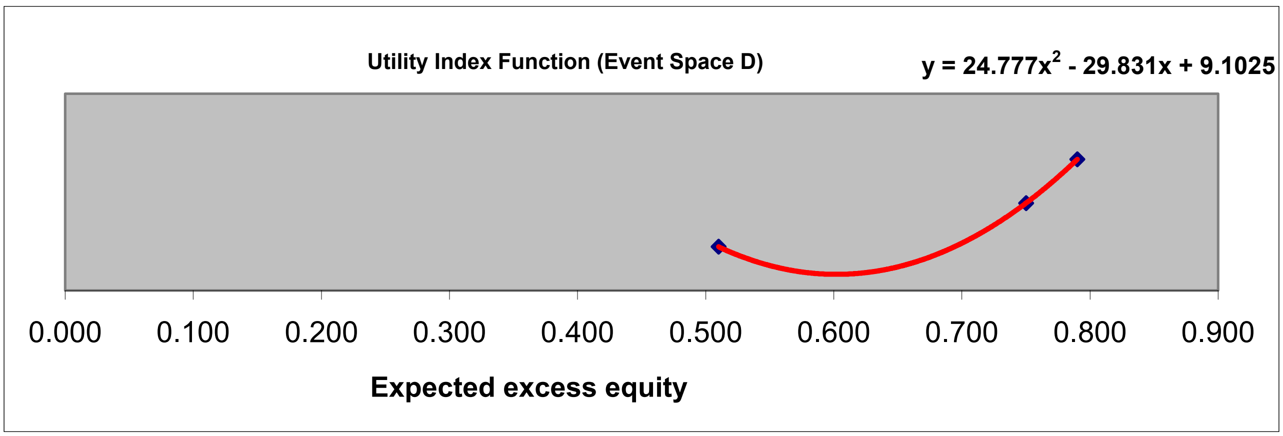
M. Khoshnevisan, S. Bhattacharya, F. Smarandache

ARTIFICIAL INTELLIGENCE AND RESPONSIVE OPTIMIZATION

(second edition)



Xiquan
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Forward

The purpose of this book is to apply the Artificial Intelligence and control systems to different real models.

In part 1, we have defined a fuzzy utility system, with different financial goals, different levels of risk tolerance and different personal preferences, liquid assets, etc. A *fuzzy system* (extendible to a *neutrosophic system*) has been designed for the evaluations of the financial objectives. We have investigated the notion of fuzzy and neutrosophiness with respect to time management of money.

In part 2, we have defined a computational model for a simple portfolio insurance strategy using a protective put and computationally derive the investor's governing utility structures underlying such a strategy under alternative market scenarios. The Arrow-Pratt measure of risk aversion has been used to determine how the investors react towards risk under the different scenarios.

In Part 3, it is proposed an artificial classification scheme to isolate truly benign tumors from those that initially start off as benign but subsequently show metastases. A non-parametric artificial neural network methodology has been chosen because of the analytical difficulties associated with extraction of closed-form stochastic-likelihood parameters given the extremely complicated and possibly non-linear behavior of the state variables we have postulated an in-depth analysis of the numerical output and model findings and compare it to existing methods of tumor growth modeling and malignancy prediction

In part 4, an alternative methodological approach has been proposed for quantifying utility in terms of expected information content of the decision-maker's choice set. It is proposed an extension to the concept of utility by incorporating extrinsic utility; which is defined as the utility derived from the element of choice afforded to the decision-maker.

This book has been designed for graduate students and researchers who are active in the applications of *Artificial Intelligence and Control Systems* in modeling. In our future research, we will address the unique aspects of Neutrosophic Logic in modeling and data analysis.

The Authors

Fuzzy and Neutrosophic Systems and Time Allocation of Money

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Abstract

Each individual investor is different, with different financial goals, different levels of risk tolerance and different personal preferences. From the point of view of investment management, these characteristics are often defined as objectives and constraints. Objectives can be the type of return being sought, while constraints include factors such as time horizon, how liquid the investor is, any personal tax situation and how risk is handled. It's really a balancing act between risk and return with each investor having unique requirements, as well as a unique financial outlook – essentially a constrained utility maximization objective. To analyze how well a customer fits into a particular investor class, one investment house has even designed a structured questionnaire with about two-dozen questions that each has to be answered with values from 1 to 5. The questions range from personal background (age, marital state, number of children, job type, education type, etc.) to what the customer expects from an investment (capital protection, tax shelter, liquid assets, etc.). A ***fuzzy logic system*** (extendible to a ***neutrosophic logic system***) has been designed for the evaluation of the answers to the above questions. We have investigated the notion of fuzzy and neutrosophiness with respect to funds allocation.

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Introduction.

In this paper we have designed our fuzzy system so that customers are classified to belong to any one of the following three categories: [1]

***Conservative and security-oriented (risk shy)**
***Growth-oriented and dynamic (risk neutral)**
***Chance-oriented and progressive (risk happy)**

A neutrosophic system has three components – that's why it may be considered as just a generalization of a fuzzy system which has only two components.

Besides being useful for clients, investor classification has benefits for the professional investment consultants as well. Most brokerage houses would value this information as it gives them a way of targeting clients with a range of financial products more effectively - including insurance, saving schemes, mutual funds, and so forth. Overall, many responsible brokerage houses realize that if they provide an effective service that is *tailored* to individual needs, in the long-term there is far more chance that they will retain their clients no matter whether the market is up or down.

Yet, though it may be true that investors can be categorized according to a limited number of types based on theories of personality already in the psychological profession's armory, it must be said that these classification systems based on the Behavioral Sciences are still very much in their infancy and they may still suffer from the problem of their meanings being similar to other related typographies, as well as of greatly oversimplifying the different investor behaviors. [2]

(I.1) Exploring the implications of utility theory on investor classification.

