  $h$ -covariant derivative with respect to  $D\Gamma(N) = (H_{jk}^i, C_i^{jk})$ .

Now, taking  $X = X^V = X_i \partial^i$ , the  $v$ -covariant derivative  $D_X^V T$  has the following form

$$D_X^V T = \dot{X}_k T_{j_1 \dots j_s}^{i_1 \dots i_r} \Big|_k \delta_{i_1} \otimes \dots \otimes \partial^{j_s} \otimes dx^{j_1} \otimes \dots \otimes \delta p_{i_r}, \quad (4.4.19)$$

where

$$\begin{aligned} T_{j_1 \dots j_s}^{i_1 \dots i_r} \Big|{}^k &= \dot{\partial}^k T_{j_1 \dots j_s}^{i_1 \dots i_r} + T_{j_1 \dots j_s}^{m \dots i_r} C_m^{i_1 k} + \dots + T_{j_1 \dots j_s}^{i_1 \dots m} C_m^{i_r k} - \\ &- T_{m \dots j_s}^{i_1 \dots i_r} C_{j_1}^{mk} - \dots - T_{j_1 \dots m}^{i_1 \dots i_r} C_{j_s}^{mk}. \end{aligned} \quad (4.4.17')$$

The operator “—” is called the  $\nu$ -covariant derivative.

**Proposition 4.4.4.** *The following properties hold:*

1°  $T_{j_1 \dots j_s | k}^{i_1 \dots i_r}$  is a  $d$ -tensor of type  $(r, s + 1)$ .

2°  $T_{j_1 \dots j_s}^{i_1 \dots i_r} \Big|{}^k$  is a  $d$ -tensor of type  $(r + 1, s)$ .

3°  $f_{|m} = \frac{\delta f}{\delta x^m}$ ,  $f^{|m} = \dot{\partial}^m f$ .

$$4^\circ X_{|k}^i = \delta_k X^i + X^m H_{mk}^i; \quad X^{i|k} = \dot{\partial}^k X^i + X^m C_m^{ik}, \quad (4.4.20)$$

$$\omega_{i|k} = \delta_k \omega_i - \omega_m H_{ik}^m; \quad \omega_1^{i|k} = \dot{\partial}^k \omega_i - \omega_m C_i^{mk}.$$

5° The operators “|” and “ $\dot{\partial}$ ” are distributive with respect to addition and verify the rule of Leibnitz with respect to tensor product of  $d$ -tensors.

Let us consider the deflection tensors of  $D\Gamma(N)$ :

$$\Delta_{ij} = p_{i|j}, \quad \check{\delta}_i^j = p_i^{j|}. \quad (4.4.21)$$

One gets

$$\Delta_{ij} = N_{ij} - p_m H_{ij}^m; \quad \check{\delta}_j^i = \delta_j^i - p_m C_i^{mj}. \quad (4.4.19')$$

**Proposition 4.4.5.** *If  $H_{jk}^i(x, p)$  is a system of differentiable functions, verifying (4.4.16), then  $N_{ij}(x, p)$  given by*

$$N_{ij} = p_m H_{ij}^m \quad (4.4.22)$$

determine a nonlinear connection in  $T^*M$ .

As in the case of tangent bundle, one can prove that if the base manifold  $M$  is paracompact, then on  $T^*M$  there exist the  $N$ -linear connections.

Indeed, on  $M$  there exists a Riemannian metric  $g_{ij}(x)$ . Then the pair  $D\Gamma(N) = (\gamma_{jk}^i, 0)$  is an  $N$ -linear connection on  $T^*M$ ,  $\gamma_{jk}^i(x)$  being the Christoffel coefficients and  $N_{ij} = p_m \gamma_{ij}^m$ , are the coefficients of a nonlinear connection on  $T^*M$ .

In the adapted basis  $(\delta_i, \partial^i)$ , the Ricci identities (4.4.12) can be written in the form

$$\begin{aligned} X^i_{|j|h} - X^i_{|h|j} &= X^m R^i_{mjh} - X^i_{|m} T^m_{jh} - X^i_{|m} R_{mjh} \\ X^i_{|j}{}^h - X^i_{|h}{}^j &= X^m P^i_{mj}{}^h - X^i_{|m} C^{mh}_j - X^i_{|m} P^h_{mj} \\ X^i_{|j}{}^h - X^i_{|h}{}^j &= X^m S_m^{ijh} - X^i_{|m} S_m^{jh}, \end{aligned} \quad (4.4.23)$$

where

$$T^i_{jh} = H^i_{jh} - H^i_{hj}, \quad S_i^{jh} = C_i^{jh} - C_i^{hj}, \quad P^i_{jh} = H^i_{jh} - \partial^i N_{hj} \quad (4.4.24)$$

and  $R_{mjh}, C_i^{mh}$  are the  $d$ -tensors of torsion and  $R^i_{kjh}, P_k{}^i{}^h, S_k^{ijh}$  are the  $d$ -tensors of curvature:

$$\begin{aligned} R^i_{kjh} &= \delta_h H^i_{kj} - \delta_j H^i_{kh} + H^m_{kj} H^i_{mh} - H^m_{kh} H^i_{kj} - C_k{}^{im} R_{mjh}, \\ P_k{}^i{}^h &= \partial^h H^i_{kj} - C_k{}^{ih}{}_{|j} + C_k{}^{im} P^h_{mj}, \quad S_k^{ijh} = \partial^h C_k{}^{ij} - \partial^j C_k{}^{ih} + \\ &+ C_k{}^{mj} C_m{}^{ih} - C_k{}^{mh} C_m{}^{ij}. \end{aligned} \quad (4.4.25)$$

Evidently, these  $d$ -tensors of curvature verify:

$$\begin{aligned} \mathbb{R}(\delta_j \delta_h) \delta_k &= R^i_{kjh} \delta_i; \quad \mathbb{R}(\delta_j, \partial^h) \delta_k = P_k{}^h{}^i \delta_i, \\ \mathbb{R}(\partial^j, \partial^h) \delta_k &= S_k^{ijh} \delta_i, \end{aligned} \quad (4.4.26)$$

and

$$\begin{aligned} \mathbb{R}(\delta_j \delta_h) \partial^k &= R^k_{ihj} \partial^i; \quad \mathbb{R}(\delta_j, \partial^h) \partial^k = -P_i{}^k{}^h \partial^i, \\ \mathbb{R}(\partial^j, \partial^h) \partial^k &= -S_i{}^{hkj} \partial^i. \end{aligned} \quad (4.4.23')$$

The Bianchi identities (4.4.13), in adapted basis  $(\delta_i, \partial^i)$  can be written without difficulties.

**Applications:**

1° For a  $d$ -tensor  $g^{ij}(x, p)$  the Ricci identities are

$$\begin{aligned} g^{ij}|_k|_h - g^{ij}|_h|_k &= g^{mj}R_m^i{}_{kh} + g^{im}R_{mkh}^j - g^{ij}|_m T_{kh}^m - g^{ij}|^m R_{mkh}, \\ g^{ij}|_k|_h - g^{ij}|_h|_k &= g^{mj}P_m^i{}_{kh} + g^{im}P_{mk}^j{}^h - g^{ij}|_m C_k^{mh} - g^{ij}|^m P_{mk}^h, \quad (4.4.27) \\ g^{ij}|^k|_h - g^{ij}|_h|^k &= g^{mj}S_m^{ikh} + g^{im}S_m^{jkh} - g^{ij}|^m S_m^{kh}. \end{aligned}$$

In particular, if  $g^{ij}$  is covariant constant with respect to  $N$ -connection  $D$ , i.e.

$$g^{ij}|_k = 0, \quad g^{ij}|^k = 0, \quad (4.4.28)$$

then, from (4.4.27) we have:

$$\begin{aligned} g^{mj}R_m^i{}_{kh} + g^{im}R_{mkh}^j &= 0, \\ g^{mj}P_m^i{}_{kh} + g^{im}P_{mk}^j{}^h &= 0, \quad (4.4.25') \\ g^{mj}S_m^{ikh} + g^{im}S_m^{jkh} &= 0. \end{aligned}$$

Such kind of equations will be used for the  $N$ -linear connections compatible with a metric structure  $G$  in (4.3.13).

The Ricci identities applied to the Liouville-Hamilton vector field  $C^* = p_i \dot{\partial}^i$  lead to the important identities, which imply the deflection tensors  $\Delta_i$ , and  $\delta_j^i$ .

**Theorem 4.4.2.** Any  $N$ -linear connection  $D$  on  $T^*M$  satisfies the following identities:

$$\begin{aligned} \Delta_{ij}|_k - \Delta_{ik}|_j &= -p_m R_i^m{}_{jk} - \Delta_{im} T_{jk}^m - \delta_i^m R_{mjk}, \\ \Delta_{ij}|^k - \delta_i^k|_j &= -p_m P_{ij}^m{}^k - \Delta_{im} C_j^{mk} - \delta_i^m P_{mj}^k, \quad (4.4.29) \\ \delta_i^j|_k - \delta_j^k|_i &= -p_m S_i^{mjk} - \delta_i^m S_m^{jk}. \end{aligned}$$

One says that are  $N$ -linear connection  $D$  is of Cartan type if  $\Delta_{ij} = 0$ ,  $\delta_j^i = \delta_j^i$ .

**Proposition 4.4.6.** Any  $N$ -linear connection  $D$  of Cartan type satisfies the identities:

$$p_m R_i^m{}_{jk} + R_{ijk} = 0, \quad p_m P_{ij}^m{}^k + P_{ij}^k = 0, \quad p_m S_i^{mjk} + S_i^{jk} = 0. \quad (4.4.30)$$

Finally, we remark that we can explicitly write, in adapted basis, the Bianchi identities (4.4.13).

## 4.5 Parallelism, paths and structure equations

Consider the Hamiltonian systems  $(T^*M, H, \theta)$ , an  $N$ -linear connection  $D\Gamma(N) = (H_{jk}^i, C_i^{jk})$ .

A curve  $c : I \rightarrow T^*M$ , locally represented by

$$x^i = x^i(t), \quad p_i = p_i(t), \quad (4.5.1)$$

has the tangent vector  $\frac{dc}{dt}$  in adapted basis  $(\delta_i, \dot{\partial}^i)$ :

$$\frac{dc}{dt} = \frac{dx^i}{dt} \delta_i + \frac{\delta p_i}{dt} \dot{\partial}^i, \quad (4.5.2)$$

with  $\frac{\delta p_i}{dt} = \frac{dp_i}{dt} - N_{ji} \frac{dx^j}{dt}$ . The operator of covariant derivative  $D$  along the curve  $c$  is

$$D_{\dot{c}} = D_{\dot{c}}^H + D_{\dot{c}}^V = \frac{dx^i}{dt} D_{\delta_i} + \frac{\delta p_i}{dt} D_{\dot{\partial}^i}.$$

If  $X = X^i \delta_i + \dot{X}_i \dot{\partial}^i$  is a vector field then the operator  $X$  of covariant differential acts on the vector fields  $D$  by the rule:

$$DX = (dx^i + X^m \omega_m^i) \delta_i + (d\dot{X}_i - \dot{X}_m \omega_i^m) \dot{\partial}^i, \quad (4.5.3)$$

where

$$\omega_j^i = H_{jk}^i dx^k + C_j^{ik} \delta p_k \quad (4.5.4)$$

is called *the connection 1-form*.

Therefore, the covariant differential  $\frac{DX}{dt}$  is:

$$\frac{DX}{dt} = \left( \frac{dX^i}{dt} + X^m \frac{\omega_m^i}{dt} \right) \delta_i + \left( \frac{d\dot{X}_i}{dt} - \dot{X}_m \frac{\omega_i^m}{dt} \right) \dot{\partial}^i, \quad (4.5.5)$$

where  $\frac{dX^i}{dt}$  is  $\frac{dX^i}{dt}(x(t), p(t))$ ,  $t \in I$  and

$$\frac{\omega_j^i}{dt} = H^i_{jk}(x(t), p(t)) \frac{dx^k}{dt} + C_j^{ik}(x(t), p(t)) \frac{\delta p_k}{dt}. \quad (4.5.6)$$

As usually, we introduce:

**Definition 4.5.1.** The vector field  $X = X^i(x, p)\delta_i + \dot{X}_m(x, p)\dot{\partial}^i$  is parallel, with respect to  $D\Gamma(N)$ , along smooth curve  $c : I \rightarrow T^*M$  if  $\frac{DX}{dt} = 0$ ,  $\forall t \in I$ .

From (4.5.5) one obtains:

**Theorem 4.5.1.** *The vector field  $X = X^i\delta_i + \dot{X}_i\dot{\partial}^i$  is parallel, with respect to the  $N$ -linear connection  $D\Gamma(H^i_{jk}, C_i^{jk})$ , along of curve  $c$  if, and only if, the functions  $X^i, \dot{X}_i$ , ( $i = 1, \dots, n$ ) are solutions of the differential equations:*

$$\frac{dX^i}{dt} + X^m \frac{\omega_m^i}{dt} = 0, \quad \frac{d\dot{X}_i}{dt} - \dot{X}_m \frac{\omega_i^m}{dt} = 0. \quad (4.5.7)$$

The proof is immediate, by means of (4.5.5).

A theorem of existence and uniqueness for the parallel vector fields along a given parametrized curve  $c$  in the manifold  $T^*M$  can be formulated in the classical manner. Let us consider the case when a vector field  $X$  is absolute parallel on  $T^*M$ , with respect to an  $N$ -linear connection  $D$ , i.e.  $DX = 0$  on  $T^*M$ , [174].

Using the formula (4.5.3),  $DX = 0$  if and only if

$$dX^i + X^m \omega_m^i = 0, \quad d\dot{X}_i - \dot{X}_m \omega_i^m = 0.$$

Remarking (4.5.4), and that we have

$$dX^i = X^i_{|k} dx^k + X^i|^k \delta p_k$$

$$d\dot{X}_i = \dot{X}_i_{|k} dx^k + \dot{X}_i|^k \delta p_k$$

it follows that the equation  $DX = 0$  along any curve  $c$  on  $T^*M$  is equivalent to

$$X^i_{|j} = 0, \quad X^i|^j = 0 \quad (4.5.8)$$

$$\dot{X}_i_{|j} = 0, \quad \dot{X}_i|^j = 0.$$

The differential consequence of the previous system are given by the Ricci identities (4.4.26), taken modulo (4.5.8). They are

$$\begin{aligned} X^h R_{h\ jk}^i &= 0, X^h P_{h\ j}^{i\ k} = 0, X^h S_h^{ijk} = 0 \\ \dot{X}_h R_i^{\ h\ jk} &= 0, \dot{X}_h P_i^{\ h\ jk} = 0, \dot{X}_h S_i^{\ hjk} = 0. \end{aligned} \quad (4.5.9)$$

But,  $\{X^i, \dot{X}^i\}$  being arbitrary, it follows

**Theorem 4.5.2.** *The  $N$ -linear connection  $D\Gamma(N)$  is with the absolute parallelism of vectors if, and only if, the curvature tensor  $\mathbb{R}$  of  $D$  vanishes.*

**Definition 4.5.2.** The curve  $c : I \rightarrow T^*M$  is called autoparallel for  $N$ -linear connection  $D$  if  $D_{\dot{c}} \cdot \dot{c} = 0$ .

Taking into account the fact that  $\dot{c} = \frac{dc}{dt} = \frac{dx^i}{dt} \delta_i + \frac{\delta p_i}{dt} \partial^i$ , one obtains

**Theorem 4.5.3.** *A curve  $c : I \rightarrow T^*M$  is autoparallel with respect to  $D\Gamma(N)$  if, and only if, the functions  $x^i(t), p_i(t), t \in I$ , are solutions of the system of differential equations*

$$\frac{d^2 x^i}{dt^2} + \frac{dx^s}{dt} \frac{\omega_s^i}{dt} = 0, \frac{d}{dt} \frac{\delta p_i}{dt} - \frac{\delta p_s}{dt} \frac{\omega_i^s}{dt} = 0. \quad (4.5.10)$$

Of course, a theorem of existence and uniqueness for the autoparallel curves can be formulated.

**Definition 4.5.3.** An horizontal path of  $D\Gamma(N)$  is an horizontal autoparallel curve.

**Theorem 4.5.4.** *The horizontal paths of  $D\Gamma(N)$  are characterized by the differential equations*

$$\frac{d^2 x^i}{dt^2} + H_{jk}^i(x, p) \frac{dx^j}{dt} \frac{dx^k}{dt} = 0. \quad (4.5.11)$$

The curve  $c : I \rightarrow T^*M$  is vertical at the point  $x_0^i = x^i(t_0), t_0 \in I$  of  $M$  if  $\dot{c}(t_0) \in V$  (vertical distribution).

A vertical path at the point  $(x_0^i) \in M$  is a vertical autoparallel curve at point  $(x_0^i)$ .



**Theorem 4.5.5.** *The vertical paths at a point  $x_0 = (x_0^i) \in M$  with respect to  $D\Gamma(N)$  are solutions of the differential equations*

$$\frac{d^2 p_i}{dt^2} - C_i^{jk}(x_0, p) \frac{dp_j}{dt} \frac{dp_k}{dt} = 0, \quad x^i = x_0^i. \quad (4.5.12)$$

Now, we can write the structure equations of  $D\Gamma(N) = (H_{jk}^i, C_i^{jk})$ , remarking that the following geometric objects are a  $d$ -vector field and a  $d$ -covector field, respectively:

$$d(dx^i) - dx^m \wedge \omega_m^i; \quad d\delta p_i + \delta p_m \wedge \omega_i^m$$

and  $d\omega_j^i - \omega_j^m \wedge \omega_m^i$  is a  $d$ -tensor of type (1,1). Here  $d$  is the operator of exterior differential.

By a straightforward calculus we can prove:

**Theorem 4.5.6.** *For any  $N$ -linear connection  $D$  with the coefficients  $D\Gamma(N) = (H_{jk}^i, C_i^{jk})$  the following structure equations hold:*

$$\begin{aligned} d(dx^i) - dx^m \wedge \omega_m^i &= -\Omega^i \\ d(\delta p_i) + \delta p_m \wedge \omega_i^m &= -\dot{\Omega}_i \\ d\omega_j^i - \omega_j^m \wedge \omega_m^i &= -\Omega_j^i, \end{aligned} \quad (4.5.13)$$

where  $\Omega^i, \dot{\Omega}_i$  are 2-forms of torsion:

$$\begin{aligned} \Omega^i &= \frac{1}{2} T_{jk}^i dx^j \wedge dx^k + C_j^{ik} dx^j \wedge \delta p_k, \\ \dot{\Omega}_i &= \frac{1}{2} R_{ijk} dx^j \wedge dx^k + P_{ij}^k dx^j \wedge \delta p_k + \frac{1}{2} S_i^{jk} \delta p_j \wedge \delta p_k, \end{aligned} \quad (4.5.14)$$

and when  $\Omega_j^i$  is 2-forms of curvature:

$$\Omega_j^i = \frac{1}{2} R_j^i{}_{km} dx^k \wedge dx^m + P_j^{im} dx^k \wedge \delta p_m + \frac{1}{2} S_i^{jkm} \delta p_k \wedge \delta p_m. \quad (4.5.15)$$

### Remarks.

1° The torsion and curvature  $d$ -tensor forms (4.5.14) and (4.5.15) are expressed in Section 4 of this chapter.

2° The Bianchi identities can be obtained exterior differentiating the structure equations modulo the same equations.

## Chapter 5

### Hamilton spaces

The notion of Hamilton space was defined and investigated by R. Miron in the papers [154], [174], [175]. It was studied by D. Hrimiuc [109], H. Shimada [223] et al. It was applied by P.L. Antonelli, S. Vacaru, D. Bao et al. [251]. The geometry of these spaces is the geometry of a Hamiltonian system  $(T^*M, \theta, H)$ , where  $H(x, p)$  is a fundamental function of an Hamilton space  $H^n = (M, H(x, p))$ . Consequently, we can apply the geometric theory of cotangent manifold  $T^*M$  presented in the previous chapter, establishing the all fundamental geometric objects of the spaces  $H^n$ .

As we see, the geometry of Hamilton spaces can be studied as dual geometry, via Legendre transformation of the geometry of Lagrange spaces.

In this chapter we study the geometry of Hamilton spaces, combining these two methods. So, we defined the notion of Hamilton space  $H^n$ , the canonical non-linear connection  $N$  of  $H^n$ , the metrical structure and the canonical metrical linear connection of  $H^n$ . The fundamental equations of  $H^n$  are the Hamilton–Jacobi equations. The notion of parallelism with respect to a  $N$ –metrical connection and its consequences are studied. As applications we study the notion of Hamilton spaces of the electrodynamics. Also the almost Kählerian model of the spaces  $H^n$  is pointed out.

#### 5.1 Notion of Hamilton space

Let  $M$  be a real differential manifold of dimension  $n$  and  $(T^*M, \pi^*, M)$  its cotangent bundle. A differentiable Hamiltonian is a mapping  $H : T^*M \rightarrow R$  differentiable on  $\widetilde{T^*M} = T^*M \setminus \{0\}$  and continuous on the null section.

The Hessian of  $H$ , with respect to the momenta variable  $p_i$  of differential Hamiltonian  $H(x, p)$  has the components

$$g^{ij} = \frac{1}{2} \frac{\partial^2 H}{\partial p_i \partial p_j}. \quad (5.1.1)$$

Of course, the Latin indices  $i, j, \dots$  run on the set  $\{1, \dots, n\}$ .

The set of functions  $g^{ij}(x, p)$  determine a symmetric contravariant of order 2 tensor field on  $\widetilde{T^*M}$ .  $H(x, p)$  is called regular if

$$\det(g^{ij}(x, p)) \neq 0 \text{ on } \widetilde{T^*M}. \quad (5.1.2)$$

Now, we can give

**Definition 5.1.1.** A Hamilton space is a pair  $H^n = (M, H(x, p))$  where  $M$  is a smooth real,  $n$ -dimensional manifold, and  $H$  is a function on  $T^*M$  having the properties

- 1°  $H : (x, p) \in T^*M \rightarrow H(x, p) \in R$  is a differentiable function on  $\widetilde{T^*M}$  and it is continuous on the null section of the natural projection  $\pi^*T^*M \rightarrow M$ .
- 2° The Hessian of  $H$  with respect to momenta  $p_i$  given by  $\|g^{ij}(x, p)\|$ , (5.1.1) is nondegenerate i.e. (5.1.2) is verified on  $\widetilde{T^*M}$ .
- 3° The  $d$ -tensor field  $g^{ij}(x, p)$  has constant signature on the manifold  $\widetilde{T^*M}$ .

One can say that the Hamilton space  $H^n = (M, H(x, p))$  has as fundamental function a differentiable regular Hamiltonian for which its *fundamental* or *metric* tensor  $g^{ij}(x, p)$  is nonsingular and has constant signature.

### Examples

- 1° If  $\mathcal{R}^n = (M, g_{ij}(x))$  is a Riemannian space then  $H^n = (M, H(x, p))$  with

$$H(x, p) = \frac{1}{mc} g^{ij}(x) p_i p_j \quad (5.1.3)$$

is an Hamilton space.

- 2° The Hamilton space of electrodynamics is defined by the fundamental function

$$H(x, p) = \frac{1}{mc} g^{ij}(x) p_i p_j - \frac{2e}{mc^2} A^i(x) p_i - \frac{e^2}{mc^2} A^i(x) A_i(x), \quad (5.1.4)$$

where  $m, c, e$  are the known physical constants,  $g_{ij}(x)$  is a pseudo-Riemannian tensor and  $A_i(x)$  are a  $d$ -covector (electromagnetic potentials) and  $A^i(x) = g^{ij}A_j$ .

*Remark 5.1.1.* The kinetic energy for a Riemannian metric  $\mathcal{R}^n = (M, g_{ij}(x))$  is given by

$$T(x, \dot{x}) = \frac{1}{2}g_{ij}(x)\dot{x}^i\dot{x}^j$$

and for the Hamilton space  $H^n = (M, H(x, p))$  with  $H(x, p) = g^{ij}(x)p_i p_j$  it is

$$T^*(x, p) = \frac{1}{2}g^{ij}p_i p_j.$$

Therefore, generally, for an Hamilton space  $H^n = (M, H(x, p))$  is convenient to introduce the energy given by the Hamiltonian

$$\mathcal{H}(x, p) = \frac{1}{2}H(x, p). \quad (5.1.5)$$

One obtain:

$$\dot{\partial}^i \dot{\partial}^j \mathcal{H} = g^{ij}(x, p). \quad (5.1.6)$$

Consider the Hamiltonian system  $(T^*M, \theta, \mathcal{H})$ , where  $\theta$  is the natural symplectic structure on  $T^*M$ ,  $\theta = dp_i \wedge dx^i$  (Ch. 4). The isomorphism  $S_\theta$ , defined by (4.1.7) can be used and Theorem 4.2 can be applied.

One obtain

**Theorem 5.1.1.** *For any Hamilton space  $H^n = (M, H(x, p))$  the following properties hold:*

1° *There exists a unique vector field  $X_H \in \mathcal{X}(T^*M)$  for which*

$$i_H \theta = -d\mathcal{H}. \quad (5.1.7)$$

2°  *$X_H$  is expressed by*

$$X_H = \frac{\partial \mathcal{H}}{\partial p_i} \frac{\partial}{\partial x^i} - \frac{\partial \mathcal{H}}{\partial x^i} \frac{\partial}{\partial p_i}. \quad (5.1.8)$$

3° *The integral curves of  $X_H$  are given by the Hamilton–Jacobi equations*

$$\frac{dx^i}{dt} = \frac{\partial \mathcal{H}}{\partial p_i}; \quad \frac{dp_i}{dt} = -\frac{\partial \mathcal{H}}{\partial x^i}. \quad (5.1.9)$$

Since

$$\frac{d\mathcal{H}}{dt} = \{\mathcal{H}, \mathcal{H}\} = 0,$$

it follows  $\frac{dH}{dt} = 0$ . Then: *The fundamental function*  $H(x, p)$  of a Hamilton space  $H^n$  is constant along the integral curves of the Hamilton - Jacobi equations (5.1.9). The Jacobi method of integration of (5.1.9) expound in chapter 4 is applicable and the variational problem, in §4.2, ch. 4, can be formulated again in the case of Hamiltonian systems  $(T^*M, \theta, \mathcal{H})$ .

## 5.2 Nonlinear connection of a Hamilton space

Let us consider a Hamilton space  $H^n \subset (M, H(x, p))$ . The theory of nonlinear connection on the manifold  $T^*M$ , in ch. 4, can be applied in the case of spaces  $H^n$ . We must determine a nonlinear connection  $N$  which depend only by the Hamilton space  $H^n$ , i.e.  $N$  must be canonical related to  $H^n$ , as in the case of canonical nonlinear connection from the Lagrange spaces. To do this, direct method is given by using the Legendre transformation, suggested by Mechanics. We will present, in the end of this chapter, this method.

Now, following a result of R. Miron we enounce without demonstration (which can be found in ch. 2, §2.3) the following result:

**Theorem 5.2.1.** 1° *The following set of functions*

$$N_{ij} = \frac{1}{4}\{g_{ij}, H\} - \frac{1}{4}\left(g_{ik}\frac{\partial^2 H}{\partial p_k \partial x^j} + g_{jk}\frac{\partial^2 H}{\partial p_k \partial x^i}\right) \quad (5.2.1)$$

*are the coefficients of a nonlinear connection of the Hamilton space*  $H^n = (M, H(x, p))$ .

2° *The nonlinear connection*  $N$  *with coefficients*  $N_{ij}$ , (5.2.1) *depends only on the fundamental function*  $H(x, p)$ .

The brackets  $\{, \}$  from (5.2.1) are the Poisson brackets  $(,)$ , §5.1.

Indeed, by a straightforward computation, it follows that, under a coordinate change  $(,)$ ,  $N_{ij}(x, p)$  from (5.2.1) obeys the rule of transformation of the coefficients of a nonlinear connection  $N$ . Evidently  $N_{ij}$  depend on the fundamental function  $H(x, p)$  only.  $N$ — will be called the canonical nonlinear connection of the Hamilton space  $H^n$ .

*Remark 5.2.2.* If the fundamental function  $H(x, p)$  of  $H^n$  is globally defined, on  $T^*M$  then the horizontal distribution determined by the canonical nonlinear connection  $N$  has the same property.

It is not difficult to prove:

**Proposition 5.2.1.** *The canonical nonlinear connection  $N$  has the properties*

$$\tau_{ij} = \frac{1}{2}(N_{ij} - N_{ji}) = 0 \quad (\text{it is symmetric}) \quad (5.2.2)$$

$$R_{ijk} + R_{jki} + R_{kij} = 0 \quad (5.2.3)$$

Taking into account that  $N$  is a horizontal distribution, we have the known direct decomposition:

$$T_u \widetilde{T^*M} = N_u \oplus V_u, \quad \forall u \in \widetilde{T^*M}, \quad (5.2.4)$$

it follows that  $(\delta_i, \dot{\partial}^i)$  is an adapted basis to the previous splitting, where

$$\delta_i = \partial_i - N_{ij} \dot{\partial}^j \quad (5.2.5)$$

and the dual basis  $(dx^i, \delta p_i)$  is:

$$\delta p_i = dp_i - N_{ij} dx^j. \quad (5.2.5')$$

Therefore we apply the theory to investigate the notion of metric  $N$ -linear connection determined only by the Hamilton space  $H^n$ .

### 5.3 The canonical metrical connection of Hamilton space $H^n$

Let us consider the  $N$ -linear connection  $D\Gamma(N) = (H_{jk}^i, C_i^{jk})$  for which  $N$  is the canonical nonlinear connection. By means of theory from chapter 4 we can prove:

**Theorem 5.3.1.** *In a Hamilton space  $H^n = (M, H(x, p))$  there exists a unique  $N$ -linear connection  $D\Gamma(N) = (H_{jk}^i, C_i^{jk})$  verifying the axioms:*

*$N$  is the canonical nonlinear connection.*

$\mathfrak{2}^\circ$  *The fundamental tensor  $g^{ij}$  is  $h$ -covariant constant*

$$g_{|k}^{ij} = \delta_k g^{ij} + g^{sj} H_{sk}^i + g^{is} H_{sk}^j = 0. \quad (5.3.1)$$

3° The fundamental tensor  $g^{ij}$  is  $\nu$ -covariant constant

$$g^{ij|k} = \dot{\partial}^k C^{ij} + g^{sj} C_s^{ik} + g^{is} C_s^{jk} = 0. \quad (5.3.2)$$

4  $D\Gamma(N)$  is  $h$ -torsion free:

$$T_{jk}^i = H_{jk}^i - H_{kj}^i = 0 \quad (5.3.3)$$

5°  $D\Gamma(N)$  is  $\nu$ -torsion free:

$$S_i^{jk} = C_i^{jk} - C_i^{kj} = 0.$$

2) The  $N$ -connection  $D\Gamma(N)$  which verify the previous axioms has the coefficients  $H_{jk}^i$  and  $C_i^{jk}$  given by the coefficients  $H_{jk}^i$  and  $C_i^{jk}$  given by the following generalized Christoffel symbols:

$$\begin{aligned} H_{jk}^i &= \frac{1}{2} g^{is} (\delta_j g_{sk} + \delta_k g_{js} - \delta_s g_{jk}), \\ C_i^{jk} &= -\frac{1}{2} g_{is} (\dot{\partial}^j g^{sk} + \dot{\partial}^k g^{js} - \partial^s g^{jk}). \end{aligned} \quad (5.3.4)$$

Clearly  $CG(N)$  with the coefficients (5.3.4) is determined only by means of the Hamilton space  $H^n$ . It is called canonical metrical  $N$ -linear connection.

Now, applying theorems from ch. 4 we have:

**Theorem 5.3.2.** *With respect to  $CG(N)$  we have the Ricci identities:*

$$\begin{aligned} X_{|j|k}^i - X_{|k|j}^i &= X^m R_{m\ jk}^i - X^i |^m R_{mjk}, \\ X_{|j}^i |^k - X^i |^k |_j &= X^m P_{m\ j}^i |^k - X_{|m}^i C_j^{mk} - X^i |^m P_{mj}^k \\ X^i |^j |^k - X^i |^k |^j &= X^m S_m^{ijk}, \end{aligned} \quad (5.3.5)$$

where the  $d$ -tensors of torsion  $R_{ijk}$ ,  $P_{jk}^i$  and  $d$  tensors of curvature  $R_{j\ kh}^i$ ,  $P_{i\ h}^j |^k$ ,  $S_i^{jkh}$  are expressed in chapter 4.

The structure equations, parallelism, autoparallel curves of the Hamilton spaces are studied exactly as in chapter 4.

**Example.** The Hamilton space of electrodynamics  $H^n = (M, H(x, p))$  where the fundamental function is expressed in (5.1.4). The funda-



mental tensor is  $\frac{1}{mc}g^{ij}(x)$ . The canonical nonlinear connection has the coefficients

$$N_{ij} = \gamma_{ij}^h(x)p_h + \frac{e}{c}(A_{i|k} + A_{k|i}) \quad (5.3.6)$$

where  $\gamma_{ij}^h(x)$  are the Christoffel symbols of the covariant tensor of metric tensor  $g^{ij}(x)$ . Evidently  $A_{i|k} = \partial_k A_i - A_s \gamma_{ik}^s$ .

Remarking that  $\frac{\delta g_{ij}}{\delta x^k} = \frac{\partial g_{ij}}{\partial x^k}$  we deduce that: the coefficients of canonical metrical  $N$ -connection  $H\Gamma(N)$  are:

$$H_{jk}^i(x) = \gamma_{jk}^i(x), C_i^{jk} = 0.$$

These geometrical object fields  $H, g_{ij}, H_{jk}^i, C_i^{jk}$  allow to develop the geometry of the Hamilton spaces of electrodynamics.

## 5.4 Generalized Hamilton Spaces $GH^n$

A straightforward generalization of the notion of Hamilton space is that of generalized Hamilton space.

**Definition 5.4.1.** A generalized Hamilton space is a pair  $GH^n = (M, g^{ij}(x, p))$  where  $M$  is a smooth real  $n$ -dimensional manifold and  $g^{ij}(x, p)$  is a  $d$ -tensor field on  $\widetilde{T^*M}$  of type  $(0, 2)$  symmetric, non-degenerate and of constant signature.

The tensor  $g^{ij}(x, p)$  is called fundamental. If  $M$  a paracompact on  $T^*M$  there exist the tensors  $g^{ij}(x, p)$  which determine a generalized Hamilton space.

From the definition 5.1.1 it follows that: Any Hamilton space  $H^n = (M, H)$  is a generalized Hamilton space.

The contrary affirmation is not true. Indeed, if  $\overset{\circ}{g}{}^{ij}(x)$  is a Riemannian tensor metric on  $M$ , then

$$g^{ij}(x, p) = e^{-2\sigma(x, p)} \overset{\circ}{g}{}^{ij}(x), \sigma \in \mathcal{F}(T^*M)$$

determine a generalized Hamilton space and  $g^{ij}(x, p)$  does not the fundamental tensor of an Hamilton space.

So, it is legitimate the following definition:

**Definition 5.4.2.** A generalized Hamilton space  $GH^n = (M, g^{ij}(x, p))$  is called reducible to a Hamilton space if there exists a Hamilton function  $H(x, p)$  on  $\widetilde{T^*M}$  such that

$$g^{ij}(x, p) = \frac{1}{2} \dot{\partial}^i \dot{\partial}^j H. \quad (5.4.1)$$

Let us consider the Cartan tensor:

$$C^{ijk} = \frac{1}{2} \dot{\partial}^i g^{jk}. \quad (5.4.2)$$

In a similar manner with the case of generalized Lagrange space (ch. 2, §2.8) we have:

**Proposition 5.4.2.** A necessary condition that a generalized Hamilton space  $GH^n = (M, g^{ij}(x, p))$  be reducible to a Hamilton one is that the Cartan tensor  $C^{ijk}(x, p)$  be totally symmetric.

There exists a particular case when the previous condition is sufficient, too.

**Theorem 5.4.1.** A generalized Hamilton space  $GH^n = (M, g^{ij}(x, p))$  for which the fundamental tensor  $g^{ij}(x, p)$  is 0-homogeneous is reducible to a Hamilton space, if and only if the Cartan tensor  $C^{ijk}(x, p)$  is totally symmetric.

Indeed, in this case  $H(x, p) = g^{ij}(x, p)p_i p_j$  is a solution of the equation (5.4.1).

Let  $g_{ij}(x, p)$  be the covariant of fundamental tensor  $g^{ij}(x, p)$ , then the following tensor field

$$C_i^{jk} = -\frac{1}{2} g_{is} (\dot{\partial}^j g^{sk} + \dot{\partial}^k g^{js} - \dot{\partial}^s g^{jk}) \quad (5.4.3)$$

determine the coefficients of a  $\nu$ -covariant derivative, which is metrical, i.e.

$$g^{ij|k} = \dot{\partial}^k g^{ij} + C_s^{ik} g^{sj} + C_s^{jk} g^{is} = 0. \quad (5.4.4)$$

The proof is very simple.

We use this  $\nu$ -derivation in the theory of canonical metrical connection of the spaces  $GH^n$ .

In general, we cannot determine a nonlinear connection from the fundamental tensor  $g^{ij}$  of  $GH^n$ . Therefore we study the geometry of

spaces  $GH^n$  when a nonlinear connection  $N$  is a priori given. In this case we can apply the methods used in the construction of geometry of spaces  $H^n$ .

Finally we remark that the class of spaces  $GH^n$  include the class of spaces  $H^n$ :

$$\{GH^n\} \supset \{H^n\}. \quad (5.4.5)$$

## 5.5 The almost Kählerian model of a Hamilton space

Let  $H^n = (M, H(x, p))$  be a Hamilton space and  $g^{ij}(x, p)$  its fundamental tensor.

The canonical nonlinear connection  $N$  has the coefficients (5.2.1). The adapted basis to the distributions  $N$  and  $V$  is  $\left(\frac{\delta}{\delta x_i} = \delta_i + N_{ij}\dot{\partial}^j, \dot{\partial}^i\right)$  and its dual basis  $(dx^i, \delta p_i = dp_i - N_{ji}dx^j)$ .

Thus, the following tensor on the cotangent manifold  $\widetilde{T^*M} = T^*M \setminus \{0\}$  can be considered:

$$\mathbb{G} = g_{ij}(x, p)dx^i \otimes dx^j + g^{ij}\delta p_i \otimes \delta p_j. \quad (5.5.1)$$

$\mathbb{G}$  determine a pseudo-Riemannian structure on  $\widetilde{T^*M}$ . If the fundamental tensor  $g^{ij}(x, p)$  is positive defined, then  $\mathbb{G}$  is a Riemannian structure on  $T^*M$ .  $\mathbb{G}$  is called the  $N$ -lift of the fundamental tensor  $g^{ij}$ . Clearly,  $\mathbb{G}$  is determined by Hamilton space  $H^N$ , only.

Some properties of  $\mathbb{G}$ :

- 1°  $\mathbb{G}$  is uniquely determined by  $g^{ij}$  and  $N_{ij}$ .
- 2° The distributions  $N$  and  $V$  are orthogonal.

Taking into account the mapping  $\check{\mathbb{F}}: \mathcal{X}(\widetilde{T^*M}) \rightarrow \mathcal{X}(T^*M)$  defined in (4.3.10) or, equivalently by:

$$\check{\mathbb{F}} = -g_{ij}\dot{\partial}^i \otimes dx^j + g^{ij}\delta_i \otimes \delta p_j, \quad (5.5.2)$$

one obtain:

**Theorem 5.5.1.** 1°  $\check{\mathbb{F}}$  is globally defined on the manifold  $\widetilde{T^*M}$ .  
2°  $\check{\mathbb{F}}$  is an almost complex structure on  $\widetilde{T^*M}$ :

$$\check{\mathbb{F}} \cdot \check{\mathbb{F}} = -I. \quad (5.5.3)$$

3°  $\check{\mathbb{F}}$  depends on the Hamilton spaces  $H^n$  only.

Finally, one obtain a particular form of the Theorem 4.3.2:

**Theorem 5.5.2.** *The following properties hold:*

- 1° *The pair  $(\mathbb{G}, \check{\mathbb{F}})$  is an almost Hermitian structure on the manifold  $\widetilde{T^*M}$ .*
- 2° *The structure  $(\mathbb{G}, \mathbb{F})$  is determined only by the fundamental function  $H(x, p)$  of the Hamilton space  $H^n$ .*
- 3° *The associated almost symplectic structure to  $(\mathbb{G}, \check{\mathbb{F}})$  is the canonical symplectic structure  $\theta = dp_i \wedge dx^i = \delta p_i \wedge dx^i$ .*
- 4° *The space  $(\widetilde{T^*M}, \mathbb{G}, \check{\mathbb{F}})$  is almost Kählerian.*

The proof is similar with that for Lagrange space (cf. Ch. 3).

The space  $\mathcal{H}^{2n} = (\widetilde{T^*M}, \mathbb{G}, \check{\mathbb{F}})$  is called the almost Kählerian model of the Hamilton space  $H^n$ . By means of  $\mathcal{H}^{2n}$  we can realize the study of gravitational and electromagnetic fields [176], [182], [183], [185], on  $\widetilde{T^*M}$ .

## Chapter 6

### Cartan spaces

A particular class of Hamilton space is given by the class of Cartan spaces. It is formed by the spaces  $H^n = (M, H(x, p))$  for which the fundamental function  $H$  is 2-homogeneous with respect to momenta  $p_i$ . It is remarkable that these spaces appear as dual of the Finsler spaces, via Legendre transformations. Using this duality, several important results in Cartan spaces can be obtained: the canonical nonlinear connection, the canonical metrical connection etc. Therefore, the theory of Cartan spaces has the same symmetry and beauty like Finsler geometry. Moreover, it gives a geometrical framework for the Hamiltonian Mechanics or Physics fields.

The modern formulation of the notion of Cartan space is due of R. Miron, but its geometry is based on the investigations of E. Cartan, A. Kawaguchi, H. Rund, R. Miron, D. Hrimiuc, and H. Shimada, P.L. Antonelli, S. Vacaru et. al. This concept is different from the notion of areal space defined by E. Cartan.

In the final part of this chapter we shortly present the notion of duality between Lagrange and Hamilton spaces.

#### 6.1 Notion of Cartan space

**Definition 6.1.1.** A Cartan space is a pair  $C^n = (M, K(x, p))$  where  $M$  is a real  $n$ -dimensional smooth manifold and  $K : T^*M \rightarrow R$  is a scalar function which satisfies the following axioms:

1.  $K$  is differentiable on  $\widetilde{T^*M}$  and continuous on the null section of  $\pi^* : T^*M \rightarrow M$ .
2.  $K$  is positive on the manifold  $T^*M$ .
3.  $K$  is positive 1-homogeneous with respect to the momenta  $p_i$ .
4. The Hessian of  $K^2$  having the components

$$g^{ij}(x, p) = \frac{1}{2} \dot{\partial}^i \dot{\partial}^j K^2 \quad (6.1.1)$$

is positive defined on the manifold  $\widetilde{T^*M}$ .

It follows that  $g^{ij}(x, p)$  is a symmetric and nonsingular  $d$ -tensor field of type  $(0, 2)$ .

So, we have

$$\text{rank} \|g^{ij}(y, p)\| = n \text{ on } \widetilde{T^*M}. \quad (6.1.2)$$

The functions  $g^{ij}(x, p)$  are 0-homogeneous with respect to momenta  $p_i$ .

For a Cartan space  $\mathcal{C}^n = (M, K(x, p))$  the function  $K$  is called fundamental function and  $g^{ij}$  the fundamental or metric tensor.

If the base  $M$  is paracompact, then on the manifold  $T^*M$  there exists real function  $K$  such that the pair  $(M, K)$  is a Cartan space.

Indeed, on  $M$  there exists a Riemann structure  $g_{ij}(x)$ ,  $x \in M$ . Then

$$K(x, p) = \{g^{ij}(x)p_i p_j\}^{1/2} \quad (6.1.3)$$

determine a Cartan space.

Other examples are given by

$$K = \alpha^* + \beta^* \quad (6.1.4)$$

$$K = \frac{(\alpha^*)^2}{\beta} \quad (6.1.5)$$

where

$$\alpha^* = \{g^{ij}(x)p_i p_j\}^{1/2}, \quad \beta^* = b^i(x)p_i. \quad (6.1.6)$$

$K$  from (6.1.4) is called a *Randers metric* and  $K$  from (6.1.5) is called a *Kropina metric*.

A first and immediate result:

**Theorem 6.1.1.** *If  $\mathcal{C}^n = (M, K)$  is a Cartan space then the pair  $H_{\mathcal{C}}^n = (M, K^2)$  is an Hamilton space.*

$H_{\mathcal{C}}^n$  is called the associate Hamilton space with  $\mathcal{C}^n$ .

This is the reason that the geometry of Cartan space include the geometry of associate Hamilton space. So, we have the sequence of inclusions

$$\{\mathcal{R}^n\} \subset \{\mathcal{C}^n\} \subset \{H^n\} \subset \{GH^n\} \quad (6.1.7)$$

where  $\{\mathcal{R}^n\}$  is the class of Riemann spaces  $\mathcal{R}^n = (M, g^{ij}(x))$  which give the Cartan spaces with the metric (6.1.3).

Now we can apply the theory from previous chapter.  
The canonical symplectic structure  $\theta$  on  $T^*M$ :

$$\theta = dp_i \wedge dx^i \quad (6.1.8)$$

and  $\mathcal{C}^n$  determine the Hamiltonian system  $(T^*M, \theta, K^2)$ . Then, setting

$$\mathcal{K} = \frac{1}{2}K^2, \quad (6.1.9)$$

we have:

**Theorem 6.1.2.** *For any Cartan space  $\mathcal{C}^n = (M, K(x, p))$ , the following properties hold:*

1° *There exists a unique vector field  $X_{K^2}$  on  $\widetilde{T^*M}$  for which*

$$i_{X_K} \theta = -d\mathcal{K}. \quad (6.1.10)$$

2° *The vector field  $X_K$  is expressed by*

$$X_K = \frac{\partial \mathcal{K}}{\partial p_i} \frac{\partial}{\partial x^i} - \frac{\partial \mathcal{K}}{\partial x^i} \frac{\partial}{\partial p_i}. \quad (6.1.11)$$

3° *The integral curves of the vector field  $X_K$  are given by the Hamilton–Jacobi equations:*

$$\frac{dx^i}{dt} = \frac{\partial \mathcal{K}}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial \mathcal{K}}{\partial x^i}. \quad (6.1.12)$$

One deduce  $\frac{d\mathcal{K}}{dt} = \{\mathcal{K}, \mathcal{K}\} = 0$ . So:

**Proposition 6.1.1.** *The fundamental function  $\mathcal{K} = \frac{1}{2}K^2$  of a Cartan space is constant along the integral curves of the Hamilton–Jacobi equations (6.1.12).*

The Jacobi method of integration of (6.1.12) can be applied.

The Hamilton–Jacobi equations of a Cartan space  $\mathcal{C}^n$  are fundamental for the geometry of  $\mathcal{C}^n$ . Therefore, the integral curves of the system of differential equations (6.1.12) are called the geodesics of Cartan space  $\mathcal{C}^n$ .

Other properties of the space  $\mathcal{C}^n$ :

1°  $p^i = \frac{1}{2} \dot{\partial}^i K^2$  is a 1-homogeneous  $d$ -vector field.

$2^\circ g^{ij} = \dot{\partial}^j p^i = \frac{1}{2} \dot{\partial}^i \dot{\partial}^j K^2$  is 0-homogeneous tensor field (the fundamental tensor of  $\mathcal{C}^n$ ).

$3^\circ C^{ijh} = -\frac{1}{4} \dot{\partial}^i \dot{\partial}^j \dot{\partial}^h K^2$  is  $(-1)$ -homogeneous with respect to  $p_i$ .

**Proposition 6.1.2.** *We have the following identities:*

$$p^i = g^{ij} p_j, \quad p_i = g_{ij} p^j, \quad (6.1.13)$$

$$K^2 = g^{ij} p_i p_j = p_i p^i, \quad (6.1.14)$$

$$C^{ijh} p_h = C^{ihj} p_h = C^{hij} p_h = 0. \quad (6.1.15)$$

**Proposition 6.1.3.** *The Cartan space  $\mathcal{C}^n = (M, K(x, p))$  is Riemannian if and only if the  $d$ -tensor  $C^{ijh}$  vanishes.*

Consider  $d$ -tensor:

$$C_i^{jh} = -\frac{1}{2} g_{is} \dot{\partial}^s g^{jh} = g_{is} C^{sjh}. \quad (6.1.16)$$

Thus  $C_i^{jh}$  are the coefficients of a  $\nu$ -metric covariant derivation:

$$\begin{aligned} g^{ij}|^h &= 0, \quad S_i^{jh} = 0 \\ p_i|^k &= \delta_i^k. \end{aligned} \quad (6.1.17)$$

## 6.2 Canonical nonlinear connection of $\mathcal{C}^n$

The canonical nonlinear connection of a Cartan space  $\mathcal{C}^n = (M, K)$  is the canonical nonlinear connection of the associate Hamilton space  $H_{\mathcal{C}}^n = (M, K^2)$ . Its coefficients  $N_{ij}$  are given by (5.2.1).

Setting

$$\gamma_{jh}^i = \frac{1}{2} g^{is} (\partial_j g_{sh} + \partial_h g_{js} - \partial_s g_{jh}) \quad (6.2.1)$$

for the Christoffel symbols of the covariant of fundamental tensor of  $\mathcal{C}^n$  and using the notations

$$\gamma_{jh}^0 = \gamma_{jh}^i p_i, \quad \gamma_{j0}^0 = \gamma_{jh}^i p_i p^h, \quad (6.2.2)$$

we obtain:



**Theorem 6.2.1 (Miron).** *The canonical nonlinear connection of the Cartan space  $\mathcal{C}^n = (M, K(x, p))$  has the coefficients*

$$N_{ij} = \gamma_{ij}^0 - \frac{1}{2} \gamma_{h0}^0 \dot{\partial}^h g_{ij}. \quad (6.2.3)$$

The proof is based on the formula (5.2.1).  
Evidently, the canonical nonlinear connection is symmetric

$$N_{ij} = N_{ji}. \quad (6.2.4)$$

Let us consider the adapted basis  $(\delta_i, \dot{\partial}^i)$  to the distributions  $N$  (determined by canonical nonlinear connection) and  $V$  (vertical distribution). We have

$$\delta_i = \partial_i + N_{ij} \dot{\partial}^j. \quad (6.2.5)$$

The adapted dual basis  $(dx^i, \delta p_i)$  has the forms  $\delta p_i$ :

$$\delta p_i = dp_i - N_{ij} dx^j. \quad (6.2.5')$$

The  $d$ -tensor of integrability of horizontal distribution  $N$  is

$$R_{ijh} = \delta_h N_{ji} - \delta_j N_{hi}. \quad (6.2.6)$$

By a direct calculus we obtain

$$R_{ijh} + R_{jki} + R_{kij} = 0. \quad (6.2.7)$$

**Proposition 6.2.4.** *The horizontal distribution  $N$  determined by the canonical nonlinear connection of a Cartan space  $\mathcal{C}^n$  is integrable if and only if the  $d$ -tensor field  $R_{ijk}$  vanishes.*

**Proposition 6.2.5.** *The canonical nonlinear connection of a Cartan space  $\mathcal{C}^n(M, K(x, p))$  depends only on the fundamental function  $K(x, p)$ .*

### 6.3 Canonical metrical connection of $\mathcal{C}^n$

Consider the canonical metrical  $N$ -linear connection of the Hamilton space  $H_e^N = (M, K^2)$ . It is the canonical metrical  $N$ -linear connection of the Cartan space  $\mathcal{C}^n = (M, K)$ . It is denoted by  $CG(N) = (H_{jk}^i, C_i^{jk})$  and shortly is named canonical metrical connection of  $\mathcal{C}^n$ .

Then, theorem 5.3.1. implies:

**Theorem 6.3.1.** 1) In a Cartan space  $\mathcal{C}^n = (M, K(x, p))$  there exists a unique  $N$ -linear connection  $CG(N) = (H_{jk}^i, C_i^{jk})$  verifying the axioms:

1°  $N$  is the canonical nonlinear connection of  $\mathcal{C}^n$ .

2°  $CG(N)$  is  $h$ -metrical

$$g_{|h}^{ij} = 0. \quad (6.3.1)$$

3°  $CG(N)$  is  $v$ -metrical:

$$g^{ij|h} = 0. \quad (6.3.2)$$

4°  $CG(N)$  is  $h$ -torsion free:  $T_{jh}^i = H_{jh}^i - H_{jh}^i = 0$ .

5°  $CG(N)$  is  $v$ -torsion free:  $S_i^{jh} = C_i^{jh} - C_i^{hj} = 0$ .

2) The connection  $CG(N)$  has the coefficients given by the generalized Christoffel symbols:

$$H_{jh}^i = \frac{1}{2} g^{is} (\delta_j g_{sh} + \delta_h g_{js} - \delta_s g_{jh}) \quad (6.3.3)$$

$$C_i^{jh} = -\frac{1}{2} g_{is} (\dot{\partial}^j g^{sh} + \dot{\partial}^h g^{js} - \dot{\partial}^s g^{jh}).$$

3)  $CG(N)$  depends only on the fundamental function  $K$  of the Cartan space  $\mathcal{C}^n$ .

The connection  $CG(N)$  is called canonical metrical connection of Cartan space  $\mathcal{C}^n$ .

We denote  $CG = (N_{ij}, H_{jk}^i, C_i^{jk})$ .

Evidently, the  $d$ -tensor  $C_i^{jh}$  has the properties (6.1.15) and the coefficients  $CG$  have 1, 0, -1 homogeneity degrees.

**Proposition 6.3.6.** The canonical metrical connections has the tensor of deflection:

$$\Delta_{ij} = p_{i|j} = 0, \quad \delta_j^i = p_j|^i = \delta_j^i. \quad (6.3.4)$$

But, the previous result allows to give a characterization of  $CG$  by a system of axioms of Matsumoto type:

**Theorem 6.3.2.** 1° For any space  $\mathcal{C}^n = (M, K(x, p))$  there exists a unique linear connection  $CG = (N_{ij}, H_{jk}^i, C_i^{jk})$  on the manifold  $\widetilde{TM}$  verifying the following axioms:

$$1'. \Delta_{ij} = 0; \quad 2'. g_{|h}^{ij} = 0; \quad 3'. g^{ij|h} = 0; \quad 4'. T_{jh}^i = 0; \quad 5'. S_i^{jh} = 0.$$

2° The previous metrical connection is exactly the canonical metrical connection  $CG$ .

The following properties of  $CG(N)$  are immediately

$$\begin{aligned} 1^\circ K_{|h} &= 0, K|h = \frac{p^h}{K}; \\ 2^\circ K_{|h}^2 &= 2p^h; \\ 3^\circ p_{i|j} &= 0, p_i|^j = \delta_i^j; \\ 4^\circ p_{|j}^i &= 0, p^i|^j = g^{ij}. \end{aligned}$$

And the  $d$ -tensors of torsion of  $CG(N)$  are

$$R_{ijh} = \delta_h N_{ij} - \delta_j N_{ih}, C_i^{jh}, T_{jh}^i = 0, S_i^{jh} = 0, P_{jh}^i = H_{jh}^i - \dot{\partial}^i N_{jh}. \quad (6.3.5)$$

Of course, we have

$$R_{ijh} = -R_{ihj}, P_{jh}^i = P_{hj}^i, C_i^{jh} = C_i^{hj}. \quad (6.3.6)$$

**Proposition 6.3.7.** The  $d$ -tensor of curvature  $S_k^{ijh}$  is given by:

$$S_k^{ijh} = C_k^{mh} C_m^{ij} - C_k^{mj} C_m^{ih}. \quad (6.3.7)$$

Denoting by

$$R_{kh}^{ij} = g^{is} R_{s kh}^j, \text{ etc.}$$

and applying the Ricci identities (5.3.5), one obtains:

**Theorem 6.3.3.** The canonical metrical connection  $CG(N)$  of the Cartan space  $\mathcal{C}^n$  satisfies the identities:

$$R_{kh}^{ij} + R_{kh}^{ji} = 0, P_k^{ijh} + P_k^{jih} = 0, S^{ijkh} + S^{jikh} = 0. \quad (6.3.8)$$

$$R_i^o{}_{jk} + R_{ijk} = 0, P_i^o{}_{jk} + P_{ij}^k = 0, S_i^{ojk} = 0. \quad (6.3.9)$$

$$R_{ojk} = 0, P_{oj}^k = 0. \quad (6.3.10)$$

Of course, the index “ $o$ ” means the contraction by  $p_i$  or  $p^i$ .

The 1-form connections of  $CG(N)$  are:

$$\omega_j^i = H_{jh}^i dx^h + C_j^{ih} \delta p_h. \quad (6.3.11)$$

Taking into account (6.2.5'), one obtains

**Theorem 6.3.4.** *The structure equations of the canonical metrical connection  $CG(N)$  of the Cartan space  $\mathcal{C}^n = (M, K(x, p))$  are*

$$\begin{aligned} d(dx^i) - dx^m \wedge \omega_m^i &= -\Omega^i; \\ d(\delta p_i) + \delta p_m \wedge \omega_i^m &= -\overset{\circ}{\Omega}_i; \\ d\omega_j^i - \omega_j^m \wedge \omega_m^i &= -\Omega_j^i, \end{aligned} \quad (6.3.12)$$

$\Omega^i, \overset{\circ}{\Omega}_i$  being the 2-forms of torsion:

$$\begin{aligned} \Omega^i &= C_j^{ik} dx^j \wedge \delta p_k \\ \overset{\circ}{\Omega}_i &= \frac{1}{2} R_{ijk} dx^j \wedge dx^k + P_{ij}^k dx^j \wedge \delta p_k \end{aligned} \quad (6.3.13)$$

and  $\Omega_j^i$  is 2-form of curvature:

$$\Omega_j^i = \frac{1}{2} R_{jkm}^i dx^k \wedge dx^m + P_{jk}^i dx^k \wedge \delta p_m + \frac{1}{2} S_j^{ikm} \delta p_k \wedge \delta p_m. \quad (6.3.14)$$

Applying Proposition 4.4.2, we determine the Bianchi identities of  $CG(N)$ .

Now, we can develop the geometry of the associated Hamilton space  $H_{\mathcal{C}}^n = (M, K^2(x, p))$ . Also, in the case of Cartan space, the geometrical model  $\mathcal{H}^{2n} = (\widetilde{T^*M}, \mathbb{G}, \mathbb{F})$  is an almost Kählerian one.

Before finish this chapter is opportune to say some words on the Legendre transformation.

## 6.4 The duality between Lagrange and Hamilton spaces

The duality, via Legendre transformation, between Lagrange and Hamilton space was formulated by R. Miron in the papers and it was developed by D. Hrimiuc, P.L. Antonelli, D. Bao, et al. Of course, it was suggested by Theoretical Mechanics.

The theory of Legendre duality is presented here follows the Chapter 7 of the book [174].

Let  $L$  be a regular Lagrangian,  $\mathcal{L} = \frac{1}{2}L$  on a domain  $\mathcal{D} \subset TM$  and let  $H$  be a regular Hamiltonian  $\mathcal{H} = \frac{1}{2}H$ , on a domain  $\mathcal{D}^* \subset T^*M$ .

Hence for  $\dot{\partial}_i = \frac{\partial}{\partial y^i}$ ,  $\dot{\partial}^i = \frac{\partial}{\partial p_i}$  the matrices with entries:

$$\begin{aligned} g_{ij}(x, y) &= \dot{\partial}_i \dot{\partial}_j \mathcal{L}(x, y), \\ g^{*ij}(x, p) &= \dot{\partial}^i \dot{\partial}^j \mathcal{H}(x, p) \end{aligned} \quad (6.4.1)$$

are nondegenerate on  $\mathcal{D}$  and on  $\mathcal{D}^*$ ,  $(i, j, \dots = 1, 2, \dots, n)$ .

Since  $\mathcal{L} \in \mathcal{F}(\mathcal{D})$  is a differentiable map consider the *fiber derivative* of  $\mathcal{L}$  locally given by

$$\varphi(x, y) = (x^i, \dot{\partial}_j \mathcal{L}(x, y)) \quad (6.4.2)$$

which is called the *Legendre transformation*. It is easy to see that:  $\mathcal{L}$  is a regular Lagrangian if and only if  $\varphi$  is a local diffeomorphism, [174]. In the same manner, for  $\mathcal{H} \in \mathcal{F}(\mathcal{D}^*)$ , the fiber derivative is locally given by

$$\psi(x, y) = (x^i, \dot{\partial}^i \mathcal{H}(x, y)), \quad (6.4.3)$$

which is a local diffeomorphism if and only if  $\mathcal{H}$  is regular.

Now, let us consider  $\frac{1}{2}L(x, y) = \mathcal{L}(x, y)$  the fundamental function of a Lagrange space. Then  $\varphi$  defined by (6.4.2) is a diffeomorphism between two open set  $U \subset \mathcal{D}$  and  $U^* \subset \mathcal{D}^*$ . In this case, we can define

$$\mathcal{H}^*(x, p) = p_i y^i - \mathcal{L}(x, y), \quad (6.4.4)$$

where  $y = (y^i)$  is solution of the equation

$$p_i = \dot{\partial}_i \mathcal{L}(x, y). \quad (6.4.4')$$

*Remark 6.4.1.* In the theory of Lagrange space the function  $\mathcal{E}(x, p) = y^i \dot{\partial}_i \mathcal{L} - \mathcal{L}(x, y)$  is the energy of Lagrange space  $L^n$ .

It follows without difficulties that  $H^{*n} = (M, H^*(x, p))$ ,  $H^* = 2\mathcal{H}^*$  is a Hamilton space. Its fundamental tensor  $g^{*ij}(x, p)$  is given by  $g^{ij}(x, \varphi^{-1}(x, p))$ .

We set  $H^{*n} = \text{Leg } L^n$  and say that  $H^{*n}$  is the dual of  $L^n$  via Legendre transformation determined by  $L^n$ .

*Remark 6.4.2.* The mapping  $Leg$  was used in chapter 2 to transform the Euler–Lagrange equations (ch. 2, part I) into the Hamilton–Jacobi equations (ch. 2, part I).

Analogously, for a Hamilton space  $H^n = (M, H(x, p))$ ,  $\mathcal{H} = \frac{1}{2}H$ , consider the function

$$\mathcal{L}^*(x, y) = p_i y^i - \mathcal{H}(x, p),$$

where  $p = (p_i)$  is the solution of the equation (6.4.4). Thus  $L^{*n} = (M, L^*(x, y))$ ,  $L^* = 2\mathcal{L}^*$ , is a Lagrange space, dual, via Legendre transformation of the Hamilton space  $H^n$ . So,  $L^{*n} = Leg H^n$ .

One proves:  $Leg(Leg L^n) = L^n$ ,  $Leg(Leg H^n) = H^n$ .

The diffeomorphisms  $\varphi$  and  $\psi$  have the property  $\varphi = \psi^{-1}$ . And they transform the fundamental object fields from  $L^n$  in the fundamental object fields of  $H^{*n} = Leg L^n$ , and conversely. For instance, the Euler–Lagrange equation of  $L^n$  are transformed in the Hamilton–Jacobi equations of  $H^n$ . The canonical nonlinear connection of  $L^n$  is transformed in the canonical nonlinear connection of  $H^{*n}$  etc.

### Examples.

1° The Lagrange space of Electrodynamics  $L_0^n = (M, L_0(x, y))$ :

$$L_0(x, y) = mc\gamma_{ij}(x)y^i y^j + \frac{2e}{m}A_i(x)y^i$$

where  $\gamma_{ij}(x)$  is a pseudo-Riemannian metric, has the Legendre transformation:

$$\varphi : x^i = x^i, p_i = \frac{1}{2} \frac{\partial L_0}{\partial y^i} = mc\gamma_{ij}(x)y^j + \frac{e}{m}A_i(x)$$

and

$$\varphi^{-1} = \psi : x^i = x^i, y^i = \frac{1}{mc} \gamma^{ij}(x) \left( p_j - \frac{e}{m} A_j(x) \right).$$

The Hamilton space  $H_0^{*n} = Leg L_0$  has the fundamental function:

$$H_0^* = \frac{1}{mc} \gamma^{ij}(x) p_i p_j - \frac{2e}{mc^2} A^i(x) p_i + \frac{e^2}{mc^3} A^i(x) A_j(x). \quad (6.4.5)$$

2° The Lagrange space of Electrodynamics  $L^n = (M, L(x, y))$  in which  $L = L_0 + U(x)$ , ((2.1.4, Ch. 2)), i.e.

$$L(x, y) = mc\gamma_{ij}(x)y^i y^j + \frac{2e}{m}A_i(x)y^i + U(x),$$

has the Legendre transformation:

$$\varphi_i : x^i = x^i, p_i = mc\gamma_{ij}(x)y^j + \frac{e}{m}A_i(x)$$

$$\varphi^{-1} = \psi : x^i = x^i, y^i = \frac{1}{mc}\gamma^{ij}(x)\left(p_j - \frac{e}{m}A_j\right).$$

And  $H^{*n} = \text{Leg } L^n$  has the fundamental function  $H^*(x, p)$  given by

$$\begin{aligned} H^*(x, p) &= \frac{1}{c(m + \frac{U}{c^2})}\gamma^{ij}(x)p_i p_j - \frac{2e}{c^2(m + \frac{U}{c^2})}p_i A^i(x) + \\ &+ \frac{e^2}{c^3(m + \frac{U}{c^2})}A_i(x)A^i(x) - \frac{1}{c}\frac{U(x)}{c}. \end{aligned}$$

3° The class of Finsler space  $\{F^n\}$  is inclosed in the calss of Lagrange space  $\{L^n\}$ , that is  $\{F^n\} \subset \{L^n\}$ , one can consider the restriction of the Legendre transformation  $\text{Leg } L^n$  to the class of Finsler spaces.

In this case, the mapping  $\varphi : (x, y) \in D \rightarrow (x, p) \in D^*$  is given by

$$\varphi(x, y) = (x, p) \text{ with}$$

$$p_i = \frac{1}{2}\dot{\partial}^i F^2 \quad (6.4.6)$$

and one obtains:  $\text{Leg}(F^n) = \mathcal{C}^{*n}$ ,  $\mathcal{C}^{*n} = (M, K^*(x, p))$  with

$$K^{*2} = 2p_i y^i - F^2(x, y) = y^i \dot{\partial}_i F^2(x, y) = F^2(x, y)$$

and  $y^i$  is solution of the equation (6.4.6).

**Theorem 6.4.1.** *The dual, via Legendre transformation, of a Finsler space  $F^n = (M, F(x, y))$  is a Cartan space  $\mathcal{C}^{*n} = (M, F(x, \varphi^{-1}(x, p)))$ .*





**Part II**  
**Lagrangian and Hamiltonian Spaces of**  
**higher order**

In this part of the book we study, the notions of Lagrange and Hamilton spaces of order  $k$ . They were introduced and investigated by the author [161]. Without explicitly formulating a clear definition of these spaces, major contributions to the edifice of these geometries have been done by M. Crampin and colab. [64], M. de Leone and colab. [138], A. Kawaguchi [120], I. Vaisman [254] etc.

For details, we refer to the books: *The Geometry of Higher Order Lagrange Spaces. Applications to Mechanics and Physics*, Kluwer Acad. Publ. FTPH, 82, 1997, [161]; *The Geometry of Higher-Order Finsler Spaces*, Hadronic Press, Inc. USA, 1998, [162]; *The Geometry of Higher Order Hamilton Spaces. Applications to Hamiltonian Mechanics*, Kluwer Acad. Publ., FTPH 132, 2003 [163]; as well as the papers [167]–[172].

This part contains: The geometry of the manifold of accelerations  $T^kM$ , Lagrange spaces of order  $k$ ,  $L^{(k)n}$ , Finsler spaces of order  $k$ ,  $F^{(k)n}$  and dual, via Legendre transformation Hamilton spaces  $H^{(k)n}$  and Cartan spaces  $\mathcal{C}^{(k)n}$ .

## Chapter 1

### The Geometry of the manifold $T^k M$

The importance of Lagrange geometries consists of the fact that the variational problems for Lagrangians have numerous applications: Mathematics, Physics, Theory of Dynamical Systems, Optimal Control, Biology, Economy etc.

But, all of the above mentioned applications have imposed also the introduction of the notions of higher order Lagrange spaces. The base manifold of this space is the bundle of accelerations of superior order. The methods used in the construction of the geometry of higher order Lagrange spaces are the natural extensions of those used in the edification of the Lagrangian geometries exposed in chapters 1, 2 and 3.

The concept of higher order Lagrange space was given by author in the books [161], [155]. The problems raised by the geometrization of Lagrangians systems of order  $k > 1$  were been investigated by many scholars: Ch. Ehresmann [82], P. Libermann [143], J. Pommaré [?], J. T. Synge [243], M. Crampin [64], P. Saunders [230], G.S. Asanov [27], D. Krupka [132], M. de Léon [141], H. Rund [218], A. Kawaguchi [119], K. Yano [256], K. Kondo [128], D. Grigore [92], R. Miron [155], [156] et al.

In this chapter we shall present, briefly, the following problems:

- 1° The geometry of total space of the bundle of higher order accelerations.
- 2° The definition of higher order Lagrange space, based on the nondegenerate Lagrangians of order  $k \geq 1$ .
- 3° The solving of the old problem of prolongation of the Riemannian structures, given on the base manifold  $M$ , to the Riemannian structures on the total space of the bundle of accelerations of order  $k \geq 1$ , we prove for the first time the existence of Lagrange spaces of order  $k \geq 1$ .
- 4° The elaboration of the geometrical ground for variational calculus involving Lagrangians which depend on higher order accelerations.
- 5° The introduction of the notion of higher order energies and proof of the law of conservation.

- 6° The notion of  $k$ -semispray. Nonlinear connection the canonical metrical connection and the structure equations.  
 7° The Riemannian  $(k-1)n$ -almost contact model of a Lagrange space of order  $k$ .

Evidently, we can not sufficiently develop these subjects. For much more informations one can see the books [161], [162].

Throughout in this chapter the differentiability of manifolds and of mappings means the class  $C^\infty$ .

## 1.1 The bundle of acceleration of order $k \geq 1$

In Analytical Mechanics a real  $n$ -dimensional differentiable manifold  $M$  is considered as space of configurations of a physical system. A point  $(x^i) \in M$  is called a material point. A mapping  $c : t \in I \rightarrow (x^i(t)) \in U \subset M$  is a law of moving (a law of evolution),  $t$  is time, a pair  $(t, x)$  is an event and the  $k$ -uple  $\left(\frac{dx^i}{dt}, \dots, \frac{1}{k!} \frac{d^k x^i}{dt^k}\right)$  gives the velocity and generalized accelerations of order 2, ...,  $k-1$ . The factors  $\frac{1}{h!}$  ( $h = 1, \dots, k$ ) are introduced here for the simplicity of calculus. In this chapter we omit the word “generalized” and say shortly, the acceleration of order  $h$ , for  $\frac{1}{h!} \frac{d^h x^i}{dt^h}$ . A law of moving  $c : t \in I \rightarrow c(t) \in U$  will be called a curve parametrized by time  $t$ .

In order to obtain the differentiable bundle of accelerations of order  $k$ , we use the accelerations of order  $k$ , by means of geometrical concept of contact of order  $k$  between two curves in the manifold  $M$ .

Two curves  $\rho, \sigma : I \rightarrow M$  in  $M$  have at the point  $x_0 \in M$ ,  $\rho(0) = \sigma(0) = x_0 \in U$  (and  $U$  a domain of local chart in  $M$ ) have a *contact* of order  $k$  if we have

$$\frac{d^\alpha (f \circ \rho)(t)}{dt^\alpha} \Big|_{t=0} = \frac{d^\alpha (f \circ \sigma)(t)}{dt^\alpha} \Big|_{t=0}, (\alpha = 1, \dots, k). \quad (1.1.1)$$

It follows that: the curves  $\rho$  and  $\sigma$  have at the point  $x_0 = \rho(0) = \sigma(0)$  a contact of order  $k$  if and only if the accelerations of order  $1, 2, \dots, k$  on the curve  $\rho$  at  $x_0$  have the same values with the corresponding accelerations on the curve  $\sigma$  at point  $x_0$ .

The relation “to have a contact of order  $k$ ” is an equivalence. Let  $[\rho]_{x_0}$  be a class of equivalence and  $T_{x_0}^k M$  the set of equivalence classes. Consider the set

$$T^k M = \bigcup_{x_0 \in M} T_{x_0}^k M \quad (1.1.2)$$

and the mapping:

$$\pi^k : [\rho]_{x_0} \in T^k M \rightarrow x_0 \in M, \quad \forall [\rho]_{x_0}. \quad (1.1.2')$$

Thus the triple  $(T^k M, \pi^k, M)$  can be endowed with a natural differentiable structure exactly as in the cases  $k = 1$ , when  $(T^1 M, \pi^1, M)$  is the tangent bundle.

If  $U \subset M$  is a coordinate neighborhood on the manifold  $M$ ,  $x_0 \in U$  and the curve  $\rho : I \rightarrow U$ ,  $\rho_0 = x_0$  is analytical represented on  $U$  by the equations  $x^i = x^i(t)$ ,  $t \in I$ , then  $T_{x_0}^k M$  can be represented by:

$$x_0^i = x^i(0), y_0^{(1)i} = \frac{dx^i}{dt}(0), \dots, y_0^{(k)i} = \frac{1}{k!} \frac{d^k x^i}{dt^k}(0). \quad (1.1.3)$$

Setting

$$\phi : ([\rho]_{x_0}) \in T^k M \rightarrow \phi([\rho]_{x_0}) = (x_0^i, y_0^{(1)i}, \dots, y_0^{(k)i}) \in R^{(k+1)n}, \quad (1.1.4)$$

it follows that the pair  $(\pi^k)^{-1}(U), \phi)$  is a local chart on  $T^k M$  induced by the local chart  $(U, \varphi)$  on the manifold  $M$ .

So a differentiable atlas of the manifold  $M$  determine a differentiable atlas on  $T^k M$  and the triple  $(T^k M, \pi^k, M)$  is a differentiable bundle. Of course the mapping  $\pi^k : T^k M \rightarrow M$  is a submersion.

$(T^k M, \pi^k, M)$  is called the  $k$  accelerations bundle or tangent bundle of order  $k$  or  $k$ -osculator bundle. A change of local coordinates  $(x^i, y^{(1)i}, \dots, y^{(k)i}) \rightarrow (\tilde{x}^i, \tilde{y}^{(1)i}, \dots, \tilde{y}^{(k)i})$  on the manifold  $T^k M$ , according with (1.1.3), is given by:

$$\left\{ \begin{array}{l} \tilde{x}^i = \tilde{x}^i(x^1, \dots, x^n), \text{rank} \left( \frac{\partial \tilde{x}^i}{\partial x^j} \right) = n \\ \tilde{y}^{(1)i} = \frac{\partial \tilde{x}^i}{\partial x^j} y^{(1)j} \\ 2\tilde{y}^{(2)i} = \frac{\partial \tilde{y}^{(1)i}}{\partial x^j} y^{(1)j} + 2 \frac{\partial \tilde{y}^{(1)i}}{\partial y^{(1)j}} y^{(2)j} \\ \dots \dots \dots \\ k\tilde{y}^{(k)i} = \frac{\partial \tilde{y}^{(k-1)i}}{\partial x^j} y^{(1)j} + 2 \frac{\partial \tilde{y}^{(k-1)i}}{\partial y^{(1)j}} y^{(2)j} + \dots + k \frac{\partial \tilde{y}^{(k-1)i}}{\partial y^{(k-1)j}} y^{(k)j}. \end{array} \right. \quad (1.1.5)$$

And remark that we have the following identities:

$$\frac{\partial \tilde{y}^{(\alpha)i}}{\partial x^j} = \frac{\partial \tilde{y}^{(\alpha+1)i}}{\partial y^{(1)j}} = \dots = \frac{\partial \tilde{y}^{(k)i}}{\partial y^{(k-\alpha)j}}, \quad (1.1.5')$$

$$(\alpha = 0, \dots, k-1; y^{(0)} = x).$$

We denote a point  $u \in T^kM$  by  $u = (x, y^{(1)}, \dots, y^{(k)})$  and its coordinates by  $(x^i, y^{(1)i}, \dots, y^{(k)i})$ .

A section of the projection  $\pi^k$  is a mapping  $S : M \rightarrow T^kM$  with the property  $\pi^k \circ S = 1_M$ . And a local section  $S$  has the property  $\pi^k \circ S|_U = 1_U$ .

If  $c : I \rightarrow M$  is a smooth curve, locally represented by  $x^i = x^i(t)$ ,  $t \in I$ , then the mapping  $\tilde{c} : I \rightarrow T^kM$  given by:

$$x^i = x^i(t), \quad y^{(1)i} = \frac{1}{1!} \frac{dx^i}{dt}(t), \dots, \quad y^{(k)i} = \frac{1}{k!} \frac{d^{(k)}x^i}{dt^k}(t), \quad t \in I \quad (1.1.6)$$

is *the extension of order  $k$  to  $T^kM$  of  $c$* . We have  $\pi^k \circ \tilde{c} = c$ .

The following property holds:

If the differentiable manifold  $M$  is paracompact, then  $T^kM$  is a paracompact manifold.

We shall use the manifold  $\widetilde{T^kM} = T^kM \setminus \{0\}$ , where  $0$  is the null section of  $\pi^k$ .

## 1.2 The Liouville vector fields

The natural basis at point  $u \in T^k M$  of  $T_u(T^k M)$  is given by

$$\left( \frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^{(1)i}}, \dots, \frac{\partial}{\partial y^{(k)i}} \right)_u.$$

A local coordinate changing (1.1.5) transform the natural basis by the following rule:

$$\begin{aligned} \frac{\partial}{\partial x^i} &= \frac{\partial \tilde{x}^j}{\partial x^i} \frac{\partial}{\partial \tilde{x}^j} + \frac{\partial \tilde{y}^{(1)j}}{\partial x^i} \frac{\partial}{\partial \tilde{y}^{(1)j}} + \dots + \frac{\partial \tilde{y}^{(k)j}}{\partial x^i} \frac{\partial}{\partial \tilde{y}^{(k)j}} \\ \frac{\partial}{\partial y^{(1)i}} &= \frac{\partial \tilde{y}^{(1)j}}{\partial y^{(1)i}} \frac{\partial}{\partial \tilde{y}^{(1)j}} + \dots + \frac{\partial \tilde{y}^{(k)j}}{\partial y^{(1)i}} \frac{\partial}{\partial \tilde{y}^{(k)j}} \\ \dots &\dots \dots \\ \frac{\partial}{\partial y^{(k)i}} &= \frac{\partial \tilde{y}^{(k)j}}{\partial y^{(k)i}} \frac{\partial}{\partial \tilde{y}^{(k)j}}, \end{aligned} \quad (1.2.1)$$

calculated at the point  $u \in T^k M$ .

The natural cobasis  $(dx^i, dy^{(1)i}, \dots, dy^{(k)i})_u$  is transformed by (1.1.5) as follows:

$$\begin{aligned} d\tilde{x}^i &= \frac{\partial \tilde{x}^i}{\partial x^j} dx^j, \\ d\tilde{y}^{(1)i} &= \frac{\partial \tilde{y}^{(1)i}}{\partial x^j} dx^j + \frac{\partial \tilde{y}^{(1)i}}{\partial y^{(1)j}} dy^{(1)j}, \\ \dots &\dots \dots \\ d\tilde{y}^{(k)i} &= \frac{\partial \tilde{y}^{(k)i}}{\partial x^j} dx^j + \frac{\partial \tilde{y}^{(k)i}}{\partial y^{(1)j}} dy^{(1)j} + \dots + \frac{\partial \tilde{y}^{(k)i}}{\partial y^{(k)j}} dy^{(k)j}. \end{aligned} \quad (1.2.1')$$

The formulae (1.2.1) and (1.2.1') allow to determine some important geometric object fields on the total space of accelerations bundle  $T^k M$ .

The vertical distribution  $V_1$  is local generated by the vector fields  $\left\{ \frac{\partial}{\partial y^{(1)i}}, \dots, \frac{\partial}{\partial y^{(k)i}} \right\}$ ,  $i = 1, \dots, n$ .  $V_1$  is integrable and of dimension

$kn$ . The distribution  $V_2$  local generated by  $\left\{ \frac{\partial}{\partial y^{(2)i}}, \dots, \frac{\partial}{\partial y^{(k)i}} \right\}$  is also

integrable, of dimension  $(k-1)n$  and it is a subdistribution of  $V_1$ . And so on.

The distribution  $V_k$  local generated by  $\left\{ \frac{\partial}{\partial y^{(k)i}} \right\}$  is integrable and of dimension  $n$ . It is a subdistribution of the distribution  $V_{k-1}$ . So we have the following sequence:

$$V_1 \supset V_2 \supset \dots \supset V_k$$

Using again (1.2.1) we deduce:

**Theorem 1.2.1.** *The following operators in the algebra of functions  $\mathcal{F}(T^k M)$  :*

$$\begin{aligned} \Gamma^1 &= y^{(1)i} \frac{\partial}{\partial y^{(k)i}}, \\ \Gamma^2 &= y^{(1)i} \frac{\partial}{\partial y^{(k-1)i}} + 2y^{(2)i} \frac{\partial}{\partial y^{(k)i}}, \\ &\dots\dots\dots \\ \Gamma^k &= y^{(1)i} \frac{\partial}{\partial y^{(1)i}} + 2y^{(2)i} \frac{\partial}{\partial y^{(2)i}} + \dots + ky^{(k)i} \frac{\partial}{\partial y^{(k)i}} \end{aligned} \quad (1.2.2)$$

are the vector fields on  $T^k M$ . They are independent on the manifold  $\widetilde{T^k M}$  and  $\Gamma^1 \subset V_k, \Gamma^2 \subset V_{k-1}, \dots, \Gamma^k \subset V_1$ .

The vector fields  $\Gamma^1, \Gamma^2, \dots, \Gamma^k$  are called the *Liouville vector fields*. Also we have:

**Theorem 1.2.2.** *For any function  $L \in \mathcal{F}(\widetilde{T^k M})$ , the following entries are 1-form fields on the manifold  $\widetilde{T^k M}$ :*

$$\begin{aligned} d_0 L &= \frac{\partial L}{\partial y^{(k)i}} dx^i, \\ d_1 L &= \frac{\partial L}{\partial y^{(k-1)i}} dx^i + \frac{\partial L}{\partial y^{(k)i}} dy^{(1)i}, \\ &\dots\dots\dots \\ d_k L &= \frac{\partial L}{\partial x^i} dx^i + \frac{\partial L}{\partial y^{(1)i}} dy^{(1)i} + \dots + \frac{\partial L}{\partial y^{(k)i}} dy^{(k)i}. \end{aligned} \quad (1.2.3)$$

Evidently,  $d_k L = dL$ .



In applications we shall use also the following nonlinear operator

$$\Gamma = y^{(1)i} \frac{\partial}{\partial x^i} + 2y^{(2)i} \frac{\partial}{\partial y^{(1)i}} + \dots + ky^{(k)i} \frac{\partial}{\partial y^{(k-1)i}}. \quad (1.2.4)$$

$\Gamma$  is not a vector field on  $\widetilde{T^k M}$ .

**Definition 1.2.1.** A  $k$ -tangent structure  $J$  on  $T^k M$  is the  $\mathcal{F}(T^k M)$ -linear mapping  $J: \mathcal{X}(T^k M) \rightarrow \mathcal{X}(T^k M)$ , defined by:

$$\begin{aligned} J\left(\frac{\partial}{\partial x^i}\right) &= \frac{\partial}{\partial y^{(1)i}}, & J\left(\frac{\partial}{\partial y^{(1)i}}\right) &= \frac{\partial}{\partial y^{(2)i}}, \dots, \\ J\left(\frac{\partial}{\partial y^{(k-1)i}}\right) &= \frac{\partial}{\partial y^{(k)i}}, & J\left(\frac{\partial}{\partial y^{(k)i}}\right) &= 0. \end{aligned} \quad (1.2.5)$$

It is not difficult to see that  $J$  has the properties:

**Proposition 1.2.1.** *We have:*

- 1°.  $J$  is globally defined on  $T^k M$ ,
- 2°.  $J$  is an integrable structure,
- 3°.  $J$  is locally expressed by

$$J = \frac{\partial}{\partial y^{(1)i}} \otimes dx^i + \frac{\partial}{\partial y^{(2)i}} \otimes dy^{(1)i} + \dots + \frac{\partial}{\partial y^{(k)i}} \otimes dy^{(k-1)i} \quad (1.2.6)$$

- 4°.  $\text{Im}J = \text{Ker}J$ ,  $\text{Ker}J = V_k$ ,
- 5°.  $\text{rank}J = kn$ ,
- 6°.  $J \overset{k}{\Gamma} = \overset{k-1}{\Gamma}$ , ...,  $J \overset{2}{\Gamma} = \overset{1}{\Gamma}$ ,  $J \overset{1}{\Gamma} = 0$ ,
- 7°.  $J \circ J \circ \dots \circ J = 0$ , ( $k+1$  factors).

In the next section we shall use the functions

$$I^1(L) = \mathcal{L}_{\overset{1}{\Gamma}} L, \dots, I^k(L) = \mathcal{L}_{\overset{k}{\Gamma}} L, \quad \forall L \in \mathcal{F}(T^k M), \quad (1.2.7)$$

where  $\mathcal{L}_{\overset{\alpha}{\Gamma}}$  is the operator of Lie derivation with respect to the Liouville vector field  $\overset{\alpha}{\Gamma}$ .

The functions  $I^1(L)$ , ...,  $I^k(L)$  are called the *main invariants* of the function  $L$ . They play an important role in the variational calculus.

### 1.3 Variational Problem

**Definition 1.3.1.** A differentiable Lagrangian of order  $k$  is a mapping  $L : (x, y^{(1)}, \dots, y^{(k)}) \in T^kM \rightarrow L(x, y^{(1)}, y^{(k)}) \in R$ , differentiable on  $\widetilde{T^kM}$  and continuous on the null section  $0 : M \rightarrow \widetilde{T^kM}$  of the projection  $\pi^k : T^kM \rightarrow M$ .

If  $c : t \in [0, 1] \rightarrow (x^i(t)) \in U \subset M$  is a curve, with extremities  $c(0) = (x^i(0))$  and  $c(1) = (x^i(1))$  and  $\tilde{c} : [0, 1] \rightarrow \widetilde{T^kM}$  from (1.1.6) is its extension. Then the integral of action of  $L \circ \tilde{c}$  is defined by

$$I(c) = \int_0^1 L \left( x(t), \frac{dx}{dt}(t), \dots, \frac{1}{k!} \frac{d^k x}{dt^k}(t) \right) dt. \quad (1.3.1)$$

*Remark 1.3.1.* One proves that if  $I(c)$  does not depend on the parametrization of curve  $c$  then the following Zermelo conditions holds:

$$I^1(L) = \dots = I^{k-1}(L) = 0, \quad I^k(L) = L. \quad (1.3.2)$$

Generally, these conditions are not verified.

The variational problem involving the functional  $I(c)$  from (1.3.1) will be studied as a natural extension of the theory expounded in §2.2, Ch. 2.

On the open set  $U$  we consider the curves

$$c_\varepsilon : t \in [0, 1] \rightarrow (x^i(t) + \varepsilon V^i(t)) \in M, \quad (1.3.3)$$

where  $\varepsilon$  is a real number, sufficiently small in absolute value such that  $\text{Im}c_\varepsilon \subset U$ ,  $V^i(t) = V^i(x(t))$  being a regular vector field on  $U$ , restricted to  $c$ . We assume all curves  $c_\varepsilon$  have the same end points  $c(0)$  and  $c(1)$  and their osculator spaces of order  $1, 2, \dots, k-1$  coincident at the points  $c(0), c(1)$ . This means:

$$V^i(0) = V^i(1) = 0; \quad \frac{d^\alpha V^i}{dt^\alpha}(0) = \frac{d^\alpha V^i}{dt^\alpha}(1) = 0, \quad (1.3.3')$$

$$(\alpha = 1, \dots, k-1).$$

The integral of action  $I(c_\varepsilon)$  of the Lagrangian  $L$  is:

$$I(c_\varepsilon) = \int_0^1 L \left( x + \varepsilon V, \frac{dx}{dt} + \varepsilon \frac{dV}{dt}, \dots, \frac{1}{k!} \left( \frac{d^k x}{dt^k} + \varepsilon \frac{d^k V}{dt^k} \right) \right) dt. \quad (1.3.4)$$

A necessary condition for  $I(c)$  to be an extremal value for  $I(c_\varepsilon)$  is

$$\frac{dI(c_\varepsilon)}{d\varepsilon}\Big|_{\varepsilon=0} = 0. \quad (1.3.5)$$

Thus, we have

$$\frac{dI(c_\varepsilon)}{d\varepsilon} = \int_0^1 \frac{d}{d\varepsilon} L\left(x + \varepsilon V, \frac{dx}{dt} + \varepsilon \frac{dV}{dt}, \dots, \frac{1}{k!} \left(\frac{d^k x}{dt^k} + \varepsilon \frac{d^k V}{dt^k}\right)\right) dt.$$

The Taylor expansion of  $L$  for  $\varepsilon = 0$ , gives:

$$\frac{dI(c_\varepsilon)}{d\varepsilon}\Big|_{\varepsilon=0} = \int_0^1 \left( \frac{\partial L}{\partial x^i} V^i + \frac{\partial L}{\partial y^{(1)i}} \frac{dV^i}{dt} + \dots + \frac{1}{k!} \frac{\partial L}{\partial y^{(k)i}} \frac{d^k V^i}{dt^k} \right) dt. \quad (1.3.6)$$

Now, with notations

$$\overset{\circ}{E}_i(L) := \frac{\partial L}{\partial x^i} - \frac{d}{dt} \frac{\partial L}{\partial y^{(1)i}} + \dots + (-1)^k \frac{1}{k!} \frac{d^k}{dt^k} \frac{\partial L}{\partial y^{(k)i}} \quad (1.3.7)$$

and

$$\begin{aligned} I_V^1 L &= V^i \frac{\partial L}{\partial y^{(k)i}}, \quad I_V^2(L) = V^i \frac{\partial L}{\partial y^{(k-1)i}} + \frac{dV^i}{dt} \frac{\partial L}{\partial y^{(k)i}}, \dots, \\ I_V^k &= V^i \frac{\partial L}{\partial y^{(1)i}} + \frac{dV^i}{dt} \frac{\partial L}{\partial y^{(2)i}} + \dots + \frac{1}{(k-1)!} \frac{d^{k-1} V^i}{dt^{k-1}} \frac{\partial L}{\partial y^{(k)i}} \end{aligned} \quad (1.3.8)$$

we obtain an important identity:

$$\begin{aligned} \frac{\partial L}{\partial x^i} V^i + \frac{\partial L}{\partial y^{(1)i}} \frac{dV^i}{dt} + \dots + \frac{1}{k!} \frac{\partial L}{\partial y^{(k)i}} \frac{d^k V^i}{dt^k} &= \overset{\circ}{E}_i(L) + \\ + \frac{d}{dt} \left\{ I_V^k(L) - \frac{1}{2!} \frac{d}{dt} I_V^{k-1}(L) + \dots + (-1)^{k-1} \frac{1}{k!} \frac{d^{k-1}}{dt^{k-1}} I_V^1(L) \right\} \end{aligned} \quad (1.3.9)$$

Now, applying (1.3.7) and taking into account (1.3.3') with

$$I_V^\alpha(L)(c(0)) = I_V^\alpha(L)c(1) = 0, \quad (\alpha = 1, 2, \dots, k)$$

we obtain

$$\left. \frac{dI(c_\varepsilon)}{d\varepsilon} \right|_{\varepsilon=0} = \int_0^1 \overset{0}{E}_i(L) V^i dt. \quad (1.3.10)$$

But  $V^i(t)$  is an arbitrary vector field. Therefore the equalities (1.3.5) and (1.3.10) lead to the following result:

**Theorem 1.3.1.** *In order that the integral of action  $I(c)$  be an extremal value for the functionals  $I(c_\varepsilon)$ , (1.3.4) is necessary that the following Euler - Lagrange equations hold:*

$$\begin{cases} \overset{0}{E}_i(L) := \frac{\partial L}{\partial x^i} - \frac{d}{dt} \frac{\partial L}{\partial y^{(1)i}} + \dots + (-1)^k \frac{1}{k!} \frac{d^k}{dt^k} \frac{\partial L}{\partial y^{(k)i}} = 0, \\ y^{(1)i} = \frac{dx^i}{dt}, \dots, y^{(k)i} = \frac{1}{k!} \frac{d^k x^i}{dt^k}. \end{cases} \quad (1.3.11)$$

One proves [161] that  $\overset{0}{E}_i(L)$  is a covector field. Consequence the equation  $\overset{0}{E}_i(L) = 0$  has a geometrical meaning.

Consider the scalar field

$$\mathcal{E}^k(L) = I^k(L) - \frac{1}{2!} \frac{d}{dt} I^{k-1}(L) + \dots + (-1)^{k-1} \frac{1}{k!} \frac{d^{k-1}}{dt^{k-1}} I^1(L) - L. \quad (1.3.12)$$

It is called the *energy of order  $k$*  of the Lagrangian  $L$ .

The next result is known, [161]:

**Theorem 1.3.2.** *For any Lagrangian  $L(x, y^{(1)}, \dots, y^{(k)})$  the energy of order  $k$ ,  $\mathcal{E}^k(L)$  is conserved along every solution curve of the Euler - Lagrange equations  $\overset{0}{E}_i(L) = 0$ ,  $y^{(1)i} = \frac{dx^i}{dt}, \dots, y^{(k)i} = \frac{1}{k!} \frac{d^k x^i}{dt^k}$ .*

*Remark 1.3.2.* Introducing the notion of energy of order  $1, 2, \dots, k-1$  we can prove a Nöther theorem for the Lagrangians of order  $k$ .

Now we remark that for any  $C^\infty$ -function  $\phi(t)$  and any differentiable Lagrangian  $L(x, y^{(1)}, \dots, y^{(k)})$  the following equality holds:

$$\overset{0}{E}_i(\phi L) = \phi \overset{0}{E}_i(L) + \frac{d\phi}{dt} \overset{1}{E}_i(L) + \dots + \frac{d^k \phi}{dt^k} \overset{k}{E}_i(L), \quad (1.3.13)$$

where  $\overset{1}{E}_i(L), \dots, \overset{k}{E}_i(L)$  are  $d$ -covector fields - called Graig - Synge covectors, [63]. We consider the covector  $\overset{k-1}{E}_i(L)$ :

$$E_i^{(k-1)}(L) = (-1)^{k-1} \frac{1}{(k-1)!} \left( \frac{\partial L}{\partial y^{(k-1)i}} - \frac{d}{dt} \frac{\partial L}{\partial y^{(k)i}} \right), \quad (1.3.14)$$

It is important in the theory of  $k$ -semisprays from the Lagrange spaces of order  $k$ .

The Hamilton - Jacobi equations, of a space  $L^n = (M, L(x, y))$  introduced in the section 4 of Chapter 2 can be extended in the higher order Lagrange spaces by using the Jacobi - Ostrogradski momenta. Indeed, the energy of order  $k$ ,  $\mathcal{E}^k(L)$  from (1.3.13) is a polynomial function in  $\frac{dx^i}{dt}, \dots, \frac{d^k x^i}{dt^k}$  given by

$$\mathcal{E}^k(L) = p_{(1)i} \frac{dx^i}{dt} + p_{(2)i} \frac{d^2 x^i}{dt^2} + \dots + p_{(k)i} \frac{d^k x^i}{dt^k} - L, \quad (1.3.15)$$

where

$$\begin{aligned} p_{(1)i} &= \frac{\partial L}{\partial y^{(1)i}} - \frac{1}{2!} \frac{d}{dt} \frac{\partial L}{\partial y^{(2)i}} + \dots + (-1)^{k-1} \frac{1}{k!} \frac{d^{k-1}}{dt^{k-1}} \frac{\partial L}{\partial y^{(k)i}}, \\ p_{(2)i} &= \frac{1}{2!} \frac{\partial L}{\partial y^{(2)i}} - \frac{1}{3!} \frac{d}{dt} \frac{\partial L}{\partial y^{(3)i}} + \dots + (-1)^{k-2} \frac{1}{k!} \frac{d^{k-2}}{dt^{k-2}} \frac{\partial L}{\partial y^{(k)i}}, \\ &\dots\dots\dots \\ p_{(k)i} &= \frac{1}{k!} \frac{\partial L}{\partial y^{(k)i}}. \end{aligned} \quad (1.3.16)$$

$p_{(1)i}, \dots, p_{(k)i}$  are called *the Jacobi-Ostrogradski momenta*.