In our present work, we have used the familiar framework of neo-classical utility theory to try and devise a structured system for investor classification according to the utility preferences of individual investors (and also possible re-ordering of such preferences).

The theory of consumer behavior in modern microeconomics is entirely founded on observable utility preferences, rejecting hedonistic and introspective aspects of utility. According to modern utility theory, utility is a representation of a set of mutually consistent choices and not an explanation of a choice. The basic approach is to ask an individual to reveal his or her personal utility preference and not to elicit any numerical measure. [1] However, the projections of the consequences of the options that we face and the subsequent choices that we make are shaped by our memories of past experiences – that "mind's eye sees the future through the light filtered by the past". However, this memory often tends to be rather selective. [9] An investor who allocates a large portion of his or funds to the risky asset in period t-1 and makes a significant gain will perhaps be induced to put an even larger portion of the available funds in the risky asset in period t. So this investor may be said to have displayed a very weak risk-aversion attitude up to period t, his or her actions being mainly determined by past happenings one-period back.

There are two interpretations of utility – normative and positive. *Normative utility* contends that *optimal* decisions do not always reflect the *best* decisions, as maximization of instant utility based on selective memory may not necessarily imply maximization of total utility. This is true in many cases, especially in the areas of health economics and social choice theory. However, since we will be applying utility theory to the very specific area of funds allocation between risky and risk-less investments (and investor classification based on such allocation), we will be concerned with *positive utility,* which considers the optimal decisions as they are, and not as what they should be. We are simply interested in using utility functions to classify an individual investor's attitude towards bearing risk at a given point of time. Given that the neo-classical utility preference approach is an objective one, we feel it is definitely more amenable to formal analysis for our purpose as compared to the philosophical conceptualizations of pure hedonism if we can accept decision utility preferences generated by selective memory.

If u is a given utility function and w is the wealth coefficient, then we have $E[u(w + k)] = u[w + E(k) – p]$, that is, $E[u(w + k)] = u(w - p)$, where k is the outcome of a risky venture given by a known probability distribution whose expected value $E(k)$ is zero. Since the outcome of the risky venture is as likely to be positive as negative, we would be willing to pay a small amount p, the *risk premium*, to avoid having to undertake the risky venture. Expanding the utilities in *Taylor series* to second order on the left-hand side and to first order on the right-hand side and subsequent algebraic simplification leads to the general formula $p = -(v/2)\, u''(w)/u'(w)$, where $v = E(k^2)$ is the variance of the possible outcomes. This shows that approximate risk premium is proportional to the variance – a notion that carries a similar implication in the mean-variance theorem of classical portfolio theory. The quantity $-u''(w)/u'(w)$ is termed the *absolute risk aversion*. [6] The nature of this absolute risk aversion depends on the form of a specific utility function. For instance, for a logarithmic utility function, the absolute risk aversion is dependent on the wealth coefficient w, such that it decreases with an increase in w. On the other hand, for an exponential utility function, the absolute risk aversion becomes a constant equal to the reciprocal of the risk premium.

(I.2) The neo-classical utility maximization approach.

In its simplest form, we may formally represent an individual investor's utility maximization goal as the following mathematical programming problem:

$$\begin{aligned} &\textbf{Maximize } U = f(x, y) \\ &\textbf{Subject to } x + y = 1, \\ &\qquad x \geq 0 \textbf{ and } y \textbf{ is unrestricted in sign} \end{aligned}$$

Here x and y stand for the proportions of investable funds allocated by the investor to the market portfolio and a risk-free asset. The last constraint is to ensure that the investor can never borrow at the market rate to invest in the risk-free asset, as this is clearly unrealistic - the market rate being obviously higher than the risk-free rate. However, *an overtly aggressive investor can borrow at the risk-free rate* to invest in the market portfolio. In investment parlance this is known as ***leverage***. [5]

As in classical microeconomics, we may solve the above problem using the *Lagrangian multiplier* technique. The transformed Lagrangian function is as follows:

$$Z = f(x, y) + \lambda(1-x-y) \qquad \dots \text{(i)}$$

By the first order (necessary) condition of maximization we derive the following system of linear algebraic equations:

$$\mathbf{Z_x = f_x - \lambda = 0} \quad (1)$$
$$\mathbf{Z_y = f_y - \lambda = 0} \quad (2)$$
$$\mathbf{Z_\lambda = 1 - x - y = 0} \quad (3) \qquad \textbf{… (ii)}$$

The investor's equilibrium is then obtained as the condition $\mathbf{f_x} = \mathbf{f_y} = \lambda^*$. λ^* may be conventionally interpreted as the *marginal utility of money* (i.e. the investable funds at the disposal of the individual investor) when the investor's utility is maximized. [2]

The individual investor's indifference curve will be obtained as the locus of all combinations of x and y that will yield a constant level of utility. Mathematically stated, this simply boils down to the following total differential:

$$\mathbf{dU = f_x dx + f_y dy = 0} \qquad \textbf{… (iv)}$$

The immediate implication of (3) is that $dy/dx = -f_x/f_y$, i.e. assuming $(f_x, f_y) > 0$; this gives the negative slope of the individual investor's indifference curve and may be equivalently interpreted as the *marginal rate of substitution of allocable funds* between the market portfolio and the risk-free asset.

A second order (sufficient) condition for maximization of investor utility may be also derived on a similar line as that in economic theory of consumer behavior, using the sign of the bordered Hessian determinant, which is given as follows:

$$\mathbf{|\overline{H}| = 2\beta_x\beta_y f_{xy} - \beta_y^2 f_{xx} - \beta_x^2 f_{yy}} \qquad \textbf{… (v)}$$

βx and βy stand for the coefficients of x and y in the constraint equation. In this case we have $\beta x = \beta y = 1$. Equation (4) therefore reduces to:

$$\mathbf{|\overline{H}| = 2f_{xy} - f_{xx} - f_{yy}} \qquad \textbf{… (vi)}$$

If $|\overline{H}| > 0$ then the stationary value of the utility function U* attains its maximum.