The following important result has been established by M. de Léon and others, [161]:

**Theorem 1.3.3.** *Along the integral curves of Euler Lagrange equations  $E_i^0(L) = 0$  the following Hamilton - Jacobi - Ostrogradski equations holds:*



A curve  $c : I \rightarrow M$  is called a  $k$ -path on  $M$  with respect to  $S$  if its extension  $\tilde{c}$  is an integral curve of  $S$ . A  $k$ -path is characterized by the  $(k+1)$ -differential equations:

$$\frac{d^{k+1}x^i}{dt^{k+1}} + (k+1)G^i\left(x, \frac{dx}{dt}, \dots, \frac{1}{k!} \frac{d^k x}{dt^k}\right) = 0. \quad (1.4.4)$$

We shall show that a  $k$ -semispray determine the main geometrical object fields on  $T^k M$  as: the nonlinear connections  $N$ , the  $N$ -linear connections  $D$  and their structure equations. Evidently,  $N$  and  $D$  are basic for the geometry of manifold  $T^k M$ .

**Definition 1.4.1.** A subbundle  $HT^k M$  of the tangent bundle  $(TT^k M, d\pi^k, T^k M)$  which is supplementary to the vertical subbundle  $V_1 T^k M$ :

$$TT^k M = HT^k M \oplus V_1 T^k M \quad (1.4.5)$$

is called a *nonlinear connection*.

The fibres of  $HT^k M$  determine a horizontal distribution

$$N : u \in T^k M \rightarrow N_u = H_u T^k M \subset T_u T^k M, \quad \forall u \in T^k M$$

supplementary to the vertical distribution  $V_1$ , i.e.

$$T_u T^k M = N_u \oplus V_{1,u}, \quad \forall u \in T^k M. \quad (1.4.5')$$

If the base manifold  $M$  is paracompact on  $T^k M$  there exist the nonlinear connections.

The local dimension of  $N$  is  $n = \dim M$ .

Consider a nonlinear connection  $N$  and denote by  $h$  and  $v$  the horizontal and vertical projectors with respect to  $N$  and  $V_1$ :

$$h + v = I, \quad hv = vh = 0, \quad h^2 = h, \quad v^2 = v.$$

As usual we denote

$$X^H = hX, \quad X^V = vX, \quad \forall X \in \mathcal{X}(T^k M).$$

An horizontal lift, with respect to  $N$  is a  $\mathcal{F}(M)$ -linear mapping  $l_h : \mathcal{X}(M) \rightarrow \mathcal{X}(T^k M)$  which has the properties

$$v \circ l_h = 0, \quad d\pi^k \circ l_h = I_d.$$

There exists an unique local basis adapted to the horizontal distribution  $N$ . It is given by

$$\frac{\delta}{\delta x^i} = l_h \left( \frac{\partial}{\partial x^i} \right), (i = 1, \dots, n). \quad (1.4.6)$$

The linearly independent vector fields of this basis can be uniquely written in the form:

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N_{(1)}^j \frac{\partial}{\partial y^{(1)j}} - \dots - N_{(k)}^j \frac{\partial}{\partial y^{(k)j}}. \quad (1.4.7)$$

The systems of differential functions on  $T^kM : (N_{(1)}^j, \dots, N_{(k)}^j)$ , gives the *coefficients* of the nonlinear connection  $N$ .

By means of (1.4.6) it follows:

**Proposition 1.4.1.** *With respect of a change of local coordinates on the manifold  $T^kM$  we have*

$$\frac{\delta}{\delta x^i} = \frac{\partial \tilde{x}^j}{\partial x^i} \frac{\delta}{\delta \tilde{x}^j}, \quad (1.4.7')$$

and

$$\tilde{N}_{(1)}^i \frac{\partial \tilde{x}^m}{\partial x^j} = N_{(1)}^m \frac{\partial \tilde{x}^i}{\partial x^m} - \frac{\partial \tilde{y}^{(1)i}}{\partial x^j}, \quad (1.4.8)$$

$$\dots \dots \dots$$

$$\tilde{N}_{(k)}^i \frac{\partial \tilde{x}^m}{\partial x^j} = N_{(k)}^m \frac{\partial \tilde{x}^i}{\partial x^m} + \dots + N_{(1)}^m \frac{\partial \tilde{y}^{(k-1)i}}{\partial x^m} - \frac{\partial \tilde{y}^{(k)i}}{\partial x^j},$$

*Remark 1.4.1.* The equalities (1.4.8) characterize a nonlinear connection  $N$  with the coefficients  $N_{(1)}^j, \dots, N_{(k)}^j$ .

These considerations lead to an important result, given by:

**Theorem 1.4.1 (I. Bucătaru [47]).** *If  $S$  is a  $k$ -semispray on  $T^kM$ , with the coefficients  $G^i$ , then the following system of functions:*



$$N_j^i = \frac{\partial G^i}{\partial y^{(k)i}}, N_j^i = \frac{\partial G^i}{\partial y^{(k-1)i}}, \dots, N_j^i = \frac{\partial G^i}{\partial y^{(1)j}} \quad (4.9)$$

gives the coefficients of a nonlinear connection  $N$ .

The  $k$ -tangent structure  $J$ , defined in (1.2.5), Ch. 4. applies the horizontal distribution  $N_0 = N$  into a vertical distribution  $N_1 \subset V_1$  of dimension  $n$ , supplementary to the distribution  $V_2$ . Then it applies the distribution  $N_1$  in distribution  $N_2 \subset V_2$ , supplementary to the distribution  $V_3$  and so on. Of course we have  $\dim N_0 = \dim N_1 = \dots = \dim N_{k-1} = n$ .

Therefore we can write:

$$N_1 = J(N_0), \quad N_2 = J(N_1), \dots, N_{k-1} = J(N_{k-2}) \quad (1.4.9)$$

and we obtain the direct decomposition:

$$T_u T^k M = N_{0,u} \oplus N_{1,u} \oplus \dots \oplus N_{k-1,u} \oplus V_{k,u}, \quad \forall u \in T^k M. \quad (1.4.10)$$

An adapted basis to  $N_0, N_1, \dots, N_{k-1}, V_k$  is given by:

$$\left\{ \frac{\delta}{\delta x^i}, \frac{\delta}{\delta y^{(1)i}}, \dots, \frac{\delta}{\delta y^{(k-1)i}}, \frac{\partial}{\partial y^{(k)i}} \right\}, (i = 1, \dots, n), \quad (1.4.11)$$

where  $\frac{\delta}{\delta x^i}$  is in (1.4.7) and

$$\left\{ \begin{array}{l} \frac{\delta}{\delta y^{(1)i}} = \frac{\partial}{\partial y^{(1)i}} - N_{(1)}^j \frac{\partial}{\partial y^{(2)j}} - \dots - N_{(k-1)}^j \frac{\partial}{\partial y^{(k)j}}, \\ \dots \\ \frac{\delta}{\delta y^{(k-1)i}} = \frac{\partial}{\partial y^{(k-1)i}} - N_{(1)}^j \frac{\partial}{\partial y^{(k)j}} \end{array} \right. \quad (1.4.12)$$

With respect to (1.1.5), Ch. 4 we have:

$$\frac{\delta}{\delta y^{(\alpha)i}} = \frac{\partial x^j}{\partial \tilde{x}^i} \frac{\delta}{\delta \tilde{y}^{(\alpha)j}}, \quad \left( \alpha = 0, 1, \dots, k; y^{(0)i} = x^i, \frac{\delta}{\delta y^{(k)i}} = \frac{\partial}{\partial y^{(k)i}} \right). \quad (1.4.13)$$

Let  $h, v_1, \dots, v_k$  be the projectors determined by (1.4.10):

$$h + \sum_1^k v_\alpha = I, h^2 = h, v_\alpha v_\alpha = v_\alpha, hv_\alpha = 0;$$

$$v_\alpha h = 0, v_\alpha v_\beta = v_\beta v_\alpha = 0, (\alpha \neq \beta).$$

If we denote

$$X^H = hX, X^{V_\alpha} = v_\alpha X, \forall X \in \mathcal{X}(T^kM) \quad (1.4.14)$$

we have, uniquely,

$$X = X^H + X^{V_1} + \dots + X^{V_k}. \quad (1.4.15)$$

In the adapted basis (1.4.12) we have:

$$X^H = X^{(0)i} \frac{\delta}{\delta x^i}, X^{V_\alpha} = X^{(\alpha)i} \frac{\delta}{\delta y^{(\alpha)i}}, (\alpha = 1, \dots, k).$$

A first result on the nonlinear connection  $N$  is as follows:

**Theorem 1.4.2.** *The nonlinear connection  $N$  is integrable if, and only if:*

$$[X^H, Y^H]^{V_\alpha} = 0, \quad \forall X, Y \in \mathcal{X}(T^kM), \quad (\alpha = 1, \dots, k).$$

## 1.5 The dual coefficients of a nonlinear connection

Consider a nonlinear connection  $N$ , with the coefficients  $(N_j^i, \dots, N_j^i)$ .

The dual basis, of the adapted basis (1.4.12) is of the form

$$(\delta x^i, \delta y^{(1)i}, \dots, \delta y^{(k)i}) \quad (1.5.1)$$

where

$$\left\{ \begin{array}{l} \delta x^i = dx^i, \\ \delta y^{(1)i} = dy^{(1)i} + M_j^i dx^j, \\ \dots \\ \delta y^{(k)i} = dy^{(k)i} + M_j^i dy^{(k-1)j} + \dots + M_j^i dy^{(1)j} + M_j^i dx^j, \end{array} \right. \quad (1.5.2)$$

and where

$$\begin{cases} M_j^i = N_j^i, M_j^i = N_j^i + N_j^m M_m^i, \dots, \\ M_j^i = N_j^i + N_j^m M_m^i + \dots + N_j^m M_m^i. \end{cases} \quad (1.5.3)$$

The system of functions  $(M_j^i, \dots, M_j^i)$  is called the system of dual coefficients of the nonlinear connection  $N$ . If the dual coefficients of  $N$  are given, then we uniquely obtain from (1.5.3) the *primal* coefficients  $(N_j^i, \dots, N_j^i)$  of  $N$ .

With respect to (4.1.4), Ch. 4 the dual coefficients of  $N$  are transformed by the rule

$$\begin{aligned} M_j^m \frac{\partial \tilde{x}^i}{\partial x^m} &= \tilde{M}_m^i \frac{\partial \tilde{x}^m}{\partial x^j} + \frac{\partial \tilde{y}^{(1)i}}{\partial x^j}, \\ \dots \dots \dots \\ M_j^m \frac{\partial \tilde{x}^i}{\partial x^m} &= \tilde{M}_m^i \frac{\partial \tilde{x}^m}{\partial x^j} + \tilde{M}_m^i \frac{\partial \tilde{y}^{(1)m}}{\partial x^j} + \dots + \tilde{M}_m^i \frac{\partial \tilde{y}^{(k-1)m}}{\partial x^j} + \frac{\partial \tilde{y}^{(k)i}}{\partial x^j}. \end{aligned} \quad (1.5.4)$$

These transformations of the dual coefficients characterize the nonlinear connection  $N$ . This property allows to prove an important result:

**Theorem 1.5.1 (R. Miron).** *For any  $k$ -semispray  $S$  with the coefficients  $G^i$  the following system of functions*

$$\begin{aligned} M_j^i &= \frac{\partial G^i}{\partial y^{(k)j}}, M_j^i = \frac{1}{2} (S M_j^i + M_m^i M_j^m), \dots, \\ M_j^i &= \frac{1}{k} (S M_j^i + M_m^i M_j^m) \end{aligned} \quad (1.5.5)$$

*gives the system of dual coefficients of a nonlinear connection, which depend on the  $k$ -semispray  $S$ , only.*

*Remark 1.5.1.* Gh. Atanasiu [168] has a real contribution in demonstration of this theorem for  $k = 2$ .

As an application we can prove:

**Theorem 1.5.2.** 1) In the adapted basis (4.4.12) Ch. 4, the Liouville vector fields  $\overset{1}{\Gamma}, \dots, \overset{k}{\Gamma}$  can be expressed in the form

$$\begin{aligned} \overset{1}{\Gamma} &= z^{(1)i} \frac{\delta}{\delta y^{(k)i}}, & \overset{2}{\Gamma} &= z^{(1)i} \frac{\delta}{\delta y^{(k-1)i}} + 2z^{(2)i} \frac{\delta}{\delta y^{(k)i}}, \\ & \dots & & \\ \overset{k}{\Gamma} &= z^{(1)i} \frac{\delta}{\delta y^{(1)i}} + 2z^{(2)i} \frac{\delta}{\delta y^{(2)i}} + \dots + kz^{(k)i} \frac{\delta}{\delta y^{(k)i}}, \end{aligned} \quad (1.5.6)$$

where

$$\begin{cases} z^{(1)i} = y^{(1)i}, & 2z^{(2)i} = 2y^{(2)i} + M_{(1)m}^i y^{(1)m}, \dots, \\ kz^{(k)i} = ky^{(k)i} + (k-1)M_{(1)m}^i y^{(k-1)m} + \dots + M_{(k-1)m}^i y^{(1)m} \end{cases} \quad (1.5.7)$$

2) With respect to (1.1.4) we have:

$$\tilde{z}^{(\alpha)i} = \frac{\partial \tilde{x}^i}{\partial x^j} z^{(\alpha)j}, \quad (\alpha = 1, \dots, k). \quad (1.5.7')$$

This is reason in which we call  $z^{(1)i}, \dots, z^{(k)i}$  the distinguished Liouville vector fields (shortly,  $d$ -vector fields). These vectors are important in the geometry of the manifold  $T^kM$ .

A field of 1-forms  $\omega \in \mathcal{X}^*(T^kM)$  can be uniquely written as

$$\omega = \omega^H + \omega^{V_1} + \dots + \omega^{V_k}$$

where

$$\omega^H = \omega \circ h, \omega^{V_\alpha} = \omega \circ v_\alpha, \quad (\alpha = 1, \dots, k).$$

For any function  $f \in \mathcal{F}(T^kM)$ , the 1-form  $df$  is

$$df = (df)^H + (df)^{V_1} + \dots + (df)^{V_k}.$$

In the adapted cobasis we have:

$$(df)^H = \frac{\delta f}{\delta x^i} dx^i, (df)^{V_\alpha} = \frac{\delta f}{\delta y^{(\alpha)i}} \delta y^{(\alpha)i}, \quad (\alpha = 1, \dots, k). \quad (1.5.8)$$

Let  $\gamma: I \rightarrow T^kM$  be a parametrized curve, locally expressed by

$$x^i = x^i(t), \quad y^{(\alpha)i} = y^{(\alpha)i}(t), \quad (t \in I), \quad (\alpha = 1, \dots, k).$$

The tangent vector field  $\frac{d\gamma}{dt}$  is given by:

$$\begin{aligned} \frac{d\gamma}{dt} &= \left(\frac{d\gamma}{dt}\right)^H + \left(\frac{d\gamma}{dt}\right)^{V_1} + \dots + \left(\frac{d\gamma}{dt}\right)^{V_k} = \\ &= \frac{dx^i}{dt} \frac{\delta}{\delta x^i} + \frac{\delta y^{(1)i}}{dt} \frac{\delta}{\delta y^{(1)i}} + \dots + \frac{\delta y^{(k)i}}{dt} \frac{\delta}{\delta y^{(k)i}}. \end{aligned}$$

The curve  $\gamma$  is called horizontal if  $\frac{d\gamma}{dt} = \left(\frac{d\gamma}{dt}\right)^H$ . It is characterized by the system of differential equations

$$x^i = x^i(t), \quad \frac{\delta y^{(1)i}}{dt} = 0, \dots, \frac{\delta y^{(k)i}}{dt} = 0. \quad (1.5.9)$$

A horizontal curve  $\gamma$  is called **autoparallel** curve of the nonlinear connection if  $\gamma = \tilde{c}$ , where  $\tilde{c}$  is the extension of a curve  $c : I \rightarrow M$ .

The *autoparallel* curves of the nonlinear connection  $N$  are characterized by the system of differential equations

$$\begin{aligned} \frac{\delta y^{(1)i}}{dt} = 0, \dots, \frac{\delta y^{(k)i}}{dt} = 0, \\ y^{(1)i} = \frac{dx^i}{dt}, \dots, y^{(k)i} = \frac{1}{k!} \frac{d^k x^i}{dt^k}. \end{aligned} \quad (1.5.9')$$

## 1.6 Prolongation to the manifold $T^k M$ of the Riemannian structures given on the base manifold $M$

Applying the previous theory of the notion of nonlinear connection on the total space of acceleration bundle  $T^k M$  we can solve the old problem of the prolongation of the Riemann (or pseudo Riemann) structure  $g$  given on the base manifold  $M$ . This problem was formulated by L. Bianchi and was studied by several remarkable mathematicians as: E. Bompiani, Ch. Ehresmann, A. Morimoto, S. Kobayashi. But the solu-

tion of this problem, as well as the solution of the prolongation to  $T^kM$  of the Finsler or Lagrange structures were been recently given by R. Miron [172]. We will expound it here with very few demonstrations.

Let  $\mathcal{R}^n = (M, g)$  be a Riemann space,  $g$  being a Riemannian metric defined on the base manifold  $M$ , having the local coordinate  $g_{ij}(x)$ ,  $x \in U \subset M$ . We extend  $g_{ij}$  to  $\pi^{-1}(U) \subset T^kM$ , setting

$$(g_{ij} \circ \pi^k)(u) = g_{ij}(x), \quad \forall u \in \pi^{-1}(U), \pi^k(u) = x.$$

$g_{ij} \circ \pi^k$  will be denoted by  $g_{ij}$ .

The problems of prolongation of the Riemannian structure  $g$  to  $T^kM$  can be formulated as follows:

The Riemannian structure  $g$  on the manifold  $M$  being a priori given, determine a Riemannian structure  $G$  on  $T^kM$  so that  $G$  be provided only by structure  $g$ .

As usually, we denoted by  $\gamma^i_{jk}(x)$  the Christoffel symbols of  $g$  and prove, according theorem 1.5.1:

**Theorem 1.6.1.** *There exists nonlinear connections  $N$  on the manifold  $\widetilde{T^kM}$  determined only by the given Riemannian structure  $g(x)$ . One of them has the following dual coefficients*

$$M_j^i = \gamma^i_{jm}(x)y^{(1)m}, \quad (1)$$

$$M_j^i = \frac{1}{2} \left\{ \Gamma_{(1)} M_j^i + M_m^i M_j^m \right\}, \quad (1.6.1)$$

$$\dots\dots\dots$$

$$M_j^i = \frac{1}{k} \left\{ \Gamma_{(k-1)} M_j^i + M_m^i M_j^m \right\}, \quad (k)$$

where  $\Gamma$  is the operator (1.2.4), Ch. 4.

*Remark 1.6.1.*  $\Gamma$  can be substituted with any  $k$ -semispray  $S$ , since  $M_j^i, \dots, M_j^i$  do not depend on the variables  $y^k$ .

(1) (k-1)

One proves, also:  $N$  is integrable if and only if the Riemann space  $\mathcal{R}^n = (M, g)$  is locally flat.

Let us consider the adapted cobasis  $(\delta x^i, \delta y^{(1)i}, \dots, \delta y^{(k)i})$  (1.5.2) to the nonlinear connection  $N$  and to the vertical distributions  $N, \dots, N^{(1)}, V_k$ . It depend on the dual coefficients (1.6.1). So it depend on the structure  $g(x)$  only.

Now, on  $T^k M$  consider the following *lift* of  $g(x)$ :

$$G = g_{ij}(x) dx^i \otimes dx^j + g_{ij}(x) \delta y^{(1)i} \otimes \delta y^{(1)j} + g_{ij}(x) \delta y^{(k)i} \otimes \delta y^{(k)j} \quad (1.6.2)$$

Finally, we obtain:

**Theorem 1.6.2.** *The pair  $\text{Prol}^k \mathcal{R}^n = (\widetilde{T^k M}, G)$  is a Riemann space of dimension  $(k+1)n$ , whose metric  $G$ , (1.6.2) depends on the a priori given Riemann structure  $g(x)$ , only.*

The announced problem is solved.

Some remarks:

- 1° The  $d$ -Liouville vector fields  $z^{(1)i}, \dots, z^{(k)i}$ , from (1.5.7) are constructed only by means of the Riemannian structure  $g$ ;
- 2° The following function

$$L(x, y^{(1)}, \dots, y^{(k)}) = g_{ij}(x) z^{(k)i} z^{(k)j} \quad (1.6.3)$$

is a regular Lagrangian which depend only on the Riemann structure  $g_{ij}(x)$ .

- 3° The previous theory, for pseudo-Riemann structure  $g_{ij}(x)$ , holds.

## 1.7 $N$ -linear connections on $T^k M$

The notion of  $N$ -linear connection on the manifold  $T^k M$  can be studied as a natural extension of that of  $N$ -linear connection on  $TM$ , given in the section 1.4.

Let  $N$  be a nonlinear connection on  $T^k M$  having the primal coefficients  $N_j^i, \dots, N_j^i$  and the dual coefficients  $M_j^i, \dots, M_j^i$ .

$$\begin{matrix} (1) & (k) & (1) & (k) \end{matrix}$$

**Definition 1.7.1.** A linear connection  $D$  on the manifold  $T^k M$  is called distinguished if  $D$  preserves by parallelism the horizontal distribution  $N$ . It is an  $N$ -connection if has the following property, too:

$$DJ = 0. \quad (1.7.1)$$

We have:

**Theorem 1.7.1.** *A linear connection  $D$  on  $T^kM$  is an  $N$ -linear connection if and only if*

$$\begin{cases} (D_X Y^H)^{V_\alpha} = 0, (\alpha = 1, \dots, k), (D_X Y^{V_\alpha})^H = 0; \\ (D_X Y^{V_\alpha})^{V_\beta} = 0 (\alpha \neq \beta) \\ D_X(JY^H) = JD_X Y^H; D_X(JV^\alpha) = JD_X V^\alpha. \end{cases} \quad (1.7.2)$$

Of course, for any  $N$ -linear connection  $D$  we have

$$Dh = 0, Dv^\alpha = 0, (\alpha = 1, \dots, k).$$

Since

$$D_X Y = D_{X^H} Y + D_{X^{V_1}} Y + \dots + D_{X^{V_k}} Y,$$

setting

$$D_X^H = D_{X^H}, D_X^{V_\alpha} = D_{X^{V_\alpha}}, (\alpha = 1, \dots, k),$$

we can write:

$$D_X Y = D_X^H Y + D_X^{V_1} Y + \dots + D_X^{V_k} Y. \quad (1.7.3)$$

The operators  $D^H, D^{V_\alpha}$  are not the covariant derivations but they have similar properties with the covariant derivations. The notion of  $d$ -tensor fields ( $d$ -means “distinguished”) can be introduced and studied exactly as in the section 1.3.

In the adapted basis (1.4.12) and in adapted cobasis (1.5.1) we represent a  $d$ -tensor field of type  $(r, s)$  in the form

$$T = T_{j_1 \dots j_s}^{i_1 \dots i_r} \frac{\delta}{\delta x^{i_1}} \otimes \dots \otimes \frac{\delta}{\delta y^{(k)i_r}} \otimes dx^{j_1} \otimes \dots \otimes \delta y^{(k)j_s}. \quad (1.7.4)$$

A transformation of coordinates (1.1.5), Ch. 4, has as effect the following rule of transformation:

$$\tilde{T}_{j_1 \dots j_s}^{i_1 \dots i_r} = \frac{\partial \tilde{x}^{i_1}}{\partial x^{h_1}} \dots \frac{\partial \tilde{x}^{i_r}}{\partial x^{h_r}} \frac{\partial x^{k_1}}{\partial x^{j_1}} \dots \frac{\partial x^{k_s}}{\partial x^{j_s}} T_{k_1 \dots k_s}^{h_1 \dots h_r}. \quad (1.7.3')$$



So,  $\left\{1, \frac{\delta}{\delta x^i}, \dots, \frac{\delta}{\delta y^{(k)i}}\right\}$  generate the tensor algebra of  $d$ -tensor fields.

The theory of  $N$ -linear connection described in the chapter 1 for case  $k = 1$  can be extended step by step for the  $N$ -linear connection on the manifold  $T^kM$ .

In the adapted basis (1.4.11) an  $N$ -linear connection  $D$  has the following form:

$$\begin{aligned} D_{\frac{\delta}{\delta x^j}} \frac{\delta}{\delta x^i} &= L_{ij}^m \frac{\delta}{\delta x^m}, \quad D_{\frac{\delta}{\delta x^j}} \frac{\delta}{\delta y^{(\alpha)i}} = L_{ij}^m \frac{\delta}{\delta y^{(\alpha)m}}, \quad (\alpha = 1, \dots, k), \\ D_{\frac{\delta}{\delta y^{(\beta)j}}} \frac{\delta}{\delta x^i} &= C_{(\beta)ij}^m \frac{\delta}{\delta x^m}, \quad D_{\frac{\delta}{\delta y^{(\beta)j}}} \frac{\delta}{\delta y^{(\alpha)i}} = C_{(\beta)ij}^m \frac{\delta}{\delta y^{(\alpha)m}}, \\ & \hspace{15em} (\alpha, \beta = 1, \dots, k). \end{aligned} \quad (1.7.5)$$

The system of functions

$$D\Gamma(N) = (L_{ij}^m, C_{(\alpha)ij}^m, \dots, C_{(\alpha)ij}^m) \quad (1.7.6)$$

(1) (k)

represents *the coefficients of  $D$* .

With respect to (1.1.5),  $L_{ij}^m$  are transformed by the same rule as the coefficients of a linear connection defined on the base manifold  $M$ . Others coefficients  $C_{(\alpha)ij}^m$ ,  $(\alpha = 1, \dots, k)$  are transformed like  $d$ -tensors

of type  $(1, 2)$ .

If  $T$  is a  $d$ -tensor field of type  $(r, s)$ , given by (1.7.4) and  $X = X^H = X^i \frac{\delta}{\delta x^i}$ , then, by means of (1.7.5),  $D_X^H T$  is:

$$D_X^H T = X^m T_{j_1 \dots j_s | m}^{i_1 \dots i_r} \frac{\delta}{\delta x^{i_1}} \otimes \dots \otimes \frac{\delta}{\delta y^{(k)i_r}} \otimes dx^{j_1} \otimes \dots \otimes \delta y^{(k)j_s}, \quad (1.7.7)$$

where

$$T_{j_1 \dots j_s | m}^{i_1 \dots i_r} = \frac{\delta T_{j_1 \dots j_s}^{i_1 \dots i_r}}{\delta x^m} + L_{hm}^{i_1} T_{j_1 \dots j_s}^{hi_2 \dots i_r} + \dots - L_{j_s m}^h T_{j_1 \dots h}^{i_1 \dots i_r}. \quad (1.7.8)$$

The operator “ $\mid$ ” will be called *the  $h$ -covariant derivative*.

Consider the  $\nu_\alpha$ -covariant derivatives  $D_X^{V\alpha}$ , for  $X = X^i \frac{\delta}{\delta y^{(\alpha)i}}$ , ( $\alpha = 1, \dots, k$ ). Then, (1.7.3) and (1.7.5) lead to:

$$D_X^{V\alpha} T = X^m T_{j_1 \dots j_s}^{i_1 \dots i_r} \Big|_m \frac{\delta}{\delta x^{i_1}} \otimes \dots \otimes \frac{\delta}{\delta y^{(k)i_r}} \otimes dx^{j_1} \otimes \dots \otimes dx^{j_s}, \quad (1.7.9)$$

where

$$T_{j_1 \dots j_s}^{i_1 \dots i_r} \Big|_m = \frac{\delta T_{j_1 \dots j_s}^{i_1 \dots i_r}}{\delta y^{(\alpha)m}} + C_{hm}^{i_1} T_{j_1 \dots j_s}^{hi_2 \dots i_r} + \dots - C_{j_s m}^h T_{j_1 \dots j_{s-1} h}^{i_1 \dots i_r}, \quad (\alpha = 1, \dots, k).$$

The operators “ $\Big|$ ”, in number of  $k$  are called  $\nu_\alpha$ -covariant derivatives.

Each of operators “ $\Big|$ ” and “ $\Big|^{(\alpha)}$ ” has the usual properties with respect to sum of  $d$ -tensor or them tensor product.

Now, in the adapted basis (1.4.11) we can determine the torsion  $T$  and curvature  $R$  of an  $N$ -linear connection  $D$ , follows the same method as in the case  $k = 1$ .

We remark the following of  $d$ -tensors of torsion:

$$T_{jk}^i = L_{jk}^i - L_{kj}^i, \quad S_{jk}^i = C_{jk}^i - C_{kj}^i = 0, \quad (\alpha = 1, \dots, k) \quad (1.7.10)$$

and  $d$ -tensors of curvature

$$R_{hjm}^i, \quad P_{(\alpha)hjm}^i, \quad S_{(\beta\alpha)hjm}^i, \quad (\alpha, \beta = 1, \dots, k) \quad (1.7.11)$$

The 1-forms connection of the  $N$ -linear connection  $D$  are:

$$\omega_j^i = L_{jh}^i dx^h + C_{jh}^i \delta y^{(1)h} + \dots + C_{jh}^i \delta y^{(k)h}. \quad (1.7.12)$$

The following important theorem holds:

**Theorem 1.7.2.** *The structure equations of an  $N$ -linear connection  $D$  on the manifold  $T^kM$  are given by:*

$$\begin{aligned}
d(dx^i) - dx^m \wedge \omega_m^i &= -\Omega^i{}^{(0)} \\
d(\delta y^{(\alpha)i}) - \delta y^{(\alpha)m} \wedge \omega_m^i &= -\Omega^i{}^{(\alpha)}, \\
d\omega_j^i - \omega_j^m \wedge \omega_m^i &= -\Omega_j^i,
\end{aligned} \tag{1.7.13}$$

where  $\Omega^i{}^{(0)}$ ,  $\Omega^i{}^{(\alpha)}$  are the 2-forms of torsion and where  $\Omega_j^i$  are the 2-forms of curvature:

$$\begin{aligned}
\Omega_j^i &= \frac{1}{2} R_j^i{}_{pq} dx^p \wedge dx^q + \\
&+ \sum_{\alpha=1}^k P_j^i{}_{pq}{}^{(\alpha)} dx^p \wedge \delta y^{(\alpha)q} + \sum_{\alpha, \beta=1}^k S_j^i{}_{pq}{}^{(\alpha\beta)} \delta y^{(\alpha)p} \wedge \delta y^{(\beta)q}.
\end{aligned}$$

Now, the Bianchi identities of  $D$  can be derived from (1.7.13).

The nonlinear connection  $N$  and the  $N$ -linear connection  $D$  allow to study the geometrical properties of the manifold  $T^kM$  equipped with these two geometrical object fields.



## Chapter 2

# Lagrange Spaces of Higher-order

The concept of higher - order Lagrange space was introduced and studied by the author of the present monograph, [155], [161].

A Lagrange space of order  $k$  is defined as a pair  $L^{(k)n} = (M, L)$  where  $L : T^k M \rightarrow R$  is a differentiable regular Lagrangian having the fundamental tensor of constant signature. Applying the variational problem to the integral of action of  $L$  we determine: a canonical  $k$ -semispray, a canonical nonlinear connection and a canonical metrical connection. All these are basic for the geometry of space  $L^{(k)n}$ .

### 2.1 The spaces $L^{(k)n} = (M, L)$

**Definition 2.1.1.** A Lagrange space of order  $k \geq 1$  is a pair  $L^{(k)n} = (M, L)$  formed by a real  $n$ -dimensional manifold  $M$  and a differentiable Lagrangian  $L : (x, y^{(1)}, \dots, y^{(k)}) \in T^k M \rightarrow L(x, y^{(1)}, \dots, y^{(k)}) \in R$  for which the Hessian with the elements:

$$g_{ij} = \frac{1}{2} \frac{\partial^2 L}{\partial y^{(k)i} \partial y^{(k)j}} \quad (2.1.1)$$

has the property

$$\text{rank}(g_{ij}) = n \text{ on } \widetilde{T^k M} \quad (2.1.2)$$

and the  $d$ -tensor  $g_{ij}$  has a constant signature.

Of course, we can prove that  $g_{ij}$  from (2.1.1) is a  $d$ -tensor field, of type (0,2), symmetric. It is called the *fundamental* (or *metric*) tensor of the space  $L^{(k)n}$ , while  $L$  is called its *fundamental function*.

The geometry of the manifold  $T^k M$  equipped with  $L(x, y^{(1)}, \dots, y^{(k)})$  is called the geometry of the space  $L^{(k)n}$ . We shall study this geometry using the theory from the last chapter. Consequently, starting from the integral of action  $I(c) = \int_0^1 L\left(x, \frac{dx}{dt}, \dots, \frac{1}{k!} \frac{d^k x}{dt^k}\right) dt$  we determine the Euler - Lagrange equations  $\overset{0}{E}_i(L) = 0$  and the Craig - Synge covectors  $\overset{1}{E}_i(L), \dots, \overset{k}{E}_i(L)$ . According to (3.1.14) one remarks that we have:

**Theorem 2.1.1.** *The equations  $g^{ij} \overset{k-1}{E}_i(L) = 0$  determine a  $k$ -semi-spray*

$$S = y^{(1)i} \frac{\partial}{\partial x^i} + 2y^{(2)i} \frac{\partial}{\partial y^{(1)i}} + \dots + ky^{(k)i} \frac{\partial}{\partial y^{(k-1)i}} - (k+1)G^i \frac{\partial}{\partial y^{(k)i}} \quad (2.1.3)$$

where the coefficients  $G^i$  are given by

$$(k+1)G^i = \frac{1}{2}g^{ij} \left\{ \Gamma \left( \frac{\partial}{\partial y^{(k)i}} \right) - \frac{\partial}{\partial y^{(k-1)i}} \right\} \quad (2.1.4)$$

$\Gamma$  being the operator (1.2.4).

The semispray  $S$  depend on the fundamental function  $L$ , only.  $S$  is called canonical. It is globally defined on the manifold  $\widetilde{T^k M}$ .

Taking into account Theorem 1.5.1, we have:

**Theorem 2.1.2.** *The systems of functions*

$$M_{(1)}^i = \frac{\partial G^i}{\partial y^{(k)j}}, \quad M_{(2)}^i = \frac{1}{2} \left( S M_{(1)}^i + M_{(1)}^i M_{(1)}^m \right), \dots, \quad (2.1.5)$$

$$M_{(k)}^i = \frac{1}{k} \left( S M_{(k-1)}^i + M_{(1)}^i M_{(k-1)}^m \right)$$

are the dual coefficients of a nonlinear connection  $N$  determined only on the fundamental function  $L$  of the space  $L^{(k)n}$ .

$N$  is the canonical nonlinear connection of  $L^{(k)n}$ .

The adapted basis  $\left\{ \frac{\delta}{\delta x^i}, \frac{\delta}{\delta y^{(1)i}}, \dots, \frac{\delta}{\delta y^{(k)i}} \right\}$  has its dual  $\{ \delta x^i, \delta y^{(1)i}, \dots, \delta y^{(k)i} \}$ . They are constructed by the canonical nonlinear connection. So, the horizontal curves are characterized by the system of differential equations Part 2, Ch. 2, and the autoparallel curves of  $N$  are given by Part II, Ch. 1.

The condition that  $N$  be integrable is expressed by  $\left[ \frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j} \right]^{V_\alpha} = 0$ , ( $\alpha = 1, \dots, k$ ).

## 2.2 Examples of spaces $L^{(k)n}$

1° Let us consider the Lagrangian:

$$L(x, y^{(1)}, \dots, y^{(k)}) = g_{ij}(x) z^{(k)i} z^{(k)j} \quad (2.2.1)$$

where  $g_{ij}(x)$  is a Riemannian (or pseudo Riemannian) metric on the base manifold  $M$  and the  $z^{(k)i}$  is the Liouville  $d$ -vector field:

$$kz^{(k)i} = ky^{(k)i} + (k-1)M_{(1)}^i y^{(k-1)m} + \dots + kM_{(k-1)}^i y^{(1)m} \quad (2.2.2)$$

constructed by means of the dual coefficients (Part II, Ch. 1) of the canonical nonlinear connection  $N$  from the problem of prolongation to  $T^k M$  of  $g_{ij}(x)$ . So that the Lagrangian (2.2.1) depend on  $g_{ij}(x)$  only.

The pair  $L^{(k)n} = (M, L)$ , (2.2.1) is a Lagrange space of order  $k$ . Its fundamental tensor is  $g_{ij}(x)$ , since the  $d$ -vector  $z^{(k)i}$  is linearly in the variables  $y^{(k)}$ .

2° Let  $\overset{\circ}{L}(x, y^{(1)})$  be the Lagrangian from electrodynamics

$$\overset{\circ}{L}(x, y^{(1)}) = mc\gamma_{ij}(x)y^{(1)i}y^{(1)j} + \frac{2e}{m}b_i(x)y^{(1)i} \quad (2.2.3)$$

Let  $N$  be the nonlinear connection given by the theorem Part II, Ch. 1, from the problem of prolongations to  $T^k M$  of the Riemannian (or pseudo Riemannian) structure  $\gamma_{ij}(x)$  and the Liouville tensor  $z^{(k)i}$  constructed by means of  $N$ . Then the pair  $L^{(k)n} = (M, L)$ , with

$$L(x, y^{(1)}, \dots, y^{(k)}) = mc\gamma_{ij}(x)z^{(k)i}z^{(k)j} + \frac{2e}{m}b_i(x)z^{(k)i} \quad (2.2.4)$$

is a Lagrange space of order  $k$ . It is the prolongation to the manifold  $T^k M$  of the Lagrangian  $\overset{\circ}{L}$  (2.2.3) of electrodynamics. These examples prove the existence of the Lagrange spaces of order  $k$ .

### 2.3 Canonical metrical $N$ -connection

Consider the canonical nonlinear connection  $N$  of a Lagrange space of order  $k$ ,  $L^{(k)n} = (M, L)$ .

An  $N$ -linear connection  $\mathbb{D}$  with the coefficients  $D\Gamma(N) = (L_{jk}^i, C_{jh}^i, \dots, C_{jh}^i)$  is called metrical with respect to metric tensor  $g_{ij}$  if

$$g_{ij|h} = 0, \quad g_{ij} \Big|_h^{(\alpha)} = 0, \quad (\alpha = 1, \dots, k). \quad (2.3.1)$$

Now we can prove the following theorem:

**Theorem 2.3.1.** *The following properties hold:*

1) *There exists a unique  $N$ -linear connection  $D$  on  $\widetilde{T^k M}$  verifying the axioms:*

- 1°  $N$ - is the canonical nonlinear connection of space  $L^{(k)n}$ .
- 2°  $g_{ij|h} = 0$ , ( $D$  is  $h$ -metrical)
- 3°  $g_{ij} \Big|_h^{(\alpha)} = 0$ , ( $\alpha = 1, \dots, k$ ), ( $D$  is  $v_\alpha$ -metrical)
- 4°  $T_{jh}^i = 0$ , ( $D$  is  $h$ -torsion free)
- 5°  $S_{jh}^i = 0$ , ( $\alpha = 1, \dots, k$ ), ( $D$  is  $v_\alpha$ -torsion free).

2) *The coefficients  $C\Gamma(N) = (L_{ij}^h, C_{ij}^h, \dots, C_{ij}^h)$  of  $D$  are given by the generalized Christoffel symbols:*



$$L_{ij}^h = \frac{1}{2} g^{hs} \left( \frac{\delta g_{is}}{\delta x^j} + \frac{\delta g_{sj}}{\delta x^i} - \frac{\delta g_{ij}}{\delta x^s} \right),$$

$$C_{ij(\alpha)}^h = \frac{1}{2} g^{hs} \left( \frac{\delta g_{is}}{\delta y^{(\alpha)j}} + \frac{\delta g_{sj}}{\delta y^{(\alpha)i}} - \frac{\delta g_{ij}}{\delta y^{(\alpha)s}} \right), \quad (\alpha = 1, \dots, k). \quad (2.3.2)$$

3)  $D$  depends only on the fundamental function  $L$  of the space  $L^{(k)n}$ .

The connection  $D$  from the previous theorem is called *canonical metrical  $N$ -connection* and its coefficients (2.3.2) are denoted by  $CG(N)$ .

Now, the geometry of the Lagrange spaces  $L^{(k)n}$  can be developed by means of these two canonical connection  $N$  and  $D$ .

## 2.4 The Riemannian $(k-1)n$ -contact model of the space $L^{(k)n}$

The almost Kählerian model of the Lagrange spaces  $L^n$  expound in the section 7, Ch. 2, can be extended in a corresponding model of the higher order Lagrange spaces. But now, it is a Riemannian almost  $(k-1)n$ -contact structure on the manifold  $\widetilde{T^k M}$ .

The canonical nonlinear connection  $N$  of the space  $L^{(k)n} = (M, L)$  determines the following  $\mathcal{F}(\widetilde{T^k M})$ -linear mapping  $\mathbb{F} : \mathcal{X}(\widetilde{T^k M}) \rightarrow \mathcal{X}(\widetilde{T^k M})$  defined on the adapted basis to  $N$  and to  $N_\alpha$ , by

$$\mathbb{F} \left( \frac{\delta}{\delta x^i} \right) = -\frac{\partial}{\partial y^{(k)i}}$$

$$F \left( \frac{\delta}{\delta y^{(\alpha)i}} \right) = 0, \quad (\alpha = 1, \dots, k-1) \quad (2.4.1)$$

$$F \left( \frac{\partial}{\partial y^{(k)i}} \right) = \frac{\delta}{\delta x^i}, \quad (i = 1, \dots, n)$$

We can prove:

**Theorem 2.4.1.** *We have:*

$1^\circ$   $\mathbb{F}$  is globally defined on  $\widetilde{T^k M}$ .

- 2°  $\mathbb{F}$  is a tensor field of type  $(1, 1)$  on  $\widetilde{T^k M}$   
 3°  $\text{Ker}\mathbb{F} = N_1 \oplus N_2 \oplus \cdots \oplus N_{k-1}$ ,  $\text{Im}\mathbb{F} = N_0 \oplus V_k$   
 4°  $\text{rank}\|\mathbb{F}\| = 2n$   
 5°  $\mathbb{F}^3 + \mathbb{F} = 0$ .

Thus  $\mathbb{F}$  is an almost  $(k-1)n$ -contact structure on  $\widetilde{T^k M}$  determined by  $N$ .

Let  $\left( \begin{matrix} \xi, \xi, \dots, \xi_a \\ 1a \quad 2a \quad (k-1)a \end{matrix} \right)$ ,  $(a = 1, \dots, n)$  be a local basis adapted to the direct decomposition  $N_1 \oplus \cdots \oplus N_{k-1}$  and  $\left( \begin{matrix} 1a \quad 2a \quad (k-1)a \\ \eta, \eta, \dots, \eta \end{matrix} \right)$ , its dual.

Thus the set

$$\left( \begin{matrix} \mathbb{F}, \xi, \dots, \xi \\ 1a \quad (k-1)a \end{matrix}, \begin{matrix} 1a \\ \eta, \dots, \eta \end{matrix}, \begin{matrix} (k-1)a \\ \eta \end{matrix} \right), \quad (a = 1, \dots, n-1) \quad (2.4.2)$$

is a  $(k-1)n$  almost contact structure.

Indeed, (2.4.1) imply:

$$\mathbb{F} \left( \begin{matrix} \xi \\ \alpha a \end{matrix} \right) = 0, \quad \begin{matrix} \alpha a \\ \eta \end{matrix} \left( \begin{matrix} \xi \\ \beta b \end{matrix} \right) = \begin{cases} \delta_b^a & \text{for } \alpha = \beta \\ 0 & \text{for } \alpha \neq \beta, (\alpha, \beta = 1, \dots, (k-1)). \end{cases}$$

$$\mathbb{F}^2(X) = -X + \sum_{a=1}^n \sum_{\alpha=1}^{k-1} \begin{matrix} \alpha a \\ \eta \end{matrix} (X) \begin{matrix} \alpha a \\ \xi \end{matrix}, \quad \forall X \in \mathcal{X}(T^k M), \quad \begin{matrix} \alpha a \\ \eta \end{matrix} \circ \mathbb{F} = 0.$$

Let  $N_{\mathbb{F}}$  be the Nijenhuis tensor of the structure  $\mathbb{F}$ .

$$N_{\mathbb{F}}(X, Y) = [\mathbb{F}X, \mathbb{F}Y] + \mathbb{F}^2[X, Y] - \mathbb{F}[\mathbb{F}X, Y] - \mathbb{F}[X, \mathbb{F}Y].$$

The structure (2.4.2) is said to be normal if:

$$N_{\mathbb{F}}(X, Y) + \sum_{a=1}^n \sum_{\alpha=1}^{k-1} d \begin{matrix} \alpha a \\ \eta \end{matrix} (X, Y) = 0, \quad \forall X, Y \in \mathcal{X}(T^k M).$$

So we obtain a characterization of the normal structure  $\mathbb{F}$  given by the following theorem:

**Theorem 2.4.2.** *The almost  $(k-1)n$ -contact structure (2.4.2) is normal if and only if for any  $X, Y \in \mathcal{X}(\widetilde{T^k M})$  we have:*

$$N_{\mathbb{F}}(X, Y) + \sum_{a=1}^n \sum_{\alpha=1}^{k-1} d(\delta y^{(\alpha)a})(X, Y) = 0.$$

The lift of fundamental tensor  $g_{ij}$  of the space  $L^{(k)n}$  with respect to  $N$  is defined by:

$$\mathbb{G} = g_{ij} dx^i \otimes dx^j + g_{ij} \delta y^{(1)i} \otimes \delta y^{(1)j} + \dots + g_{ij} \delta y^{(k)i} \otimes \delta y^{(k)j}. \quad (2.4.3)$$

Evidently,  $\mathbb{G}$  is a pseudo-Riemannian structure on the manifold  $\widetilde{T^k M}$ , determined only by space  $L^{(k)n}$ .

Now, it is not difficult to prove:

**Theorem 2.4.3.** *The pair  $(\mathbb{G}, \mathbb{F})$  is a Riemannian  $(k-1)n$ -almost contact structure on  $\widetilde{T^k M}$ .*

In this case, the next condition holds:

$$\mathbb{G}(\mathbb{F}X, Y) = -\mathbb{G}(\mathbb{F}Y, X), \quad \forall X, Y \in \mathcal{X}(\widetilde{T^k M}).$$

Therefore the triple  $(\widetilde{T^k M}, \mathbb{G}, \mathbb{F})$  is an metrical  $(k-1)n$ -almost contact space named the *geometrical model of the Lagrange space of order  $k$ ,  $L^{(k)n}$* .

Using this space we can study the electromagnetic and gravitational fields in the spaces  $L^{(k)n}$ , [161].

## 2.5 The generalized Lagrange spaces of order $k$

The notion of generalized Lagrange space of higher order is a natural extension of that studied in chapter 2.

**Definition 2.5.1.** A generalized Lagrange space of order  $k$  is a pair  $GL^{(k)n} = (M, g_{ij})$  formed by a real differentiable  $n$ -dimensional manifold  $M$  and a  $C^\infty$ -covariant of type  $(0,2)$ , symmetric  $d$ -tensor field  $g_{ij}$  on  $\widetilde{T^k M}$ , having the properties:

- a.  $g_{ij}$  has a constant signature on  $\widetilde{T^k M}$ ;

b.  $\text{rank}(g_{ij}) = n$  on  $\widetilde{T^k M}$ .

$g_{ij}$  is called the *fundamental tensor* of  $GL^{(k)n}$ .