To illustrate the application of classical utility theory in investor classification, let the utility function of a rational investor be represented by the following utility function:

$$\mathbf{U\ (x, y) = ax^2 - by^2}\text{; where}$$

x = proportion of funds invested in the market portfolio; and
y = proportion of funds invested in the risk-free asset.

Quite obviously, x + y = 1 since the efficient portfolio must consist of a combination of the market portfolio with the risk-free asset. The problem of funds allocation within the efficient portfolio then becomes that of maximizing the given utility function subject to the efficient portfolio constraint. As per J. Tobin's ***Separation Theorem***; which states that investment is a two-phased process with the problem of portfolio selection which is considered independent of an individual investor's utility preferences (i.e. the first phase) to be treated *separately* from the problem of funds allocation within the selected portfolio which is dependent on the individual investor's utility function (i.e. the second phase). Using this concept we can mathematically categorize all individual investor attitudes towards bearing risk into any one of three distinct classes:

Class A+: "Overtly Aggressive"(no risk aversion attitude)
Class A: "Aggressive" (weak risk aversion attitude)
Class B: "Neutral"(balanced risk aversion attitude)
Class C: "Conservative"(strong risk aversion attitude)

The problem is then to find the general point of maximum investor utility and subsequently derive a mathematical basis to categorize the investors into one of the three classes depending upon the optimum values of x and y. The original problem can be stated as a *classical non-linear programming* with a single equality constraint as follows:

Maximize $\mathbf{U\ (x, y) = ax^2 - by^2}$
Subject to:

$x + y = 1,$

$x \geq 0$ and y is unrestricted in sign

We set up the following transformed Lagrangian objective function:

Maximize $Z = ax^2 - by^2 + \lambda\,(1 - x - y)$

Subject to:

$x + y = 1,$

$x \geq 0$ and y is unrestricted in sign, (where λ is the Lagrangian multiplier)

By the usual first-order (necessary) condition we therefore get the following system of linear algebraic equations:

$$Z_x = 2ax - \lambda = 0 \quad (1)$$

$$Z_y = -2by - \lambda = 0 \quad (2)$$

$$Z_\lambda = 1 - x - y = 0 \quad (3) \qquad \ldots \text{(vii)}$$

Solving the above system we get $x/y = -b/a$. But $x + y = 1$ as per the funds constraint. Therefore $(-b/a)\,y + y = 1$ i.e. $\mathbf{y^* = [1 + (-b/a)]^{-1} = [(a-b)/a]^{-1} = a/(a-b)}$. Now substituting for y in the constraint equation, we get $\mathbf{x^* = 1-a/(a-b) = -b/(a-b)}$. Therefore the stationary value of the utility function is $\mathbf{U^* = a\,[-b/(a-b)]^2 - b\,[a/(a-b)]^2 = -ab/(a - b)}$.

Now, $f_{xx} = 2a$, $f_{xy} = f_{yx} = 0$ and $f_{yy} = -2b$. Therefore, by the second order (sufficient) condition, we have:

$$|\overline{H}| = 2f_{xy} - f_{xx} - f_{yy} = 0 - 2a - (-2b) = 2\,(b - a) \qquad \ldots \text{(viii)}$$

Therefore, the bordered Hessian determinant will be positive in this case if and only if we have $(a - b) < 0$. That is, given that $a < b$, our chosen utility function will be maximized at $U^* = ax^{*2} - by^{*2}$. However, the satisfaction of the non-negativity constraint on x^* would require that $b > 0$ so that $-b < 0$; thus yielding $[-b / (a - b)] > 0$.

Classification of investors:

Class	Basis of determination
A+	**($y^* < x^*$) and ($y^* \leq 0$)**
A	**($y^* < x^*$) and ($y^* > 0$)**
B	($y^* = x^*$)
C	($y^* > x^*$)

(I.3) Effect of a risk-free asset on investor utility.

The possibility to lend or borrow money at a risk-free rate widens the range of investment options for an individual investor. The inclusion of the risk-free asset makes it possible for the investor to select a portfolio that *dominates* any other portfolio made up of only risky securities. This implies that an individual investor will be able to attain a higher *indifference curve* than would be possible in the absence of the risk-free asset. The risk-free asset makes it possible to *separate* the investor's decision-making process into two distinct phases – identifying the market portfolio and funds allocation. The market portfolio is the portfolio of risky assets that includes each and every available risky security. As all investors who hold any risky assets at all will choose to hold the market portfolio, this choice is independent of an individual investor's utility preferences.

Now, the expected return on a two-security portfolio involving a risk-free asset and the market portfolio is given by $E(R_p) = xE(R_m) + yR_f$, where $E(R_p)$ is the expected return on the optimal portfolio, $E(R_m)$ = expected return on the market portfolio; and R_f is the return on the risk-free asset. Obviously, $x + y = 1$. Substituting for x and y with x^* and y^* from our illustrative case, we therefore get:

$$\mathbf{E(R_p)^* = [-b/(a-b)]\, E(R_m) + [a/(a-b)]\, R_f} \qquad \text{… (ix)}$$

As may be verified intuitively, if $b = 0$ then of course we have $E(R_p) = R_f$, as in that case the optimal value of the utility function too is reduced to $U^* = -a0/(a-0) = 0$.

The equation of the Capital Market Line in the original version of the CAPM may be recalled as **$E(R_p) = R_f + [E(R_m) - Rf](s_p/s_m)$**; where $E(R_p)$ is expected return on the efficient portfolio, $E(R_m)$ is the expected return on the market portfolio, R_f is the return on the risk-free asset, s_m is the standard deviation of the market portfolio returns; and s_p is the standard deviation of the efficient portfolio returns. Equating $E(R_p)$ with $E(R_p)^*$ we therefore get:

$$
\begin{aligned}
&\mathbf{R_f + [E(R_m) - Rf](s_p/s_m) = [-b/(a-b)]\,E(R_m) + [a/(a-b)]\,R_f,\ i.e.}\\
&\mathbf{s_p^* = s_m\,[R_f\,\{a/(a-b) - 1\} + \{-b/(a-b)\}\,E(R_m)] / [E(R_m) - R_f]}\\
&\quad\mathbf{= s_m\,[E(R_m) - R_f][-b/(a-b)] / [E(R_m) - R_f]}\\
&\quad\mathbf{= s_m\,[-b/(a - b)]} \qquad \mathbf{\ldots (x)}
\end{aligned}
$$

This mathematically demonstrates that a rational investor having a quadratic utility function of the form $U = ax^2 - by^2$, at his or her point of maximum utility (i.e. affinity to return coupled with averseness to risk), assumes a given efficient portfolio risk (standard deviation of returns) equivalent to $Sp^* = Sm\,[-b/(a - b)]$; when the efficient portfolio consists of the market portfolio coupled with a risk-free asset.

The investor in this case, will be classified within a particular category A, B or C according to whether $-b/(a-b)$ is greater than, equal in value or lesser than $a/(a-b)$, given that $a < b$ and $b > 0$.

Case I: (b > a, b > 0 and a >0)

Let b = 3 and a = 2. Thus, we have (b > a) and (-b < a). Then we have $x^* = -3/(2-3) = 3$ and $y^* = 2/(2-3) = -2$. Therefore ($x^*>y^*$) and ($y^*<0$). So the investor can be classified as Class A+.

Case II: (b > a, b > 0, a < 0 and b > |a|)

Let b = 3 and a = - 2. Thus, we have (b > a) and (- b < a). Then, $x^* = -3/(-2-3) = 0.60$ and $y^* = -2/(-2-3) = 0.40$. Therefore ($x^* > y^*$) and ($y^*>0$). So the investor can be re-classified as Class A!

Case III: (b > a, b > 0, a < 0 and b = |a|)

Let $b = 3$ and $a = -3$. Thus, we have $(b > a)$ and $(b = |a|)$. Then we have $x^* = -3/(-3-3) = 0.5$ and $y^* = -3/(-3-3) = 0.5$. Therefore we have $(x^* = y^*)$. So now the investor can be re-classified as Class B!

Case IV: (b > a, b > 0, a < 0 and b < |a|)

Let $b = 3$ and $a = -5$. Thus, we have $(b>a)$ and $(b<|a|)$. Then we have $x^* = -3/(-5-3) = 0.375$ and $y^* = -5/(-5-3) = 0.625$. Therefore we have $(x^* < y^*)$. So, now the investor can be re-classified as Class C!

So we may see that even for this relatively simple utility function, the final classification of the investor permanently into any one risk-class would be unrealistic as the range of values for the coefficients a and b could be switching dynamically from one range to another as the investor tries to adjust and re-adjust his or her risk-bearing attitude. This makes the neo-classical approach insufficient in itself to arrive at a classification. Here lies the justification to bring in a complimentary ***fuzzy modeling*** approach which may be further extended to ***neutrosophic modeling***. Moreover, if we bring in time itself as an independent variable into the utility maximization framework, then one choice variable (weighted in favour of risk-avoidance) could be viewed as a *controlling factor* on the other choice variable (weighted in favour of risk-acceptance). Then the resulting problem could be gainfully explored in the light of ***optimal control theory***.

(II.1) Modeling fuzziness in the funds allocation behavior of an individual investor.

The boundary between the preference sets of an individual investor, for funds allocation between a risk-free asset and the risky market portfolio, tends to be rather fuzzy as the investor continually evaluates and shifts his or her position; unless it is a passive ***buy-and-hold*** kind of portfolio.

Thus, if the universe of discourse is **U = {C, B, A and A+}** where C, B, A and A+ are our four risk classes "conservative", "neutral", "aggressive" and "overtly aggressive" respectively, then the fuzzy subset of U given by $\mathbf{P = \{x_1/C, x_2/B, x_3/A, x_4/A+\}}$ is the *true* preference set for our purposes; where we have $\mathbf{0 \leq (x_1, x_2, x_3, x_4) \leq 1}$, all the symbols having their usual meanings. Although theoretically any of the $P(x_i)$ values could be equal to unity, in reality it is far more likely that $P(x_i) < 1$ for $i = 1, 2, 3, 4$ i.e. the fuzzy subset P is most likely to be *subnormal*. Also, similarly, in most real-life cases it is expected that P (x_i) > 0 for i = 1, 2, 3, 4 i.e. all the elements of P will be included in its *support*: **supp (P) = {C, B, A, A+} = U**.

The critical point of analysis is definitely the individual investors preference ordering i.e. whether an investor is ***primarily conservative*** or ***primarily aggressive***. It is understandable that a primarily conservative investor could behave aggressively at times and vice versa but in general, their behavior will be in line with their classification. So the classification often depends on the height of the fuzzy subset P: **height (P) = $\mathbf{Max_x P (x)}$**. So one would think that the risk-neutral class becomes largely superfluous, as investors in general will tend to get classified as either primarily conservative or primarily aggressive. However, as already said, in reality, the element B will also generally have a *non-zero degree of membership* in the fuzzy subset and hence cannot be dropped.