Evidently, any Lagrange space of order  $k$ ,  $L^{(k)n} = (M, L)$  determines a space  $GL^{(k)n}$  with fundamental tensor

$$g_{ij} = \frac{1}{2} \frac{\partial^2 L}{\partial y^{(k)i} \partial y^{(k)j}}. \quad (2.5.1)$$

But not and conversely. If  $g_{ij}(x, y^{(1)}, \dots, y^{(k)})$  is a priori given, it is possible that the system of differential partial equations (2.5.1) does not admit any solution in  $L(x, y^{(1)}, \dots, y^{(k)})$ . A necessary condition that the system (2.5.1) admits solutions in the function  $L$  is that  $d$ -tensor field

$$C_{(k)ijh} = \frac{1}{2} \frac{\partial g_{ij}}{\partial y^{(k)h}} \quad (2.5.2)$$

be completely symmetric.

If the system (2.5.1) has solutions, with respect to  $L$  we say that the space  $GL^{(k)n}$  is reducible to a Lagrange space of order  $k$ . If this property is not true, then  $GL^n$  is said to be nonreducible to a Lagrange space  $L^{(k)n}$ .

### Examples

1° Let  $\mathcal{R} = (M, \gamma_{ij}(x))$  be a Riemannian space and  $\sigma \in \mathcal{F}(T^k M)$ . Consider the  $d$ -tensor field:

$$g_{ij} = e^{2\sigma} (\gamma_{ij} \circ \pi^k). \quad (2.5.3)$$

If  $\frac{\partial \sigma}{\partial y^{(k)h}}$  is a nonvanishes  $d$ -covector on the manifold  $\widetilde{T^k M}$ , then the pair  $GL^{(k)n} = (M, g_{ij})$  is a generalized Lagrange space of order  $k$  and it is not reducible to a Lagrange space  $L^{(k)n}$ .

2° Let  $\mathcal{R}^n = (M, \gamma_{ij}(x))$  be a Riemann space and  $\text{Prol}^k \mathcal{R}^n$  be its prolongation of order  $k$  to  $\widetilde{T^k M}$ .

Consider the Liouville  $d$ -vector field  $z^{(k)i}$  of  $\text{Prol}^k \mathcal{R}^n$ . It is expressed in the formula (2.2.2). We can introduce the  $d$ -covector field  $z_i^{(k)} = \gamma_{ij} z^{(k)j}$ .

We assume that exists a function  $n(x, y^{(1)}, \dots, y^{(k)}) \geq 1$  on  $\widetilde{T^k M}$ .

Thus

$$g_{ij} = \gamma_{ij} + \left(1 - \frac{1}{n^2}\right) z_i^{(k)} z_j^{(k)} \quad (2.5.4)$$

is the fundamental tensor of a space  $GL^{(k)n}$ . Evidently this space is not reducible to a space  $L^{(k)n}$ , if the function  $n \neq 1$ .

These two examples prove the existence of the generalized Lagrange space of order  $k$ .

In the last example,  $k = 1$  leads to the metric Part I, Ch. 2 of the Relativistic Optics, ( $n$  being the refractive index).

In a generalized Lagrange space  $GL^{(k)n}$  is difficult to find a nonlinear connection  $N$  derived only by the fundamental tensor  $g_{ij}$ . Therefore, assuming that  $N$  is a priori given, we shall study the pair  $(N, GL^{(k)n})$ . Thus of theorem of the existence and uniqueness metrical  $N$ -linear connection holds:

**Theorem 2.5.1.** *We have:*

1° *There exists an unique  $N$ -linear connection  $D$  for which*

$$g_{ij|h} = 0, \quad g_{ij} \Big|_h^{(\alpha)} = 0, \quad (\alpha = 1, \dots, k),$$

$$T^i_{jk} = 0, \quad S^i_{jk} = 0, \quad (\alpha = 1, \dots, k).$$

2° *The coefficients of  $D$  are given by the generalized Christoffel symbols Part II, Ch. 2.*

3°  *$D$  depends on  $g_{ij}$  and  $N$  only.*

Using this theorem it is not difficult to study the geometry of Generalized Lagrange spaces.



## Chapter 3

### Higher-Order Finsler spaces

The notion of Finsler spaces of order  $k$ , introduced by the author of this monograph and presented in the book *The Geometry of Higher-Order Finsler Spaces*, Hadronic Press, 1998, is a natural extension to the manifold  $\widetilde{T^k M}$  of the theory of Finsler spaces given in the Part I, Ch. 3. A substantial contribution in the studying of these spaces have H. Shimada and S. Sabău [223].

The impact of this geometry in Differential Geometry, Variational Calculus, Analytical Mechanics and Theoretical Physics is decisive. Finsler spaces play a role in applications to Biology, Engineering, Physics or Optimal Control. Also, the introduction of the notion of Finsler space of order  $k$  is demanded by the solution of problem of prolongation to  $T^k M$  of the Riemannian or Finslerian structures defined on the base manifold  $M$ .

#### 3.1 Notion of Finsler space of order $k$

In order to introduce the Finsler space of order  $k$  are necessary some considerations on the concept of homogeneity of functions on the manifold  $T^k M$ , [161].

A function  $f : T^k M \rightarrow R$  of  $C^\infty$ -class on  $\widetilde{T^k M}$  and continuous on the null section of  $\pi^k : T^k M \rightarrow M$  is called homogeneous of degree  $r \in Z$  on the fibres of  $T^k M$  (briefly  $r$ -homogeneous) if for any  $a \in R^+$  we have

$$f(x, ay^{(1)}, a^2 y^{(2)}, \dots, a^k y^{(k)}) = a^r f(x, y^{(1)}, \dots, y^{(k)}).$$

An Euler Theorem hold:

A function  $f \in \mathcal{F}(T^k M)$ , differentiable on  $\widetilde{T^k M}$  and continuous on the null section of  $\pi^k$  is  $r$ -homogeneous if and only if

$$\mathcal{L}_{\Gamma}^k f = rf, \quad (3.1.1)$$

$\mathcal{L}_{\Gamma}^k$  being the Lie derivative with respect to the Liouville vector field  $\Gamma^k$ .

A vector field  $X \in \mathcal{X}(T^k M)$  is  $r$ -homogeneous if

$$\mathcal{L}_{\Gamma}^k X = (r-1)X \quad (3.1.1')$$

**Definition 3.1.1.** A Finsler space of order  $k$ ,  $k \geq 1$ , is a pair  $F^{(k)n} = (M, F)$  determined by a real differentiable manifold  $M$  of dimension  $n$  and a function  $F : T^k M \rightarrow R$  having the following properties:

- 1°  $F$  is differentiable on  $\widetilde{T^k M}$  and continuous on the null section on  $\pi^k$ .
- 2°  $F$  is positive.
- 3°  $F$  is  $k$ -homogeneous on the fibres of the bundle  $T^k M$ .
- 4° The Hessian of  $F^2$  with the elements:

$$g_{ij} = \frac{1}{2} \frac{\partial^2 F^2}{\partial y^{(k)i} \partial y^{(k)j}} \quad (3.1.2)$$

is positively defined on  $\widetilde{T^k M}$ .

From this definition it follows that *the fundamental tensor*  $g_{ij}$  is nonsingular and 0-homogeneous on the fibres of  $T^k M$ .

Also, we remark: Any Finsler space  $F^{(k)n}$  can be considered as a Lagrange space  $L^{(k)n} = (M, L)$ , whose fundamental function  $L$  is  $F^2$ .

By means of the solution of the problem of prolongation of a Finsler structure  $F(x, y^{(1)})$  to  $T^k M$  we can construct some important examples of spaces  $F^{(k)n}$ .

A Finsler space with the property  $g_{ij}$  depend only on the points  $x \in M$  is called a Riemann space of order  $k$  and denoted by  $\mathcal{R}^{(k)n}$ .

Consequently, we have the following sequence of inclusions, similar with that from Ch. 3:

$$\left\{ \mathcal{R}^{(k)n} \right\} \subset \left\{ F^{(k)n} \right\} \subset \left\{ L^{(k)n} \right\} \subset \left\{ GL^{(k)n} \right\}. \quad (3.1.3)$$



So, the Lagrange geometry of order  $k$  is the geometrical theory of the sequence (1.1.3).

Of course the geometry of  $F^{(k)n}$  can be studied as the geometry of Lagrange space of order  $k$ ,  $L^{(k)n} = (M, F^2)$ . Thus the canonical nonlinear connection  $N$  is *the Cartan nonlinear connection of  $F^{(k)n}$*  and the metrical  $N$ -linear connection  $D$  is the Cartan  $N$ -metrical connection of the space  $F^{(k)n}$ , [161].



## Chapter 4

### The Geometry of $k$ -cotangent bundle

#### 4.1 Notion of $k$ -cotangent bundle, $T^{*k}M$

The  $k$ -cotangent bundle  $(T^{*k}M, \pi^{*k}, M)$  is a natural extension of that of cotangent bundle  $(T^*M, \pi^*, M)$ . It is basic for the Hamilton spaces of order  $k$ . The manifold  $T^{*k}M$  must have some important properties:

- 1°  $T^{*1}M = T^*M$ ;
- 2°  $\dim T^{*k}M = \dim T^kM = (k+1)n$ ;
- 3°  $T^{*k}M$  carries a natural Poisson structure;
- 4°  $T^{*k}M$  is local diffeomorphic to  $T^kM$ .

These properties are satisfied by considering the differentiable bundle  $(T^{*k}M, \pi^{*k}, M)$  as the fibered bundle  $(T^{k-1}M x_M T^*M, \pi^{k-1} x_M \pi^*, M)$ . So we have

$$T^{*k}M = T^{k-1}M x_M T^*M, \quad \pi^{*k} = \pi^{k-1} x_M \pi^*. \quad (4.1.1)$$

A point  $u \in T^{*k}M$  is of the form  $u = (x, y^{(1)}, \dots, y^{(k-1)}, p)$ . It is determined by the point  $x = (x^j) \in M$ , the acceleration  $y^{(1)i} = \frac{dx^i}{dt}$ , ...,  $y^{(k-1)i} = \frac{1}{(k-1)!} \frac{d^{k-1}x^i}{dt^{k-1}}$  and the momenta  $p = (p_i)$ . The geometries of the manifolds  $T^kM$  and  $T^{*k}M$  are dual via Legendre transformation. Then,  $(x^i, y^{(1)i}, \dots, y^{(k-1)i}, p_i)$  are the local coordinates of a point  $u \in T^{*k}M$ .

The change of local coordinates on  $T^{*k}M$  is:

$$\begin{aligned} \tilde{x}^i &= \tilde{x}^i(x^1, \dots, x^n), \det \left( \frac{\partial \tilde{x}^i}{\partial x^j} \right) \neq 0, \\ \tilde{y}^{(1)i} &= \frac{\partial \tilde{x}^i}{\partial x^j} y^{(1)j}, \\ &\dots\dots\dots \\ (k-1)\tilde{y}^{(k-1)i} &= \frac{\partial \tilde{y}^{(k-1)i}}{\partial x^j} y^{(1)j} + \dots + (k-1) \frac{\partial \tilde{y}^{(k-2)i}}{\partial y^{(k-2)j}} y^{(k-1)j} \\ \tilde{p}_i &= \frac{\partial x^j}{\partial \tilde{x}^i} p_j \end{aligned} \tag{4.1.2}$$

where the following equalities hold

$$\frac{\partial \tilde{y}^{(\alpha)i}}{\partial x^j} = \frac{\partial \tilde{y}^{(\alpha+1)i}}{\partial y^{(1)j}} = \dots = \frac{\partial \tilde{y}^{(k-1)i}}{\partial y^{(k-1-\alpha)j}}; \quad (\alpha = 0, \dots, k-2; y^{(0)} = x). \tag{4.1.3}$$

The Jacobian matrix  $J_k$  of (4.1.2) have the property

$$\det J_k(u) = \left[ \det \left( \frac{\partial \tilde{x}^j}{\partial x^i}(u) \right) \right]^{k-1}. \tag{4.1.4}$$

The vector fields  $\partial^i = (\partial^1, \dots, \partial^n) = \frac{\partial}{\partial p_i}$  generate a vertical distribution  $W_k$ , while  $\left\{ \frac{\partial}{\partial y^{(k-1)i}} \right\}$  determine a vertical distribution  $V_{k-1}$ , ...,  $\left\{ \frac{\partial}{\partial y^{(1)i}}, \dots, \frac{\partial}{\partial y^{(k-1)i}} \right\}$  determine a vertical distribution  $V_1$ . We have the sequence of inclusions:

$$V_{k-1} \subset V_{k-2} \subset \dots \subset V_1 \subset V,$$

$$V_u = V_{1,u} \oplus W_{k,u}, \quad \forall u \in T^{*k}M.$$

We obtain without difficulties:

**Theorem 4.1.1.**<sup>o</sup> *The following operators in the algebra of functions on  $T^{*k}M$  are the independent vector field on  $T^{*k}M$ :*

$$\begin{aligned}
\overset{1}{\Gamma} &= y^{(1)i} \frac{\partial}{\partial y^{(k-1)i}} \\
\overset{2}{\Gamma} &= y^{(1)i} \frac{\partial}{\partial y^{(k-2)i}} + 2y^{(2)i} \frac{\partial}{\partial y^{(k-1)i}} \\
&\dots\dots\dots \\
\overset{k-1}{\Gamma} &= y^{(1)i} \frac{\partial}{\partial y^{(1)i}} + \dots + (k-1)y^{(k-1)i} \frac{\partial}{\partial y^{(k-1)i}} \\
\mathbb{C}^* &= p_i \dot{\partial}^i.
\end{aligned} \tag{4.1.5}$$

2° The function

$$\varphi = p_i y^{(1)i} \tag{4.1.6}$$

is a scalar function on  $T^{*k}M$ .

$\overset{1}{\Gamma}, \dots, \overset{k-1}{\Gamma}$  are called the **Liouville vector fields**.

**Theorem 4.1.2.1**° For any differentiable function  $H : \widetilde{T^{*k}M} \rightarrow R$ ,  $\widetilde{T^{*k}M} = T^{*k}M \setminus \{0\}$ ,  $d_0H, \dots, d_{k-2}H$  defined by

$$\begin{aligned}
d_0H &= \frac{\partial H}{\partial y^{(k-1)i}} dx^i \\
d_1H &= \frac{\partial H}{\partial y^{(k-2)i}} dx^i + \frac{\partial H}{\partial y^{(k-1)i}} dy^{(1)i} \\
&\dots\dots\dots \\
d_{k-2}H &= \frac{\partial H}{\partial y^{(1)i}} dx^i + \frac{\partial H}{\partial y^{(2)i}} dy^{(1)i} + \dots + \frac{\partial H}{\partial y^{(k-1)i}} dy^{(k-2)i}
\end{aligned} \tag{4.1.7}$$

are fields of 1-forms on  $\widetilde{T^{*k}M}$ .

2° While

$$d_{k-1}H = \frac{\partial H}{\partial x^i} dx^i + \dots + \frac{\partial H}{\partial y^{(k-1)i}} dy^{(k-1)i} \tag{4.1.8}$$

is not a field of 1-form.

3° We have

$$dH = d_{k-1}H + \dot{\partial}^i H p_i. \tag{4.1.9}$$

If  $H = \varphi = p_i y^{(1)i}$ , then  $d_0H = \dots = d_{k-1}H = 0$  and

$$\omega = d_{k-2}\varphi = p_i dx^i \quad (4.1.10)$$

$\omega$  is called the Liouville 1-form. Its exterior differential is expressed by

$$\theta = d\omega = dp_i \wedge dx^i. \quad (4.1.11)$$

$\theta$  is a 2-form of rank  $2n < (k+1)n = \dim T^{*k}M$ , for  $k > 1$ . Consequence  $\theta$  is a presymplectic structure on  $T^{*k}M$ .

Let us consider the tensor field of type  $(1, 1)$  on  $T^{*k}M$ :

$$J = \frac{\partial}{\partial y^{(1)i}} \otimes dx^i + \frac{\partial}{\partial y^{(2)i}} \otimes dy^{(1)i} + \dots + \frac{\partial}{\partial y^{(k-1)i}} \otimes dy^{(k-2)i}. \quad (4.1.12)$$

**Theorem 4.1.3.** *We have:*

- 1°  $J$  is globally defined.
- 2°  $J$  is integrable.
- 3°  $J \circ J \circ \dots \circ J = J^k = 0$ .
- 4°  $\ker J = V_{k-1} \oplus W_k$ .
- 5°  $\text{rank} J = (k-1)n$ .
- 6°  $J(\Gamma^1) = 0, J(\Gamma^2) = \Gamma^1, \dots, J(\Gamma^{k-1}) = \Gamma^{k-2}, J(\mathbb{C}^*) = 0$ .

**Theorem 4.1.4.** 1° For any vector field  $X \in \mathcal{X}(T^{*k}M)$ ,  $\overset{1}{X}, \dots, \overset{k-1}{X}$  given by

$$\overset{1}{X} = J(X), \overset{2}{X} = J^2 X, \dots, \overset{k-1}{X} = J^{k-1} X \quad (4.1.13)$$

are vector fields.

2° If  $X = \overset{(0)i}{X} \frac{\partial}{\partial x^i} + \overset{(1)i}{X} \frac{\partial}{\partial y^{(1)i}} + \dots + \overset{(k-1)i}{X} \frac{\partial}{\partial y^{(k-1)i}} + X_i \dot{\partial}^i$ . Then

$$\overset{1}{X} = J(X) = \overset{(0)i}{X} \frac{\partial}{\partial y^{(1)i}} + \dots + \overset{(k-2)i}{X} \frac{\partial}{\partial y^{(k-1)i}}, \dots, \overset{k-1}{X} = \overset{0i}{X} \frac{\partial}{\partial y^{(k-1)i}}. \quad (4.1.14)$$

On the manifold  $T^{*k}M$  there exists a Poisson structure given by

$$\{f, g\}_0 = \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial x^i} \quad (4.1.15)$$

$$\{f, g\}_\alpha = \frac{\partial f}{\partial y^{(\alpha)i}} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial y^{(\alpha)i}} \quad (\alpha = 1, \dots, k-1).$$

It is not difficult to study the properties of this structure.  
For details see the book [163].





**Part III**  
**Analytical Mechanics of Lagrangian and  
Hamiltonian Mechanical Systems**

This part is devoted to applications of the Lagrangian and Hamiltonian geometries of order  $k = 1$  and  $k > 1$  to Analytical Mechanics.

Firstly, to classical Mechanics of Riemannian mechanical systems  $\Sigma_{\mathcal{R}} = (M, T, Fe)$  for which the external forces  $Fe$  depend on the ma-

terial point  $x \in M$  and on the velocities  $\frac{dx^i}{dt}$ . So, in general,  $\Sigma_{\mathcal{R}}$  are the

nonconservative systems. Then we are obliged to take  $Fe$  as a vertical field on the phase space  $TM$  and apply the Lagrange geometry for study the geometrical theory of  $\Sigma_{\mathcal{R}}$ . More general, we introduce the

notion on Finslerian mechanical system  $\Sigma_F = (M, F, Fe)$ , where  $M$  is the configuration space,  $TM$  is the velocity space,  $F$  is the fundamen-

tal function of a given Finsler space  $F^n = (M, F(x, y))$  and  $Fe \left( x, \frac{dx}{dt} \right)$

are the external forces defined as vertical vector field on the velocity space  $TM$ . The fundamental equations of  $\Sigma_F$  are the *Lagrange equations*

$$\frac{d}{dt} \frac{\partial F^2}{\partial y^i} - \frac{\partial F^2}{\partial x^i} = F_i(x, y), \quad y^i = \frac{dx^i}{dt},$$

$F^2$  being the energy of Finsler space  $F^n$ .

More general, consider a triple  $\Sigma_L = (M, L, Fe)$ , where  $M$  is space of configurations,  $L^n = (M, L)$  is a Lagrange space, and  $Fe$  are the external forces. The fundamental equations are the Lagrange equations, too.

The dual theory leads to the Cartan and Hamiltonian mechanical systems and is based on the Hamilton equations.

Finally, we remark the extension of such kind of analytical mechanics, to the higher order. Some applications will be done.

These considerations are based on the paper [175] and on the papers of J. Klein [123], M. Crampin [64], Manuel de Leon [138]. Also, we use the papers of R. Miron, M. Anastasiei, I. Bucataru [166], and of R. Miron, H. Shimada, S. Sabau and M. Roman [174], [180].

# Chapter 1

## Riemannian mechanical systems

### 1.1 Riemannian mechanical systems

Let  $g_{ij}(x)$  be Riemannian tensor field on the configuration space  $M$ . So its kinetic energy is

$$T = \frac{1}{2}g_{ij}(x)y^i y^j, \quad y^i = \frac{dx^i}{dt} = \dot{x}^i. \quad (1.1.1)$$

Following J. Klein [123], we can give:

**Definition 1.1.1.** A Riemannian Mechanical system (shortly RMS) is a triple  $\Sigma_{\mathcal{R}} = (M, T, Fe)$ , where

- 1°  $M$  is an  $n$ -dimensional, real, differentiable manifold (called configuration space).
- 2°  $T = \frac{1}{2}g_{ij}(x)\dot{x}^i \dot{x}^j$  is the kinetic energy of an a priori given Riemannian space  $\mathcal{R}^n = (M, g_{ij}(x))$ .
- 3°  $Fe(x, y) = F^i(x, y) \frac{\partial}{\partial y^i}$  is a vertical vector field on the velocity space  $TM$  ( $Fe$  are called external forces).

Of course,  $\Sigma_{\mathcal{R}}$  is a scleronomic mechanical system. The covariant components of  $Fe$  are:

$$F_i(x, y) = g_{ij}(x)F^j(x, y). \quad (1.1.2)$$

#### Examples:

1. RMS - for which  $Fe(x, y) = a(x, y)\mathbb{C}$ ,  $a \neq 0$ . Thus  $F^i = a(x, y)y^i$  and  $\Sigma_{\mathcal{R}}$  is called a Liouville RMS.

2. The RMS  $\Sigma_{\mathcal{R}}$ , where  $Fe(x, y) = F^i(x) \frac{\partial}{\partial y^i}$ , and  $F_i(x) = \text{grad}_i f(x)$ , called conservative systems.
3. The RMS  $\Sigma_{\mathcal{R}}$ , where  $Fe(x, y) = F^i(x) \frac{\partial}{\partial y^i}$ , but  $F_i(x) \neq \text{grad}_i f(x)$ , called non-conservative systems.

*Remark 1.1.1.* 1. A conservative system  $\Sigma_{\mathcal{R}}$  is called by J. Klein [123] a *Lagrangian system*.

2. One should pay attention to not make confusion of this kind of mechanical systems with the ‘‘Lagrangian mechanical systems’’  $\Sigma_L = (M, L(x, y), Fe(x, y))$  introduced by R. Miron [175], where  $L: TM \rightarrow \mathbb{R}$  is a regular Lagrangian.

Starting from Definition 1.1.1, in a very similar manner as in the geometrical theory of mechanical systems, one introduces

**Postulate.** The evolution equations of a RSM  $\Sigma_{\mathcal{R}}$  are the *Lagrange equations*:

$$\frac{d}{dt} \frac{\partial L}{\partial y^i} - \frac{\partial L}{\partial x^i} = F_i(x, y), \quad y^i = \frac{dx^i}{dt}, \quad L = 2T. \quad (1.1.3)$$

This postulate will be geometrically justified by the existence of a semispray  $S$  on  $TM$  whose integral curves are given by the equations (1.1.3). Therefore, the integral curves of Lagrange equations will be called the *evolution curves* of the RSM  $\Sigma_{\mathcal{R}}$ .

The Lagrangian  $L = 2T$  has the fundamental tensor  $g_{ij}(x)$ .

*Remark 1.1.2.* In classical Analytical Mechanics, the coordinates  $(x^i)$  of a material point  $x \in M$  are denoted by  $(q^i)$ , and the velocities  $y^i = \frac{dx^i}{dt}$  by  $\dot{q}^i = \frac{dq^i}{dt}$ . However, we prefer to use the notations  $(x^i)$  and  $(y^i)$  which are often used in the geometry of the tangent manifold  $TM$ .

The external forces  $Fe(x, y)$  give rise to the one-form

$$\sigma = F_i(x, y) dx^i. \quad (1.1.4)$$

Since  $Fe$  is a vertical vector field it follows that  $\sigma$  is semibasic one form. Conversely, if  $\sigma$  from (1.1.4) is semibasic one form, then  $Fe = F^i(x, y) \frac{\partial}{\partial y^i}$ , with  $F^i = g^{ij} F_j$ , is a vertical vector field on the manifold  $TM$ . J Klein introduced the the external forces by means of a one-form  $\sigma$ , while R. Miron [175] defined  $Fe$  as a vertical vector field on  $TM$ .

The RMS  $\Sigma_{\mathcal{R}}$  is a regular mechanical system because the Hessian matrix with elements  $\frac{\partial^2 T}{\partial y^i \partial y^j} = g_{ij}(x)$  is nonsingular.

We have the following important result.

**Proposition 1.1.1.** *The system of evolution equations (1.1.3) are equivalent to the following second order differential equations:*

$$\frac{d^2x^i}{dt^2} + \gamma^i_{jk}(x) \frac{dx^j}{dt} \frac{dx^k}{dt} = \frac{1}{2} F^i(x, \frac{dx}{dt}), \quad (1.1.5)$$

where  $\gamma^k_{ij}(x)$  are the Christoffel symbols of the metric tensor  $g_{ij}(x)$ .

In general, for a RMS  $\Sigma_{\mathcal{R}}$ , the system of differential equations (1.1.5) is not autoadjoint, consequently, it can not be written as the Euler-Lagrange equations for a certain Lagrangian.

In the case of conservative RMS, with  $\frac{1}{2}F_i(x) = -\frac{\partial U(x)}{\partial x^i}$ , here  $U(x)$  a potential function, the equations (1.1.2) can be written as Euler-Lagrange equations for the Lagrangian  $T + U$ . They have  $T + U = \text{constant}$  as a prime integral.

This is the reason that the nonconservative RMS  $\Sigma_{\mathcal{R}}$ , with  $F_i$  depending on  $y^i = \frac{dx^i}{dt}$  cannot be studied by the methods of classical mechanics. A good geometrical theory of the RMS  $\Sigma_{\mathcal{R}}$  should be based on the geometry of the velocity space  $TM$ .

From (1.1.4) we can see that in the canonical parametrization  $t = s$  ( $s$  being the arc length in the Riemannian space  $\mathcal{R}^n$ ), we obtain the following result:

**Proposition 1.1.2.** *If the external forces are identically zero, then the evolution curves of the system  $\Sigma_{\mathcal{R}}$  are the geodesics of the Riemannian space  $\mathcal{R}^n$ .*

In the following we will study how the evolution equations change when the space  $\mathcal{R}^n = (M, g)$  is replaced by another Riemannian space  $\bar{\mathcal{R}}^n = (M, \bar{g})$  such that:

- 1°  $\mathcal{R}^n$  and  $\bar{\mathcal{R}}^n$  have the same parallelism of directions;
- 2°  $\mathcal{R}^n$  and  $\bar{\mathcal{R}}^n$  have same geodesics;
- 3°  $\bar{\mathcal{R}}^n$  is conformal to  $\mathcal{R}^n$ .

In each of these cases, the Levi-Civita connections of these two Riemannian spaces are transformed by the rule:

$$1^\circ \bar{\gamma}^i_{jk}(x) = \gamma^i_{jk}(x) + \delta^i_j \alpha_k(x);$$

$$2^\circ \bar{\gamma}^i_{jk}(x) = \gamma^i_{jk}(x) + \delta^i_j \alpha_k(x) + \delta^i_k \alpha_j(x);$$

$$3^\circ \bar{\gamma}^i_{jk}(x) = \gamma^i_{jk}(x) + \delta^i_j \alpha_k(x) + \delta^i_k \alpha_j(x) - g_{jk}(x) \alpha^i(x),$$

where  $\alpha_k(x)$  is an arbitrary covector field on  $M$ , and  $\alpha^i(x) = g^{ij}(x) \alpha_j(x)$ .

It follows that the evolution equations (1.1.3) change to the evolution equations of the system  $\Sigma_{\bar{\mathcal{R}}}$  as follows:

1° In the first case we obtain

$$\frac{d^2x^i}{dt^2} + \gamma^j{}_{jk}(x) \frac{dx^j}{dt} \frac{dx^k}{dt} = -\alpha \frac{dx^i}{dt} + \frac{1}{2} F^i(x, \frac{dx}{dt}); \alpha = \alpha_k(x) \frac{dx^k}{dt} \quad (1.1.6)$$

Therefore, even though  $\Sigma_{\bar{\mathcal{R}}}$  is a conservative system, the mechanical system  $\Sigma_{\bar{\mathcal{R}}}$  is nonconservative system having the external forces

$$\bar{F}e = \left( -\alpha(x, y)y^i + \frac{1}{2} F^i(x, y) \right) \frac{\partial}{\partial y^i}, \quad \alpha = \alpha_k(x) \frac{dx^k}{dt}.$$

2° In the second case we have

$$\frac{d^2x^i}{dt^2} + \gamma^j{}_{jk}(x) \frac{dx^j}{dt} \frac{dx^k}{dt} = -2\alpha \frac{dx^i}{dt} + \frac{1}{2} F^i \left( x, \frac{dx}{dt} \right); \quad \alpha = \alpha_k(x) \frac{dx^k}{dt} \quad (1.1.7)$$

and

$$\bar{F}e = \left( -2\alpha(x, y)y^i + \frac{1}{2} F^i(x, y) \right) \frac{\partial}{\partial y^i}, \quad \alpha = \alpha_k(x) \frac{dx^k}{dt}.$$

3° In the third case  $\bar{F}e$  is

$$\bar{F}e = \left\{ 2(-\alpha y^i + T\alpha^i) + \frac{1}{2} F^i \right\} \frac{\partial}{\partial y^i}, \quad \alpha = \alpha_k(x) \frac{dx^k}{dt},$$

$$\alpha^i(x) = g^{ij}(x) \alpha_j(x).$$

The previous properties lead to examples with very interesting properties.

## 1.2 Examples of Riemannian mechanical systems

Recall that in the case of classical *conservative mechanical systems* we have  $2Fe = \text{grad } \mathcal{U}$ , where  $\mathcal{U}(x)$  is a potential function. Therefore, the Lagrange equations are given by

$$\frac{d}{dt} \frac{\partial}{\partial y^i} (T + \mathcal{U}) - \frac{\partial}{\partial x^i} (T + \mathcal{U}) = 0.$$

We obtain from here a prime integral  $T + \mathcal{U} = h$  (constant) which give us *the energy conservation law*.

In the nonconservative case we have numerous examples suggested by 1°, 2°, 3° from the previous section, where we take  $F^i(x, y) = 0$ .

Other examples of RMS can be obtained as follows

1.

$$2Fe = -\beta(x, y)y^i \frac{\partial}{\partial y^i}, \quad (1.2.1)$$

where  $\beta = \beta_i(x)y^i$  is determined by the electromagnetic potentials  $\beta_i(x)$ , ( $i = 1, \dots, n$ ).

2.

$$Fe = (T - \beta)y^i \frac{\partial}{\partial y^i}, \quad (1.2.2)$$

where  $\beta = \beta_i(x)y^i$  and  $T$  is the kinetic energy.

3. In the three-body problem, M. Bărbosu [36] applied the following conformal transformation:

$$d\bar{s}^2 = (T + \mathcal{U})ds^2$$

to the classic Lagrange equations and had obtained a nonconservative mechanical system with external force field

$$Fe = (T + U)y^i \frac{\partial}{\partial y^i}.$$

4. The external forces  $Fe = F^i(x) \frac{\partial}{\partial y^i}$  lead to classical nonconservative Riemannian mechanical systems. For instance, for  $F_i = -\text{grad}_i \mathcal{U} + R_i(x)$  where  $R_i(x)$  are the resistance forces, and the configuration space  $M$  is  $\mathbb{R}^3$ .

5. If  $M = \mathbb{R}^3$ ,  $T = \frac{1}{2}m\delta_{ij}y^i y^j$  and  $Fe = 2F^i(x) \frac{\partial}{\partial y^i}$ , then the evolution equations are

$$m \frac{d^2 x^i}{dt^2} = F^i(x),$$

which is *the Newton's law*.

6. The harmonic oscillator.

$M = \mathbb{R}^n$ ,  $g_{ij} = \delta_{ij}$ ,  $2F_i = -\omega_i^2 x^i$  (the summation convention is not applied) and  $\omega_i$  are positive numbers, ( $i = 1, \dots, n$ ).

The functions

$$h_i = (x^i)^2 + \omega_i^2 x^i, \text{ and } H = \sum_{i=1}^n h_i$$

are prime integrals.

7. Suggested by the example 6°, we consider a system  $\Sigma_{\mathcal{R}}$  with  $Fe = -2\omega(x)\mathbb{C}$ , where  $\omega(x)$  is a positive function and  $\mathbb{C}$  is the Liouville vector field.

The evolution equations, in the case  $M = \mathbb{R}^n$ , are given by

$$\frac{d^2 x^i}{dt^2} + \omega(x) \frac{dx^i}{dt} = 0.$$

Putting  $y^i = \frac{dx^i}{dt}$ , we can write

$$\frac{dy^i}{dt} + \omega(x)y^i = 0, \quad (i = 1, \dots, n).$$

So, we obtain  $y^i = C^i e^{-\int \omega(x(t)) dt}$  and therefore  $x^i = C_0^i + C^i \int e^{-\int \omega(x(t)) dt} dt$ .

8. We can consider the systems  $\Sigma_{\mathcal{R}}$  having

$$Fe = 2a_{jk}^i(x)y^j y^k \frac{\partial}{\partial y^i},$$

where  $a_{jk}^i(x)$  is a symmetric tensor field on  $M$ . The external force field  $Fe$  has homogeneous components of degree 2 with respect to  $y^i$ .

9. Relativistic nonconservative mechanical systems can be obtained for a Minkowski metric in the space-time  $\mathbb{R}^4$ .
10. A particular case of example 1° above [223] is the case when the external force field coefficients  $F^i(x, y)$  are linear in  $y^i$ , i.e.

$$Fe = 2F^i(x, y) \frac{\partial}{\partial y^i} = 2Y_k^i(x)y^k \frac{\partial}{\partial y^i},$$

where  $Y : TM \rightarrow TM$  is a fiber diffeomorphism called *Lorentz force*, namely for any  $x \in M$ , we have

$$Y_x : T_x M \rightarrow T_x M, \quad Y_x \left( \frac{\partial}{\partial x^i} \right) = Y_i^j(x) \frac{\partial}{\partial x^j}.$$



Let us remark that in this case, formally, we can write the Lagrange equations of this RMS in the form

$$\nabla_{\dot{\gamma}} \dot{\gamma} = Y(\dot{\gamma}),$$

where  $\nabla$  is the Levi-Civita connection of the Riemannian space  $(M, g)$  and  $\dot{\gamma}$  is the tangent vector along the evolution curves  $\gamma : [a, b] \rightarrow M$ .

This type of RMS is important because of the global behavior of its evolution curves.

Let us denote by  $S$  the evolutionary semispray, i.e.  $S$  is a vector field on  $TM$  which is tangent to the canonical lift  $\hat{\gamma} = (\gamma, \dot{\gamma})$  of the evolution curves (see the following section for a detailed discussion on the evolution semispray).

We will denote by  $T^a$  the energy levels of the Riemannian metric  $g$ , i.e.

$$T^a = \left\{ (x, y) \in TM : T(x, y) = \frac{a^2}{2} \right\},$$

where  $T$  is the kinetic energy of  $g$ , and  $a$  is a positive constant. One can easily see that  $T^a$  is the hypersurface in  $TM$  of constant Riemannian length vectors, namely for any  $X = (x, y) \in T^a$ , we must have  $|X|_g = a$ , where  $|X|_g$  is the Riemannian length of the vector field  $X$  on  $M$ .

If we restrict ourselves for a moment to the two dimensional case, then it is known that for sufficiently small values of  $c$  the restriction of the flow of the semispray  $S$  to  $T^c$  contains no less than two closed curves when  $M$  is the 2-dimensional sphere, and at least three otherwise. These curves projected to the base manifold  $M$  will give closed evolution curves for the given Riemannian mechanical system.

### 1.3 The evolution semispray of the mechanical system

$\Sigma_{\mathcal{R}}$

Let us assume that  $Fe$  is global defined on  $M$ , and consider the mechanical system  $\Sigma_{\mathcal{R}} = (M, T, Fe)$ . We have

**Theorem 1.3.1 ([175]).** *The following properties hold good:*

1° *The quantity  $S$  defined by*

$$\begin{cases} S = y^i \frac{\partial}{\partial x^i} - (2 \overset{\circ}{G}^i - \frac{1}{2} F^i) \frac{\partial}{\partial y^i}, \\ 2 \overset{\circ}{G}^i = \gamma^i_{jk} y^j y^k, \end{cases} \quad (1.3.1)$$

is a vector field on the velocity space  $TM$ .

2°  $S$  is a semispray, which depends on  $\Sigma_{\mathcal{R}}$  only.

3° The integral curves of the semispray  $S$  are the evolution curves of the system  $\Sigma_{\mathcal{R}}$ .

*Proof.* 1° Writing  $S$  in the form

$$S = \overset{\circ}{S} + \frac{1}{2} Fe, \quad (1.3.2)$$

where  $\overset{\circ}{S}$  is the canonical semispray with the coefficients  $\overset{\circ}{G}^i$ , we can see immediately that  $S$  is a vector field on  $TM$ .

2° Since  $\overset{\circ}{S}$  is a semispray and  $Fe$  a vertical vector field, it follows  $S$  is a semispray. From (1.3.1) we can see that  $S$  depends on  $\Sigma_{\mathcal{R}}$ , only.

3° The integral curves of  $S$  are given by

$$\frac{dx^i}{dt} = y^i; \quad \frac{dy^i}{dt} + 2 \overset{\circ}{G}^i(x, y) = \frac{1}{2} F^i(x, y). \quad (1.3.3)$$

Replacing  $y^i$  in the second equation we obtain (1.1.2)

$S$  will be called *the evolution* or *canonical semispray* of the non-conservative Riemannian mechanical system  $\Sigma_{\mathcal{R}}$ . In the terminology of J. Klein [123],  $S$  is the dynamical system of  $\Sigma_{\mathcal{R}}$ .

Based on  $S$  we can develop the geometry of the mechanical system  $\Sigma_{\mathcal{R}}$  on  $TM$ .

Let us remark that  $S$  can also be written as follows:

$$S = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y) \frac{\partial}{\partial y^i}, \quad (1.3.1')$$

with the coefficients

$$2G^i = 2 \overset{\circ}{G}^i - \frac{1}{2} F^i. \quad (1.3.4)$$

We point out that  $S$  is homogeneous of degree 2 if and only if  $F^i(x, y)$  is 2-homogeneous with respect to  $y^i$ . This property is not sat-

isfied in the case  $\frac{\partial F^i}{\partial y^j} \equiv 0$ , and it is satisfied for examples 1 and 7 from section 1.2.

We have

**Theorem 1.3.2.** *The variation of the kinetic energy  $T$  of a mechanical system  $\Sigma_{\mathcal{R}}$ , along the evolution curves (1.1.2), is given by:*

$$\frac{dT}{dt} = \frac{1}{2} F_i \frac{dx^i}{dt}. \quad (1.3.5)$$

*Proof.* A straightforward computation gives

$$\begin{aligned} \frac{dT}{dt} &= \frac{\partial T}{\partial x^i} \frac{dx^i}{dt} + \frac{\partial T}{\partial y^i} \frac{dy^i}{dt} = \left( \frac{d}{dt} \frac{\partial T}{\partial y^i} - \frac{1}{2} F_i \right) \frac{dx^i}{dt} + \frac{\partial T}{\partial y^i} \frac{dy^i}{dt} = \\ &= \frac{d}{dt} \left( y^i \frac{\partial T}{\partial y^i} \right) - \frac{1}{2} F_i \frac{dx^i}{dt} = 2 \frac{dT}{dt} - \frac{1}{2} F_i \frac{dx^i}{dt}, \end{aligned}$$

and the relation (1.3.5) holds good.

**Corollary 1.3.1.**  *$T = \text{constant}$  along the evolution curves if and only if the Liouville vector  $\mathbb{C}$  and the external force  $Fe$  are orthogonal vectors along the evolution curves of  $\Sigma$ .*

**Corollary 1.3.2.** *If  $F_i = \text{grad}_i \mathcal{U}$  then  $\Sigma_{\mathcal{R}}$  is conservative and  $T + \mathcal{U} = h$  (constant) on the evolution curves of  $\Sigma_{\mathcal{R}}$ .*

If the external forces  $Fe$  are dissipative, i.e.  $\langle \mathbb{C}, Fe \rangle < 0$ , then from the previous theorem, it follows a result of Bucataru-Miron (see [49]):

**Corollary 1.3.3.** *The kinetic energy  $T$  decreases along the evolution curves if and only if the external forces  $Fe$  are dissipative.*

Since the energy of  $\Sigma_{\mathcal{R}}$  is  $T$  (the kinetic energy), the Theorem 1.3.2 holds good in this case. The variation of  $T$  is given by (1.3.5) and hence we obtain:  $T$  is conserved along the evolution curves of  $\Sigma_{\mathcal{R}}$  if and only if the vector field  $Fe$  and the Liouville vector field  $\mathbb{C}$  are orthogonal.

## 1.4 The nonlinear connection of $\Sigma_{\mathcal{R}}$

Let us consider the evolution semispray  $S$  of  $\Sigma_{\mathcal{R}}$  given by

$$S = y^j \frac{\partial}{\partial x^i} - \left( 2 \overset{\circ}{G}^i - \frac{1}{2} F^i \right) \frac{\partial}{\partial y^i} \quad (1.4.1)$$

with the coefficients

$$2G^i = 2 \overset{\circ}{G}^i - \frac{1}{2} F^i. \quad (1.4.2)$$

Consequently, the evolution nonlinear connection  $N$  of the mechanical system  $\Sigma_{\mathcal{R}}$  has the coefficients:

$$N^i_j = \overset{\circ}{N}^i_j - \frac{1}{4} \frac{\partial F^i}{\partial y^j} = \gamma^i_{jk} y^k - \frac{1}{4} \frac{\partial F^i}{\partial y^j}. \quad (1.4.3)$$

If the external forces  $Fe$  does not depend by velocities  $y^i = \frac{dx^i}{dt}$ , then  $N = \overset{\circ}{N}$ .

Let us consider the *helicoidal vector field* (see Bucataru-Miron, [49], [50])

$$P_{ij} = \frac{1}{2} \left( \frac{\partial F_i}{\partial y^j} - \frac{\partial F_j}{\partial y^i} \right) \quad (1.4.4)$$

and the symmetric part of tensor  $\frac{\partial F_i}{\partial y^j}$ :

$$Q_{ij} = \frac{1}{2} \left( \frac{\partial F_i}{\partial y^j} + \frac{\partial F_j}{\partial y^i} \right). \quad (1.4.5)$$

On  $TM$ ,  $P$  gives rise to the 2-form:

$$(1.4.4') \quad P = P_{ij} dx^i \wedge dx^j$$

and  $Q$  is the symmetric vertical tensor:

$$(1.4.5') \quad Q = Q_{ij} dx^i \otimes dx^j.$$

Denoting by  $\nabla$  the dynamical derivative with respect to the pair  $(S, N)$ , one proves the theorem of Bucataru-Miron:

**Theorem 1.4.1.** *For a Riemannian mechanical system  $\Sigma_{\mathcal{R}} = (M, T, Fe)$  the evolution nonlinear connection is the unique nonlinear connection that satisfies the following conditions:*

$$(1)\nabla g = -\frac{1}{2}Q$$

$$(2)\theta_L(hX, hY) = \frac{1}{2}P(X, Y), \quad \forall X, Y \in \chi(TM),$$

where

$$\theta_L = 4g_{ij} \overset{\circ}{\delta} y^j \wedge dx^i \quad (1.4.6)$$

is the symplectic structure determined by the metric tensor  $g_{ij}$  and the nonlinear connection  $\overset{\circ}{N}$ .

The adapted basis of the distributions  $N$  and  $V$  is given by  $\left(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\right)$ , where

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N^j_i \frac{\partial}{\partial y^j} = \frac{\overset{\circ}{\delta}}{\delta x^i} + \frac{1}{4} \frac{\partial F^j}{\partial y^i} \frac{\partial}{\partial y^j} \quad (1.4.7)$$

and its dual basis  $(dx^i, \delta y^i)$  has the 1-forms  $\delta y^i$  expressed by

$$\delta y^i = dy^i + N^i_j dx^j = \overset{\circ}{\delta} y^i - \frac{1}{4} \frac{\partial F^i}{\partial y^j} dx^j. \quad (1.4.8)$$

It follows that the curvature tensor  $\mathcal{R}^i_{jk}$  of  $N$  (from (1.4.3) is

$$\mathcal{R}^k_{ij} = \frac{\delta N^k_i}{\delta x^j} - \frac{\delta N^k_j}{\delta x^i} = \left( \frac{\delta}{\delta x^j} \frac{\partial}{\partial y^i} - \frac{\delta}{\delta x^i} \frac{\partial}{\partial y^j} \right) \left( G^k - \frac{1}{4} F^k \right) \quad (1.4.9)$$

and the torsion tensor of  $N$  is:

$$t^k_{ij} = \frac{\partial N^k_i}{\partial y^j} - \frac{\partial N^k_j}{\partial y^i} = \left( \frac{\partial}{\partial y^j} \frac{\partial}{\partial y^i} - \frac{\partial}{\partial y^i} \frac{\partial}{\partial y^j} \right) \left( G^k - \frac{1}{4} F^k \right) = 0. \quad (1.4.10)$$

These formulas have the following consequences.

1. The evolution nonlinear connection  $N$  of  $\Sigma_{\mathcal{R}}$  is integrable if and only if the curvature tensor  $\mathcal{R}^i_{jk}$  vanishes.
2. The nonlinear connection is torsion free, i.e.  $t^k_{ij} = 0$ .

The autoparallel curves of the evolution nonlinear connection  $N$  are given by the system of differential equations

$$\frac{d^2x^i}{dt^2} + N^i_j \left( x, \frac{dx}{dt} \right) \frac{dx^j}{dt} = 0,$$

which is equivalent to

$$\frac{d^2x^i}{dt^2} + \overset{\circ}{N}^i_j \left( x, \frac{dx}{dt} \right) \frac{dx^j}{dt} = \frac{1}{4} \frac{\partial F^i}{\partial y^j} \frac{dx^j}{dt}. \quad (1.4.11)$$

In the initial conditions  $\left( x_0, \left( \frac{dx}{dt} \right)_0 \right)$ , locally one uniquely determines the autoparallel curves of  $N$ .

If  $F^i$  is 2-homogeneous with respect to  $y^i$ , then the previous system coincides with the Lagrange equations (1.1.4).

Therefore, we have:

**Theorem 1.4.2.** *If the external forces  $Fe$  are 2-homogeneous with respect velocities  $y^i = \frac{dx^i}{dt}$ , then the evolution curves of  $\Sigma_{\mathcal{R}}$  coincide to the autoparallel curves of the evolution nonlinear connection  $N$  of  $\Sigma_{\mathcal{R}}$ .*

In order to proceed further, we need the exterior differential of 1-forms  $\delta y^i$ .

One obtains

$$d(\delta y^i) = dN^i_j \wedge dx^j = \frac{1}{2} R^i_{kj} dx^j \wedge dx^k + B^i_{kj} \delta y^j \wedge dx^k, \quad (1.4.12)$$

where

$$B^i_{kj} = B^i_{jk} = \frac{\partial^2 G^i}{\partial y^k \partial y^j} \quad (1.4.13)$$

are the coefficients of the Berwald connection determined by the nonlinear connection  $N$ .

## 1.5 The canonical metrical connection $CG(N)$

The coefficients of the canonical metrical connection  $CG(N) = (F^i_{jk}, C^i_{jk})$  are given by the generalized Christoffel symbols [166]:

$$\begin{cases} F^i_{jk} = \frac{1}{2}g^{is} \left( \frac{\delta g_{sk}}{\delta x^j} + \frac{\delta g_{js}}{\delta x^k} - \frac{\delta g_{jk}}{\delta x^s} \right), \\ C^i_{jk} = \frac{1}{2}g^{is} \left( \frac{\partial g_{sk}}{\partial y^j} + \frac{\partial g_{js}}{\partial y^k} - \frac{\partial g_{jk}}{\partial y^s} \right) \end{cases} \quad (1.5.1)$$

where  $g_{ij}(x)$  is the metric tensor of  $\Sigma_{\mathcal{R}}$ .

On the other hand, we have  $\frac{\delta g_{jk}}{\delta x^i} = \frac{\partial g_{jk}}{\partial x^i}$  and  $\frac{\partial g_{jk}}{\partial y^i} = 0$ , and therefore we obtain:

**Theorem 1.5.1.** *The canonical metrical connection  $CG(N)$  of the mechanical system  $\Sigma_{\mathcal{R}}$  has the coefficients*

$$F^i_{jk}(x, y) = \gamma^i_{jk}(x), \quad C^i_{jk}(x, y) = 0. \quad (1.5.2)$$

Let  $\omega^i_j$  be the connection forms of  $CG(N)$ :

$$\omega^i_j = F^i_{jk} dx^k + C^i_{jk} \delta y^k = \gamma^i_{jk}(x) dx^k. \quad (1.5.3)$$

Then, we have ([166]):

**Theorem 1.5.2.** *The structure equation of  $CG(N)$  can be expressed by*

$$\begin{cases} d(dx^i) - dx^k \wedge \omega^i_k = -\overset{1}{\Omega}^i, \\ d(\delta y^i) - \delta y^k \wedge \omega^i_k = -\overset{2}{\Omega}^i, \\ d\omega^i_j - \omega^k_j \wedge \omega^i_k = -\overset{1}{\Omega}^i_j, \end{cases} \quad (1.5.4)$$

where the 2-forms of torsion  $\overset{1}{\Omega}^i$ ,  $\overset{2}{\Omega}^i$  are as follows

$$\begin{aligned} \overset{1}{\Omega}^i &= C^i_{jk} dx^j \wedge \delta y^k = 0, \\ \overset{2}{\Omega}^i &= R^i_{jk} dx^j \wedge dx^k + P^i_{jk} dx^j \wedge \delta y^k. \end{aligned} \quad (1.5.5)$$

Here  $R^i_{jk}$  is the curvature tensor of  $N$  and  $P^i_{jk} = \gamma^i_{jk} - \gamma^i_{kj} = 0$ .

The curvature 2-form  $\Omega^i_j$  is given by

$$\Omega^i_j = \frac{1}{2} R^i_{jkh} dx^k \wedge dx^h + P^i_{jkh} dx^k \wedge \delta y^h + \frac{1}{2} S^i_{jkh} \delta y^k \wedge \delta y^h, \quad (1.5.6)$$

where

$$\begin{aligned} R_{hjk}^i &= \frac{\delta F_{hj}^i}{\delta x^k} - \frac{\delta F_{hk}^i}{\delta x^j} + F_{hj}^s F_{sk}^i - F_{hk}^s F_{sj}^i + C_{hs}^i R_{jk}^s = \\ &= \frac{\partial \gamma_{hj}^i}{\partial x^k} - \frac{\partial \gamma_{hk}^i}{\partial x^j} + \gamma_{hj}^s \gamma_{sk}^i - \gamma_{hk}^s \gamma_{sj}^i = \mathbf{r}_{hjk}^i \end{aligned} \quad (1.5.7)$$

is the Riemannian tensor of curvature of the Levi-Civita connection  $\gamma_{jk}^i(x)$  and the curvature tensors  $P_j^{ikh}, S_j^{ikh}$  vanish.

Therefore, the tensors of torsion of  $C\Gamma(N)$  are

$$R_{jk}^i, T_{jk}^i = 0, S_{jk}^i = 0, P_{jk}^i = 0, C_{jk}^i = 0 \quad (1.5.8)$$

and the curvature tensors of  $C\Gamma(N)$  are

$$R_{jkh}^i(x, y) = \mathbf{r}_{jkh}^i(x), P_{jkh}^i(x, y) = 0, S_{jkh}^i(x, y) = 0. \quad (1.5.9)$$

The Bianchi identities can be obtained directly from (1.5.4), taking into account the conditions (1.5.8) and (1.5.9).

The  $h$ - and  $v$ -covariant derivatives of  $d$ -tensor fields with respect to  $C\Gamma(N) = (\gamma_{jk}^i, 0)$  are expressed, for instance, by

$$\begin{aligned} \nabla_k t_{ij} &= \frac{\delta t_{ij}}{\delta x^k} - \gamma_{ik}^s t_{sj} - \gamma_{jk}^s t_{is}, \\ \dot{\nabla}_k t_{ij} &= \frac{\partial t_{ij}}{\partial y^k} - C_{ik}^s t_{sj} - C_{jk}^s t_{is} = \frac{\partial t_{ij}}{\partial y^k}. \end{aligned}$$

Therefore,  $C\Gamma(N)$  being a metric connection with respect to  $g_{ij}(x)$ , we have

$$\nabla_k g_{ij} = \overset{\circ}{\nabla}_k g_{ij} = 0, \quad (1.5.10)$$

( $\overset{\circ}{\nabla}$  is the covariant derivative with respect to Levi-Civita connection of  $g_{ij}$ ) and

$$\dot{\nabla}_k g_{ij} = 0. \quad (1.5.10')$$

The deflection tensors of  $C\Gamma(N)$  are

$$D_j^i = \nabla_j y^i = \frac{\delta y^i}{\delta x^j} + y^s \gamma_{sj}^i = -N_j^i + y^s \gamma_{sj}^i$$



and

$$d^i_j = \dot{\nabla}_j y^i = \delta^i_j.$$

The evolution nonlinear connection of a Riemannian mechanical system  $\Sigma_{\mathcal{R}}$  given by (1.4.3) implies

$$D^i_j = -\overset{\circ}{N}_j^i + \frac{1}{4} \frac{\partial F^i}{\partial y^j} + y^s \gamma_{sj}^i = \frac{1}{4} \frac{\partial F^i}{\partial y^j},$$

where we have used  $\overset{\circ}{D}_j y^i = \overset{\circ}{D}_j^i = 0$ . It follows

**Proposition 1.5.1.** *For a Riemannian mechanical system the deflection tensors  $D^i_j$  and  $d^i_j$  of the connection  $C\Gamma(N)$  are expressed by*

$$D^i_j = \frac{1}{4} \frac{\partial F^i}{\partial y^j}, \quad d^i_j = \delta^i_j. \quad (1.5.11)$$

## 1.6 The electromagnetism in the theory of the Riemannian mechanical systems $\Sigma_{\mathcal{R}}$

In a Riemannian mechanical system  $\Sigma_{\mathcal{R}} = (M, T, Fe)$  whose external forces  $Fe$  depend on the point  $x$  and on the velocity  $y^i = \frac{dx^i}{dt}$ , the electromagnetic phenomena appears because the deflection tensors  $D^i_j$  and  $d^i_j$  from (1.5.11) nonvanish. Hence the  $d$ -tensors  $D_{ij} = g_{ih} D^h_j$ ,  $d_{ij} = g_{ih} d^h_j = g_{ij}$  determine the  $h$ -electromagnetic tensor  $\mathcal{F}_{ij}$  and  $v$ -electromagnetic tensor  $f_{ij}$  by the formulas, [166]:

$$\mathcal{F}_{ij} = \frac{1}{2}(D_{ij} - D_{ji}), \quad f_{ij} = \frac{1}{2}(d_{ij} - d_{ji}). \quad (1.6.1)$$

By means of (1.5.11), we have

**Proposition 1.6.1.** *The  $h$ - and  $v$ -tensor fields  $\mathcal{F}_{ij}$  and  $f_{ij}$  are given by*

$$\mathcal{F}_{ij} = \frac{1}{4} P_{ij}, \quad f_{ij} = 0. \quad (1.6.2)$$

where  $P_{ij}$  is the helicoidal tensor (1.4.4) of  $\Sigma_{\mathcal{R}}$ .

Indeed, we have

$$\mathcal{F}_{ij} = \frac{1}{4}(D_{ij} - D_{ji}) = \frac{1}{8} \left( \frac{\partial F^i}{\partial y^j} - \frac{\partial F^j}{\partial y^i} \right) = \frac{1}{4} P_{ij}.$$

If we denote  $R_{ijk} := g_{ih} R^h{}_{jk}$ , then we can prove:

**Theorem 1.6.1.** *The electromagnetic tensor  $\mathcal{F}_{ij}$  of the mechanical system  $\Sigma_{\mathcal{R}} = (M, T, Fe)$  satisfies the following generalized Maxwell equations:*

$$\nabla_k \mathcal{F}_{ji} + \nabla_i \mathcal{F}_{kj} + \nabla_j \mathcal{F}_{ik} = -(R_{kji} + R_{ikj} + R_{jik}). \quad (1.6.3)$$

$$\dot{\nabla}_k \mathcal{F}_{ji} = 0. \quad (1.6.3')$$

*Proof.* Applying the Ricci identities to the Liouville vector field, we obtain

$$\nabla_i D^k{}_j - \nabla_j D^k{}_i = y^h \mathbf{r}_{hij}{}^k - R_{ij}{}^k, \quad \nabla_i d^k{}_j - \dot{\nabla}_j D^k{}_i = 0$$

and this leads to

$$\nabla_i D_{kj} - \nabla_j D_{ki} = y^h \mathbf{r}_{hki}{}_j - R_{ki}{}_j, \quad (1.6.4)$$

$$\dot{\nabla}_j D_{ki} = 0. \quad (1.6.4')$$

By taking cyclic permutations of the indices  $i, k, j$  and adding in (1.6.4), by taking into account the identity  $\mathbf{r}_{hijk} + \mathbf{r}_{hjki} + \mathbf{r}_{hkij} = 0$  we deduce (1.6.3), and analogously (1.6.3').

From equations (1.6.2) and (1.6.4') we obtain as consequences:

**Corollary 1.6.4.** *The electromagnetic tensor  $\mathcal{F}_{ij}$  of the mechanical system  $\Sigma_{\mathcal{R}} = (M, T, Fe)$  does not depend on the velocities  $y^i = \frac{dx^i}{dt}$ .*

Indeed, by means of (1.6.3') we have  $\dot{\nabla}_j \mathcal{F}_{ik} = \frac{\partial \mathcal{F}_{ik}}{\partial y^j} = 0$ .

In other words, the helicoidal tensor  $P_{ij}$  of  $\Sigma_{\mathcal{R}}$  does not depend on the velocities  $y^i = \frac{dx^i}{dt}$ .

We end the present section with a remark: this theory has applications to the mechanical systems given by example 1° in Section 1.2.

## 1.7 The almost Hermitian model of the RMS $\Sigma_{\mathcal{R}}$

Let us consider a RMS  $\Sigma_{\mathcal{R}} = (M, T(x, y), Fe(x, y))$  endowed with the evolution nonlinear connection with coefficients  $N_j^i$  from (1.4.3) and with the canonical  $N$ -metrical connection  $CG(N) = (\gamma_{jk}^i(x), 0)$ . Thus, on the velocity space  $\widetilde{TM} = TM \setminus \{0\}$  we can determine an almost Hermitian structure  $H^{2n} = (\widetilde{TM}, \mathbb{G}, \mathbb{F})$  which depends on the RMS only.

Let  $\left(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\right)$  be the adapted basis to the distributions  $N$  and  $V$  and its adapted cobasis  $(dx^i, \delta y^i)$ , where

$$\begin{aligned}\frac{\delta}{\delta x^i} &= \overset{\circ}{\delta} \frac{\partial}{\partial x^i} + \frac{1}{4} \frac{\partial F^s}{\partial y^i} \frac{\partial}{\partial y^s} \\ \delta y^i &= \overset{\circ}{\delta} y^i - \frac{1}{4} \frac{\partial F^i}{\partial y^s} dx^s\end{aligned}\tag{1.7.1}$$

The lift of the fundamental tensor  $g_{ij}(x)$  of the Riemannian space  $\mathcal{R}^n = (M, g_{ij}(x))$  is defined by

$$\mathbb{G} = g_{ij} dx^i \otimes dx^j + g_{ij} \delta y^i \otimes \delta y^j,\tag{1.7.2}$$

and the almost complex structure  $\mathbb{F}$ , determined by the nonlinear connection  $N$ , is expressed by

$$\mathbb{F} = -\frac{\partial}{\partial y^i} \otimes dx^i + \frac{\delta}{\delta x^i} \otimes \delta y^i.\tag{1.7.3}$$

Thus, the following theorems hold good.

**Theorem 1.7.1.** *We have:*

1. The pair  $(\widetilde{TM}, \mathbb{G})$  is a pseudo-Riemannian space.
2. The tensor  $\mathbb{G}$  depends on  $\Sigma_{\mathcal{R}}$  only.
3. The distributions  $N$  and  $V$  are orthogonal with respect to  $\mathbb{G}$ .

**Theorem 1.7.2.** 1. The pair  $(\widetilde{TM}, \mathbb{F})$  is an almost complex space.

2. The almost complex structure  $\mathbb{F}$  depends on  $\Sigma_{\mathcal{R}}$  only.

3.  $\mathbb{F}$  is integrable on the manifold  $\widetilde{TM}$  if and only if the  $d$ -tensor field  $R_{jk}^i(x, y)$  vanishes.

Also, it is not difficult to prove

**Theorem 1.7.3.** *We have*

1. *The triple  $H^{2n} = (\widetilde{TM}, \mathbb{G}, \mathbb{F})$  is an almost Hermitian space.*
2. *The space  $H^{2n}$  depends on  $\Sigma_{\mathcal{R}}$  only.*
3. *The almost symplectic structure of  $H^{2n}$  is*

$$\theta = g_{ij} \delta y^i \wedge dx^j. \quad (1.7.4)$$

If the almost symplectic structure  $\theta$  is a symplectic one (i.e.  $d\theta = 0$ ), then the space  $H^{2n}$  is almost Kählerian.

On the other hand, using the formulas (1.7.4), (1.4.12) one obtains

$$\begin{aligned} d\theta &= \frac{1}{3!} (R_{ijk} + R_{jki} + R_{kij}) dx^i \wedge dx^j \wedge dx^k \\ &\quad + \frac{1}{2} (g_{is} B_{jk}^s - g_{js} B_{ik}^s) \delta y^k \wedge dx^j \wedge dx^i. \end{aligned}$$

Therefore, we deduce

**Theorem 1.7.4.** *The almost Hermitian space  $H^{2n}$  is almost Kählerian if and only if the following relations hold good*

$$R_{ijk} + R_{jki} + R_{kij} = 0, \quad g_{is} B_{jk}^s - g_{js} B_{ik}^s = 0. \quad (1.7.5)$$

The space  $H^{2n} = (\widetilde{TM}, \mathbb{G}, \mathbb{F})$  is called the *almost Hermitian model* of the RMS  $\Sigma_{\mathcal{R}}$ .

One can use the almost Hermitian model  $H^{2n}$  to study the geometrical theory of the mechanical system  $\Sigma_{\mathcal{R}}$ . For instance the Einstein equations of the RMS  $\Sigma_{\mathcal{R}}$  are the Einstein equations of the pseudo-Riemannian space  $(\widetilde{TM}, \mathbb{G})$  (cf. Chapter 6, part 1).

*Remark 1.7.3.* The previous theory can be applied without difficulties to the examples 1-8 in Section 1.2.

## Chapter 2

### Finslerian Mechanical systems

The present chapter is devoted to the Analytical Mechanics of the Finslerian Mechanical systems. These systems are defined by a triple  $\Sigma_F = (M, F^2, Fe)$  where  $M$  is the configuration space,  $F(x, y)$  is the fundamental function of a semidefinite Finsler space  $F^n = (M, F(x, y))$  and  $Fe(x, y)$  are the external forces. Of course,  $F^2$  is the kinetic energy of the space. The fundamental equations are the Lagrange equations:

$$E_i(F^2) \equiv \frac{d}{dt} \frac{\partial F^2}{\partial \dot{x}^i} - \frac{\partial F^2}{\partial x^i} = F_i(x, \dot{x}).$$

We study here the canonical semispray  $S$  of  $\Sigma_F$  and the geometry of the pair  $(TM, S)$ , where  $TM$  is velocity space.

One obtains a generalization of the theory of Riemannian Mechanical systems, which has numerous applications and justifies the introduction of such a new kind of analytical mechanics.

#### 2.1 Semidefinite Finsler spaces

**Definition 2.1.1.** A Finsler space with semidefinite Finsler metric is a pair  $F^n = (M, F(x, y))$  where the function  $F : TM \rightarrow \mathbb{R}$  satisfies the following axioms:

- 1°  $F$  is differentiable on  $\widetilde{TM}$  and continuous on the null section of  $\pi : TM \rightarrow M$ ;
- 2°  $F \geq 0$  on  $TM$ ;
- 3°  $F$  is positive 1-homogeneous with respect to velocities  $\dot{x}^i = y^i$ .
- 4° The fundamental tensor  $g_{ij}(x, y)$

$$g_{ij} = \frac{1}{2} \frac{\partial^2 F^2}{\partial y^i \partial y^j} \quad (2.1.1)$$

has a constant signature on  $\widetilde{TM}$ ;

5° The Hessian of fundamental function  $F^2$  with elements  $g_{ij}(x, y)$  is nonsingular:

$$\det(g_{ij}(x, y)) \neq 0 \text{ on } \widetilde{TM}. \quad (2.1.2)$$

**Example.** If  $g_{ij}(x)$  is a semidefinite Riemannian metric on  $M$ , then

$$F = \sqrt{|g_{ij}(x)y^i y^j|} \quad (2.1.3)$$

is a function with the property  $F^n = (M, F)$  is a semidefinite Finsler space.

Any Finsler space  $F^n = (M, F(x, y))$ , in the sense of definition 3.1.1, part I, is a definite Finsler space. In this case the property 5° is automatical verified.

But, these two kind of Finsler spaces have a lot of common properties. Therefore, we will speak in general on Finsler spaces. The following properties hold:

1° The fundamental tensor  $g_{ij}(x, y)$  is 0-homogeneous;

2°  $F^2 = g_{ij}(x, y)y^i y^j$ ;

3°  $p_i = \frac{1}{2} \frac{\partial F^2}{\partial y^i}$  is  $d$ -covariant vector field;

4° The Cartan tensor

$$C_{ijk} = \frac{1}{4} \frac{\partial^3 F^2}{\partial y^i \partial y^j \partial y^k} = \frac{1}{2} \frac{\partial g_{ij}}{\partial y^k} \quad (2.1.4)$$

is totally symmetric and

$$y^i C_{ijk} = C_{0jk} = 0. \quad (2.1.5)$$

5°  $\omega = p_i dx^i$  is 1-form on  $\widetilde{TM}$  (the Cartan 1-form);

6°  $\theta = d\omega = dp_i \wedge dx^i$  is 2-form (the Cartan 2-form);

7° The Euler-Lagrange equations of  $F^n$  are

$$E_i(F^2) = \frac{d}{dt} \frac{\partial F^2}{\partial y^i} - \frac{\partial F^2}{\partial x^i} = 0, \quad y^j = \frac{dx^j}{dt} \quad (2.1.6)$$

8° The energy  $\mathcal{E}_F$  of  $F^n$  is

$$\mathcal{E}_F = y^i \frac{\partial F^2}{\partial y^i} - F^2 = F^2 \quad (2.1.7)$$

9° The energy  $\mathcal{E}_F$  is conserved along to every integral curve of Euler-Lagrange equations (2.1.6);

10° In the canonical parametrization, the equations (2.1.6) give the geodesics of  $F^n$ ;

11° The Euler-Lagrange equations (2.1.6) can be written in the equivalent form

$$\frac{d^2 x^i}{dt^2} + \gamma_{jk}^i(x, \frac{dx}{dt}) \frac{dx^j}{dt} \frac{dx^k}{dt} = 0, \quad (2.1.8)$$

where  $\gamma_{jk}^i(x, \frac{dx}{dt})$  are the Christoffel symbols of the fundamental tensor  $g_{ij}(x, y)$ .

12° The canonical semispray  $S$  is

$$S = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y) \frac{\partial}{\partial y^i} \quad (2.1.9)$$

with the coefficients:

$$2G^i(x, y) = \gamma_{jk}^i(x, y) y^j y^k = \gamma_{00}^i(x, y), \quad (2.1.9')$$

(the index “0” means the contraction with  $y^i$ ).

13° The canonical semispray  $S$  is 2-homogeneous with respect to  $y^i$ . So,  $S$  is a spray.

14° The nonlinear connection  $N$  determined by  $S$  is also canonical and it is exactly the famous Cartan nonlinear connection of the space  $F^n$ . Its coefficients are

$$N^i_j(x, y) = \frac{\partial G^i(x, y)}{\partial y^j} = \frac{1}{2} \frac{\partial}{\partial y^j} \gamma_{00}^i(x, y). \quad (2.1.10)$$

An equivalent form for the coefficients  $N^i_j$  is as follows

$$N^i_j = \gamma_{j0}^i(x, y) - C_{jk}^i(x, y) \gamma_{00}^k(x, y). \quad (2.1.10')$$

Consequently, we have

$$N^i_0 = \gamma_{00}^i = 2G^i. \quad (2.1.11)$$

Therefore, we can say: The semispray  $S'$  determined by the Cartan nonlinear connection  $N$  is the canonical spray  $S$  of space  $F^n$ .

15° The Cartan nonlinear connection  $N$  determines a splitting of vector space  $T_uTM$ ,  $\forall u \in TM$  of the form:

$$T_uTM = N_u \oplus V_u, \quad \forall u \in TM \quad (2.1.12)$$

Thus, the adapted basis  $\left(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\right)$ ,  $(i = 1, \dots, n)$ , to the previous splitting has the local vector fields  $\frac{\delta}{\delta x^i}$  given by:

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N^j{}_i(x, y) \frac{\partial}{\partial y^j}, \quad i = 1, \dots, n, \quad (2.1.13)$$

with the coefficients  $N^i{}_j(x, y)$  from (2.1.6).

Its dual basis is  $(dx^i, \delta y^i)$ , where

$$\delta y^i = dy^i + N^i{}_j(x, y) dx^j. \quad (2.1.14)$$

The autoparallel curves of the nonlinear connection  $N$  are given by, [166],

$$\frac{d^2 x^i}{dt^2} + N^i{}_j(x, \frac{dx}{dt}) \frac{dx^j}{dt} = 0. \quad (2.1.15)$$

Using the dynamic derivative  $\nabla$  defined by  $N$ , the equations (2.1.11) can be written as follows

$$\nabla\left(\frac{dx^i}{dt}\right) = 0. \quad (2.1.11')$$

16° The variational equations of the autoparallel curves (2.1.11) give the Jacobi equations:

$$\nabla^2 \xi^i + \left(\frac{\partial N^i{}_j}{\partial y^k} \frac{dx^j}{dt} - N^i{}_k\right) \nabla \xi^k + R^i{}_{jk} \frac{dx^j}{dt} \frac{dx^k}{dt} = 0. \quad (2.1.16)$$

The vector field  $\xi^i(t)$  along a solution  $c(t)$  of the equations (2.1.11) and which verifies the previous equations is called a Jacobi field.

In the Riemannian case,  $\frac{\partial g_{ij}}{\partial y^k} = 0$ , the Jacobi equations (2.1.12) are exactly the classical Jacobi equations:



$$\nabla^2 \xi^i + R^i{}_{jlk}(x) \frac{dx^l}{dt} \frac{dx^j}{dt} \xi^k = 0 \quad (2.1.17)$$

17° A distinguished metric connections  $D$  with the coefficients  $CG(N) = (F^i{}_{jk}, C^i{}_{jk})$  is defined as a  $N$ -linear connection on  $TM$ , metric with respect to the fundamental tensor  $g_{ij}(x, y)$  of Finsler space  $F^n$ , i.e. we have

$$\begin{aligned} g_{ij|k} &= \frac{\delta g_{ij}}{\delta x^k} - F^s{}_{ik} g_{sj} - F^s{}_{jk} g_{is} = 0, \\ g_{ij|k} &= \frac{\partial g_{ij}}{\partial y^k} - C^s{}_{ik} g_{sj} - C^s{}_{jk} g_{is} = 0. \end{aligned} \quad (2.1.18)$$

18° The following theorem holds:

**Theorem 2.1.1.** 1° *There is an unique  $N$ -linear connection  $D$ , with coefficients  $CG(N)$  which satisfies the following system of axioms:  $A_1$ .  $N$  is the Cartan nonlinear connection of Finsler space  $F^n$ .*

*$A_2$ .  $D$  is metrical, (i.e.  $D$  satisfies (2.1.14)).*

*$A_3$ .  $T^i{}_{jk} = F^i{}_{jk} - F^i{}_{kj} = 0$ ,  $S^i{}_{jk} = C^i{}_{jk} - C^i{}_{kj} = 0$ .*

2° *The metric  $N$ -linear connection  $D$  has the coefficients  $CG(N) = (F^i{}_{jk}, C^i{}_{jk})$  given by the generalized Christoffel symbols*

$$\begin{aligned} F^i{}_{jk} &= \frac{1}{2} g^{is} \left( \frac{\delta g_{sj}}{\delta x^k} + \frac{\delta g_{sk}}{\delta x^j} - \frac{\delta g_{jk}}{\delta x^s} \right), \\ C^i{}_{jk} &= \frac{1}{2} g^{is} \left( \frac{\partial g_{sj}}{\partial y^k} + \frac{\partial g_{sk}}{\partial y^j} - \frac{\partial g_{jk}}{\partial y^s} \right). \end{aligned} \quad (2.1.19)$$

20° By means of this theorem, it is not difficult to see that we have

$$C^i{}_{jk} = g^{is} C_{sjk} \quad (2.1.20)$$

and

$$y^i|_k = 0. \quad (2.1.21)$$

21° The Cartan nonlinear connection  $N$  determines on  $\widetilde{TM}$  an almost complex structure  $\mathbb{F}$ , as follows:

$$\mathbb{F}\left(\frac{\delta}{\delta x^i}\right) = -\frac{\partial}{\partial y^i}, \quad \mathbb{F}\left(\frac{\partial}{\partial y^i}\right) = \frac{\delta}{\delta x^i}, \quad i = 1, \dots, n. \quad (2.1.22)$$

But one can see that  $\mathbb{F}$  is the tensor field on  $\widetilde{TM}$ :

$$\mathbb{F} = -\frac{\partial}{\partial y^i} \otimes dx^i + \frac{\delta}{\delta x^i} \otimes \delta y^i, \quad (2.1.22')$$

with the 1-forms  $\delta y^i$  and the vector field  $\frac{\delta}{\delta x^i}$  given by (2.1.10), (2.1.9), (2.1.6).

It is not difficult to prove that: The almost complex structure  $\mathbb{F}$  is integrable if and only if the distribution  $N$  is integrable on  $TM$ .

22° The Sasaki-Matsumoto lift of the fundamental tensor  $g_{ij}$  of Finsler space  $F^n$  is

$$\mathbb{G}(x, y) = g_{ij}(x, y) dx^i \otimes dx^j + g_{ij}(x, y) \delta y^i \otimes \delta y^j. \quad (2.1.23)$$

The tensor field  $\mathbb{G}$  determines a pseudo-Riemannian structure on  $TM$ .

23° The following theorem is known:

**Theorem 2.1.2.** 1° *The pair  $(\mathbb{G}, \mathbb{F})$  is an almost Hermitian structure on  $\widetilde{TM}$  determined only by the Finsler space  $F^n$ .*

2° *The symplectic structure associate to the structure  $(\mathbb{G}, \mathbb{F})$  is the Cartan 2-form:*

$$\theta = 2g_{ij} \delta y^i \wedge dx^j. \quad (2.1.24)$$

3° *The space  $(\widetilde{TM}, \mathbb{G}, \mathbb{F})$  is almost Kählerian.*

The space  $H^{2n} = (\widetilde{TM}, \mathbb{G}, \mathbb{F})$  is called *the almost Kählerian model* of the Finsler space  $F^n$ .

G.S. Asanov in the paper [27] proved that the metric  $\mathbb{G}$  from (2.1.23) does not satisfies the principle of the Post-Newtonian Calculus. This is due to the fact that the horizontal and vertical terms of  $\mathbb{G}$  do not have the same physical dimensions.

This is the reason for R. Miron to introduce a new lift of the fundamental tensor  $g_{ij}$ , [166], in the form:

$$\widetilde{\mathbb{G}}(x, y) = g_{ij}(x, y) dx^i \otimes dx^j + \frac{a^2}{\|y\|^2} g_{ij}(x, y) \delta y^i \otimes \delta y^j$$

where  $a > 0$  is a constant imposed by applications in Theoretical Physics and where  $\|y\|^2 = g_{ij}(x, y) y^i y^j = F^2$  has the property  $F^2 > 0$ . The lift  $\mathbb{G}$  is 2-homogeneous with respect to  $y^i$ . The Sasaki-Matsumoto lift  $\mathbb{G}$  has not the property of homogeneity.

**Two examples:** 1. *Randers spaces.* They have been defined by R. S. Ingarden as a triple  $RF^n = (M, \alpha + \beta, N)$ , where  $\alpha + \beta$  is a Randers metric and  $N$  is the Cartan nonlinear connection of the Finsler space  $F^n = (M, \alpha + \beta)$ , [175].

2. *Ingarden spaces.* These spaces have been defined by R. Miron, [166], as a triple  $IF^n = (M, \alpha + \beta, N_L)$ , where  $\alpha + \beta$  is a Randers metric and  $N_L$  is the Lorentz nonlinear connection of  $F^n = (M, \alpha + \beta)$  having the coefficients

$$N_j^i(x, y) = \overset{\circ}{\gamma}_{jk}^i(x) y^k - \overset{\circ}{F}_j^i(x), \quad \overset{\circ}{F}_j^i = \frac{1}{2} a^{is}(x) \left( \frac{\partial b_s}{\partial x^j} - \frac{\partial b_j}{\partial x^s} \right). \quad (2.1.25)$$

The Christoffel symbols are constructed with the Riemannian metric tensor  $a_{ij}(x)$  of the Riemann space  $(M, \alpha^2)$  and  $\overset{\circ}{F}_j^i(x)$  is the electromagnetic tensor determined by the electromagnetic form  $(\alpha + \beta)$ .

## 2.2 The notion of Finslerian mechanical system

As we know from the previous chapter, the Riemannian mechanical systems  $\Sigma_{\mathcal{R}} = (M, T, Fe)$  is defined as a triple in which  $M$  is the configuration space,  $T$  is the kinetic energy and  $Fe$  are the external forces, which depend on the material point  $x \in M$  and depend on velocities

$$y^i = \frac{dx^i}{dt}.$$

Extending the previous ideas, we introduce the notion of Finslerian Mechanical System, studied by author in the paper [175]. The shortly theory of this analytical mechanics can be find in the joint book *Finsler-Lagrange Geometry. Applications to Dynamical Systems*, by Ioan Bucataru and Radu Miron, Romanian Academy Press, Bucharest, 2007.

In a different manner, M. de Leon and colab. [138], M. Crampin et colab. [66], have studied such kind of new Mechanics.

A Finslerian mechanical system  $\Sigma_F$  is defined as a triple

$$\Sigma_F = (M, \mathcal{E}_{F^2}, Fe) \quad (2.2.1)$$

where  $M$  is a real differentiable manifold of dimension  $n$ , called *the configuration space*,  $\mathcal{E}_{F^2}$  is *the energy* of an a priori given Finsler space  $F^n = (M, F(x, y))$ , which can be positive defined or semidefined, and  $Fe(x, y)$  are the external forces given as a vertical vector

field on the tangent manifold  $TM$ . We continue to say that  $TM$  is the velocity space of  $M$ .

Evidently, the Finslerian mechanical system  $\Sigma_F$  is a straightforward generalization of the known notion of Riemannian mechanical system  $\Sigma_{\mathcal{R}}$  obtained for  $\mathcal{E}_{F^2}$  as kinetic energy of a Riemann space  $\mathcal{R}^n = (M, g)$ .

Therefore, we can introduce the evolution (or fundamental) equations of  $\Sigma_F$  by means of the following Postulate:

**Postulate.** *The evolution equations of the Finslerian mechanical system  $\Sigma_F$  are the Lagrange equations:*

$$\frac{d}{dt} \frac{\partial \mathcal{E}_{F^2}}{\partial y^i} - \frac{\partial \mathcal{E}_{F^2}}{\partial x^i} = F_i(x, y), \quad y^i = \frac{dx^i}{dt} \quad (2.2.2)$$

where the energy is

$$\mathcal{E}_{F^2} = y^i \frac{\partial F^2}{\partial y^i} - F^2 = F^2, \quad (2.2.3)$$

and  $F_i(x, y)$ , ( $i = 1, \dots, n$ ), are the covariant components of the external forces  $Fe$ :

$$\begin{cases} Fe(x, y) = F^i(x, y) \frac{\partial}{\partial y^i} \\ F_i(x, y) = g_{ij}(x, y) F^j(x, y), \end{cases} \quad (2.2.4)$$

and

$$g_{ij}(x, y) = \frac{1}{2} \frac{\partial^2 F^2}{\partial y^i \partial y^j}, \quad \det(g_{ij}(x, y)) \neq 0, \quad (2.2.5)$$

is the fundamental (or metric) tensor of Finsler space  $F^n = (M, F(x, y))$ .

Finally, the Lagrange equations of the Finslerian mechanical system are:

$$\frac{d}{dt} \frac{\partial F^2}{\partial y^i} - \frac{\partial F^2}{\partial x^i} = F_i(x, y), \quad y^i = \frac{dx^i}{dt}. \quad (2.2.6)$$

A more convenient form of the previous equations is given by:

**Theorem 2.2.1.** *The Lagrange equations (2.2.6) are equivalent to the second order differential equations:*

$$\frac{d^2 x^i}{dt^2} + \gamma_{jk}^i(x, \frac{dx}{dt}) \frac{dx^j}{dt} \frac{dx^k}{dt} = \frac{1}{2} F^i(x, \frac{dx}{dt}), \quad (2.2.7)$$

where  $\gamma_{jk}^i(x, y)$  are the Christoffel symbols of the metric tensor  $g_{ij}(x, y)$  of the Finsler space  $F^n$ .

*Proof.* Writing the kinetic energy  $F^2(x, y)$  in the form:

$$F^2(x, y) = g_{ij}(x, y)y^i y^j, \quad (2.2.8)$$

the equivalence of the systems of equations (2.2.6) and (2.2.7) is not difficult to establish.

But, the form (2.2.7) is very convenient in applications. So, we obtain a first result expressed in the following theorems:

**Theorem 2.2.2.** *The trajectories of the Finslerian mechanical system  $\Sigma_F$ , without external forces ( $Fe \equiv 0$ ), are the geodesics of the Finsler space  $F^n$ .*

Indeed,  $F^i(x, y) \equiv 0$  and the SODE (2.2.7) imply the equations (2.2.4) of geodesics of space  $F^n$ .

A second important result is a consequence of the Lagrange equations, too.

**Theorem 2.2.3.** *The variation of kinetic energy  $\mathcal{E}_{F^2} = F^2$  of the mechanical system  $\Sigma_F$  along the evolution curves (2.2.6) is given by*

$$\frac{d\mathcal{E}_{F^2}}{dt} = \frac{dx^i}{dt} F_i. \quad (2.2.9)$$

Consequently:

**Theorem 2.2.4.** *The kinetic energy  $\mathcal{E}_{F^2}$  of the system  $\Sigma_F$  is conserved along the evolution curves (2.2.6) if the external forces  $Fe$  are orthogonal to the evolution curves.*

The external forces  $Fe$  are called *dissipative* if the scalar product  $\langle \mathbb{C}, Fe \rangle$  is negative, [175].

The formula (2.2.9) leads to the following property expressed by:

**Theorem 2.2.5.** *The kinetic energy  $\mathcal{E}_{F^2}$  decreases along the evolution curves of the Finslerian mechanical system  $\Sigma_F$  if and only if the external forces  $Fe$  are dissipative.*

### Some examples of Finslerian mechanical systems

1° The systems  $\Sigma_F = (M, \mathcal{E}_{F^2}, Fe)$  given by  $F^n = (M, \alpha + \beta)$  as a Randers space and  $Fe = \beta\mathbb{C} = \beta y^i \frac{\partial}{\partial y^i}$ . Evidently  $Fe$  is 2-homogeneous with respect to  $y^i$ .

2°  $\Sigma_F$  determined by  $F^n = (M, \alpha + \beta)$  and  $Fe = \alpha\mathbb{C}$ .

3°  $\Sigma_F$  with  $F^n = (M, \alpha + \beta)$  and  $Fe = (\alpha + \beta)\mathbb{C}$ .

4°  $\Sigma_F$  defined by a Finsler space  $F^n = (M, F)$  and  $Fe = a^i_{jk}(x)y^j y^k \frac{\partial}{\partial y^i}$ ,  $a^i_{jk}(x)$  being a symmetric tensor on the configuration space  $M$  of type  $(1, 2)$ .

### 2.3 The evolution semispray of the system $\Sigma_F$

The Lagrange equations (2.2.6) give us the integral curves of a remarkable semispray on the velocity space  $TM$ , which governed the geometry of Finslerian mechanical system  $\Sigma_F$ . So, if the external forces  $Fe$  are global defined on the manifold  $TM$ , we obtain:

**Theorem 2.3.1 (Miron, [175]).** *For the Finslerian mechanical systems  $\Sigma_F$ , the following properties hold good:*

1° The operator  $S$  defined by

$$S = y^i \frac{\partial}{\partial x^i} - \left( 2 \overset{\circ}{G}^i - \frac{1}{2} F^i \right) \frac{\partial}{\partial y^i}; \quad 2 \overset{\circ}{G}^i = \gamma^i_{jk}(x, y) y^j y^k \quad (2.3.1)$$

is a vector field, global defined on the phase space  $TM$ .

2°  $S$  is a semispray which depends only on  $\Sigma_F$  and it is a spray if  $Fe$  is 2-homogeneous with respect to  $y^i$ .

3° The integral curves of the vector field  $S$  are the evolution curves given by the Lagrange equations (2.2.7) of  $\Sigma_F$ .

*Proof.* 1° Let us consider the canonical semispray  $\overset{\circ}{S}$  of the Finsler space  $F^n$ . Thus from (2.3.1) we have

$$S = \overset{\circ}{S} + \frac{1}{2} Fe. \quad (2.3.2)$$

It follows that  $S$  is a vector field on  $TM$ .

2° Since  $Fe$  is a vertical vector field, then  $S$  is a semispray. Evidently,  $S$  depends on  $\Sigma_F$ , only.

3° The integral curves of  $S$  are given by:

$$\frac{dx^i}{dt} = y^i; \quad \frac{dy^i}{dt} + 2 \overset{\circ}{G}^i(x, y) = \frac{1}{2} F^i(x, y). \quad (2.3.3)$$

The previous system of differential equations is equivalent to system (2.2.7).

In the book of I. Bucataru and R. Miron [49], one proves the following important result, which extend a known J. Klein theorem:

**Theorem 2.3.2.** *The semispray  $S$ , given by the formula (2.3.1), is the unique vector field on  $\widetilde{TM}$ , solution of the equation:*

$$i_S \overset{\circ}{\omega} = -dT + \sigma, \quad (2.3.4)$$

where  $\overset{\circ}{\omega}$  is the symplectic structure of the Finsler space  $F^n = (M, F)$ ,  $T = \frac{1}{2} F^2 = \frac{1}{2} g_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt}$  and  $\sigma$  is the 1-form of external forces:

$$\sigma = F_i(x, y) dx^i. \quad (2.3.5)$$

In the terminology of J. Klein, [123],  $S$  is the dynamical system of  $\Sigma_F$ , defined on the tangent manifold  $TM$ . We will say that  $S$  is the evolution semispray of  $\Sigma_F$ .

By means of semispray  $S$  (or spray  $S$ ) we can develop the geometry of the Finslerian mechanical system  $\Sigma_F$ . So, all geometrical notion derived from  $S$ , as nonlinear connections, N-linear connections etc. will be considered as belong to the system  $\Sigma_F$ .

## 2.4 The canonical nonlinear connection of the Finslerian mechanical systems $\Sigma_F$

The evolution semispray  $S$  (2.3.1) has the coefficients  $G^i$  expressed by

$$2G^i(x, y) = 2 \overset{\circ}{G}^i(x, y) - \frac{1}{2} F^i(x, y) = \gamma_{jk}^i(x, y) y^j y^k - \frac{1}{2} F^i(x, y). \quad (2.4.1)$$

Thus, the evolution nonlinear connection  $N$  (or canonical nonlinear connection) of the Finslerian mechanical system  $\Sigma_F$  has the coeffi-

cients:

$$N^i_j = \frac{\partial G^i}{\partial y^j} = \overset{\circ}{N}^i_j - \frac{1}{4} \frac{\partial F^i}{\partial y^j} \quad (2.4.2)$$

where  $\overset{\circ}{N}$  with coefficients  $\overset{\circ}{N}^i_j$  is the Cartan nonlinear connection of Finsler space  $F^n = (M, F(x, y))$ .

$N$  depends on the mechanical system  $\Sigma_F$ , only. It is called canonical for  $F^n$ .

The nonlinear connection  $N$  determines the horizontal distribution, denoted by  $N$  too, with the property

$$T_u TM = N_u \oplus V_u, \quad \forall u \in TM, \quad (2.4.3)$$

$V_u$  being the natural vertical distribution on the tangent manifold  $TM$ .

A local adapted basis to the horizontal and vertical vector spaces  $N_u$  and  $V_u$  is given by  $\left( \frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i} \right)$ , ( $i = 1, \dots, n$ ), where

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N^j_i \frac{\partial}{\partial y^j} = \frac{\partial}{\partial x^i} - \left( \overset{\circ}{N}^j_i - \frac{1}{4} \frac{\partial F^j}{\partial y^i} \right) \frac{\partial}{\partial y^j}, \quad i = 1, \dots, n, \quad (2.4.4)$$

and the adapted cobasis  $(dx^i, \delta y^i)$  with

$$\delta y^i = dy^i + N^i_j dx^j = dy^i + \left( \overset{\circ}{N}^i_j - \frac{1}{4} \frac{\partial F^i}{\partial y^j} \right) dx^j, \quad i = 1, \dots, n. \quad (2.4.4')$$

From (2.4.4) and (2.4.4') it follows

$$\begin{cases} \frac{\delta}{\delta x^i} = \frac{\overset{\circ}{\delta}}{\delta x^i} + \frac{1}{4} \frac{\partial F^j}{\partial y^i} \frac{\partial}{\partial y^j}, \\ \delta y^i = \overset{\circ}{\delta} y^i - \frac{1}{4} \frac{\partial F^i}{\partial y^j} dx^j. \end{cases} \quad (2.4.5)$$

Now we easy determine the curvature  $\mathcal{R}^i_{jk}$  and torsion  $t^i_{jk}$  of the canonical nonlinear connection  $N$ . One obtains



$$\mathcal{R}^i_{jk} = \frac{\delta N^i_j}{\delta x^k} - \frac{\delta N^i_k}{\delta x^j} = \left( \frac{\delta}{\delta x^k} \frac{\partial}{\partial y^j} - \frac{\delta}{\delta x^j} \frac{\partial}{\partial y^k} \right) (G^i - \frac{1}{2}F^i), \quad (2.4.6)$$

$$t^i_{jk} = \frac{\partial N^i_j}{\partial y^k} - \frac{\partial N^i_k}{\partial y^j} = 0$$

such that:

1° The torsion of the canonical nonlinear connection  $N$  vanishes.

2° The condition  $\mathcal{R}^i_{jk} = 0$  is necessary and sufficient for  $N$  to be integrable.

Another important geometric object field determined by the canonical nonlinear connection  $N$  is the Berwald connection  $B\Gamma(N) =$

$(B^i_{jk}, 0) = \left( \frac{\partial N^i_j}{\partial y^k}, 0 \right)$ . Its coefficients are:

$$B^i_{jk} = \overset{\circ}{B}^i_{jk} - \frac{1}{4} \frac{\partial^2 F^i}{\partial y^j \partial y^k}, \quad (2.4.7)$$

where  $\overset{\circ}{B}^i_{jk}$  are coefficients of Berwald connection of Finsler space  $F^n$ . The coefficients (2.4.7) are symmetric.

We can prove:

**Theorem 2.4.1.** *The autoparallel curves of the evolution nonlinear connection  $N$  are given by the following SODE:*

$$y^i = \frac{dx^i}{dt}, \quad \frac{\delta y^i}{dt} = \frac{\overset{\circ}{\delta} y^i}{dt} - \frac{1}{4} \frac{\partial F^i}{\partial y^j} \frac{dx^j}{dt} = 0. \quad (2.4.8)$$

**Corollary 2.4.1.** *If the external forces  $Fe$  vanish, then the evolution nonlinear connection  $N$  is the Cartan nonlinear connection of Finsler space  $F^n$ .*

**Corollary 2.4.2.** *If the external forces  $Fe$  are 2-homogeneous with respect to velocities  $y^i$ , then the equations (2.4.8) coincide with the evolution equations of the Finslerian mechanical system  $\Sigma_F$ .*

It is not difficult to determine the evolution nonlinear connection  $N$  from examples in section 2.3.

**Lemma 2.4.1.** *The exterior differential of the 1-forms  $\delta y^i$  are given by formula:*

$$d(\delta y^i) = dN^i_j \wedge dx^j = \frac{1}{2} \mathcal{R}^i_{kj} dx^j \wedge dx^k + B^i_{kj} \delta y^j \wedge dx^k. \quad (2.4.9)$$

## 2.5 The dynamical derivative determined by the evolution nonlinear connection $N$

The dynamical covariant derivation induced by the evolution nonlinear connection  $N$  is expressed by (?) using the coefficients  $N^i_j$  from (2.4.2). It is given by

$$\nabla \left( X^i \frac{\partial}{\partial y^i} \right) = (S X^i + X^j N^i_j) \frac{\partial}{\partial y^i}. \quad (2.5.1)$$

Applied to a  $d$ -vector field  $X^i(x, y)$ , we have the formula:

$$\nabla X^i = S X^i| = S X^i + X^j N^i_j \quad (2.5.2)$$

and for a  $d$ -tensor  $g_{ij}$ , we have

$$\nabla g_{ij} = g_{ij}| = S g_{ij} - g_{sj} N^s_i - g_{is} N^s_j, \quad (2.5.2')$$

where

$$S = \overset{\circ}{S} + \frac{1}{2} F e \quad (2.5.3)$$

$$N^i_j = \overset{\circ}{N}^i_j - \frac{1}{4} \frac{\partial F^i}{\partial y^j}.$$

Remarking that  $\frac{\partial F^i}{\partial y^j}$  is a  $d$ -tensor field of type (1,1), one can introduce two  $d$ -tensors, important in the geometrical theory of the Finslerian mechanical systems  $\Sigma_F$  :

$$P_{ij} = \frac{1}{2} \left( \frac{\partial F_i}{\partial y^j} - \frac{\partial F_j}{\partial y^i} \right), \quad Q_{ij} = \frac{1}{2} \left( \frac{\partial F_i}{\partial y^j} + \frac{\partial F_j}{\partial y^i} \right). \quad (2.5.4)$$

The first one  $P_{ij}$  is called the helicoidal tensor of the Finslerian mechanical system  $\Sigma_F$ , [166].

Also, on the phase space  $TM$ , the elicoidal  $d$ -tensor  $P_{ij}$  give rise to the 2-form

$$P = P_{ij} dx^i \wedge dx^j \quad (2.5.5)$$

and  $Q_{ij}$  allows to consider the symmetric tensor

$$Q = Q_{ij} dx^i \otimes dx^j. \quad (2.5.6)$$

The following Bucataru-Miron theorem holds:

**Theorem 2.5.1.** *For a Finslerian mechanical system  $\Sigma_F = (M, \mathcal{E}_{F^2}, Fe)$  the evolution nonlinear connection  $N$  is the unique nonlinear connection that satisfies the conditions:*

$$\nabla g = -\frac{1}{2} Q, \quad \overset{\circ}{\omega}(hX, hY) = \frac{1}{2} P(X, Y), \quad \forall X, Y \in \chi(TM), \quad (2.5.7)$$

where  $\overset{\circ}{\omega}$  is the symplectic structure of the Finsler space  $F^n$ :

$$\overset{\circ}{\omega} = 2g_{ij} \overset{\circ}{\delta} y^j \wedge dx^i. \quad (2.5.8)$$

If  $Fe$  does not depend on velocities  $y^i = \frac{dx^i}{dt}$  we have  $P \equiv 0$ ,  $Q \equiv 0$ . One obtains the the case of Riemannian mechanical systems  $\Sigma_R$ , studied in the previous chapter.