The fuzziness surrounding investor classification stems from the fuzziness in the preference relations regarding the allocation of funds between the risk-free and the risky assets in the optimal portfolio. It may be mathematically described as follows:

Let **M** be the set of allocation options open to the investor. Then, the fuzzy preference relation is a fuzzy subset of the **M x M** space identifiable by the following membership function:

$\mathbf{\mu_R (m_i, m_j) = 1;}$ m_i is definitely preferred to m_j

$\mathbf{c \in (0.5, 1);}$ m_i is somewhat preferred to m_j

0.5; point of perfect neutrality

$\mathbf{d \in (1, 0.5);}$ mj is somewhat preferred to m_i; and

0; m_j is definitely preferred to m_i **… (xi)**

The neutrosophic preference relation is obtained by including an intermediate neutral value of m_n between m_i and $m_{j.}$ The preference relations are assumed to meet the necessary conditions of reciprocity and transitivity. However, owing to substantial confusion regarding acceptable working definition of transitivity in a fuzzy set-up, it is often entirely neglected thereby leaving only the reciprocity property. This property may be succinctly represented as follows:

$$\mathbf{\mu_R (m_i, m_j) = 1 - \mu_R (m_j, m_i), \forall i \neq j} \quad \textbf{… (xii)}$$

If we are to further assume a reasonable cardinality of the set **M**, then the preference relation R_v of an individual investor v may also be written in a matrix form as follows: [12]

$$[r_{ij}^{v}] = [\mu_R (m_i, m_j)], \forall i, j, v \qquad \text{… (xiii)}$$

Classically, given the efficient frontier and the risk-free asset, there can be one and only one optimal portfolio corresponding to the point of tangency between the risk-free rate and the convex efficient frontier. Then fuzzy logic modeling framework does not in any way disturbs this bit of the classical framework. The fuzzy modeling, like the classical Lagrangian multiplier method, comes in only *after* the optimal portfolio has been identified and the problem facing the investor is that of allocating the available funds between the risky and the risk-free assets subject to a governing budget constraint. The investor is theoretically faced with an infinite number of possible combinations of the risk-free asset and the market portfolio but the ultimate allocation depends on the investor's utility function to which we now extend the fuzzy preference relation.

The available choices to the investor given his or her utility preferences determine the universe of discourse. The more uncertain are the investor's utility preferences, the wider is the range of available choices and the greater is the degree of fuzziness involved in the preference relation, which would then extend to the investor classification. Also, wider the range of available choices to the investor the higher is the expected information content or *entropy* of the allocation decision.

(II.2) Entropy as a measure of fuzziness.

The term entropy arises in analogy with *thermodynamics* where the defining expression has the following mathematical form:

$$S = k \log_b \omega \qquad \text{… (xiv)}$$

In thermodynamics, entropy is related to the *degree of disorder* or configuration probability ω of the canonical assembly. Its use involves an analysis of the microstates' distribution in the canonical assembly among the available energy levels for both isothermal reversible and isothermal irreversible (spontaneous) processes (with an attending modification). The physical scale factor k is the *Boltzmann constant*. [7]

However, the thermodynamic form has a different sign and the word *negentropy* is therefore sometimes used to denote expected information. Though Claude Shannon originally conceptualized the entropy measure of expected information, it was DeLuca and Termini who brought this concept in the realms of fuzzy mathematics when they sought to derive a universal mathematical measure of fuzziness.

Let us consider the fuzzy subset $F = \{r_1/X, r_2/Y\}$, $0 \leq (r_1, r_2) \leq 1$, where X is the event $(y<x)$ and Y is the event $(y \geq x)$, x being the proportion of funds to be invested in the market portfolio and y being the proportion of funds to be invested in the risk-less security. Then the ***DeLuca-Termini conditions*** for measure of fuzziness may be stated as follows: [3]

- **FUZ (F) = 0** if F is a crisp set i.e. if the investor classified under a particular risk category *always* invests entire funds either in the risk-free asset (conservative attitude) or in the market portfolio (aggressive attitude)
- **FUZ (F) = Max FUZ (F)** when **F = (0.5/X, 0.5/Y)**
- **FUZ (F) ≥ FUZ (F*)** if F* is a *sharpened version* of F, i.e. if F* is a fuzzy subset satisfying $F^*(r_i) \geq F(r_i)$ given that $F(r_i) \geq 0.5$ and $F(r_i) \geq F^*(r_i)$ given that $0.5 \geq F(r_i)$

The second condition is directly derived from the concept of entropy. Shannon's measure of entropy for an n – events case is given as follows: [10]

$$\mathbf{H = - k\, \Sigma(p_i \log p_i)}, \text{ where we have } \Sigma p_i = 1 \qquad \dots \textbf{(xv)}$$

The Lagrangian form of the above function is as follows:

$$\mathbf{H_L = - k\, \Sigma(pi \log p_i) + \lambda\, (1 - \Sigma p_i)} \qquad \dots \textbf{(xvi)}$$

Taking partial derivatives with respect to p_i and setting equal to zero as per the necessary condition of maximization, we have the following stationary condition:

$$\mathbf{\partial H_L/\partial p_i = -k\, [\log p_i + 1] - \lambda = 0} \qquad \dots \textbf{(xvii)}$$

It may be derived from (16) that at the point of maximum entropy, log p_i = -[(λ/k)+1], i.e. log p_i becomes a constant. This means that at the point of maximum entropy, p_i becomes independent of the i and equalized to a constant value for i = 1, 2 ... n. In an n-events case therefore, at the point of maximum entropy we necessarily have:

$$\mathbf{p_1 = p_2 = \ldots = p_i = \ldots = p_n = 1/n} \qquad \ldots \text{(xviii)}$$

For n = 2 therefore, we obviously have the necessary condition for entropy maximization as $p_1 = p_2 = ½ = 0.5$. In terms of the fuzzy preference relation, this boils down to exactly the second DeLuca-Termini condition. Keeping this close relation with mathematical information theory in mind, DeLuca and Termini even went on to incorporate Shannon's entropy measure as their chosen measure of fuzziness. For our portfolio funds allocation model, this measure could simply be stated as follows:

$$\mathbf{FUZ\ (F) = -\ k\ [\{F(r_1)\ \log F(r_i) + (1\text{-}F(r_1))\ \log\ (1\text{-}F\ (r_1))\} + \{F(r_2)\ \log F(r_2) + (1\text{-}F(r_2))\ \log\ (1\text{-}F(r_2))\}]} \qquad \ldots\text{(xix)}$$

(II.3) Metric measures of fuzziness.