## 2.6 Metric N-linear connection of $\Sigma_F$

The metric, or canonic,  $N$ -linear connection  $D$ , with coefficients  $CG(N) = (F^i_{jk}, C^i_{jk})$  of the Finslerian mechanical system  $\Sigma_F$  is uniquely determined by the following axioms, (Miron [161]):

A<sub>1</sub>.  $N$  is the canonical nonlinear connection of  $\Sigma_F$ .

A<sub>2</sub>.  $D$  is  $h$ -metric, i.e.  $g_{ij|k} = 0$ .

A<sub>3</sub>.  $D$  is  $h$ -symmetric, i.e.  $T^i_{jk} = F^i_{jk} - F^i_{kj} = 0$ .

A<sub>4</sub>.  $D$  is  $v$ -metric, i.e.  $g_{ij|k} = 0$ .

A<sub>5</sub>.  $D$  is  $v$ -symmetric, i.e.  $S^i_{jk} = C^i_{jk} - C^i_{kj} = 0$ .

The following important result holds:

**Theorem 2.6.1.** *The local coefficients  $D\Gamma(N) = (F^i_{jk}, C^i_{jk})$  of the canonical  $N$ -connection  $D$  of the Finslerian mechanical system  $\Sigma_F$  are given by the generalized Christoffel symbols:*

$$\begin{cases} F^i_{jk} = \frac{1}{2} g^{is} \left( \frac{\delta g_{sk}}{\delta x^j} + \frac{\delta g_{js}}{\delta x^k} - \frac{\delta g_{jk}}{\delta x^s} \right), \\ C^i_{jk} = \frac{1}{2} g^{is} \left( \frac{\partial g_{sk}}{\partial y^j} + \frac{\partial g_{js}}{\partial y^k} - \frac{\partial g_{jk}}{\partial y^s} \right). \end{cases} \quad (2.6.1)$$

Using the expression (?) of the operator  $\frac{\delta}{\delta x^i}$ , one obtains:

**Theorem 2.6.2.** *The coefficients  $F^i_{jk}$ ,  $C^i_{jk}$  of canonical  $N$ -connection  $D$  have the expressions:*

$$\begin{aligned} F^i_{jk} &= \overset{\circ}{F}^i_{jk} + \check{C}^i_{jk}, & C^i_{jk} &= \overset{\circ}{C}^i_{jk} \\ \check{C}^i_{jk} &= \frac{1}{4} g^{is} \left( \overset{\circ}{C}_{skh} \frac{\partial F^h}{\partial y^j} + \overset{\circ}{C}_{jsh} \frac{\partial F^h}{\partial y^k} - \overset{\circ}{C}_{jkh} \frac{\partial F^h}{\partial y^s} \right), \end{aligned} \quad (2.6.2)$$

where  $CG(\overset{\circ}{N}) = (\overset{\circ}{F}^i_{jk}, \overset{\circ}{C}^i_{jk})$  are the coefficients of canonic Cartan metric connection of Finsler space  $F^n$ .

A consequence of the previous formulas is the following relation:

$$y^j \check{C}^i_{jh} = \frac{1}{4} \overset{\circ}{C}^i_{hky^s} \frac{\partial F^k}{\partial y^s}. \quad (2.6.3)$$

Let  $\omega^i_j$  be the connection forms of  $CG(N)$ :

$$\omega^i_j = F^i_{jk} dx^k + C^i_{jk} \delta y^k. \quad (2.6.4)$$

Then, we have

**Theorem 2.6.3.** *The structure equations of the canonical connection  $CG(N)$  are given by:*

$$\begin{cases} d(dx^i) - dx^k \wedge \omega^i_k = -\overset{1}{\Omega}^i, \\ d(\delta y^i) - \delta y^k \wedge \omega^i_k = -\overset{2}{\Omega}^i, \\ d\omega^i_j - \omega^k_j \wedge \omega^i_k = -\Omega^{ij}, \end{cases} \quad (2.6.5)$$

where the 2-forms of torsion  $\overset{1}{\Omega}^i$ ,  $\overset{2}{\Omega}^i$  are as follows

$$\begin{aligned} \overset{1}{\Omega}^i &= C^i_{jk} dx^j \wedge \delta y^k, \\ \overset{2}{\Omega}^i &= \frac{1}{2} R^i_{jk} dx^j \wedge dx^k + P^i_{jk} dx^j \wedge \delta y^k, \end{aligned} \quad (2.6.6)$$

with

$$R^i_{jk} = \frac{\delta N^i_j}{\delta x^k} - \frac{\delta N^i_k}{\delta x^j}, \quad P^i_{jk} = \frac{\partial N^i_j}{\partial y^k} - F^i_{kj} \quad (2.6.7)$$

and the 2-form of curvature  $\Omega^i_j$  is given by

$$\Omega^i_j = \frac{1}{2} R^i_{jkh} dx^k \wedge dx^h + P^i_{jkh} dx^k \wedge \delta y^h + \frac{1}{2} S^i_{jkh} \delta y^k \wedge \delta y^h, \quad (2.6.8)$$

where

$$\begin{cases} R^i_{jkh} = \frac{\delta F^i_{jk}}{\delta x^h} - \frac{\delta F^i_{jh}}{\delta x^k} + F^s_{jk} F^i_{sh} - F^s_{jh} F^i_{sk} + C^i_{hs} R^s_{kh}, \\ P^i_{jkh} = \frac{\partial F^i_{jk}}{\partial y^h} - C^i_{jh|k} + C^i_{js} P^s_{kh}, \\ S^i_{jkh} = \frac{\partial C^i_{jk}}{\partial y^h} - \frac{\partial C^i_{jh}}{\partial y^k} + C^s_{jk} C^i_{sh} - C^s_{jh} C^i_{jk}. \end{cases} \quad (2.6.9)$$

Taking into account that the coefficients  $F^i_{jk}$  and  $C^i_{jk}$  are expressed in the formulas (2.6.2), the calculus of curvature tensors is not difficult.

Exterior differentiating (2.6.5) and using them again one obtains the Bianchi identities of  $CI(N)$ .

The  $h$ - and  $v$ -covariant derivatives, denoted by “ $|$ ” and “ $\cdot$ ”, with respect to canonical connection  $D$  have the properties given by the axioms  $A_1 - A_4$ .

So, we obtain

$$\begin{cases} g_{ij|k} = \frac{\delta g_{ij}}{\delta x^k} - g_{sj} F^s_{ik} - g_{is} F^s_{jk} = 0, \\ g_{ij} \cdot k = \frac{\partial g_{ij}}{\partial y^k} - g_{sj} C^s_{ik} - g_{is} C^s_{jk} = 0. \end{cases} \quad (2.6.10)$$

The Ricci identities applied to the fundamental tensor  $g_{ij}$  give us:

$$R_{ijkh} + R_{jikh} = 0, \quad P_{ijkh} + P_{jikh} = 0, \quad S_{ijkh} + S_{jikh} = 0,$$

where  $R_{ijkh} = g_{js} R^s_{ikh}$ , etc.

Also,  $P_{ijk} = g_{is} P^s_{jk}$  is totally symmetric.

The deflection tensors of  $D$  are

$$D^i_j = y^i|_j = \frac{\delta y^i}{\delta x^j} + y^s F^i_{sj}; \quad d^i_j = y^i|_j. \quad (2.6.11)$$

Taking into account (2.6.11), we obtain:

$$D^i_j = y^s \check{C}^i_{sj} + \frac{1}{4} \frac{\partial F^i}{\partial y^j}, \quad d^i_j = \delta^i_j. \quad (2.6.12)$$

## 2.7 The electromagnetism in the theory of the Finslerian mechanical systems $\Sigma_F$

For a Finslerian mechanical system  $\Sigma_F = (M, \mathcal{E}_{F^2}, Fe)$  whose external forces  $Fe$  depend on the material point  $x \in M$  and on velocity  $y^i = \frac{dx^i}{dt}$ , the electromagnetic phenomena appears because the deflection tensors  $D^i_j$  and  $d^i_j$  nonvanish.

Setting  $D_{ij} = g_{ih} D^h_j$ ,  $d_{ij} = g_{ih} d^h_j$ , the  $h$ -electromagnetic tensor  $\mathcal{F}_{ij}$  and the  $v$ -electromagnetic tensor  $f_{ij}$  are defined by

$$\mathcal{F}_{ij} = \frac{1}{2}(D_{ij} - D_{ji}), \quad f_{ij} = \frac{1}{2}(d_{ij} - d_{ji}). \quad (2.7.1)$$

By using the equalities (2.6.11), we have

$$\mathcal{F}_{ij} = \frac{1}{4} P_{ij}, \quad f_{ij} = 0. \quad (2.7.2)$$

If we denote  $R_{ijk} := g_{ih} R^h_{jk}$ , then one proves:

**Theorem 2.7.1.** *The electromagnetic tensor  $\mathcal{F}_{ij}$  of the Finslerian mechanical system  $\Sigma_F = (M, T, Fe)$  satisfies the following generalized Maxwell equations:*

$$\begin{aligned}
\mathcal{F}_{ij}|_k + \mathcal{F}_{jk}|_i + \mathcal{F}_{ki}|_j &= \frac{1}{2} \{y^s (R_{sijk} + R_{sjki} + R_{skij}) - \\
&\quad - (R_{ijk} + R_{jki} + R_{kij})\}, \\
\mathcal{F}_{ij}|_k + \mathcal{F}_{jk}|_i + \mathcal{F}_{ki}|_j &= \frac{1}{2} \{y^s [(P_{sijk} - P_{sikj}) + \\
&\quad + (P_{sjki} - P_{sjik}) + (P_{skij} - P_{skji})]\}
\end{aligned} \tag{2.7.3}$$

where  $\mathcal{F}_{ij}$  is expressed in (2.7.2).

**Corollary 2.7.3.** *If the external forces  $Fe$  does not depend on the velocity  $y^i$  then the electromagnetic fields  $\mathcal{F}_{ij}$  and  $f_{ij}$  vanish.*

*Remark 2.7.1.* 1° The application of the previous theory to the examples from section 2.2 is immediate.

2° The theory of the gravitational field  $g_{ij}$  and the Einstein equations can be realized by same method as in the papers, [166].

## 2.8 The almost Hermitian model on the tangent manifold $TM$ of the Finslerian mechanical systems $\Sigma_F$

Consider a Finslerian mechanical system  $\Sigma_F = (M, F(x, y), Fe(x, y))$  endowed with the evolution nonlinear connection  $N$  and also endowed with the canonical  $N$ -metrical connection having the coefficients  $C\Gamma(N) = (F^i{}_{jk}, C^i{}_{jk})$ . On the velocity manifold  $\widetilde{TM} = TM \setminus \{0\}$  we can see that the previous geometrical object fields determine an almost Hermitian structure  $H^{2n} = (\widetilde{TM}, \mathbb{G}, \mathbb{F})$ . Moreover, the theory of gravitational and electromagnetic fields can be geometrically studied much better on such model, since the symplectic structure  $\theta$ , the almost complex structure  $\mathbb{F}$  and the Riemannian structure  $\mathbb{G}$  on  $\widetilde{TM}$  are well determined by the Finslerian mechanical system  $\Sigma_F$ .

One obtains:

$$\begin{aligned}
\mathbb{G} &= g_{ij} dx^i \otimes dx^j + g_{ij} \delta y^i \otimes \delta y^j \\
\mathbb{F} &= -\frac{\partial}{\partial y^i} \otimes dx^i + \frac{\delta}{\delta x^i} \otimes \delta y^i \\
\mathbb{P} &= -\frac{\partial}{\partial y^i} \otimes \delta y^i + \frac{\delta}{\delta x^i} \otimes dx^i,
\end{aligned} \tag{2.8.1}$$

where

$$\begin{cases} \frac{\delta}{\delta x^i} = \overset{\circ}{\delta} \frac{\partial}{\partial x^i} + \frac{1}{4} \frac{\partial F^s}{\partial y^i} \frac{\partial}{\partial y^s} \\ \delta y^i = \overset{\circ}{\delta} y^i - \frac{1}{4} \frac{\partial F^i}{\partial y^s} dx^s. \end{cases} \tag{2.8.2}$$

Thus, the following theorem holds:

**Theorem 2.8.1.** *We have:*

- 1° *The pair  $(\widetilde{TM}, \mathbb{G})$  is a pseudo Riemannian space.*
- 2° *The tensor  $\mathbb{G}$  depends on  $\Sigma_F$ , only.*
- 3° *The distributions  $N$  and  $V$  are orthogonal with respect to  $\mathbb{G}$ .*

**Theorem 2.8.2.** 1° *The pair  $(\widetilde{TM}, \mathbb{F})$  is an almost complex space.*

- 2°  *$\mathbb{F}$  depends on  $\Sigma_R$ , only.*
- 3°  *$\mathbb{F}$  is integrable on  $\widetilde{TM}$  iff the tensors  $R^i{}_{jk}$  vanishes.*

**Theorem 2.8.3.** 1° *The pair  $(\widetilde{TM}, \mathbb{P})$  is an almost product space.*

- 2°  *$\mathbb{P}$  depends on  $\Sigma_F$ , only.*
- 3°  *$\mathbb{P}$  is integrable iff the tensors  $R^i{}_{jk}$  vanishes.*

The integrability of  $\mathbb{F}$  and  $\mathbb{P}$  are studied by means of Nijenhuis tensors  $\mathcal{N}_{\mathbb{P}}$  and  $\mathcal{N}_{\mathbb{F}}$ :

$$\begin{aligned}
\mathcal{N}_{\mathbb{F}}(X, Y) &= \mathbb{F}^2(X, Y) + [\mathbb{F}X, \mathbb{F}Y] - \mathbb{F}[\mathbb{F}X, Y] - \\
&\quad - \mathbb{F}[X, \mathbb{F}Y], \quad \forall X, Y \in \mathcal{X}(TM).
\end{aligned}$$

In the adapted basis  $\left( \frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i} \right)$   $\mathcal{N}_{\mathbb{F}} = 0$  if, and only if  $R^i{}_{jk} = 0$ ,  $t^i{}_{jk} =$

0. But the torsion tensor  $t^i{}_{jk}$  vanishes.

It is not difficult to prove the following results:

**Theorem 2.8.4.** 1° *The triple  $(\widetilde{TM}, \mathbb{G}, \mathbb{F})$  is an almost Hermitian space.*



2°  $(\widetilde{TM}, \mathbb{G}, \mathbb{F})$  depends on Finslerian mechanical system  $\Sigma_F$ , only.

3° The almost symplectic structure  $\theta$  of the space  $(\widetilde{TM}, \mathbb{G}, \mathbb{F})$  is

$$\theta = g_{ij} \delta y^i \wedge dx^j, \quad (2.8.3)$$

with  $\delta y^i$  from (2.8.2).

If the almost symplectic structure  $\theta$  is a symplectic one (i.e.  $d\theta = 0$ ), then the space  $H^{2n} = (\widetilde{TM}, \mathbb{G}, \mathbb{F})$  is almost Kählerian.

But, using the formulas (2.8.3) and (2.4.9) one obtains:

$$\begin{aligned} d\theta = & \frac{1}{3!} (R_{ijk} + R_{jki} + R_{kij}) dx^i \wedge dx^j \wedge dx^k + \\ & + \frac{1}{2} (g_{is} B^s_{jk} - g_{js} B^s_{ik}) \delta y^k \wedge dx^j \wedge dx^i. \end{aligned} \quad (2.8.4)$$

Therefore we deduce

**Theorem 2.8.5.** *The space  $H^{2n} = (\widetilde{TM}, \mathbb{G}, \mathbb{F})$  is almost Kählerian if, and only if the following equations hold:*

$$R_{ijk} + R_{jki} + R_{kij} = 0; \quad g_{is} B^s_{jk} - g_{js} B^s_{ik} = 0. \quad (2.8.5)$$

The space  $H^{2n} = (\widetilde{TM}, \mathbb{G}, \mathbb{F})$  is called the almost Hermitian model of the Finslerian Mechanical System  $\Sigma_F$ .

**Remark.** We can study the space  $H^{2n} = (\widetilde{TM}, \mathbb{G}, \mathbb{P})$  by similar way.

One can use the model  $H^{2n} = (\widetilde{TM}, \mathbb{G}, \mathbb{F})$  to study the geometrical theory of Finslerian Mechanical system  $\Sigma_F = (M, F(x, y), Fe(x, y))$ . For instance, the Einstein equations of the pseudo Riemannian space  $(\widetilde{TM}, \mathbb{G})$  can be considered as the Einstein equations of the Finslerian Mechanical system  $\Sigma_F$ .

*Remark.* G.S. Asanov showed, [27], that the metric  $\mathbb{G}$  given by the lift (2.8.1) does not satisfy the principle of the Post-Newtonian calculus. This is due to the fact that the horizontal and vertical terms of the metric  $\mathbb{G}$  do not have the same physical dimensions. This is the reason for the author to introduce a new lift, [174], of the fundamental tensor  $g_{ij}(x, y)$  of  $\Sigma_F$  that can be used in a gauge theory. This lift is similar with that introduced in section 2.1, adapted to  $\Sigma_R$ .

It is expressed by

$$\tilde{G}(x, y) = g_{ij}(x, y) dx^i \otimes dx^j + \frac{a^2}{F^2} g_{ij}(x, y) \delta y^i \otimes \delta y^j, \quad (2.8.6)$$

where  $\delta y^i = \overset{\circ}{\delta} y^i + \frac{1}{4} \frac{\partial F^i}{\partial y^j} dx^j$  and  $a > 0$  is a constant imposed by applications. This is to preserve the physical dimensions to the both terms of  $\tilde{G}$ .

Let us consider also the tensor field on  $\widetilde{TM}$

$$\tilde{F} = -\frac{F}{a} \frac{\partial}{\partial y^i} \otimes dx^i + \frac{a}{F} \frac{\delta}{\delta x^i} \otimes \delta y^i \quad (2.8.7)$$

and the 2-form

$$\tilde{\theta} = \frac{a}{F} \theta. \quad (2.8.8)$$

Then, we have:

**Theorem 2.8.6.**

- 1° The triple  $(\widetilde{TM}, \tilde{\mathbb{G}}, \tilde{\mathbb{F}})$  is an almost Hermitian space.
- 2° The 2-form  $\tilde{\theta}$  given by (2.8.8) is the almost symplectic structure determined by  $(\tilde{\mathbb{G}}, \tilde{\mathbb{F}})$ .
- 3°  $\tilde{\theta}$  is conformal to  $\theta$ .

The space  $H^{2n} = (\widetilde{TM}, \tilde{\mathbb{G}}, \tilde{\mathbb{F}})$  can be used to study the geometrical theory of the Finslerian mechanical system  $\Sigma_F$ , too.

## Chapter 3

### Lagrangian Mechanical systems

A natural extension of the notion of the Finslerian mechanical system is that of the Lagrangian mechanical system. It is defined as a triple  $\Sigma_L = (M, L(x, y), Fe(x, y))$  where:  $M$  is a real  $n$ -dimensional  $C^\infty$  manifold called the configuration space;  $L(x, y)$  is a regular Lagrangian with the property that the pair  $L^n = (M, L(x, y))$  is a Lagrange space;  $Fe(x, y)$  is an a priori given vertical vector field on the velocity space  $TM$  called the external forces. The number  $n$  is the number of freedom degree of  $\Sigma_L$ . The equations of evolution, or fundamental equations of Lagrangian mechanical system  $\Sigma_L$  are the Lagrange equations:

$$(I) \quad \frac{d}{dt} \frac{\partial L}{\partial y^i} - \frac{\partial L}{\partial x^i} = F_i(x, y), \quad y^i = \frac{dx^i}{dt} \quad (i = 1, 2, \dots, n)$$

where  $F_i$  are the covariant components of the external forces  $Fe$ . These equations determine the integral curves of a canonical semispray  $S$ . Thus, the geometrical theory of the semispray  $S$  is the geometrical theory of the Lagrangian mechanical system  $\Sigma_L$ . The Lagrangian  $L(x, y)$  and the external forces  $Fe(x, y)$  do not explicitly depend on the time  $t$ . Therefore  $\Sigma_L$  is a scleronomic Lagrangian mechanical system.

#### 3.1 Lagrange Spaces. Preliminaries

Let  $L^n = (M, L(x, y))$  be a Lagrange space (see ch. 2, part I).  $L : TM \rightarrow R$  being a regular Lagrangian for which

$$g_{ij}(x, y) = \frac{1}{2} \dot{\partial}_i \dot{\partial}_j L(x, y) \quad (3.1.1)$$

is the fundamental tensor. Thus  $g_{ij}(x, y)$  is a covariant of order 2 symmetric  $d$ -tensor field of constant signature and non singular:

$$\det(g_{ij}(x, y)) \neq 0 \text{ on } \widetilde{TM} = TM \setminus \{0\}. \quad (3.1.2)$$

The Euler-Lagrange equations of the space  $L^n$  are:

$$\frac{d}{dt} \frac{\partial L}{\partial y^i} - \frac{\partial L}{\partial x^i} = 0, \quad y^j = \frac{dx^j}{dt}. \quad (3.1.3)$$

As we know, the system of differential equations (3.1.3) can be written in the equivalent form:

$$\frac{d^2 x^i}{dt^2} + 2\overset{\circ}{G}^i \left( x, \frac{dx}{dt} \right) = 0 \quad (3.1.4)$$

with

$$2\overset{\circ}{G}^i = \frac{1}{2} g^{is} \{ (\partial_s \partial_h L) y^h - \partial_s L \} \quad (3.1.4')$$

$$\left( \overset{\circ}{\partial}_i = \frac{\partial}{\partial y^i}, \partial_i = \frac{\partial}{\partial x^i} \right).$$

$\overset{\circ}{G}^i$  are the coefficients of a semispray  $\overset{\circ}{S}$ :

$$\overset{\circ}{S} = y^i \partial_i - 2\overset{\circ}{G}^i(x, y) \overset{\circ}{\partial}_i \quad (3.1.5)$$

and  $\overset{\circ}{S}$  depend on the space  $L^n$ , only.

$\overset{\circ}{S}$  is called the canonical semispray of Lagrange space  $L^n$ .

The energy of the Lagrangian  $L(x, y)$  is:

$$\mathcal{E}_L = y^i \overset{\circ}{\partial}_i L - L. \quad (3.1.6)$$

In the chapter 2, part I, we prove the Law of conservation energy:

**Theorem 3.1.1.** *Along the solution curves of Euler-Lagrange equations (3.1.3) the energy  $\mathcal{E}_L$  is conserved.*

The following properties hold:

1° The canonical nonlinear connection  $\overset{\circ}{N}$  of Lagrange space  $L^n$  has the coefficients

$$\overset{\circ}{N}_j^i = \partial_j \overset{\circ}{G}^i \quad (3.1.7)$$

2°  $\overset{\circ}{N}$  determine a differentiable distribution on the velocity space  $TM$  supplementary to the vertical distribution  $V$ :

$$T_u TM = \overset{\circ}{N}_u \oplus V_u, \quad \forall u \in TM. \quad (3.1.8)$$

3° An adapted basis to (3.1.8) is  $(\overset{\circ}{\delta}_i, \overset{\circ}{\partial}_i)_u$ ,  $(i = 1, \dots, n)$ , where

$$\overset{\circ}{\delta}_i = \partial_i - \overset{\circ}{N}_i^j(x, y) \partial_j \quad (3.1.9)$$

and an adapted cobasis  $(dx^i, \overset{\circ}{\delta}y^i)_u$ ,  $(i = 1, \dots, n)$  with

$$\overset{\circ}{\delta}y^i = dy^i + N_j^i(x, y) dx^j \quad (3.1.9')$$

4° The canonical metrical  $\overset{\circ}{N}$ -connection  $C\Gamma(\overset{\circ}{N}) = (\overset{\circ}{L}_{jk}, \overset{\circ}{C}_{jk})$  has the coefficients expressed by the generalized Christoffel symbols:

$$\overset{\circ}{L}_{jh}^i = \frac{1}{2} g^{is} (\overset{\circ}{\delta}_j g_{sh} + \overset{\circ}{\delta}_h g_{js} - \overset{\circ}{\delta}_s g_{jh}) \quad (3.1.10)$$

$$\overset{\circ}{C}_{jh}^i = \frac{1}{2} g^{is} (\overset{\circ}{\partial}_j g_{sh} + \overset{\circ}{\partial}_h g_{js} - \overset{\circ}{\partial}_s g_{jh})$$

5° The Cartan 1-form of  $L^n$  is

$$\overset{\circ}{\omega} = \frac{1}{2} (\overset{\circ}{\partial}_i L) dx^i \quad (3.1.11)$$

6° The Cartan-Poincaré 2-form of  $L^n$  is

$$\overset{\circ}{\theta} = d\overset{\circ}{\omega} = g_{ij}(x, y) \overset{\circ}{\delta}y^i \wedge dx^j \quad (3.1.12)$$

7° The 2-form  $\overset{\circ}{\theta}$  determine a symplectic structure on the velocity manifold  $TM$ .

**Example 3.1.1.** The function

$$L(x, y) = mc\gamma_{ij}(x)y^i y^j + \frac{2e}{m}A_i(x)y^i + \mathcal{U}(x) \quad (3.1.13)$$

with  $m, c, e$  the known physical constants,  $\gamma_{ij}(x)$  are the gravitational potentials,  $\gamma_{ij}(x)$  being a Riemannian metric,  $A_i(x)$  are the electromagnetic potentials and  $\mathcal{U}(x)$  is a potential function. Thus,  $L^n = (M, L(x, y))$  is a Lagrange space. It is the Lagrange space of electrodynamics (cd. ch. 2, part I).

### 3.2 Lagrangian Mechanical systems, $\Sigma_L$

**Definition 3.2.1.** A Lagrangian mechanical system is a triple:

$$\Sigma_L = (M, L(x, y), Fe(x, y)), \quad (3.2.1)$$

where  $L^n = (M, L(x, y))$  is a Lagrange space,  $Fe(x, y)$  is an a priori given vertical vector field:

$$Fe(x, y) = F^i(x, y)\dot{\partial}_i, \quad (3.2.2)$$

the number  $n = \dim M$  is the number of freedom degree of  $\Sigma_L$ ; the manifold  $M$  is real  $C^\infty$  differentiable manifold called the configuration space;  $Fe(x, y)$  is the external forces and  $F^i(x, y)$  are the contravariant components of  $Fe$ . Of course,  $F^i(x, y)$  is a vertical vector field and

$$F_i(x, y) = g_{ij}(x, y)F^j(x, y) \quad (3.2.3)$$

are the covariant components of  $Fe$ .  $F_i(x, y)$  is a  $d$ -covector field.

The 1-form:

$$\sigma = F_i(x, y)dx^i \quad (3.2.4)$$

is a vertical 1-form on the velocity space  $TM$ .

The Riemannian mechanical systems  $\Sigma_{\mathcal{R}}$  and the Finslerian mechanical systems  $\Sigma_F$  are the particular Lagrangian mechanical systems  $\Sigma_L$ .

*Remark 3.2.1.* The notion of Lagrangian mechanical system  $\Sigma_L$  is different of that of ‘‘Lagrangian mechanical system’’ defined by Joseph Klein [123], which is a Riemannian conservative mechanical system.

The fundamental equations of  $\Sigma_L$  are an extension of the Lagrange equations of a Finslerian mechanical system  $\Sigma_F$ , from the previous chapter.

So, we introduce the following Postulate:

**Postulate.** *The evolution equations of the Lagrangian mechanical system  $\Sigma_L = (M, L, Fe)$  are the following Lagrange equations:*

$$\frac{d}{dt} \left( \frac{\partial L}{\partial y^i} \right) - \frac{\partial L}{\partial x^i} = F_i(x, y), \quad y^i = \frac{dx^i}{dt}. \quad (3.2.5)$$

But the both members of the Lagrange equations (3.2.5) are  $d$ -covectors. Consequently, we have:

**Theorem 3.2.1.** *The Lagrange equations (3.2.5) of a Lagrangian mechanical system  $\Sigma_L = (M, L, Fe)$  have a geometrical meaning.*

**Theorem 3.2.2.** *The trajectories without external forces of the Lagrangian mechanical system  $\Sigma_L = (M, L, Fe)$  are the geodesics of the Lagrange space  $L^n = (M, L)$ .*

Indeed,  $F_i(x, y) \equiv 0$  implies the previous affirmations.

For a regular Lagrangian  $L(x, y)$  (in this case (3.1.2) holds), the Lagrange equations (3.2.5) are equivalent to the second order differential equations (SODE):

$$\frac{d^2 x^i}{dt^2} + 2\overset{\circ}{G}{}^i \left( x, \frac{dx}{dt} \right) = \frac{1}{2} F^i \left( x, \frac{dx}{dt} \right), \quad (3.2.6)$$

where the functions  $\overset{\circ}{G}{}^i(x, y)$  are the local coefficients (3.1.4') of canonical semispray  $\overset{\circ}{S}$  of Lagrange space  $L^n = (M, L(x, y))$ .

The equations (3.2.6) are called fundamental equations of  $\Sigma_L$ , too.

### 3.3 The evolution semispray of $\Sigma_L$

The Lagrange equations (3.2.6) determine a semispray  $S$  which depend on the Lagrangian mechanical system  $\Sigma_L$ , only.

Indeed, the vector field on  $TM$ :

$$S = y^i \partial_i - 2G^i(x, y) \dot{\partial}_i \quad (3.3.1)$$

with

$$2G^i(x, y) = 2\overset{\circ}{G}{}^i(x, y) - \frac{1}{2} F^i(x, y) \quad (3.3.2)$$

has the property  $JS = \mathbb{C}$ . So it is a semispray depending only on  $\Sigma_L$ .

**Theorem 3.3.1 (Miron).** *For a Lagrangian mechanical system  $\Sigma_L$  the following properties hold:*

1° *The semispray  $S$  is given by*

$$S = \overset{\circ}{S} + \frac{1}{2}Fe \quad (3.3.1')$$

2°  *$S$  is a dynamical system on the velocity space  $TM$ .*

3° *The integral curves of  $S$  are the evolution curves (3.2.6) of  $\Sigma_L$ .*

In fact, 1° derives from the formulas (3.3.1) and (3.3.2).

2°  $S$  being a vector field on  $TM$ , compatible with the geometric structure of  $TM$ , (i.e.  $JS = \mathbb{C}$ ), it is a dynamical system on the manifold  $TM$ .

3° The integral curves of  $S$  are determined by the system of differential equations

$$\frac{dx^i}{dt} = y^i, \quad \frac{dy^i}{dt} + 2G^i(x, y) = 0. \quad (3.3.3)$$

By means of the expression (3.3.2) of the coefficients  $G^i(x, y)$  the system (3.3.4) is coincident to (3.2.6).

The vector field  $S$  is called the *evolution* (or *canonical*) semispray of the Lagrangian mechanical system  $\Sigma_L$ . Exactly as in the case of Riemannian or Finslerian mechanical systems, one can prove:

**Theorem 3.3.2 ([49]).** *The evolution semispray  $S$  is the unique vector field, on the velocity space  $TM$ , solution of the equation*

$$i_S \overset{\circ}{\theta} = -d\mathcal{E}_L + \sigma \quad (3.3.4)$$

with  $\mathcal{L}$  from (3.1.6) and  $\sigma$  from (3.2.4).

But  $S$  being a solution of the previous equations, we get:

$$S(\Sigma_L) = d\mathcal{E}_L(S) = \sigma(S) = F_i y^i = g_{ij} F^i y^j.$$

So, we have:

**Theorem 3.3.3.** *The variation of energy  $\mathcal{E}_L$  along the evolution curves of mechanical system  $\Sigma_L$  is given by*

$$\frac{d\mathcal{E}_L}{dt} = F_i \left( x, \frac{dx}{dt} \right) \frac{dx^i}{dt}. \quad (3.3.5)$$



The external forces field  $Fe$  is called dissipative if  $g(\mathbb{C}, Fe) = g_{ij}F^i y^j \leq 0$ . Thus, the previous theorem implies:

**Theorem 3.3.4.** *The energy of Lagrange space  $L^n = (M, L)$  is decreasing along the evolution curves of the mechanical system  $\Sigma_L$  if and only if the external forces field  $Fe$  is dissipative.*

Evidently, the semispray  $S$  being a dynamical system on the velocity space  $TM$  it can be used for study the important problems, as the stability of evolution curves of  $\Sigma_L$ , the equilibrium points etc.

### 3.4 The evolution nonlinear connection of $\Sigma_L$

The geometrical theory of the Lagrangian mechanical system  $\Sigma_L$  is based on the evolution semispray  $S$ , with the coefficients  $G^i(x, y)$  expressed in formula (3.3.2). Therefore, all notions or properties derived from  $S$  will be considered belonging to the mechanical system  $\Sigma_L$ . Such that, the *evolution nonlinear connection*  $N$  of  $\Sigma_L$  is characterized by the coefficients:

$$N_j^i(x, y) = \dot{\partial}_j G^i(x, y) = \dot{\partial} \overset{\circ}{G}(x, y) - \frac{1}{4} \dot{\partial} F^i(x, y) = \overset{\circ}{N}_j^i(x, y) - \frac{1}{4} \dot{\partial}_j F^i(x, y). \quad (3.4.1)$$

Since  $\dot{\partial}_j F_i$  is a  $d$ -tensor field, consider its symmetric and skewsymmetric parts:

$$P_{ij} = \frac{1}{2}(\dot{\partial}_j F_i + \dot{\partial}_i F_j), \quad F_{ij} = \frac{1}{2}(\dot{\partial}_j F_i - \dot{\partial}_i F_j) \quad (3.4.2)$$

The  $d$ -tensor  $F_{ij}$  is the *elicoidal tensor* field of  $\Sigma_L$ , [166].

The evolution nonlinear connection  $N$  allows to determine the dynamic derivative of the fundamental tensor  $g_{ij}(x, y)$  of  $\Sigma_L$ :

$$g_{ij|} = S(g_{ij}) - g_{i|s} N_j^s - g_{sj} N_i^s. \quad (3.4.3)$$

It is not difficult to prove the following formula:

$$g_{ij|} = \frac{1}{2} P_{ij}. \quad (3.4.4)$$

$N$  is metric nonlinear connection if  $g_{ij|} = 0$ . So, we have

**Proposition 3.4.1.** *The evolution nonlinear connection  $N$  is metric if, and only if the  $d$ -tensor field  $P_{ij}$  vanishes.*

The adapted basis  $(\delta_i, \dot{\partial}_i)$  to the distributions  $N$  and  $V$  has the operators  $\delta_i$  of the form:

$$\delta_i = \partial_i - N_j^i \dot{\partial}_j = \overset{\circ}{\delta}_i + \frac{1}{4} \dot{\partial}_i F^s \dot{\partial}_s. \quad (3.4.5)$$

The dual adapted cobasis  $(dx^i, \delta y^i)$  has 1-forms  $\delta y^i$ :

$$\delta y^i = dy^i + N_j^i dx^j = \overset{\circ}{\delta} y^i - \frac{1}{4} \dot{\partial}_j F^i dx^j \quad (3.4.5')$$

In the adapted basis the evolution semispray  $S$  has the expression:

$$S = y^i \frac{\delta}{\delta x^i} - \left[ 2\overset{\circ}{G} - \overset{\circ}{N}_j^i y^j - \frac{1}{2} \left( F^i - \frac{1}{2} \dot{\partial}_j F^i y^j \right) \right] \frac{\partial}{\partial y^i}. \quad (3.4.5'')$$

Consequences:

1°  $S$  cannot be a vertical vector field.

2° If the coefficients  $\overset{\circ}{G}$  are 2-homogeneous with respect to the vertical variables  $y^i$  and the contravariant components  $F^i(x, y)$  are 2-homogeneous in  $y^i$ , then the canonical semispray  $S$  belongs to the horizontal distribution  $N$ .

The tensor of integrability of the distribution  $N$  is

$$R_{jh}^i = \delta_h N_j^i - \delta_j N_h^i = (\delta_h \dot{\partial}_j - \delta_j \dot{\partial}_h) \left( \overset{\circ}{G} - \frac{1}{4} F^i \right) \quad (3.4.6)$$

Thus:  $N$  is integrable iff  $R_{jh}^i = 0$ .

The  $d$ -tensor of torsion of the nonlinear connection  $N$  vanishes.

Indeed:

$$t_{jh}^i = \dot{\partial}_h N_j^i - \dot{\partial}_j N_h^i = 0. \quad (3.4.7)$$

The Berwald connection  $B\Gamma(N) = (B_{jk}^i, 0)$  of the canonical nonlinear connection  $N$  has the coefficients

$$B_{jh}^i = \overset{\circ}{B}_{jh}^i - \frac{1}{4} \dot{\partial}_j \dot{\partial}_h F^i, \quad (3.4.8)$$

where  $(\overset{\circ}{B}_{jh}, 0)$  are the coefficients of Berwald connection of Lagrange space  $L^n$ .

It follows that  $B\Gamma(N)$  is symmetric:  $B_{jh}^i - B_{hj}^i = t_{jh}^i = 0$ .

In the following section we need:

**Lemma 3.4.1.** *The exterior differential of 1-forms  $\delta y^i$  are given by*

$$d\delta y^i = \frac{1}{2}R_{jh}^i dx^h \wedge dx^j + B_{jh}^i \delta y^j \wedge dx^h. \quad (3.4.9)$$

Indeed, from (3.4.5') we have:

$$d\delta y^i = dN_j^i \wedge dx^j = \delta_h N_j^i dx^h \wedge dx^j + \dot{\partial}_h N_j^i \delta y^h \wedge dx^j,$$

which are exactly (3.4.9).

By means of the formula (3.4.5') one gets:

**Theorem 3.4.1.** *The autoparallel curves of the canonical nonlinear connection  $N$  are given by the differential system of equations:*

$$y^i = \frac{dx^i}{dt}, \quad \frac{\delta y^i}{dt} = \frac{\overset{\circ}{\delta} y^i}{dt} - \frac{1}{4} \partial_j F^i \frac{dx^j}{dt} = 0. \quad (3.4.10)$$

Evidently, if  $Fe = 0$ , then the canonical nonlinear connection  $N$  coincides with the canonical nonlinear connection  $\overset{\circ}{N}$  of the Lagrange space  $L^n$ .

In particular, if  $\Sigma_L$  is a Finslerian mechanical system  $\Sigma_F$ , then the previous theory reduces to that studied in the previous chapter.

### 3.5 Canonical $N$ -metrical connection of $\Sigma_L$ . Structure equations

Taking into account the results from part I, the canonical  $N$ -metrical connection  $C\Gamma(N)$  is characterized by:

**Theorem 3.5.1.** *The local coefficients  $D\Gamma(N) = (L_{jk}^i, C_{jk}^i)$  of the canonical  $N$ -metrical connection of Lagrangian mechanical system  $\Sigma_L$  are given by the generalized Christoffel symbols:*

$$\begin{aligned}
L_{jh}^i &= \frac{1}{2} g^{is} (\delta_j g_{sh} + \delta_h g_{sj} - \delta_s g_{jk}) \\
C_{jh}^i &= \frac{1}{2} g^{is} (\dot{\delta}_j g_{sh} + \dot{\delta}_h g_{sj} - \dot{\delta}_s g_{jh}).
\end{aligned} \tag{3.5.1}$$

Using the expression (3.4.5) of the operator  $\delta_k$  and developing the terms  $\delta_h g_{ij}$  from (3.5.1), we get:

**Theorem 3.5.2.** *The coefficients  $L_{jh}^i, C_{jh}^i$  of  $D\Gamma(N)$  can be set in the form*

$$\begin{aligned}
L_{jk}^i &= \overset{\circ}{L}_{jk}^i + \check{C}_{jk}^i, \quad C_{jk}^i = \overset{\circ}{C}_{jk}^i, \\
\check{C}_{jk}^i &= \frac{1}{4} g^{is} \left( \overset{\circ}{C}_{skh} \dot{\delta} F^h + \overset{\circ}{C}_{jsh} \dot{\delta} F^h - \overset{\circ}{C}_{jkh} \dot{\delta} F^h \right),
\end{aligned} \tag{3.5.2}$$

where  $C\Gamma(\overset{\circ}{N}) = (\overset{\circ}{L}_{jk}, \overset{\circ}{C}_{jk})$  is the canonical  $N$ -metrical connection of the Lagrange space  $L^n = (M, L(x, y))$ .

Let  $\omega_j^i$  be the connection 1-forms of the canonical  $N$ -metrical connection  $D\Gamma(N)$ :

$$\omega_j^i = L_{jk}^i dx^k + C_{jk}^i \delta y^k. \tag{3.5.3}$$

Then, we have:

**Theorem 3.5.3.** *The structure equations of canonical  $N$ -metrical connection  $D\Gamma(N)$  of Lagrangian mechanical system  $\Sigma_L$  are given by:*

$$\begin{aligned}
d(dx^i) - dx^k \wedge \omega_k^i &= -\overset{1}{\Omega}{}^i \\
d(\delta y^i) - \delta y^k \wedge \omega_k^i &= -\overset{2}{\Omega}{}^i \\
d\omega_j^i - \omega_j^k \wedge \omega_k^i &= -\Omega^i,
\end{aligned} \tag{3.5.4}$$

where the 1-form of torsions  $\overset{1}{\Omega}{}^i$  and  $\overset{2}{\Omega}{}^i$  are:

$$\overset{1}{\Omega}{}^i = C_{jk}^i dx^j \wedge \delta y^k \quad (3.5.5)$$

$$\overset{2}{\Omega}{}^i = \frac{1}{2} R_{jk}^i dx^j \wedge dx^k + P_{jk}^i dx^j \wedge \delta y^k$$

and the 2-forms of curvature  $\Omega_j^i$  is as follows:

$$\Omega_j^i = \frac{1}{2} R_{jkh}^i dx^k \wedge dx^h + P_{jkh}^i dx^k \wedge \delta y^h + \frac{1}{2} S_{jkh}^i \delta y^k \wedge \delta y^h \quad (3.5.6)$$

where the  $d$ -tensors of torsions are:

$$T_{jk}^i = L_{jk}^i - L_{kj}^i = 0; \quad S_{jk}^i = C_{jk}^i - C_{kj}^i = 0, \quad (3.5.7)$$

$$C_{jk}^i = \overset{\circ}{C}_{jk}^i, \quad R_{jk}^i = \delta_k N_j^i - \delta_j N_k^i, \quad P_{jk}^i = B_{jk}^i - L_{jk}^i$$

and the  $d$ -tensors of curvature are:

$$\begin{aligned} R_{jkh}^i &= \delta_k L_{jk}^i - \delta_k L_{jh}^i + L_{jk}^s L_{sh}^i - L_{jh}^s L_{sk}^i + C_{js}^i R_{kh}^s \\ P_{jkh}^i &= \dot{\partial}_h L_{jk}^i - C_{jh|k}^i + C_{js}^i P_{kh}^s \\ S_{jkh}^i &= \dot{\partial}_h C_{jk}^i - \dot{\partial}_k C_{jh}^i + C_{jk}^s C_{sh}^i - C_{jh}^s C_{sk}^i. \end{aligned} \quad (3.5.8)$$

Here  $C_{jk|h}^i$  is the  $h$ -covariant derivative of the tensor  $C_{jk}^i$  with respect to  $D\Gamma(N)$ .

So, we have

$$\begin{aligned} g_{ij|h} &= \delta_h g_{ij} - g_{sj} L_{ih}^s - g_{is} L_{jh}^s = 0, \\ g_{ij|h} &= \dot{\partial}_h g_{ij} - g_{sj} C_{ih}^s - g_{is} C_{jh}^s = 0. \end{aligned} \quad (3.5.9)$$

Applying the Ricci identities to the fundamental tensor  $g_{ij}$ , it is not difficult to prove the identities

$$R_{ijkh} + R_{jikh} = 0, \quad P_{ijkh} + P_{jikh} = 0, \quad S_{ijkh} + S_{jikh} = 0, \quad (3.5.10)$$

where  $R_{ijkh} = g_{js} R_{ikh}^s$ , etc.

Also,  $P_{ijk} = g_{is} P_{kh}^s$  is totally symmetric.

Taking into account the form (3.5.2) of the coefficients of  $D\Gamma(N)$ , the calculus of  $d$ -tensors of torsion (3.5.7) and of  $d$ -tensors of curvature (3.5.6) can be obtained.

The exterior differentiating the structure equations (3.5.4) modulo the same system of equations (3.5.4) and using Lemma (3.4.1) one obtains the Bianchi identities of the  $N$ -metrical connection  $D\Gamma(N)$ .

The  $h$ - and  $v$ -covariant derivative of Liouville vector field  $\mathbb{C} = y^i \frac{\partial}{\partial y^i}$  lead to the deflection tensors of  $D\Gamma(N)$ :

$$D^i_j = y^i|_j = \delta_j y^i + y^s L^i_{sj}, \quad d^i_j = y^i|_j = \dot{\partial}_j y^i + y^s C^i_{sj}. \quad (3.5.11)$$

**Theorem 3.5.4.** *The expression of  $h$ -tensor of deflection  $D^i_j$  and  $v$ -tensor of deflection  $d^i_j$  are given by*

$$D^i_j = y^s L^i_{sj} - N^i_j, \quad d^i_j = \delta^i_j + y^s \overset{\circ}{C}^i_{sj}. \quad (3.5.12)$$

Finally, we determine the horizontal paths, vertical paths of  $D\Gamma(N)$  and its autoparallel curves.

The *horizontal paths* of metrical connection  $D\Gamma(N) = (L^i_{jk}, C^i_{jk})$  are given by

$$\frac{d^2 x^i}{dt^2} + L^i_{jk} \left( x, \frac{dx}{dt} \right) \frac{dx^j}{dt} \frac{dx^k}{dt} = 0, \quad y^i = \frac{dx^i}{dt}, \quad (3.5.13)$$

where the coefficients  $L^i_{jk}$  are expressed by formula (3.5.2).

In the initial conditions  $x^i = x^i_0, y^i = \left( \frac{dx^i}{dt} \right)_0$ , local, this system of differential equation has a unique solution  $x^i = x^i(t), y^i = y^i(t), t \in I$ .

The *vertical paths*, at a point  $(x^i_0) \in M$  are characterized by system of differential equations:

$$\frac{dy^i}{dt} + \overset{\circ}{C}^i_{jk}(x_0, y) y^j y^k = 0, \quad x^i = x^i_0. \quad (3.5.14)$$

In the initial conditions  $x^i = x^i_0, y^i = y^i_0$  the previous system of differential equations, locally, has a unique solution.

### 3.6 Electromagnetic field

For Lagrangian mechanical system  $\Sigma_L = (M, L, Fe)$  the electromagnetic phenomena appear because the deflection tensor  $D_j^i$  nonvanishes.

The covariant components  $D_{ij} = g_{is}D_j^s$ ,  $d_{ij} = g_{is}d_j^s$  determine  $h$ -electromagnetic field  $\mathcal{F}_{ij}$  and  $v$ -electromagnetic field  $f_{ij}$  as follows:

$$\mathcal{F}_{ij} = \frac{1}{2}(D_{ij} - D_{ji}), \quad f_{ij} = \frac{1}{2}(d_{ij} - d_{ji}). \quad (3.6.1)$$

Taking into account the formula (3.5.11) one obtains:

$$\mathcal{F}_{ij} = \overset{\circ}{\mathcal{F}}_{ij} + \frac{1}{4}F_{ij} + \frac{1}{4}\check{F}_{ij} \quad (3.6.2)$$

where  $\overset{\circ}{\mathcal{F}}_{ij}$  is the electromagnetic tensor field of the Lagrange space  $L^n$ ,  $F_{ij}$  is the elicoidal tensor field of mechanical system  $\Sigma_L$  and  $\check{F}_{ij}$  is the skewsymmetric tensor

$$\check{F}_{ij} = \{\dot{\partial}_j F^s C_{irs} - \dot{\partial}_i F^r C_{jrs}\} y^s. \quad (3.6.3)$$

In the case of Finslerian mechanical system  $\Sigma_F$  the electromagnetic tensor  $\mathcal{F}_{ij}$  is equal to  $\frac{1}{4}F_{ij}$ .