Perhaps the best method of measuring fuzziness will be through measurement of the distance between F and F^c, as fuzziness is mathematically equivalent to the lack of distinction between a set and its complement. In terms of our portfolio funds allocation model, this is equivalent to the ambivalence in the mind of the individual investor regarding whether to put a larger or smaller proportion of available funds in the risk-less asset. The higher this ambivalence, the closer F is to F^c and greater is the fuzziness.

This measure may be constructed for our case by considering the fuzzy subset F as a *vector with 2 components*. That is, F (r_i) is the i^{th} component of a vector representing the fuzzy subset F and (1 – F (r_i)) is the i^{th} component of a vector representing the complementary fuzzy subset F^c. Thus letting D be a metric in *2* space; we have the distance between F and F^c as follows: [11]

$$\mathbf{D\rho\ (F, F^c) = [\Sigma|F\ (r_i) - F^c\ (r_i)|^{\rho}]^{1/\rho}}, \text{ where } \rho = 1, 2 \ldots \qquad \ldots \mathbf{(xx)}$$

For Euclidean Space with ρ=2, this metric becomes very similar to the statistical variance measure RMSD (root mean square deviation). Moreover, as $F^c (r_i) = 1 – F (r_i)$, the above formula may be written in a simplified manner as follows:

$$\mathbf{D\rho\ (F, F^c) = [\Sigma|2F\ (r_i) - 1|^{\rho}]^{1/\rho}}, \text{ where } \rho = 1, 2 \ldots \qquad \ldots \mathbf{(xxi)}$$

For ρ=1, this becomes the *Hamming metric* having the following form:

$$\mathbf{D_1\ (F, F^c) = \Sigma|2F(r_i) - 1|} \qquad \ldots \mathbf{(xxii)}$$

If the investor always puts a greater proportion of funds in either the risk-free asset or the market portfolio, then F is reduced to a crisp set and $|2F (r_i) – 1| = 1$.
Based on the above metrics, a universal measure of fuzziness may now be defined as follows for our portfolio funds allocation model. This is done as follows:

For a crisp set F, F^c is truly complementary, meaning that the metric distance becomes:

$$\mathbf{D\rho\ {*}(F, F^c) = 2^{1/\rho}}, \text{ where } \rho = 1, 2 \qquad \ldots \mathbf{(xxiii)}$$

An effective measure of fuzziness could therefore be as follows:

$$\mathbf{FUZ_\rho\ (F) = [2^{1/\rho} - D\rho\ (F, F^c)]/\ 2^{1/\rho} = 1 - D\rho\ (F, F^c)]/\ 2^{1/\rho}} \qquad \ldots \mathbf{(xxiv)}$$

For the Euclidean metric we would then have:

$$\mathbf{FUZ_2\ (F) = 1 - \frac{[\Sigma(2F\ (r_i) - 1)^2]^{1/2}}{\sqrt{2}}}$$

$$\mathbf{= 1 - (\sqrt{2})\ (RMSD),} \text{ where } \mathbf{RMSD = ([\Sigma(2F\ (r_i) - 1)^2]^{1/2})/2} \qquad \ldots \mathbf{(xxv)}$$

For the Hamming metric, the formula will simply be as follows:

$$\mathbf{FUZ_1 (F) = 1 - \frac{\Sigma|2F (r_i) - 1|}{2}} \qquad \text{… (xxvi)}$$

Having worked on the applicable measure for the degree of fuzziness of our governing preference relation, we devote the next section of our present paper to the possible application of optimal control theory to model the temporal dynamics of funds allocation behavior of an individual investor.

(IV) Exploring time-dependent funds allocation behavior of individual investor in the light of optimal control theory.

If the inter-temporal utility of an individual viewed from time t is recursively defined as $\mathbf{U_t = W [c_t, \mu (U_{t+1}| I_t)]}$, then the aggregator function W makes current inter-temporal utility a function of current consumption ct and of a certainty equivalent of next period's random utility It that is computed using information up to t. Then, the individual could choose a control variable xt in period t to maximize Ut. [4] In the context of the mean-variance model, a suitable candidate for the control variable could well be the proportion of funds set aside for investment in the risk-free asset. So, the objective function would incorporate the investor's total temporal utility in a given time range [0, T]. Given that we include time as a continuous variable in the model, we may effectively formulate the problem applying classical optimal control theory. The plausible methodology for formulating this model is what we shall explore in this section.

The basic optimal control problem can be stated as follows: [8]

Find the control vector $\mathbf{u} = (u_1, u_2 \dots u_m)$ which optimizes the functional, called the *performance index,* $\mathbf{J = \int f_0 (x, u, t) dt}$ over the range (0, T), where $\mathbf{x} = (x_1, x_2 \dots x_n)$ is called the state vector, t is the time parameter, T is the terminal time and f_0 is a specified function of **x**, **u** and t. The state variables xi and the control variables ui are related as $dx_i/dt = f_i (x_1, x_2 \dots x_n; u_1, u_2 \dots u_m; t)$, i = 1, 2 … n.

In many control problems, the system is linearly expressible as $\mathbf{x} \textbf{(.)} = [A]_{nxn}\, \mathbf{x} + [B]_{nxm}\, \mathbf{u}$, where all the symbols have their usual connotations. As an illustrative example, we may again consider the quadratic function that we used earlier f_0 (x, y) = $ax^2 - by^2$. Then the problem is to find the control vector that makes the performance index given by the integral $\mathbf{J = \int(ax^2 - by^2)\, dt}$ stationary with x = 1 – y in the range (0, T).

The ***Hamiltonian*** may be expressed as $\boldsymbol{H} = \mathbf{f_0 + \lambda y = (ax^2 - by^2) + \lambda y}$. The standard solution technique yields $-H_x = \lambda(.)$ … (i) and $H_u = 0$ … (ii) whereby we have the following system of equations: -2ax = λ(.) … (iii) and -2y + λ = 0 … (iv). Differentiation of (iv) leads to –2y(.) + λ(.) = 0 … (v). Solving (iii) and (v) simultaneously, we get 2ax = -2y(.) = -λ(.) i.e. y(.) = -ax … (vi). Transforming (iii) in terms of x and solving the resulting ordinary differential equation would yield the state trajectory x (t) and the optimal control u (t) for the specified quadratic utility function, which can be easily done by most standard mathematical computing software packages.

So, given a particular form of a utility function, we can trace the dynamic time-path of an individual investor's fund allocation behavior (and hence; his or her classification) within the ambit of the mean-variance model by obtaining the state trajectory of x – the proportion of funds invested in the market portfolio and the corresponding control variable y – the proportion of funds invested in the risk-free asset using the standard techniques of optimal control theory.

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Neutrosophical Computational Exploration of Investor Utilities Underlying a Portfolio Insurance Strategy

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Abstract

In this paper we take a look at a simple portfolio insurance strategy using a protective put and computationally derive the investor's governing utility structures underlying such a strategy under alternative market scenarios. Investor utility is deemed to increase with an increase in the excess equity generated by the portfolio insurance strategy over a simple investment strategy without any insurance. Three alternative market scenarios (probability spaces) have been explored – "Down", "Neutral" and "Up", categorized according to whether the price of the underlying security is most likely to go down, stay unchanged or go up. The methodology used is computational, primarily based on simulation and numerical extrapolation. The Arrow-Pratt measure of risk aversion has been used to determine how the investors react towards risk under the different scenarios. We have further proposed an extension of the classical computational modeling to a neutrosophical one

Keywords: Option pricing, investment risk, portfolio insurance, utility theory, behavioral economics

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Introduction:

Basically, a derivative financial asset is a legal contract between two parties – a buyer and a seller, whereby the former receives a rightful claim on an underlying asset while the latter has the corresponding liability of making good that claim, in exchange for a mutually agreed consideration. While many derivative securities are traded on the floors of exchanges just like ordinary securities, some derivatives are not exchange-traded at all. These are called OTC (Over-the-Counter) derivatives, which are contracts not traded on organized exchanges but rather negotiated privately between parties and are especially tailor-made to suit the nature of the underlying assets and the pay-offs desired therefrom. While countless papers have been written on the mathematics of option pricing formulation, surprisingly little work has been done in the area of exploring the exact nature of investor utility structures that underlie investment in derivative financial assets. This is an area we deem to be of tremendous interest both from the point of view of mainstream financial economics as well as from the point of view of a more recent and more esoteric perspective of *behavioral economics*.

The basic building blocks of derivative assets:

<u>Forward Contract</u>

A contract to buy or sell a specified amount of a designated commodity, currency, security, or financial instrument at a known date in the future and at a price set at the time the contract is made. Forward contracts are negotiated between the contracting parties and are not traded on organized exchanges.

<u>Futures Contract</u>

Quite similar to a forwards contract – this is a contract to buy or sell a specified amount of a designated commodity, currency, security, or financial instrument at a known date in the future and at a price set at the time the contract is made. What primarily distinguishes forward contracts from futures contracts is that the latter are traded on organized exchanges and are thus standardized. These contracts are marked to market daily, with profits and losses settled in cash at the end of the trading day.

Swap Contract

A private contract between two parties to exchange cash flows in the future according to some prearranged formula. The most common type of swap is the "plain vanilla" interest rate swap, in which the first party agrees to pay the second party cash flows equal to interest at a predetermined f*ixed* rate on a notional principal. The second party agrees to pay the first party cash flows equal to interest at a *floating* rate on the same notional principal. Both payment streams are denominated in the same currency. Another common type of swap is the currency swap. This contract calls for the counter-parties to exchange specific amounts of two different currencies at the outset, which are repaid over time according to a prearranged formula that reflects amortization and interest payments.

Option Contract

A contract that gives its owner the right, but not the obligation, to buy or sell a specified asset at a stipulated price, called the strike price. Contracts that give owners the right to buy are referred to as *call* options and contracts that give the owner the right to sell are called *put* options. Options include both standardized products that trade on organized exchanges and customized contracts between private parties.

In our present analysis we will be restricted exclusively to portfolio insurance strategy using a long position in put options and explore the utility structures derivable therefrom.

The simplest option contracts (also called *plain vanilla options*) are of two basic types – ***call*** and ***put***. The call option is a right to buy (or call up) some underlying asset at or within a specific future date for a specific price called the strike price. The put option is a right to sell (or put through) some underlying asset at or within a specified date – again for a pre-determined strike price. The options come with no obligations attached – it is totally the discretion of the option holder to decide whether or not to exercise the same.

The pay-off function (from an option buyer's viewpoint) emanating from a call option is given as $\boldsymbol{P_{call} = Max\ [(S_T - X),\ 0]}$. Here, S_T is the price of the underlying asset on maturity and X is the strike price of the option. Similarly, for a put option, the pay-off function is given as $\boldsymbol{P_{put} = Max\ [(X - S_T),\ 0]}$. The implicit assumption in this case is that the options can only be exercised on the maturity date and not earlier. Such options are called ***European options***. If the holder of an option contract is allowed to exercise the same any time on or before the day of maturity, it is termed an ***American option***. A third, not-so-common category is one where the holder can exercise the option only on specified dates prior to its maturity. These are termed ***Bermudan options***. The options we refer to in this paper will all be European type only but methodological extensions are possible to extend our analysis to also include American or even Bermudan options.