Applying the method given in the chapter 2, part I, one gets the Maxwell equations of the Lagrangian mechanical system  $\Sigma_L$ .

### 3.7 The almost Hermitian model of the Lagrangian mechanical system $\Sigma_L$

Let  $N$  be the canonical nonlinear connection of the mechanical system  $\Sigma_L$  and  $(\delta_i, \dot{\partial}_i)$  the adapted basis to the distribution  $N$  and  $V$ . Its dual basis is  $(dx^i, \delta y^i)$ . The  $N$ -lift  $\mathbb{G}$  of fundamental tensor  $g_{ij}$  on the velocity space  $\widetilde{TM} = TM \setminus \{0\}$  is given by

$$\mathbb{G} = g_{ij} dx^i \otimes dx^j + g_{ij} \delta y^i \otimes \delta y^j. \quad (3.7.1)$$

The almost complex structure  $\mathbb{F}$  determined by the nonlinear connection  $N$  is expressed by

$$\mathbb{F} = \delta_i \otimes dx^j - \dot{\partial}_i \otimes dx^i. \quad (3.7.2)$$

Thus, one proves:

**Theorem 3.7.1.** *We have:*

1°  $\mathbb{G}$  is a pseudo-Riemannian structure and  $\mathbb{F}$  is an almost complex structure on the manifold  $TM$ . They depend only on the Lagrangian mechanical system  $\Sigma_L$ .

2° The pair  $(\mathbb{G}, \mathbb{F})$  is an almost Hermitian structure.

3° The associated 2-form of  $(\mathbb{G}, \mathbb{F})$  is given by

$$\theta = g_{ij} \delta y^i \wedge dx^j.$$

4°  $\theta$  is an almost symplectic structure on the velocity space  $\widetilde{TM}$ .

5° The following equality holds:

$$\theta = \overset{\circ}{\theta} - \frac{1}{4} F_{ij} dx^i \wedge dx^j,$$

$\overset{\circ}{\theta}$  being the symplectic structure of the Lagrange space  $L^n$ , i.e.  $\overset{\circ}{\theta} = g_{ij} \delta y^i \wedge dx^j$ .

Since  $\overset{\circ}{\theta}$  is the symplectic structure of Lagrange space  $L^n$ , we have  $d\overset{\circ}{\theta} = 0$ . So the exterior differential  $d\theta$  is as follows:

$$d\theta = -\frac{1}{4} dF_{ij} \wedge dx^i \wedge dx^j. \quad (3.7.3)$$

But  $dF_{ij} = \delta_k F_{ij} dx^k + \dot{\partial}_k F_{ij} \delta y^k$ . Consequently one obtain from (2.2):

$$d\theta = -\frac{1}{12} (F_{ij|k} + F_{jk|i} + F_{ki|j}) dx^k \wedge dx^i \wedge dx^j - \frac{1}{4} \dot{\partial}_k F_{ij} \delta y^k \wedge dx^i \wedge dx^j. \quad (3.7.4)$$

Consequently:

1°  $\theta = \overset{\circ}{\theta}$ , if and only if the helicoidal tensor  $F_{ij}$  vanish.

2° The almost symplectic structure  $\theta$  of the Lagrangian mechanical system  $\Sigma_L$  is integrable, if and only if the helicoidal tensor  $F_{ij}$  satisfies the following tensorial equations:



$$F_{ij|k} + F_{jk|i} + F_{ki|j} = 0, \quad \dot{\partial}_k F_{ij} = 0, \quad (3.7.5)$$

where the operator “ $|$ ” is  $h$ -covariant derivation with respect to canonical  $N$ -linear connection  $D\Gamma(N)$ .

Now we observe that some good applications of this theory can be done for the Lagrangian mechanical systems  $\Sigma_L = (M, L(x, y), Fe(x, y))$  when the external forces  $Fe$  are of Liouville type:  $Fe = a(x, y)\mathbb{C}$ .

### 3.8 Generalized Lagrangian mechanical systems

As we know, a generalized Lagrange space  $GL^n = (M, g_{ij})$  is given by the configuration space  $M$  and by a  $d$ -tensor field  $g_{ij}(x, y)$  on the velocity space  $\widetilde{TM} = TM \setminus \{0\}$ ,  $g_{ij}$  being symmetric, nonsingular and of constant signature.

There are numerous examples of spaces  $GL^n$  given by Miron R. [175], Anastasiei M. [14], R.G. Beil [40], [41], T. Kawaguchi [176] etc.

1° The  $GL^n$  space with the fundamental tensor

$$g_{ij}(x, y) = \alpha_{ij}(x) + y_i y_j, \quad y_i = \alpha_j y^j \quad (3.8.1)$$

and  $\alpha_{ij}(x)$  a semidefinite Riemann tensor.

2°  $g_{ij}(x, y) = \alpha_{ij}(x) + \left(1 - \frac{1}{n(x, y)}\right) y_i y_j$ ,  $y_i = \alpha_{ij} y^j$  where  $n(x, y) \geq 1$  is a  $C^\infty$  function (the refractive index in Relativistic optics), [176].

3°  $g_{ij}(x, y) = \alpha_{ij}(x, y) + a y_i y_j$ ,  $a \in R_+$ ,  $y_i = \alpha_{ij} y^j$  and where  $\alpha_{ij}(x, y)$  is the fundamental tensor of a Finsler space.

The space  $GL^n = (M, g_{ij}(x, y))$  is called reducible to a Lagrange space if there exists a regular Lagrangian  $L(x, y)$  such that

$$\frac{1}{2} \frac{\partial^2 L}{\partial y^i \partial y^j} = g_{ij}(x, y). \quad (3.8.2)$$

A necessary condition that the space  $GL^n = (M, g_{ij}(x, y))$  be reducible to a Lagrange space is that the  $d$ -tensor field

$$C_{ijk} = \frac{1}{2} \frac{\partial g_{ij}}{\partial y^k} \quad (3.8.3)$$

be totally symmetric.

The following property holds:

**Theorem 3.8.1.** *If we have a generalized Lagrange space  $GL^n$  for which the fundamental tensor  $g_{ij}(x, y)$  is 0-homogeneous with respect to  $y^i$  and it is reducible to a Lagrange space  $L^n = (M, L)$ , then the function  $L(x, y)$  is given by:*

$$L(x, y) = g_{ij}(x, y)y^i y^j + 2A_i(x)y^i + \mathcal{U}(x). \quad (3.8.4)$$

The prove does not present some difficulties, [166].

Also, we remark that in some conditions the fundamental tensor  $g_{ij}(x, y)$  of a space  $GL^n$  determine a non linear connection, [241], [242].

For instance, in the cases  $1^\circ, 2^\circ, 3^\circ$  of the previous examples we can take:  $1^\circ, 2^\circ, \overset{\circ}{N}_j = \frac{1}{2}\gamma_{jk}^i(x)y^j y^k$ ,  $\gamma_{jk}^i(x)$  being the Christoffel symbols of the Riemannian metric  $\gamma_{ij}(x)$ . For example  $3^\circ$  we take  $N_j^i = \frac{1}{2}\gamma_{jk}^i(x, y)y^j y^k$  where  $\gamma_{jk}^i(x, y)$  are the Christoffel symbols of  $\gamma_{ij}(x, y)$  (see ch. 2, part I).

Let us consider the function  $\mathcal{E}(x, y)$  on  $\widetilde{TM}$ :

$$\mathcal{E}(x, y) = g_{ij}(x, y)y^i y^j \quad (3.8.5)$$

called the *absolute energy* [161] of the space  $GL^n$ . The fundamental tensor  $g_{ij}(x, y)$  is say to be weakly regular if the absolute energy  $\mathcal{E}(x, y)$  is a regular Lagrangian. That means: the tensor  $\check{g}_{ij}(x, y) = \frac{1}{2}\dot{\partial}_i \dot{\partial}_j \mathcal{E}$  is nonsingular. Thus the pair  $(M, \mathcal{E}(x, y))$  is a Lagrange space.

**Definition 3.8.1.** A generalized Lagrangian mechanical system is a triple  $\Sigma_{GL} = (M, g_{ij}, Fe)$ , where  $GL = (M, g_{ij})$  is a generalized Lagrange space and  $Fe$  is the vector field of external forces.  $Fe$  being a vertical vector field we can write:

$$Fe = F^i(x, y)\dot{\partial}_i. \quad (3.8.6)$$

In the following we assume that the fundamental tensor  $g_{ij}(x, y)$  of  $\Sigma_{GL}$  is weakly regular. Therefore we can give the following Postulate [49]:

**Postulate.** *The evolution equations (or Lagrange equations) of a generalized Lagrange mechanical system  $\Sigma_{GL}$  are:*

$$\frac{d}{dt} \frac{\partial \mathcal{E}}{\partial y^i} - \frac{\partial \mathcal{E}}{\partial x^i} = F_i(x, y), \quad y^i = \frac{dx^i}{dt} \quad (3.8.7)$$

with the covariant components of  $\mathbb{F}e$ :

$$F_i(x, y) = g_{ij}(x, y) F^j(x, y). \quad (3.8.8)$$

The Lagrange equations (3.8.7) are equivalent to the following system of second order differential equations:

$$\begin{cases} \frac{d^2 x^i}{dt^2} + 2\check{G}^i \left( x, \frac{dx}{dt} \right) = \frac{1}{2} F^i \left( x, \frac{dx}{dt} \right) \\ 2\check{G}^i = \frac{1}{2} g^{is} \left( \frac{\partial^2 \mathcal{E}}{\partial y^s \partial x^j} y^j - \frac{\partial \mathcal{E}}{\partial x^s} \right). \end{cases} \quad (3.8.9)$$

Therefore, one can apply the theory from this chapter for the Lagrangian mechanical system  $\Sigma_L = (M, \mathcal{E}(x, y), Fe)$ . So we have:

**Theorem 3.8.2.** 1° *The operator*

$$\check{S} = y^i \partial_i - 2 \left( \check{G}^i - \frac{1}{2} F^i \right) \dot{\partial}_i \quad (3.8.10)$$

*is a semispray on the velocity space  $\widetilde{TM}$ .*

2° *The integral curves of  $\check{S}$  are the evolution curves of  $\Sigma_{GL}$ .*

3°  *$\check{S}$  is determined only by the generalized mechanical system  $\Sigma_{GL}$ .*

**Theorem 3.8.3.** *The variation of energy  $E_{\mathcal{E}} = y^i \frac{\partial \mathcal{E}}{\partial y^i} - \mathcal{E}$  are given by*

$$\frac{dE_{\mathcal{E}}}{dt} = \frac{dx^i}{dt} F_i \left( x, \frac{dx}{dt} \right). \quad (3.8.11)$$

*$Fe$  are called dissipative if  $g_{ij} F^i y^j \leq 0$ .*

**Theorem 3.8.4.** *The energy  $E_{\mathcal{E}}$  is decreasing on the evolution curves (3.8.9) of system  $\Sigma_{GL}$  if and only if the external forces  $Fe$  are dissipative.*

Other results:

4° The canonical nonlinear connection  $\check{N}$  of  $\Sigma_{G,L}$  has the local coefficients

$$\check{N}_j^i = \dot{\partial}_j \overset{\circ}{G}^i - \frac{1}{4} \dot{\partial}_j F^i. \quad (3.8.12)$$

5° The canonical  $\check{N}$ -linear metric connection of mechanical system  $\Sigma_{GL}$ ,  $D\Gamma(\check{N}) = (L_{jk}, C_{jk})$  has the coefficients:

$$\begin{cases} \check{L}_{jk}^i = \frac{1}{2} g^{is} \left( \frac{\check{\delta} g_{js}}{\delta x^k} + \frac{\check{\delta} g_{sk}}{\delta x^j} - \frac{\check{\delta} g_{jk}}{\delta x^s} \right) \\ \check{C}_{jk}^i = \frac{1}{2} g^{is} \left( \frac{\partial g_{js}}{\partial y^k} + \frac{\partial g_{sk}}{\partial y^j} - \frac{\partial g_{jk}}{\partial y^s} \right). \end{cases} \quad (3.8.13)$$

By using the geometrical object fields  $g_{ij}$ ,  $\check{S}$ ,  $\check{N}$ ,  $D\Gamma(\check{N})$  we can study the theory of Generalized Lagrangian mechanical systems  $\Sigma_{G,L}$ .

## Chapter 4

### Hamiltonian and Cartanian mechanical systems

The theory of Hamiltonian mechanical systems can be constructed step by step following the theory of Lagrangian mechanical system, using the differential geometry of Hamiltonian spaces expound in part I. The legality of this theory is proved by means of Legendre duality between Lagrange and Hamilton spaces. The Cartanian mechanical system appears as a particular case of the Hamiltonian mechanical systems. We develop here these theories using the author's papers [154] and the book of R. Miron, D. Hrimiuc, H. Shimada and S. Sabau [174].

#### 4.1 Hamilton spaces. Preliminaries

Let  $M$  be a  $C^\infty$ -real  $n$ -dimensional manifold, called configuration space and  $(T^*M, \pi^*, M)$  be the cotangent bundle,  $T^*M$  is called momentum (or phase) space. A point  $u^* = (x, p) \in T^*M$ ,  $\pi^*(u^*) = x$ , has the local coordinate  $(x^i, p_i)$ .

As we know from the previous chapter, a changing of local coordinate  $(x, p) \rightarrow (\tilde{x}, \tilde{p})$  of point  $u^*$  is given by

$$\begin{cases} \tilde{x}^i = \tilde{x}^i(x^1, \dots, x^n), \det \left( \frac{\partial \tilde{x}^i}{\partial x^j} \right) \neq 0 \\ \tilde{p}_i = \frac{\partial x^j}{\partial \tilde{x}^i} p_j \end{cases} \quad (4.1.1)$$

The tangent space  $T_{u^*} T^*M$  has a natural basis  $\left( \frac{\partial}{\partial x^i} = \partial_i, \frac{\partial}{\partial p_i} = \dot{\partial}^i \right)$  with respect to (4.1.1) this basis is transformed as follows

$$\frac{\partial}{\partial x^i} = \frac{\partial \tilde{x}^j}{\partial x^i} \frac{\partial}{\partial \tilde{x}^j} + \frac{\partial \tilde{p}_j}{\partial x^i} \frac{\partial}{\partial \tilde{p}_j} \quad (4.1.2)$$

$$\dot{\partial}^i = \frac{\partial x^i}{\partial \tilde{x}^j} \tilde{\partial}^j$$

Thus we have:

1° On  $T^*M$  there are globally defined the Liouville 1-form:

$$\check{p} = p_i dx^i \quad (4.1.3)$$

and the natural symplectic structure

$$\overset{\circ}{\theta} = d\check{p} = dp_i \wedge dx^i. \quad (4.1.4)$$

2° The vertical distribution  $V$  has a local basis  $(\dot{\partial}^1, \dots, \dot{\partial}^n)$ . It is of dimension  $n$  and is integrable.

3° A supplementary distribution  $N$  to the distribution  $V$  is given by a splitting:

$$T_{u^*}T^*M = N_{u^*} \oplus V_{u^*}, \quad \forall u^* \in T^*M. \quad (4.1.5)$$

$N$  is called a horizontal distribution or a *nonlinear connection* on the momentum space  $T^*M$ . The dimension of  $N$  is  $n$ .

4° A local adapted basis to  $N$  and  $V$  are given by  $(\delta_i, \dot{\partial}^i)$ ,  $(i = 1, \dots, n)$ , with

$$\delta_i = \partial_i + N_{ji} \dot{\partial}^j. \quad (4.1.6)$$

The system of functions  $N_{ji}(x, p)$  are the coefficients of the nonlinear connection  $N$ .

5°

$$t_{ij} = N_{ij} - N_{ji} \quad (4.1.7)$$

is a  $d$ -tensor on  $T^*M$  - called the torsion tensor of the nonlinear connection  $N$ . If  $t_{ij} = 0$  we say that  $N$  is a symmetric nonlinear connection.

6° The dual basis  $(dx^i, \delta p_i)$  of the adapted basis  $(\delta_i, \dot{\partial}^i)$  has 1-form  $\delta p_i$  expressed by

$$\delta p_i = dp_i - N_{ij} dx^j. \quad (4.1.8)$$

7°

**Proposition 4.1.1.** *If  $N$  is a symmetric nonlinear connection then the symplectic structure  $\overset{\circ}{\theta}$  can be written:*

$$\overset{\circ}{\theta} = \delta p_i \wedge dx^i. \quad (4.1.9)$$

8° The integrability tensor of the horizontal distribution  $N$  is

$$R_{kij} = \delta_i N_{kj} - \delta_j N_{ki} \quad (4.1.10)$$

and the equations  $R_{kij} = 0$  gives the necessary and sufficient condition for integrability of the distribution  $N$ .

The notion of  $N$ -linear connection can be taken from the ch. 2, part I.

**Definition 4.1.1.** A Hamilton space is a pair  $H^n = (M, H(x, p))$  where  $H(x, p)$  is a real scalar function on the momentum space  $T^*M$  having the following properties:

- 1°  $H : T^*M \rightarrow R$  is differentiable on  $\widetilde{T^*M} = T^*M \setminus \{0\}$  and continuous on the null section of projection  $\pi_*$ .
- 2° The Hessian of  $H$ , with the elements

$$g^{ij} = \frac{1}{2} \dot{\partial}^i \dot{\partial}^j H \quad (4.1.11)$$

is nonsingular, i.e.

$$\det(g^{ij}) \neq 0 \text{ on } \widetilde{T^*M}. \quad (4.1.12)$$

- 3° The 2-form  $g^{ij}(x, p)\eta_i\eta_j$  has a constant signature on  $\widetilde{T^*M}$ .

In the ch. 2, part I it is proved the following Miron's result:

**Theorem 4.1.1.** 1° In a Hamilton space  $H^n = (M, H(x, p))$  for which  $M$  is a paracompact manifold, there exist nonlinear connections determined only by the fundamental function  $H(x, p)$ .

- 2° One of the,  $\overset{\circ}{N}$ , has the coefficients

$$\overset{\circ}{N}_{ij} = -\frac{1}{2} g_{jh} \left[ \frac{1}{4} g_{ik} \dot{\partial}^k \{H, \dot{\partial}^h H\} + \dot{\partial}^h \partial_i H \right]. \quad (4.1.13)$$

- 3° The nonlinear connection  $\overset{\circ}{N}$  is symmetric.

In the formula (4.1.13),  $\{, \}$  is the Poisson brackets.  $\overset{\circ}{N}$  is called the canonical nonlinear connection of  $H^n$ .

The variational problem applied to the integral of action of  $H^n$ :

$$I(c) = \int_0^1 \left[ p_i(t) \frac{dx^i}{dt} - \mathcal{H}(x(t), p(t)) \right] dt, \quad (4.1.14)$$

where

$$\mathcal{H}(x, p) = \frac{1}{2}H(x, p) \quad (4.1.15)$$

leads to the following results:

**Theorem 4.1.2.** *The necessary conditions as the functional  $I(c)$  be an extremal value of the functionals  $I(\tilde{c})$  imply that the curve  $c(t) = (x^i(t), p_i(t))$  is a solution of the Hamilton–Jacobi equations:*

$$\frac{dx^i}{dt} - \frac{\partial \mathcal{H}}{\partial p_i} = 0, \quad \frac{dp_i}{dt} + \frac{\partial \mathcal{H}}{\partial x^i} = 0. \quad (4.1.16)$$

If  $\overset{\circ}{N}$  is the canonical nonlinear connection then the Hamilton–Jacobi equations can be written in the form

$$\frac{dx^i}{dt} - \frac{\partial \mathcal{H}}{\partial p_i} = 0, \quad \frac{\delta p_i}{dt} + \frac{\delta \mathcal{H}}{\delta x^i} = 0. \quad (4.1.17)$$

Evidently, the equations (4.1.16) have a geometrical meaning.

The curves  $c(t)$  which verify the Hamilton–Jacobi equations are called the extremal curves (or geodesics) of  $H^n$ .

Other important results:

**Theorem 4.1.3.** *The fundamental function  $H(x, p)$  of a Hamilton space  $H^n = (M, H)$  is constant along to every extremal curves.*

**Theorem 4.1.4.** *The following properties hold:*

1° For a Hamilton space  $H^n$  there exists a vector field  $\overset{\circ}{\xi} \in \mathcal{X}(\widetilde{T^*M})$  with the property

$$i_{\overset{\circ}{\xi}} \overset{\circ}{\theta} = -dH. \quad (4.1.18)$$

2°  $\overset{\circ}{\xi}$  is given by

$$\overset{\circ}{\xi} = \frac{1}{2}(\partial^i H \partial_i - \partial_i H \partial^i). \quad (4.1.19)$$



3° The integral curve of  $\overset{\circ}{\xi}$  is given by the Hamilton–Jacobi equations (4.1.16).

The vector field  $\overset{\circ}{\xi}$  from the formula (4.1.19) is called the Hamilton vector of the space  $H^n$ .

**Proposition 4.1.2.** *In the adapted basis of the canonical nonlinear connection  $\overset{\circ}{N}$  the Hamilton vector  $\overset{\circ}{\xi}$  is expressed by*

$$\overset{\circ}{\xi} = \frac{1}{2}(\partial^i H \delta_i - \delta_i H \partial^i). \quad (4.1.20)$$

By means of the Theorem 4.1.4 we can say that the Hamilton vector field  $\overset{\circ}{\xi}$  is a dynamical system of the Hamilton space  $H^n$ .

## 4.2 The Hamiltonian mechanical systems

Following the ideas from the first part, ch. ..., we can introduce the next definition:

**Definition 4.2.1.** A Hamiltonian mechanical system is a triple:

$$\Sigma_H = (M, H(x, p), Fe(x, p)), \quad (4.2.1)$$

where  $H^n = (M, H(x, p))$  is a Hamilton space and

$$Fe(x, p) = F_i(x, p) \partial^i \quad (4.2.2)$$

is a given vertical vector field on the momenta space  $T^*M$ .

$Fe$  is called the external forces field.

The evolution equations of  $\Sigma_H$  can be defined by means of equations (4.1.16) from the variational problem.

**Postulate 4.2.1.** *The evolution equations of the Hamiltonian mechanical system  $\Sigma_H$  are the following **Hamilton equations**:*

$$\frac{dx^i}{dt} - \partial^i calH = 0, \quad \frac{dp_i}{dt} + \partial_i \mathcal{H} = \frac{1}{2} F_i(x, p), \quad \mathcal{H} = \frac{1}{2} H. \quad (4.2.3)$$

Evidently, for  $Fe = 0$ , the equations (4.2.3) give us the geodesics of the Hamilton space  $H^n$ .

Using the canonical nonlinear connection  $\overset{\circ}{N}$  we can write in an invariant form the Hamilton equations, which allow to prove the geometrical meaning of these equations.

**Examples.** 1° Consider  $H^n = (M, H(x, p))$  the Hamilton spaces of electrodynamics, [19]:

$$H(x, p) = \frac{1}{mc} \gamma^{jj}(x) p_i p_j - \frac{2e}{mc^2} A^i(x) p_i + \frac{e^3}{mc^3} A_i(x) A^i(x)$$

and  $Fe = p_i \dot{\partial}^i$ . Then  $\Sigma_H$  is a Hamiltonian mechanical system determined only by  $H^n$ .

2°  $H^n = (M, K^2(x, p))$  is a Cartan space and  $Fe = p_i \dot{\partial}^i$ .

3°  $H^n = (M, \mathcal{E}(x, p))$  with  $\mathcal{E}(x, p) = \gamma^{jj}(x) p_i p_j$  and  $Fe = a(x) p_i \dot{\partial}^i$ .

Returning to the general theory, we can prove:

**Theorem 4.2.1.** *The following properties hold:*

1°  $\xi$  given by

$$\xi = \frac{1}{2} [\dot{\partial}^i H \partial_i - (\partial_i H - F_i) \dot{\partial}^i] \quad (4.2.4)$$

is a vector field on  $\widetilde{T^*M}$ .

2°  $\xi$  is determined only by the Hamiltonian mechanical system  $\Sigma_H$ .

3° The integral curves of  $\xi$  are given by the Hamilton equation (4.2.3).

The previous Theorem is not difficult to prove if we remark the following expression of  $\xi$ :

$$\xi = \overset{\circ}{\xi} + \frac{1}{2} Fe. \quad (4.2.5)$$

Also we have:

**Proposition 4.2.1.** *The variation of the Hamiltonian  $H(x, p)$  along the evolution curves of  $\Sigma_H$  is given by:*

$$\frac{dH}{dt} = F_i \frac{dx^i}{dt}. \quad (4.2.6)$$

As we know, the external forces  $Fe$  are dissipative if  $\langle Fw, \mathbb{C} \rangle \geq 0$ . Looking at the formula (4.2.6) one can say:

**Proposition 4.2.2.** *The fundamental function  $H(x, p)$  of the Hamiltonian mechanical system  $\Sigma_H$  is decreasing on the evolution curves of  $\Sigma_H$ , if and only if, the external forces  $Fe$  are dissipative.*

The vector field  $\xi$  on  $\widetilde{T^*M}$  is called the canonical dynamical system of the Hamilton mechanical system  $\Sigma_H$ .

Therefore we can say: The geometry of  $\Sigma_H$  is the geometry of pair  $(H^n, \xi)$ .

### 4.3 Canonical nonlinear connection of $\Sigma_H$

The fundamental tensor  $g^{ij}(x, p)$  of the Hamilton space  $H^n$  is the fundamental or metric tensor of the mechanical system  $\Sigma_H$ . But others fundamental geometric notions, as the canonical nonlinear connection of  $\Sigma_H$  cannot be introduced in a straightforward manner. They will be defined by means of  $\mathcal{L}$ -duality between the Lagrangian and the Hamiltonian mechanical systems  $\Sigma_L$  and  $\Sigma_H$  (see ch. ..., part I).

Let  $\Sigma_L = (M, L(x, y), Fe(x, y))$ ,  $Fe = F^i_1(x, y)\dot{\partial}_i$  be a Lagrangian mechanical system. The mapping

$$\varphi : (x, y) \in TM \rightarrow (x, p) \in T^*M, \quad p_i = \frac{1}{2}\dot{\partial}_i L$$

is a local diffeomorphism. It is called the *Legendre transformation*.

Let  $\psi$  be the inverse of  $\varphi$  and

$$H(x, p) = 2p_i y^i - L(x, y), \quad y = \psi(x, p). \quad (4.3.1)$$

One prove that  $H^n = (M, H(x, p))$  is an Hamilton space. It is the  $\mathcal{L}$ -dual of Lagrange space  $L^n = (M, L(x, y))$ .

One proves that  $\varphi$  transform:

1° The canonical semispray  $\overset{\circ}{S}$  of  $L^n$  in the Hamilton vector  $\overset{\circ}{\xi}$  of  $H^n$ .

2° The canonical nonlinear connection  $\overset{\circ}{N}_L$  of  $L^n$  into the canonical nonlinear connection  $\overset{\circ}{N}_H$  of  $H^n$ .

3° The external forces  $Fe$  of  $\Sigma_L$  into external forces  $Fe$  of  $\Sigma_H$ , with

$$F_i(x, p) = g_{ij}(x, p)F^j_1(x, \psi(x, p)).$$

4° The canonical nonlinear connection  $N_L$  of  $\Sigma_L$  with coefficients into the canonical nonlinear connection  $N_H$  of  $\Sigma_H$  with the coefficients

$$N_{ij}(x, p) = \overset{\circ}{N}_{ij}(x, p) + \frac{1}{4}g_{ih}\dot{\partial}^h F_j. \quad (4.3.2)$$

$\overset{\circ}{N}_{ij}$  are given by (...., ch. ...).

Therefore we have:

**Proposition 4.3.1.** *The canonical nonlinear connection  $N$  of the Hamiltonian mechanical system  $\Sigma_H$  has the coefficients  $N_{ij}$ , (4.3.2).*

Of course, directly we can prove that  $N_{ij}$  are the coefficients of a nonlinear connection. It is canonical for  $\Sigma_H$ , since  $N$  depend only on the mechanical system  $\Sigma_H$ .

The torsion of  $N$  is

$$t_{ij} = \frac{1}{4}(g_{ih}\dot{\partial}^h F_j - g_{jh}\dot{\partial}^h F_i). \quad (4.3.3)$$

Evidently  $\dot{\partial}^i F_j = 0$ , implies that  $N$  is symmetric nonlinear connection on the momenta space  $T^*M$ .

Let  $(\delta_i, \dot{\partial}^i)$  be the adapted basis to  $N$  and  $V$  and  $(dx^i, \delta p_i)$  its adapted cobasis:

$$\begin{aligned} \delta_i &= \partial_i + N_{ji}\dot{\partial}^j = \delta_i + \frac{1}{4}g_{jh}\dot{\partial}^h F_i\dot{\partial}^j, \\ \delta p_i &= dp_i - N_{ij}dx^j = \overset{\circ}{\delta}p_i - \frac{1}{4}g_{ih}\dot{\partial}^h F_i dx^j. \end{aligned} \quad (4.3.4)$$

The tensor of integrability of the canonical nonlinear connection  $N$  is

$$R_{kij} = \delta_i N_{kj} - \delta_j N_{ki}. \quad (4.3.5)$$

$R_{kij} = 0$  characterize the integrability of the horizontal distribution  $N$ .

The canonical  $N$ -metrical connection  $CG(N) = (H_{jk}^i, C_i^{jk})$  of the Hamiltonian mechanical system  $\Sigma_H$  is given by the following Theorem:

**Theorem 4.3.1.** *The following properties hold:*

1) *There exists only one  $N$ -linear connection  $CG = (N_{ij}, H_{jk}^i, C_i^{jk})$  which depend on the Hamiltonian system  $\Sigma_H$  and satisfies the axioms:*

1°  $N_{ij}$  from (4.3.2) is the canonical nonlinear connection.

2°  $CG$  is  $h$ -metric:

$$g_{|k}^{ij} = 0. \quad (4.3.6)$$

3°  $CF$  is  $v$ -metric:

$$g^{ij|k} = 0. \quad (4.3.6')$$

4°  $CF$  is  $h$ -torsion free:

$$T_{jk}^i = H_{jk}^i - H_{kj}^i = 0. \quad (4.3.7)$$

5°  $CF$  is  $v$ -torsion free

$$S_i^{jk} = C_i^{jk} - C_i^{kj} = 0. \quad (4.3.7')$$

2) The coefficients of  $CF$  are given by the generalized Christoffel symbols:

$$H_{jk}^i = \frac{1}{2} g^{is} (\delta_j g_{sk} + \delta_k g_{js} - \delta_s g_{jk}) \quad (4.3.8)$$

$$C_i^{jk} = -\frac{1}{2} g_{is} (\dot{\partial}^j g^{sk} + \dot{\partial}^k g^{js} - \dot{\partial}^s g^{jk})$$

Now, we are in possession of all data to construct the geometry of Hamilton mechanical system  $\Sigma_H$ . Such that we can investigate the electromagnetic and gravitational fields of  $\Sigma_H$ .

## 4.4 The Cartan mechanical systems

An important class of systems  $\Sigma_H$  is obtained when the Hamiltonian  $H(x, p)$  is 2-homogeneous with respect to momenta  $p_i$ .

**Definition 4.4.1.** A Cartan mechanical system is a set

$$\Sigma_{\mathcal{C}} = (M, K(x, p), Fe(x, p)) \quad (4.4.1)$$

where  $\mathcal{C}^n = (M, K(x, p))$  is a Cartan space and

$$Fe = F_i(x, p) \dot{\partial}^i \quad (4.4.2)$$

are the external forces.

The fact that  $\mathcal{C}^n$  is a Cartan spaces implies:

1°  $K(x, p)$  is a positive scalar function on  $T^*M$ .

2°  $K(x, p)$  is a positive 1-homogeneous with respect to momenta  $p_i$ .

3° The pair  $H^n = (M, K^2(x, p))$  is a Hamilton space.

Consequently:

a. The fundamental tensor  $g^{ij}(x, p)$  of  $\mathcal{C}^n$  is

$$g^{ij} = \frac{1}{2} \partial^i \partial^j K^2. \quad (4.4.3)$$

b. We have

$$K^2 = g^{ij} p_i p_j. \quad (4.4.4)$$

c. The Cartan tensor is

$$C^{ijk} = \frac{1}{4} \partial^i \partial^j \partial^k K^2, \quad p_i C^{ijk} = 0.$$

d.  $\mathcal{C}^n = (M, K(x, p))$  is  $\mathcal{L}$ -dual of a Finsler space  $F^n = (M, F(x, y))$ .

e. The canonical nonlinear connection  $\overset{\circ}{N}$ , established by R. Miron [175], has the coefficients

$$\overset{\circ}{N}_{ij} = \gamma_{ij}^h p_h - \frac{1}{2} (\gamma_{sr}^h p_h p^r) \partial^s g_{ij}, \quad (4.4.5)$$

$\gamma_{jk}^i(x, p)$  being the Christoffel symbols of  $g_{ij}(x, p)$ .

Clearly, the geometry of  $\Sigma_{\mathcal{C}}$  is obtained from the geometry of  $\Sigma_H$  taking  $H = K^2(x, p)$ .

So, we have:

**Postulate 4.4.1.** *The evolution equations of the Cartan mechanical system  $\Sigma_{\mathcal{C}}$  are the Hamilton equations:*

$$\frac{dx^i}{dt} - \frac{1}{2} \partial^i K^2 = 0, \quad \frac{dp_i}{dt} + \frac{1}{2} \partial_i K^2 = \frac{1}{2} F_i(x, p). \quad (4.4.6)$$

A first result is given by

**Proposition 4.4.1.** *1° The energy of the Hamiltonian  $K^2$  is given by  $\mathcal{E}_{K^2} = p_i \dot{\partial}^i K^2 - K^2 = K^2$ .*

*2° The variation of energy  $\mathcal{E}_{K^2} = K^2$  along to every evolution curve (4.4.6) is*

$$\frac{dK^2}{dt} = F_i \frac{dx^i}{dt}. \quad (4.4.7)$$

**Example.**  $\Sigma_{\mathcal{C}}$ , with  $K(x, p) = \{\gamma^{ij}(x) p_i p_j\}^{1/2}$ ,  $(M, \gamma_{ij}(x))$  being a Riemann spaces and  $F_i = a(x, p) p_i \dot{\partial}^i$ .

Theorem ... of part ... can be particularized in:

**Theorem 4.4.1.** *The following properties hold good:*

1°  $\xi$  given by

$$\xi = \frac{1}{2}[\dot{\partial}^i K^2 \partial_i - (\partial_i K^2 - F_i) \dot{\partial}^i] \quad (4.4.8)$$

is a vector field on  $\widetilde{T^*M}$ .

2°  $\xi$  is determined only by the Cartan mechanical system  $\Sigma_{\mathcal{L}}$ .

3° The integral curves of  $\xi$  are given by the evolution equations (4.4.6) of  $\Sigma_{\mathcal{L}}$ .

The vector  $\xi$  is Hamiltonian vector field on  $T^*M$  or of the Cartan mechanical system  $\Sigma_{\mathcal{L}}$ .

Therefore, the geometry of  $\Sigma_{\mathcal{L}}$  is the differential geometry of the pair  $(\mathcal{C}^n, \xi)$ .

The fundamental object fields of this geometry are  $\mathcal{C}^n, \xi, Fe$ , the canonical nonlinear connection  $N$  with the coefficients

$$N_{ij} = \overset{\circ}{N}_{ij} + \frac{1}{4} g_{ih} \dot{\partial}^h F_j, \quad (4.4.9)$$

Taking into account that the vector fields  $\delta_i = \partial_i - N_{ji} \dot{\partial}^j$  determine an adapted basis to the nonlinear connection  $N$ , we can get the canonical  $N$ -metrical connection  $CF(N)$  of  $\Sigma_{\mathcal{L}}$ .

**Theorem 4.4.2.** *The canonical  $N$ -metrical connection  $CF(N)$  of the Cartan mechanical system  $\Sigma_{\mathcal{L}}$  has the coefficients*

$$H_{jk}^i = \frac{1}{2} g^{is} (\delta_j g_{sk} + \delta_k g_{js} - \delta_s g_{jk}), \quad C_i^{jk} = g_{is} C^{sjk}. \quad (4.4.10)$$

Using the canonical connections  $N$  and  $CF(N)$  one can study the electromagnetic and gravitational fields on the momenta space  $T^*M$  of the Cartan mechanical systems  $\Sigma_{\mathcal{L}}$ , as well as the dynamical system  $\xi$  of  $\Sigma_{\mathcal{L}}$ .





## Chapter 5

### Lagrangian, Finslerian and Hamiltonian mechanical systems of order $k \geq 1$

The notion of Lagrange or Finsler spaces of higher order allows us to introduce the Lagrangian and Finslerian mechanical systems of order  $k \geq 1$ . They will be defined as a natural extension of the Lagrangian and Finslerian mechanical systems. Now, the geometrical theory of these analytical mechanics is constructed by means of the differential geometry of the manifold  $T^kM$ -space of accelerations of order  $k$  and of the notion of  $k$ -semispray, presented in the part II.

#### 5.1 Lagrangian Mechanical systems of order $k \geq 1$

We repeat shortly some fundamental notion about the differential geometry of the acceleration space  $T^kM$ .

Let  $T^kM$  be the acceleration space of order  $k$  (see ch. 1, part II) and  $u \in T^kM$ ,  $u = (x, y^{(1)}, \dots, y^{(k)})$  a point with the local coordinates  $(x^i, y^{(1)i}, \dots, y^{(k)i})$ .

A smooth curve  $c : I \rightarrow M$  represented on a local chart  $\mathcal{U} \subset M$  by  $x^i = x^i(t)$ ,  $t \in I$  can be extended to  $(\pi^k)^{-1}(\mathcal{U}) \subset T^kM$  by  $\tilde{c} : I \rightarrow T^kM$ , given as follows:

$$x^i = x^i(t), \quad y^{(1)i} = \frac{1}{1!} \frac{dx^i(t)}{dt}, \dots, y^{(k)i} = \frac{1}{k!} \frac{d^k x^i}{dt^k}(t), \quad t \in I. \quad (5.1.1)$$

Consider the vertical distribution  $V_1, V_2, \dots, V_k$ . They are integrable and have the properties  $V_1 \supset V_2 \supset \dots \supset V_k$ . The dimension of  $V_k$  is  $n$ ,  $\dim V_{k-1} = 2n, \dots, \dim V_1 = kn$ .

The Liouville vector fields on  $T^kM$  are:

$$\begin{aligned}
\overset{1}{\Gamma} &= y^{(1)i} \frac{\partial}{\partial y^{(k)i}} \\
\overset{2}{\Gamma} &= y^{(1)i} \frac{\partial}{\partial y^{(k-1)i}} + 2y^{(2)i} \frac{\partial}{\partial y^{(k)i}} \\
&\dots\dots\dots \\
\overset{k}{\Gamma} &= y^{(1)i} \frac{\partial}{\partial y^{(1)i}} + 2y^{(2)i} \frac{\partial}{\partial y^{(2)i}} + \dots + ky^{(k)i} \frac{\partial}{\partial y^{(k)i}}.
\end{aligned} \tag{5.1.2}$$

Thus  $\overset{1}{\Gamma} \in V_k, \overset{2}{\Gamma} \in V_{k-1}, \dots, \overset{k}{\Gamma} \in V_1$ .

The following operator:

$$\Gamma = y^{(1)i} \frac{\partial}{\partial x^i} + 2y^{(2)i} \frac{\partial}{\partial y^{(1)i}} + \dots + ky^{(k)i} \frac{\partial}{\partial y^{(k-1)i}} \tag{5.1.3}$$

is not a vector field on  $T^k M$ , but it is frequently used. On  $T^k M$  there exists a  $k$  tangent structure  $J$  defined in ch. 2.  $J$  is integrable and has the properties

$$\text{Im}J = V, \quad \ker J = V_k, \quad \text{rank}J = k^n \tag{5.1.4}$$

and  $J(\overset{1}{\Gamma}) = 0, J(\overset{2}{\Gamma}) = \overset{1}{\Gamma}, \dots, J(\overset{k}{\Gamma}) = \overset{k-1}{\Gamma}, J \circ J \circ \dots \circ J = 0$  ( $k$  times).

A  $k$ -semispray is a vector field  $S$  on the acceleration space  $T^k M$  with the property

$$JS = \overset{k}{\Gamma}. \tag{5.1.5}$$

Also,  $S$  is called a *dynamical system* on  $T^k M$ .

The local expression of a  $k$ -semispray  $S$  is (ch. 1, part II)

$$S = \Gamma - (k+1)G^i \frac{\partial}{\partial y^{(k)i}}. \tag{5.1.6}$$

The system of functions  $G^i(x, y^{(1)}, \dots, y^{(k)})$  are the *coefficients* of  $S$ .

The integral curves of the  $k$ -semispray  $S$  are given by the following system of differential equations

$$\begin{cases} \frac{d^{k-1}x^i}{dt^{k-1}} + (k+1)G^i(x, y^{(1)}, \dots, y^{(k)}) = 0 \\ y^{(1)i} = \frac{dx^i}{dt}, \dots, y^{(k)i} = \frac{1}{k!} \frac{d^k x^i}{dt^k}. \end{cases} \tag{5.1.7}$$

If  $Fe = F^i(x, y^{(1)}, \dots, y^{(k)}) \frac{\partial}{\partial y^{(k)i}}$  is an arbitrary vertical vector field into the vertical distribution  $V_k$  and  $S$  is a  $k$ -semispray, thus

$$S' = S + Fe \quad (5.1.8)$$

is a  $k$ -semispray, because  $JS' = JS = \overset{k}{\Gamma}$ .

Recall the notion of nonlinear connection (ch. 1, part II):

A nonlinear connection on the manifold  $T^kM$  is a distribution  $N$  supplementary to the vertical distribution  $V = V_1$ :

$$T_u T^k M = N(u) + V(u), \quad \forall u \in T^k M. \quad (5.1.9)$$

An adapted basis to the distribution  $N$  is (ch. 1, part II)

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N_{1i}^j \frac{\partial}{\partial y^{(1)j}} - \dots - N_{ki}^j \frac{\partial}{\partial y^{(k)j}}. \quad (5.1.10)$$

The systems of functions  $N_{1i}^j, \dots, N_{ki}^j$  are the coefficients of nonlinear connection  $N$ .

Let  $N_1, \dots, N_{k-1}, V_k$  the vertical distributions defined by

$$N_1 = J(N), N_2 = J(N_1), \dots, N_{k-1} = J(N_{k-2}), V_k = J(N_{k-1}) \text{ and } N_0 = N. \quad (5.1.11)$$

Thus,  $\dim N_0 = \dim N_1 = \dots = \dim N_k = n$  and we have

$$T_u T^k N = N_0(u) \oplus N_1(u) \oplus \dots \oplus N_{k-1}(u) \oplus V_k, \quad \forall u \in T^k M. \quad (5.1.12)$$

The adapted basis to the direct decomposition (5.1.12) is as follows:

$$\left( \frac{\delta}{\delta x^i}, \frac{\delta}{\delta y^{(1)i}}, \dots, \frac{\delta}{\delta y^{(k)i}} \right), \quad (5.1.13)$$

where  $\frac{\delta}{\delta x^i}$  is given by (5.1.10) and

$$\begin{aligned} \frac{\delta}{\delta y^{(1)i}} &= J \left( \frac{\delta}{\delta x^i} \right), \frac{\delta}{\delta y^{(2)i}} = J \left( \frac{\delta}{\delta y^{(1)i}} \right), \dots, \frac{\delta}{\delta y^{(k)i}} = \\ &= J \left( \frac{\delta}{\delta y^{(k-1)i}} \right) = \frac{\partial}{\partial y^{(k)i}}. \end{aligned} \quad (5.1.14)$$

The expressions of the basis (5.1.13) can be easily obtained from the formulas (5.1.10), (5.1.14) (ch. 1, part II).

The dual basis of the adapted basis (5.1.13) is:

$$(\delta x^i, \delta y^{(1)i}, \dots, \delta y^{(k)i}), \tag{5.1.15}$$

where

$$\begin{aligned} \delta x^i &= dx^i \\ \delta y^{(1)i} &= dy^{(1)i} + M^i_{1j} dx^j \\ \delta y^{(2)i} &= dy^{(2)i} + M^i_{1j} dy^{(1)j} + M^i_{2j} dx^j \\ \dots\dots\dots \\ \delta y^{(k)i} &= dy^{(k)i} + M^i_{1j} dy^{(k-1)j} + \dots + M^i_{kj} dx^j \end{aligned} \tag{5.1.16}$$

and the systems of functions  $M^i_{1j}, \dots, M^i_{kj}$  are the dual coefficients of the nonlinear connection  $N$ . The relations between dual coefficients  $M^i_{1j}, \dots, M^i_{kj}$  and (primal) coefficients  $N^i_{1j}, \dots, N^i_{kj}$  are (ch. 1, part II):

$$\begin{aligned} M^i_{1j} &= N^i_{1j}, \\ M^i_{2j} &= N^i_{2j} + N^m_{1j} M^i_{1m}, \\ \dots\dots\dots \\ M^i_{kj} &= N^i_{kj} + N^m_{k-1j} M^i_{1m} + \dots + N^m_{1j} N^i_{k-1m}. \end{aligned} \tag{5.1.17}$$

This recurrence formulas allow us to calculate the primal coefficients  $N^i_{1j}, \dots, N^i_{kj}$  functions by the dual coefficients  $M^i_{1j}, \dots, M^i_{kj}$ .

In the adapted basis (5.1.13) the Liouville vector fields  $\Gamma^1, \dots, \Gamma^k$  can be expressed:

$$\begin{aligned}
\overset{1}{\Gamma} &= z^{(1)i} \frac{\delta}{\delta y^{(k)i}} \\
\overset{2}{\Gamma} &= z^{(1)i} \frac{\delta}{\delta y^{(k-1)i}} + 2z^{(2)i} \frac{\delta}{\delta y^{(k)i}} \\
&\dots\dots\dots \\
\overset{k}{\Gamma} &= z^{(1)i} \frac{\delta}{\delta y^{(1)i}} + 2z^{(2)i} \frac{\delta}{\delta y^{(2)i}} + \dots + kz^{(k)i} \frac{\delta}{\delta y^{(k)i}},
\end{aligned} \tag{5.1.18}$$

where  $z^{(1)i}, \dots, z^{(k)i}$  are the  $d$ -Liouville vector fields:

$$\begin{aligned}
z^{(1)i} &= y^{(1)i} \\
2z^{(2)i} &= 2y^{(2)i} + M_{1m}^i y^{(1)m} \\
&\dots\dots\dots \\
kz^{(k)i} &= ky^{(k)i} + M_{(1)m}^{(k-1)i} y^{(k-1)m} + \dots + M_{(1)m}^{(k-1)i} y^{(1)m}.
\end{aligned} \tag{5.1.19}$$

To a change of local coordinates (ch. 1, part II) on the acceleration manifold  $T^k M$ , every one of the Liouville  $d$ -vectors transform on the rule:

$$\tilde{z}^{(\alpha)i} = \frac{\partial \tilde{x}^i}{\partial x^j} z^{(\alpha)j}, \quad (\alpha = 1, \dots, k). \tag{5.1.20}$$

So, they have a geometric meaning.

## 5.2 Lagrangian mechanical system of order $k$ , $\Sigma_{L^k}$

**Definition 5.2.1.** A Lagrangian mechanical system of order  $k \geq 1$  is a triple:

$$\Sigma_{L^k} = (M, L(x, y^{(1)}, \dots, y^{(k)}), Fe(x, y^{(1)}, \dots, y^{(k)})), \tag{5.2.1}$$

where

$$L^{(k)n} = (M, L(x, y^{(1)}, \dots, y^{(k)})) \tag{5.2.2}$$

is a Lagrange space of order  $k \geq 1$ , and

$$Fe(x, y^{(1)}, \dots, y^{(k)}) = F^i(x, y^{(1)}, \dots, y^{(k)}) \frac{\partial}{\partial y^{(k)i}} \tag{5.2.3}$$

is an a priori given vertical vector field.

Clearly,  $Fe$  is a vector field belonging to the vertical distribution  $V_k$  and its contravariant components  $F^i$  is a distinguished vector field.  $Fe$  or  $F^i$  are called the external forces of  $\Sigma_{L^n}$ .

Applying the theory of Lagrange spaces of order  $k \geq 1$  to the space  $L^{(k)n}$  we have the fundamental (or metric) tensor field of  $\Sigma_{L^k}$ :

$$g_{ij} = \frac{1}{2} \frac{\partial^2 L}{\partial y^{(k)i} \partial y^{(k)j}}, \quad (5.2.4)$$

and the Euler–Lagrange equations

$$\overset{\circ}{E}_i(L) \stackrel{det}{=} \frac{\partial L}{\partial x^i} - \frac{d}{dt} \frac{\partial L}{\partial y^{(1)i}} + \dots + (-1)^k \frac{d^k}{dt^k} \frac{\partial L}{\partial y^{(k)i}} = 0 \quad (5.2.5)$$

$$y^{(1)i} = \frac{dx^i}{dt}, \dots, y^{(k)1} = \frac{1}{k!} \frac{d^k x^i}{dt^k}.$$

This is a system of differential equations of order  $2k$ , autoadjoint. It allows us to determine a canonical semispray of order  $k$  on  $T^k M$ .

Indeed, since  $\overset{\circ}{E}_i(L)$  is a  $d$ -covector field on  $T^k M$ , we can calculate  $\overset{\circ}{E}(\phi(t)L(x, y^{(1)}, \dots, y^{(k)}))$  and obtain (ch. 1, part II):

$$\overset{\circ}{E}_i(\phi L) = \phi \overset{\circ}{E}_i + \frac{d\phi}{dt} \overset{1}{E}_i(L) + \dots + \frac{d^k \phi}{dt^k} \overset{k}{E}_i(L). \quad (5.2.6)$$

The coefficients  $\overset{1}{E}_i(L), \dots, \overset{k}{E}_i(L)$  are  $d$ -covector fields discovered by Craig and Synge (we refer to books [161]).

So, we have:

$$\overset{k-1}{E}_i(L) = (-1)^{k-1} \frac{1}{(k-1)!} \left[ \frac{\partial L}{\partial y^{(k-1)i}} - \frac{d}{dt} \frac{\partial L}{\partial y^{(k)i}} \right]. \quad (5.2.7)$$

**Theorem 5.2.1.** *For any Lagrange spaces  $L^{(k)n} = (M, L)$  the following properties hold:*

1° *the system of differential equations  $g^{ij} \overset{k-1}{E}_i(L) = 0$  has the form*

$$\frac{d^{k+1} x^i}{dt^{k+1}} + (k+1)! G^{\circ i} \left( x, \frac{dx}{dt}, \dots, \frac{1}{k!} \frac{d^k x}{dt^k} \right) = 0 \quad (5.2.8)$$

with

$$(k+1)G^{\circ i} = \frac{1}{2}g^{ij} \left[ \Gamma \frac{\partial L}{\partial y^{(k)j}} - \frac{\partial L}{\partial y^{(k-1)j}} \right]. \quad (5.2.9)$$

2° The system of functions  $G^{\circ i}$  from (5.2.9) are the coefficients of  $k$ -semispray  $\overset{\circ}{S}$  which depends only on the Lagrangian  $L(x, y^{(1)}, \dots, y^{(k)})$ .

So,  $\overset{\circ}{S}$  is the **canonical semispray** of  $L^{(k)n}$ . But it leads to a canonical nonlinear connection  $\overset{\circ}{N}$  for  $L^{(k)n}$ .

Indeed, we have

**Theorem 5.2.2 (Miron [161]).**  $\overset{\circ}{S}$  being the canonical  $k$ -semispray of Lagrange space  $L^{(k)n}$ , there exists a nonlinear connection  $\overset{\circ}{N}$  of the space  $L^{(k)n}$  which depend only on the fundamental function  $L$ . The non-connection  $\overset{\circ}{N}$  has the dual coefficients:

$$\begin{aligned} M^i_{1j} &= \frac{\partial G^i}{\partial y^{(k)j}}, \\ M^i_{2j} &= \frac{1}{2}(\overset{\circ}{S}M^i_{1j} + M^i_{1m} M^m_{1j}), \\ &\dots\dots\dots \\ M^i_{kj} &= \frac{1}{k}(\overset{\circ}{S}M^i_{k-1j} + M^i_{k-1m} M^m_{k-1j}). \end{aligned} \quad (5.2.10)$$

Now, we can develop the geometry of Lagrange space  $L^{(k)n}$  by means of the geometrical object fields  $L, \overset{\circ}{S}, \overset{\circ}{N}$  and  $g_{ij}$  (ch. 2, part II). We apply these considerations to the geometrization of Lagrange mechanical system  $\Sigma_{L^k}$ .

### 5.3 Canonical $k$ -semispray of mechanical system $\Sigma_{L^k}$

The fundamental equations of mechanical system  $\Sigma_k$  are given by:

**Postulate 5.3.1.** The evolution equations of the Lagrangian mechanical system of order  $k$ ,  $\Sigma_{L^k} = (M, L, Fe)$  are the following Lagrange equations:

$$\begin{cases} \frac{d}{dt} \frac{\partial L}{\partial y^{(k)i}} - \frac{\partial L}{\partial y^{(k-1)i}} = F_i, \\ y^{(1)i} = \frac{dx^i}{dt}, \dots, y^{(k)i} = \frac{1}{k!} \frac{d^k x^i}{dt^k}, \end{cases} \quad (5.3.1)$$

where

$$F_i(x, y^{(1)}, \dots, y^{(k)}) = g_{ij}(x, y^{(1)}, \dots, y^{(k)}) F^j(x, y^{(1)}, \dots, y^{(k)}) \quad (5.3.2)$$

are the covariant components of external forces  $Fe$ .

Evidently, in the case  $k = 1$ , the previous equations are the evolution equations of Lagrangian mechanical system  $\Sigma_L = (M, L(x, y), Fe(x, y))$ . Also, the equation (5.3.1) for  $k = 1$  are the Lagrange equations for Riemannian or Finslerian mechanical systems. These arguments justify the introduction of previous Postulate.

**Examples.**  $1^\circ$  Let  $L^{(k)n} = (M, L(x, y^{(1)}, \dots, y^{(k)}))$  be a Lagrange space of order  $k \geq 1$ , and  $\overset{\circ}{z}^{(k)i}$  be its Liouville  $d$ -vector field of order  $k$ . Consider the external forces

$$Fe = h \overset{\circ}{z}^{(k)i} \frac{\partial}{\partial y^{(k)i}}, \quad h \in \mathbb{R}^*. \quad (5.3.3)$$

Thus,  $\Sigma_{L^k} = (M, L(x, y^{(1)}, \dots, y^{(k)}), Fe(x, y^{(1)}, \dots, y^{(k)}))$  is a Lagrangian mechanical system of order  $k$ .

The covariant component  $F_i$  of  $Fe$  is  $F_i = h g_{ij} \overset{\circ}{z}^{(k)j}$ , which substitute in (5.3.1) give us the Lagrange equations of order  $k$  for the considered mechanical system.

In the general case of mechanical systems  $\Sigma_{L^k}$  is important to prove the following property:

**Theorem 5.3.1.** *The Lagrange equations (5.3.1) are equivalent to the following system of differential equations of order  $k + 1$ :*

$$\frac{d^{k+1} x^i}{dt^{k+1}} + (k+1)! \overset{\circ}{G}^i \left( x, \frac{dx}{dt}, \dots, \frac{1}{k} \frac{d^k x}{dt^k} \right) = \frac{k!}{2} F^i \quad (5.3.4)$$

with the coefficients  $\overset{\circ}{G}^i(x, y^{(1)}, \dots, y^{(k)})$  from (5.2.9).

*Proof.* We have



$$\frac{d}{dt} \frac{\partial L}{\partial y^{(k)i}} - \frac{\partial L}{\partial y^{(k-1)i}} = \Gamma \frac{\partial L}{\partial y^{(k)i}} - \frac{\partial L}{\partial y^{(k-1)i}} + \frac{2}{k!} g^{ij} \frac{d^{k+1}x^j}{dt^{k+1}} - F_i.$$

It follows

$$\frac{d^{k+1}x^i}{dt^{k+1}} + \frac{k!}{2} g^{ij} \left( \Gamma \frac{\partial L}{\partial y^{(k)j}} - \frac{\partial L}{\partial y^{(k-1)j}} \right) = \frac{k!}{2} F^i. \quad (5.3.4')$$

But the previous equations is exactly (5.3.4), with the coefficients  $(k+1)\overset{\circ}{G}^i$  given by (5.2.9).

In initial conditions:  $(x^i)_0 = x_0^i$ ,  $\left(\frac{dx^i}{dt}\right)_0 = y_0^{(1)i}, \dots, \frac{1}{k!} \left(\frac{d^k x^i}{dt^k}\right)_0 = y_0^{(k)i}$ , locally there exists a unique solution of evolution equations (5.3.4):  $x^i = x^i(t)$ ,  $y^{(1)i} = y^{(1)i}(t), \dots, y^{(k)i} = y^{(k)i}(t)$ ,  $t \in (a, b)$  which express the moving of the mechanical system  $\Sigma_{L^k}$ .

But, it is convenient to write the  $k$ -Lagrange equations (5.3.4) in the form

$$\begin{aligned} y^{(1)i} &= \frac{dx^i}{dt}, y^{(2)i} = \frac{1}{2} \frac{d^2 x^i}{dt^2}, \dots, y^{(k)i} = \frac{1}{k!} \frac{d^k x^i}{dt^k} \\ \frac{dy^{(k)i}}{dt} + (k+1)\overset{\circ}{G}^i(x, y^{(1)}, \dots, y^{(k)}) &= \frac{1}{2} F^i(x, y^{(1)}, \dots, y^{(k)}). \end{aligned} \quad (5.3.5)$$

These equations determine the trajectories of the vector field  $S$  on the acceleration space  $T^k M$ :

$$\begin{aligned} S &= y^{(1)i} \frac{\partial}{\partial x^i} + 2y^{(2)i} \frac{\partial}{\partial y^{(1)i}} + \dots + ky^{(k)i} \frac{\partial}{\partial y^{(k-1)i}} - \\ &-(k+1)\overset{\circ}{G}^i \frac{\partial}{\partial y^{(k)i}} + \frac{1}{2} F^i \frac{\partial}{\partial y^{(k)i}} \end{aligned} \quad (5.3.6)$$

or

$$S = \overset{\circ}{S} + \frac{1}{2} F e \quad (5.3.6')$$

where  $\overset{\circ}{S}$  is the canonical  $k$ -semispray of the Lagrange space of order  $k$ ,  $L^{(k)n} = (M, L)$ . Consequently, we have:

**Theorem 5.3.2.** 1° For the Lagrangian mechanical system of order  $k$ ,  $\Sigma_{L^k}$  the operator  $S$  from (5.3.6) is a  $k$ -semispray which depend only on the mechanical system  $\Sigma_{L^k}$ .

2° The integral curves of  $S$  are the evolution curves given by Lagrange equations (5.3.5).

$S$  is called the canonical semispray of  $\Sigma_{L^k}$  or it is called the dynamical system of  $\Sigma_{L^k}$ , too.

The coefficients  $G^i$  of the canonical  $k$ -semispray  $S$  are:

$$(k+1)G^i = (k+1)\overset{\circ}{G}^i - \frac{1}{2}F^i. \quad (5.3.7)$$

In the example (5.3.3) the canonical  $k$ -semispray has the coefficients

$$(k+1)G^i = (k+1)\overset{\circ}{G}^i - \frac{h}{2}z^{(k)i}. \quad (5.3.8)$$

Of course, the vector field  $S$  on the manifold of the acceleration of order  $k \geq 1$ ,  $T^kM$  being a dynamical system it can be used for investigate the qualitative problems, as stability of solutions, equilibrium points etc.

Now, we can study the canonical nonlinear connection of the mechanical system  $\Sigma_{L^k}$ .

## 5.4 Canonical nonlinear connection of mechanical system $\Sigma_{L^k}$

Let  $G^i$  be the coefficients of the canonical semispray  $S$  of mechanical system  $\Sigma_{L^k}$ . Thus

$$G^i = \overset{\circ}{G}^i - \frac{1}{2(k+1)}F^i, \quad (5.4.1)$$

where

$$\overset{\circ}{G}^i = \frac{1}{2(k+1)}g^{ij} \left[ \Gamma \frac{\partial L}{\partial y^{(k)j}} - \frac{\partial L}{\partial y^{(k-1)j}} \right]. \quad (5.4.2)$$

The dual coefficients  $\overset{\circ}{M}_{1j}^i, \dots, \overset{\circ}{M}_{kj}^i$  of the canonical nonlinear connection

$\overset{\circ}{N}$  of Lagrange space  $L^{(k)n} = (M, L)$  are given by (5.2.10):

$$\begin{aligned} \overset{\circ}{M}_{1j}^i &= \frac{\partial \overset{\circ}{G}}{\partial y^{(k)j}}, \overset{\circ}{M}_{2j}^i = \frac{1}{2}(\overset{\circ}{S}M_{1j}^i + \overset{\circ}{M}_{1m}^i \overset{\circ}{M}_{1j}^m), \dots, \\ \overset{\circ}{M}_{kj}^i &= \frac{1}{k}(\overset{\circ}{S}M_{k-1j}^i + \overset{\circ}{M}_{1mk-1j}^i \overset{\circ}{M}_{1j}^m). \end{aligned} \tag{5.4.3}$$

By virtue of (5.3.6’):

$$S = \overset{\circ}{S} + \frac{1}{2}Fe \tag{5.4.4}$$

the dual coefficients  $M_{1j}^i, \dots, M_{kj}^i$  of canonical nonlinear connection  $N$  of  $\Sigma_{L^k}$  can be written:

$$\begin{aligned} M_{1j}^i &= \overset{\circ}{M}_{1j}^i - \frac{1}{2(k+1)} \frac{\partial F^i}{\partial y^{(k+1)j}} \\ M_{2j}^i &= \frac{1}{2}(\overset{\circ}{S}M_{1j}^i + \overset{\circ}{M}_{1m}^i \overset{\circ}{M}_{1j}^m) \\ \dots\dots\dots \\ M_{(k)j}^i &= \frac{1}{k}(\overset{\circ}{S}M_{k-1j}^i + \overset{\circ}{M}_{1mk-1j}^i \overset{\circ}{M}_{1j}^m) \end{aligned} \tag{5.4.5}$$

where

$$SM_{\alpha j}^i = \overset{\circ}{S}M_{\alpha j}^i + \frac{1}{2}Fe(M_{\alpha j}^i), \quad \alpha = 1, 2, \dots, k-1. \tag{5.4.6}$$

The coefficients  $N_{1j}^i, \dots, N_{(k)j}^i$  of  $N$  are given by the formulas

$$N_{1j}^i = M_{1j}^i, \quad N_{(\alpha)j}^i = M_{\alpha j}^i - N_{(\alpha-1)j(1)m}^m M_{1j}^i - \dots - N_{(1)j(\alpha-1)m}^m M_{1j}^i, \quad (\alpha = 2, \dots, k). \tag{5.4.7}$$

The adapted basis  $\frac{\delta}{\delta x^i}$  ( $i = 1, \dots, n$ ) to the distribution  $N$  and the adapted basis to the vertical distributions  $V_1, \dots, V_k$ :

$$\left( \frac{\delta}{\delta x^i}, \frac{\delta}{\delta y^{(1)i}}, \dots, \frac{\delta}{\delta y^{(k)i}} \right) \tag{5.4.8}$$

can be explicitly written.

Its dual adapted basis

$$(\delta x^i, \delta y^{(1)i}, \dots, \delta y^{(k)i}) \quad (5.4.9)$$

has the following expression:

$$\begin{aligned} \delta x^i = dx^i, \quad \delta y^{(1)i} = dy^{(1)i} + M_{(1)j}^i dx^j, \dots, \delta y^{(k)i} = dy^{(k)i} + \\ + M_{(1)j}^i dy^{(k-1)j} + \dots + M_{(k)j}^i dx^j. \end{aligned} \quad (5.4.9')$$

By using the Definition from part II, equations (1.5.9), we have

**Theorem 5.4.1.** *The autoparallel curves of the canonical nonlinear connection  $N$  of the mechanical system  $\Sigma_{L^k}$  are given by the following system of differential equation*

$$\begin{aligned} \frac{\delta y^{(1)i}}{dt} = 0, \dots, \frac{\delta y^{(k)i}}{dt} = 0 \\ \frac{dx^i}{dt} = y^{(1)i}, \dots, \frac{1}{k!} \frac{d^k x^i}{dt^k} = y^{(k)i}. \end{aligned} \quad (5.4.10)$$

**Theorem 5.4.2.** *The variation of the Lagrangian  $L$  along the autoparallel curves of the canonical nonlinear connection  $N$  are given by*

$$\frac{dL}{dt} = \frac{\delta L}{\delta x^i} \frac{dx^i}{dt}. \quad (5.4.11)$$

**Corollary 5.4.1.** *The Lagrangian  $L$  of the space  $L^{(k)n} = (M, L)$  of a mechanical system  $\Sigma_{L^k}$  is conserved along the autoparallel curves of the canonical nonlinear connection  $N$  iff the scalar product  $\frac{\delta L}{\delta x^i} \frac{dx^i}{dt}$  vanishes.*

**Remark 5.4.1.** The Bucataru's nonlinear connection  $\overset{b}{N}$  of the mechanical system  $\Sigma_{L^k}$  has the coefficients

$$\begin{aligned}
 N_{(1)j}^i &= \frac{\partial \overset{\circ}{G}^i}{\partial y^{(k)j}} + \frac{1}{2(k+1)} \frac{\partial F^i}{\partial y^{(k)j}} \\
 N_{(2)j}^i &= \frac{\partial \overset{\circ}{G}^i}{\partial y^{(k-1)j}} + \frac{1}{2(k+1)} \frac{\partial F^i}{\partial y^{(k-1)j}} \\
 &\dots\dots\dots \\
 N_{(k)j}^i &= \frac{\partial \overset{\circ}{G}^i}{\partial y^{(1)j}} + \frac{1}{2(k+1)} \frac{\partial F^i}{\partial y^{(1)j}}.
 \end{aligned}
 \tag{5.4.12}$$

This nonlinear connection depend only on the mechanical system  $\Sigma_{L^k}$ , too. Sometime  $N$  is more convenient to solve the problems of the geometrization of mechanical system  $\Sigma_{L^k}$ .

### 5.5 Canonical metrical $N$ -connection

Let  $N$  be the canonical nonlinear connection with the coefficients (5.4.7) or  $N$  being the Bucataru's connection  $N_b$  with the coefficients (5.4.12). An  $N$ -linear connection  $D$  with the coefficients  $D\Gamma(N) = (L_{jh}^i, C_{(1)jh}^i, \dots, C_{(k)jh}^i)$  is metrical with respect to the metric tensor  $g_{ij}$  of the mechanical system  $\Sigma_{L^k}$  (cf. ch. 2, part II) if the following equations hold:

$$g_{ij|h} = 0, \quad g_{ij} \Big|_h^{(\alpha)} = 0, \quad (\alpha = 1, \dots, k). \tag{5.5.1}$$

Thus, the theorem ... implies:

**Theorem 5.5.1.** *The we have:*

(I) *There exists a unique  $N$ -linear connection  $D$  on  $\widetilde{T^kM}$  satisfying the axioms:*

- (1)  *$N$  is the canonical nonlinear connection of the Lagrange space  $L^{(k)n}$  (or  $N$  is Bucataru nonlinear connection);*
- (2)  *$g_{ij|h} = 0$ , ( $D$  is  $h$ -metrical);*
- (3)  *$g_{ij} \Big|_h^{(\alpha)} = 0$ , ( $\alpha = 1, \dots, k$ ) ( $D$  is  $v_\alpha$ -metrical);*
- (4)  *$T_{jh}^i = 0$  ( $D$  is  $h$ -torsion free);*
- (5)  *$S_{(\alpha)jh}^i = 0$  ( $D$  is  $v_\alpha$ -torsion free).*

(II) The coefficients  $C\Gamma(N) = (L_{ij}^h, C_{(1)ij}^h, \dots, C_{(k)ij}^h)$  of  $D$  are given by the generalized Christoffel symbols

$$L_{ij}^h = \frac{1}{2} g^{hs} \left( \frac{\delta g_{is}}{\delta x^j} + \frac{\delta g_{sj}}{\delta x^i} - \frac{\delta g_{ij}}{\delta x^s} \right)$$

$$C_{(\alpha)ij}^h = \frac{1}{2} g^{hs} \left( \frac{\delta g_{is}}{\delta y^{(\alpha)j}} + \frac{\delta g_{sj}}{\delta y^{(\alpha)i}} - \frac{\delta g_{ij}}{\delta y^{(\alpha)s}} \right), \quad (\alpha = 1, \dots, k).$$

(III)  $D$  depends only on the mechanical system  $\Sigma_{L^k}$ .

The  $N$ -metrical connection  $D$  is called the *canonical metrical connection* of the Lagrangian mechanical system of order  $k \geq 1$   $\Sigma_{L^k}$ .

Now, one can affirm: *the geometrization of  $\Sigma_{L^k}$  can be developed by means of the canonical semispray  $S$  and by the geometrical object fields:  $L, g_{ij}, N$ , and  $C\Gamma(N)$ .*

## 5.6 The Riemannian $(k-1)n$ almost contact model of the Lagrangian mechanical system of order $k$ , $\Sigma_{L^k}$

$N$  being the nonlinear connection with the coefficients (5.4.7) (or with the coefficients (5.4.12)) and  $(\delta x^i, \delta y^{(1)i}, \dots, \delta y^{(k)i})$  be the adapted cobasis to the direct decomposition (5.1.12).

Consider the  $N$ -lift  $\mathbb{G}$  of the metric tensor  $g_{ij}$ :

$$\mathbb{G} = g_{ij} dx^i \otimes dx^j + g_{ij} \delta y^{(1)i} \otimes \delta y^{(1)j} + \dots + g_{ij} \delta y^{(k)j} \otimes \delta y^{(k)i}. \quad (5.6.1)$$

Thus,  $\mathbb{G}$  is a semidefinite Riemannian structure on the manifold  $\widetilde{T^k M}$  depending only on  $\Sigma_{L^k}$ .

The distributions  $\{N_0, N_1, \dots, N_{k-1}, V_k\}$  are two by two orthogonal with respect to  $\mathbb{G}$ .

The nonlinear connection  $N$  uniquely determines the  $(k-1)n$ -almost contact structure  $\mathbb{F}$  by:

$$\mathbb{F} \left( \frac{\delta}{\delta x^i} \right) = -\frac{\partial}{\partial y^{(k)i}}; \quad \mathbb{F} \left( \frac{\delta}{\delta y^{(\alpha)i}} \right) = 0, \quad \alpha = 1, \dots, k-1; \quad \mathbb{F} \left( \frac{\partial}{\partial y^{(k)i}} \right) = \frac{\delta}{\delta x^i}. \quad (5.6.2)$$

Indeed, it follows:

**Theorem 5.6.1.** (1)  $\mathbb{F}$  is globally defined on the manifold  $\widetilde{T^k M}$  and depend on  $\Sigma_{T^k}$ , only.

(2)  $\ker \mathbb{F} = N_1 \oplus N_2 \oplus \dots \oplus N_{k-2}$ ,  $\text{Im} \mathbb{F} = N_0 \oplus V_k$ .

(3)  $\text{rank} \|\mathbb{F}\| = 2n$ .

(4)  $\mathbb{F}^3 + \mathbb{F} = 0$ .

So,  $\mathbb{F}$  is an almost  $(k-1)n$ -contact structure on the bundle of acceleration of order  $k$ ,  $T^k M$ .

Let  $\left( \begin{matrix} \xi \\ \xi \\ \dots \\ \xi \\ \xi \\ \xi \end{matrix} \right)_{\substack{1a \\ 2a \\ \dots \\ (k-1)a}}$ ,  $a = 1, \dots, n$ , be a local basis adapted to the

direct decomposition  $N_1 \oplus \dots \oplus N_{k-1}$  and  $\begin{matrix} 1a & 2a & \dots & (k-1)a \\ \eta & \eta & \dots & \eta \end{matrix}$  its dual basis.

Thus the set

$$\left( \begin{matrix} \mathbb{F}, \xi, \dots, \xi \\ \xi \\ \xi \end{matrix} \right)_{\substack{1a \\ \dots \\ (k-1)a}}, \begin{matrix} 1a & \dots & (k-1)a \\ \eta & \dots & \eta \end{matrix}, (a = 1, \dots, n) \quad (5.6.3)$$

is a  $(k-1)n$ -almost contact structure.

Indeed, we have

$$\mathbb{F} \left( \begin{matrix} \xi \\ \alpha a \end{matrix} \right) = 0, \quad \begin{matrix} a \\ \eta \end{matrix} \left( \begin{matrix} \xi \\ \beta b \end{matrix} \right) = \begin{cases} \delta_b^a, & \text{for } \alpha = \beta \\ 0, & \text{for } \alpha \neq \beta, \end{cases} \quad (\alpha, \beta = 1, \dots, n-1) \quad (5.6.4)$$

and

$$\begin{cases} \mathbb{F}^2(X) = -X + \sum_{a=1}^n \sum_{\alpha=1}^{k-1} \begin{matrix} a \\ \eta \end{matrix} \begin{matrix} \alpha \\ \xi \end{matrix} (X), \quad \forall X \in \mathcal{X}(T^k M) \\ \begin{matrix} a \\ \eta \end{matrix} \circ \mathbb{F} = 0. \end{cases} \quad (5.6.5)$$

The Nijenhuis tensor  $N_{\mathbb{F}}$  is expressed by

$$N_{\mathbb{F}}(X, Y) = [\mathbb{F}X, \mathbb{F}Y] + \mathbb{F}^2(X, Y) - \mathbb{F}[\mathbb{F}X, Y] - \mathbb{F}[X, \mathbb{F}Y]. \quad (5.6.6)$$

The structure (5.6.5) is normal if:

$$N_{\mathbb{F}}(X, Y) + \sum_{a=1}^n \sum_{\alpha=1}^{k-1} d \begin{matrix} a \\ \eta \end{matrix} \begin{matrix} \alpha \\ \xi \end{matrix} (X, Y) = 0, \quad \forall X, Y \in \mathcal{X}(\widetilde{T^k M}). \quad (5.6.7)$$

A characterization of the normality of structure (4.6.7) is as follows:

**Theorem 5.6.2.** *The almost  $(k-1)n$ -contact structure  $\left(\mathbb{F}, \xi, \overset{a}{\underset{\alpha a}{\eta}}\right)$  is normal, if and only if, for any vector fields  $X, Y$  on  $\widetilde{T^k M}$  we have*

$$N_{\mathbb{F}}(X, Y) + \sum_{a=1}^n \sum_{\alpha=1}^{k-1} d(y^{(\alpha)a})(X, Y) = 0. \quad (5.6.8)$$

Thus, we can prove:

**Theorem 5.6.3.** *The pair  $(\mathbb{G}, \mathbb{F})$  is a Riemannian  $(k-1)n$ -almost contact structure on  $\widetilde{T^k M}$ , depending only on the Lagrangian mechanical system of order  $k \geq 1$ ,  $\Sigma_{L^k}$ .*

Indeed, the following condition holds:

$$\mathbb{G}(\mathbb{F}X, Y) = -\mathbb{G}(\mathbb{F}Y, X), \quad \forall X, Y \in \mathcal{X}(\widetilde{T^k M}). \quad (5.6.9)$$

Therefore, the triple  $(\widetilde{T^k M}, \mathbb{G}, \mathbb{F})$  is called the Riemannian  $(k-1)n$ -almost contact model of the mechanical system  $\Sigma_{L^k}$ . It can be used to solve the problem of geometrization of mechanical system  $\Sigma_{L^k}$ .

## 5.7 Classical Riemannian mechanical system with external forces depending on higher order accelerations

Let us consider the classical Riemannian mechanical systems

$$\Sigma_{\mathcal{R}} = (M, T, Fe), \quad (5.7.1)$$

with  $T = \frac{1}{2} \gamma_{ij}(x) \frac{dx^i}{dt} \frac{dx^j}{dt}$  as kinetic energy and the external forces depending on the acceleration of order  $1, 2, \dots, k$ , i.e.  $Fe(x, y^{(1)}, \dots, y^{(k)})$ . The problem is *what kind of evolution equations can be postulate in order to study the movings of these systems?*

Of course, in nature we have such kind of mechanical systems. For instance, the friction forces, which action on a rocket in the time of its moving are the external forces depending by higher order accelerations. But, it is clear that the classical Lagrange equations:



$$\frac{d}{dt} \left( \frac{\partial(2T(x, y^{(1)}))}{\partial y^{(1)i}} \right) - \frac{\partial(2T(x, y^{(1)}))}{\partial x^i} = F_i(x, y^{(1)}, \dots, y^{(1)})$$

have the form

$$\frac{d^2 x^i}{dt^2} + \gamma_{jh}^i(z) \frac{dx^j}{dt} \frac{dx^h}{dt} = \frac{1}{2} \gamma^{is}(x) F_s \left( x, \frac{dx}{dt}, \dots, \frac{1}{k!} \frac{d^k x}{dt^k} \right)$$

and for  $k > 2$  it is not a differential system of equations of order 2, like in the classical Lagrange equations.

We solve this problem by means of the prolongation of order  $k$  of the Riemannian space  $\mathcal{R}^n = (M, \gamma_{ij}(x))$ .

Let  $\Sigma_{\mathcal{R}} = (M, 2T, Fe)$  a mechanical system of form (5.7.1) with  $T = \frac{1}{2} \gamma_{ij}(x) y^{(1)i} y^{(1)j}$ ,  $\left( y^{(1)i} = \frac{dx^i}{dt} \right)$  as kinetic energy. Thus the Riemannian space  $\mathcal{R}^n = (M, \gamma_{ij}(x))$  can be prolonged to the Riemannian space of order  $k \geq 1$   $\text{Prol}^k \mathcal{R}^n = (T^k M, \mathbb{G})$ , where

$$\mathbb{G} = \gamma_{ij}(x) \delta x^i \otimes \delta x^j + \gamma_{ij}(x) \delta y^{(1)i} \otimes \delta y^{(1)j} + \dots + \gamma_{ij}(x) \delta y^{(k)i} \otimes \delta y^{(k)j}. \quad (5.7.2)$$

The 1-forms  $\delta x^i, \delta y^{(1)i}, \dots, \delta y^{(k)i}$  being determined as follows:

$$\begin{aligned} \delta x^i &= dx^i, \quad \delta y^{(1)i} = dy^{(1)i} + M_{(1)j}^i dx^j, \dots, \delta y^{(k)i} = dy^{(k)i} + \\ &+ M_{(1)j}^i dy^{(k-1)i} + \dots + M_{kj}^i dx^j. \end{aligned} \quad (5.7.3)$$

In these formulas  $M_{(\alpha)j}^i$  ( $\alpha = 1, \dots, k$ ) are the dual coefficients of a nonlinear connection  $\overset{\circ}{N}$  determined only on the Riemannian structure  $\gamma_{ij}(x)$ . That are

$$\begin{aligned}
 M^i_{1j}(x, y^{(1)}) &= \gamma^i_{jh}(x) y^{(1)m}, \\
 M^i_{2j}(x, y^{(1)}, y^{(2)}) &= \frac{1}{2} \left( \Gamma_{(1)j} M^i + M^i_{(1)m} M^m_{(1)j} \right) \\
 \dots\dots\dots \\
 M^i_{kj}(x, y^{(1)}, \dots, y^{(k)}) &= \frac{1}{2} \left( \Gamma_{(k-1)j} M^i + M^i_{(1)m} M^m_{(k-1)j} \right),
 \end{aligned}
 \tag{5.7.4}$$

$\Gamma$  being the nonlinear operator:

$$\Gamma = y^{(1)i} \frac{\partial}{\partial x^i} + 2y^{(2)i} \frac{\partial}{\partial y^{(1)i}} + \dots + ky^{(k)i} \frac{\partial}{\partial y^{(k-1)i}}.
 \tag{5.7.5}$$

Thus, (cf. ch. ...) the dual coefficients  $M_{(\alpha)}$  (5.7.4) and the primal coefficients  $N_{(\alpha)}$  (5.4.7) depend on the Riemannian structure  $\gamma_{ij}(x)$ , only.

Now the Lagrangian mechanical system of order  $k$ :

$$\begin{aligned}
 \Sigma_{Prol^k \mathcal{R}^n} &= (M, L(x, y^{(1)}, \dots, y^{(k)}), Fe(x, y^{(1)}, y^{(k)}), \\
 L &= \gamma_{ij}(x) z^{(k)i} z^{(k)j}, \\
 kz^{(k)i} &= ky^{(k)i} + M^i_{(1)m} y^{(k-1)m} + \dots + M^i_{(k-1)m} y^{(1)m},
 \end{aligned}
 \tag{5.7.6}$$

depend only on  $\Sigma_{\mathcal{R}^n}$ . It is the prolongation of order  $k$  of the Riemannian mechanical system (4.7.1),  $\Sigma_{\mathcal{R}^n}$ .

We can postulate:

*The Lagrange equations of  $\Sigma_{\mathcal{R}^n}$  are the Lagrange equations of the prolonged Lagrange equations of order  $k$ ,  $\Sigma_{Prol^k \mathcal{R}^n}$ :*

$$\begin{aligned}
 \frac{d}{dt} \left( \frac{\partial L(x, y^{(1)}, \dots, y^{(k)})}{\partial y^{(k)i}} \right) - \frac{\partial L(x, y^{(1)}, \dots, y^{(k)})}{\partial y^{(k-1)i}} &= F_i(x, y^{(1)}, \dots, y^{(k)}) \\
 y^{(1)i} = \frac{dx^i}{dt}, \dots, y^{(k)i} = \frac{1}{k!} \frac{d^k x^i}{dt^k}.
 \end{aligned}
 \tag{5.7.7}$$

Applying the general theory from this chapter, we have

**Theorem 5.7.1.** *The Lagrange equations of the classical Riemannian systems  $\Sigma_{\mathcal{R}}$  with external forces depending on higher order accelera-*

tions are given by

$$\begin{aligned} \frac{d^{k+1}x^i}{dt^{k+1}} + (k+1)! \overset{\circ}{G}^i(x, y^{(1)}, \dots, y^{(k)}) &= \frac{1}{2} k! F^i(x, y^{(1)}, \dots, y^{(k)}) \\ (k+1) \overset{\circ}{G}^i &= \frac{1}{2} \gamma^{ij} \left[ \Gamma \frac{\partial L}{\partial y^{(k)j}} - \frac{\partial L}{\partial y^{(k-1)i}} \right] \end{aligned} \quad (5.7.8)$$

$$L = \gamma_{ij}(x) z^{(k)i} z^{(k)j}.$$

The canonical semispray of  $\Sigma_{\mathcal{R}}$  is expressed by

$$S = y^{(1)i} \frac{\partial}{\partial x^i} + \dots + k y^{(k)i} \frac{\partial}{\partial y^{(k-1)i}} - (k+1) \overset{\circ}{G}^i \frac{\partial}{\partial y^{(k)i}} + \frac{1}{2} F^i \frac{\partial}{\partial y^{(k)i}}. \quad (5.7.9)$$

The integral curves of  $S$  are given by the Lagrange equations (5.7.15).

The canonical semispray  $S$  is the dynamical system of the classical Lagrangian mechanical system  $\Sigma_{\mathcal{R}}$  with the external forces depending on the higher order accelerations.

## 5.8 Finslerian mechanical systems of order $k$ , $\Sigma_{F^k}$

The theory of the Finslerian mechanical systems of order  $k$  is a natural particularization of that of Lagrangian mechanical systems of order  $k \geq 1$ , described in the previous sections of the present chapter.

**Definition 5.8.1.** A Finslerian mechanical system of order  $k \geq 1$  is a triple:

$$\Sigma_{F^k} = (M, F(x, y^{(1)}, \dots, y^{(k)})), Fe(x, y^{(1)}, \dots, y^{(k)}), \quad (5.8.1)$$

where

$$F^{(k)n} = (M, F(x, y^{(1)}, \dots, y^{(k)})) \quad (5.8.2)$$

is a Finsler space of order  $k \geq 1$ , and

$$Fe(x, y^{(1)}, \dots, y^{(k)}) = F^i(x, y^{(1)}, \dots, y^{(k)}) \frac{\partial}{\partial y^{(k)i}} \quad (5.8.3)$$

is an a priori given vertical vector field.

$F(x, y^{(1)}, \dots, y^{(k)})$  is the fundamental function of  $\Sigma_{F^k}$ ,

$$g_{ij} = \frac{1}{2} \frac{\dot{\partial}}{\partial y^{(k)i}} \frac{\partial}{\partial y^{(k)j}} F^2 \quad (5.8.4)$$

is the fundamental tensor of  $\Sigma_{F^k}$  and  $Fe$  are the external forces.

The Euler–Lagrange equations of  $F^{(k)n}$  are given by (5.2.5)  $\overset{\circ}{E}_i(F^2) = 0$ ,  $y^{(1)i} = \frac{dx^i}{dt}$ , ...,  $y^{(k)i} = \frac{1}{k!} \frac{d^k x^i}{dt^k}$  and the Craig–Synge covector field is  $E^{k-1}(F^2)$  from (5.2.7). Theorem 5.2.1 for  $L = F^2$  can be applied.

Thus, the canonical semispray  $\overset{\circ}{S}$  of  $F^{(k)n}$  is:

$$\overset{\circ}{S} = y^{(1)i} \frac{\partial}{\partial x^i} + 2y^{(2)i} \frac{\partial}{\partial y^{(1)i}} + \dots + ky^{(k)i} \frac{\partial}{\partial y^{(k-1)i}} - (k+1) \overset{\circ}{G} \frac{\partial}{\partial y^{(k)i}}, \quad (5.8.5)$$

with the coefficients:

$$(k+1) \overset{\circ}{G} = \frac{1}{2} g^{ij} \left[ \Gamma \frac{\partial F^2}{\partial y^{(k)i}} - \frac{\partial F^2}{\partial y^{(k-1)i}} \right] \quad (5.8.6)$$

and with nonlinear operator

$$\Gamma = y^{(1)i} \frac{\partial}{\partial x^i} + 2y^{(2)i} \frac{\partial}{\partial y^{(1)i}} + \dots + ky^{(k)i} \frac{\partial}{\partial y^{(k-1)i}}. \quad (5.8.7)$$

The canonical nonlinear connection  $\overset{\circ}{N}$  of  $F^{(k)n}$  has dual coefficients (5.2.10) particularized in the following form:

$$\begin{aligned} \overset{\circ}{M}_{1j}^i &= \frac{\partial \overset{\circ}{G}}{\partial y^{(k)j}} \\ \overset{\circ}{M}_{2j}^i &= \frac{1}{2} (\overset{\circ}{S} \overset{\circ}{M}_{1j}^i + \overset{\circ}{M}_{1m}^i \overset{\circ}{M}_{1j}^m), \\ \dots \dots \dots \\ \overset{\circ}{M}_{kj}^i &= \frac{1}{k} (\overset{\circ}{S} \overset{\circ}{M}_{k-1j}^i + \overset{\circ}{M}_{k-1m}^i \overset{\circ}{M}_{k-1j}^m). \end{aligned} \quad (5.8.8)$$

The fundamental equations of the Finslerian mechanical systems of order  $n \geq 1$ ,  $\Sigma_{F^n}$  are given by

**Postulate 4.8.1.** *The evolution equations of the Finslerian mechanical system  $\Sigma_{F^k} = (M, F, Fe)$  are the following Lagrange equations*

$$\frac{d}{dt} \frac{\partial F^2}{\partial y^{(k)i}} - \frac{\partial F^2}{\partial y^{(k-1)i}} = F_i, \quad F_i = g_{ij} F^j, \quad (5.8.9)$$

$$y^{(1)i} = \frac{dx^i}{dt}, \dots, y^{(k)i} = \frac{1}{k!} \frac{d^k x^i}{dt^k}.$$

*Remark 5.8.1.* For  $k = 1$ , (5.8.8) are the Lagrange equations of a Finslerian mechanical system  $\Sigma_F = (M, F, Fe)$ .

The Lagrange equations are equivalent to the following system of differential equations of order  $k + 1$ :

$$\frac{d^{k+1} x^i}{dt^{k+1}} + \frac{k!}{2} g^{ij} \left[ \Gamma \frac{\partial F^2}{\partial y^{(k)i}} - \frac{\partial F^2}{\partial y^{(k-1)i}} \right] = \frac{k!}{2} F^i. \quad (5.8.10)$$

Integrating this system in initial conditions we obtain an unique solution  $x^i = x^i(t)$ , which express the moving of Finslerian mechanical system  $\Sigma_{F^k}$ .

But it is convenient to write the system (5.8.9) in the form (5.3.5):

$$y^{(1)i} = \frac{dx^i}{dt}, \quad y^{(2)i} = \frac{1}{2} \frac{d^2 x^i}{dt^2}, \dots, y^{(k)i} = \frac{1}{k!} \frac{d^k x^i}{dt^k} \quad (5.8.11)$$

$$\frac{dy^{(k)i}}{dt} + (k+1) \overset{\circ}{G}{}^i(x, y^{(1)}, \dots, y^{(k)}) = \frac{1}{2} F^i(x, y^{(1)}, \dots, y^{(k)}).$$

These equations determine the trajectories of the vector field

$$S = \overset{\circ}{S} + \frac{1}{2} Fe. \quad (5.8.12)$$

Consequently, we have:

**Theorem 5.8.1.**  $1^\circ$  For the Finslerian mechanical system of order  $k \geq 1$ ,  $\Sigma_{F^k}$  the operator  $S$  from (5.8.12) is a  $k$ -semispray which depend only on  $\Sigma_{F^k}$ .

$2^\circ$  The integral curves of  $S$  are the evolution curves of  $\Sigma_{F^k}$ .

$S$  is the canonical semispray of the mechanical system  $\Sigma_{F^k}$ . It is named the dynamical system of  $\Sigma_{F^k}$ , too.

The geometrical theory of the Finslerian mechanical system  $\Sigma_{F^k}$  can be developed by means of the canonical semispray  $S$ , applying the theory from the sections 5.4, 5.5, 5.6 from this chapter.

*Remark 5.8.2.* Let  $\Sigma_F = (M, F, Fe)$  be a Finslerian mechanical system, where  $Fe$  depend on the higher order accelerations. What kind of Lagrange equations we have for  $\Sigma_F$ ?

The problem can be solved by means of the prolongation of Finsler space  $F^n = (M, F)$  to the manifold  $T^k M$ , following the method used in the section 5.7 of the present chapter.

## 5.9 Hamiltonian mechanical systems of order $k \geq 1$

The Hamiltonian mechanical systems of order  $k \geq 1$  can be studied as a natural extension of that of Hamiltonian mechanical systems of order  $k = 1$ , given by  $\Sigma_H = (M, H(x, p), Fe(x, p))$ , ch. 4, part II.

So, a Hamiltonian mechanical system of order  $k \geq 1$  is a triple

$$\Sigma_{H^k} = (M, H(x, y^{(1)}, \dots, y^{(k-1)}), p), Fe(x, y^{(1)}, \dots, y^{(k-1)}, p) \quad (5.9.1)$$

where

$$H^{(k)n} = (M, H(x, y^{(1)}, \dots, y^{(k-1)}), p) \quad (5.9.2)$$

is a Hamilton space of order  $k \geq 1$  (see definition 5.1.1) and

$$Fe(, y^{(1)}, \dots, y^{(k-1)}, p) = F_i(x, y^{(1)}, \dots, y^{(k-1)}, p) \dot{\partial}^i \quad (5.9.3)$$

are the external forces.

Of course,  $\dot{\partial}^i = \frac{\partial}{\partial p_i}$ .

Looking to the Hamilton–Jacobi equation of the space  $H^{(k)n}$  we can formulate

**Postulate 5.9.1.** *The Hamilton equations of the Hamilton mechanical system  $\Sigma_{H^k}$  are as follows:*

$$\left\{ \begin{array}{l} \mathcal{H} = \frac{1}{2}H \\ \frac{dx^i}{dt} = \frac{\partial \mathcal{H}}{\partial p_i} \\ \frac{dp_i}{dt} = -\frac{\partial \mathcal{H}}{\partial x_i} + \frac{d}{dt} \frac{\partial \mathcal{H}}{\partial y^{(1)i}} + \dots + (-1) \frac{1}{(k-1)!} \frac{d}{dt} \frac{d^{k-1}}{dt^{k-1}} \frac{\partial \mathcal{H}}{\partial y^{k-1}} = \frac{1}{2}F_i \\ y^{(1)i} = \frac{dx^i}{dt}, y^{(2)i} = \frac{1}{2} \frac{d^2 x^i}{dt^2}, \dots, y^{(k-1)} = \frac{1}{(k-1)!} \frac{d^{k-1} x^i}{dt^{k-1}}. \end{array} \right. \quad (5.9.4)$$

It is not difficult to see that these equations have a geometric meaning.

For  $k = 1$  the equations (5.9.4) are the Hamilton equations of a Hamiltonian mechanical system  $\Sigma_H = (M, H(x, p), Fe(x, p))$ .

These reasons allow us to say that the Hamilton equations (5.9.4) are *the fundamental equations* for the evolution of the Hamiltonian mechanical systems of order  $k \geq 1$ .

**Example 5.9.1.** The mechanical system  $\Sigma_{H^k}$  with

$$H(x, y^{(1)}, \dots, y^{(k-1)}, p) = \alpha \gamma^{ij}(x, y^{(1)}, \dots, y^{(k-1)}) p_i p_j, \quad (5.9.5)$$

$\gamma^{ij}$  being a Riemannian metric tensor on the manifold  $T^{(k-1)n}$  and with

$$Fe = \beta_i(x, y^{(1)}, \dots, y^{(k-1)}) \dot{\partial}^i \quad (5.9.5')$$

$\beta_i$  is a  $d$ -covector field on  $T^{k-1}M$  and  $\alpha > 0$  a constant.

Let us consider the energy of order  $k - 1$  of

$$\begin{aligned} \mathcal{E}^{k-1}(H) &= I^{k-1}(H) - \frac{1}{2!} \frac{d}{dt} I^{k-2}(H) + \dots + \\ &+ (-1)^{k-2} \frac{1}{(k-1)!} \frac{d^{k-2}}{dt^{k-2}}(H) - H, \end{aligned} \quad (5.9.6)$$

where

$$I^{(1)}H = \mathcal{L}_{\Gamma} H, \dots, I^{k-1}H = \mathcal{L}_{\Gamma}^{k-1} H.$$

**Theorem 5.9.1.** *The variation of energy  $\mathcal{E}^{k-1}(H)$  along the evolution curves (5.9.4) can be calculate without difficulties.*

Thus, the canonical nonlinear connection  $N^*$  of  $\Sigma_{H^k}$  is given by the direct decomposition (5.1.11), and has the coefficients  $N_{(1)j}^i, \dots, N_{(k-1)j}^i$ ,  $N_{ij}$  from (5.1.12') and its dual coefficients  $M_{(1)}^i, \dots, M_{(k-1)}^i$  from (5.1.14).

We can calculate the Liouville vector fields  $z^{(1)}, \dots, z^{(k-1)}$  from (5.1.17).

The metrical  $N^*$ -linear connection has the coefficients  $D\Gamma(N^*) = (H_{ij}^s, C_{(k)jk}^i, C_i^{js})$  given by (5.5.2).

Now we can say: The geometries of the Hamiltonian mechanical systems of order  $k \geq 1$ ,  $\Sigma_{H^k}$  can be constructed only by the fundamental equations (5.9.4), by the nonlinear connection of  $\Sigma_{H^k}$  and by the  $N^*$ -metrical connection  $C\Gamma(N^*)$ .

*Remark 5.9.1.* The homogeneous case of Cartanian mechanical systems of order  $k$ ,  $\Sigma_{\mathcal{E}^k} = (M, K(x, y^{(1)}, \dots, y^{(k-1)}), p), Fe(x, y^{(1)}, \dots, y^{(k-1)}, p)$  is a direct particularization of the previous theory for  $H(x, y^{(1)}, \dots, y^{(k-1)}, p) = K^2(x, y^{(1)}, \dots, y^{(k-1)}, p)$ , where  $K(x, y^{(1)}, \dots, y^{(k-1)}, p)$  is  $k$ -homogeneous with respect to variables  $(y^{(1)}, \dots, y^{(k-1)}, p)$ .



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