Investor's utility structures governing the purchase of plain vanilla option contracts:

Let us assume that an underlying asset priced at S at time t will go up or down by Δs or stay unchanged at time T either with probabilities p_U (u), p_U (d) and p_U (n) respectively contingent upon the occurrence of event U, or with probabilities p_D (u), p_D (d) and p_D (n) respectively contingent upon the occurrence of event D, or with probabilities p_N (u), p_N (d) and p_N (n) respectively contingent upon the occurrence of event N, in the time period (T – t). This, by the way, is comparable to the analytical framework that is exploited in option pricing using the numerical method of ***trinomial trees***. The trinomial tree algorithm is mainly used in the pricing of the non-European options where no closed-form pricing formula exists.

Theorem:

Let P_U, P_D and P_N be the three probability distributions contingent upon events U, D and N respectively. Then we have *a consistent preference relation* for a call buyer such that ***P_U is strictly preferred to P_N and P_N is strictly preferred to P_D*** and *a corresponding consistent preference relation* for a put buyer such that ***P_D is strictly preferred to P_N and P_N is strictly preferred to P_U***.

Proof:

Case I: Investor buys a call option for \$C maturing at time T having a strike price of \$X on the underlying asset. We modify the call pay-off function slightly such that we now have the pay-off function as: $\boldsymbol{P_{call} = Max\ (S_T - X - C_{price}, - C_{price})}$.

Event U:

$$E_U\ (Call) = [(S + e^{-r\ (T-t)}\ \Delta s)\ p_U\ (u) + (S - e^{-r\ (T-t)}\ \Delta s)\ p_U\ (d)\ + S\ p_U\ (n)] - C - Xe^{-r\ (T-t)}$$

$$= [S + e^{-r\ (T-t)}\ \Delta s\ \{p_U\ (u) - p_U\ (d)\}] - C - Xe^{-r\ (T-t)} \ldots\ p_U\ (u) > p_U\ (d)$$

Therefore, $E\ (P_{call}) = Max\ [S + e^{-r\ (T-t)}\ \{\Delta s\ (p_U\ (u) - p_U\ (d)) - X\} - C, - C]$ **… (i)**

Event D:

$$E_D\ (Call) = [(S + e^{-r\ (T-t)}\ \Delta s)\ p_D\ (u) + (S - e^{-r\ (T-t)}\ \Delta s)\ p_D\ (d)\ + S\ p_D\ (n)] - C - Xe^{-r\ (T-t)}$$

$$= [S + e^{-r\ (T-t)}\ \Delta s\ \{p_D\ (u) - p_D\ (d)\}] - C - Xe^{-r\ (T-t)} \ldots\ p_D\ (u) < p_D\ (d)$$

Therefore, $E\ (P_{call}) = Max\ [S - e^{-r\ (T-t)}\ \{\Delta s\ (p_D\ (d) - p_D\ (u)) + X\} - C, - C]$ **… (ii)**

Event N:

$$E_N\ (Call) = [(S + e^{-r\ (T-t)}\ \Delta s)\ p_N\ (u) + (S - e^{-r\ (T-t)}\ \Delta s)\ p_N\ (d)\ + S\ p_N\ (n)] - C - Xe^{-r\ (T-t)}$$

$$= [S + e^{-r\ (T-t)}\ \Delta s\ \{p_N\ (u) - p_N\ (d)\}] - C - Xe^{-r\ (T-t)}$$

$$= S - C - Xe^{-r\ (T-t)} \ldots\ p_N\ (u) = p_N\ (d)$$

Therefore, $E\ (P_{call}) = Max\ [S - Xe^{-r\ (T-t)} - C, - C]$ **… (iii)**

Case II: Investor buys a put option for \$P maturing at time T having a strike price of \$X on the underlying asset. Again we modify the pay-off function such that we now have the pay-off function as: $\boldsymbol{P_{put} = Max\ (X - S_T - P_{price}, - P_{price})}$.

Event U:

$$E_U\ (Put) = Xe^{-r\ (T-t)} - [\{(S + e^{-r\ (T-t)}\ \Delta s)\ p_U\ (u) + (S - e^{-r\ (T-t)}\ \Delta s)\ p_U\ (d)\ + S\ p_U\ (n)\} + P]$$

$$= Xe^{-r\ (T-t)} - [S + e^{-r\ (T-t)}\ \Delta s\ \{p_U\ (u) - p_U\ (d)\} + P]$$

$= Xe^{-r(T-t)} - [S + e^{-r(T-t)} \Delta s \{p_U(u) - p_U(d)\} + (C + Xe^{-r(T-t)} - S)]$ … ***put-call parity***

$= - e^{-r(T-t)} \Delta s \{p_U(u) - p_U(d)\} - C$

Therefore, $E(P_{put}) = \text{Max}\,[- e^{-r(T-t)} \Delta s \{p_U(u) - p_U(d)\} - C, - P]$

$= \text{Max}\,[- e^{-r(T-t)} \Delta s \{p_U(u) - p_U(d)\} - C, - (C + Xe^{-r(T-t)} - S)]$ **… (iv)**

Event D:

$E_D(\text{Put}) = Xe^{-r(T-t)} - [\{(S + e^{-r(T-t)} \Delta s)\, p_D(u) + (S - e^{-r(T-t)} \Delta s)\, p_D(d) + S\, p_D(n)\} + P]$

$= Xe^{-r(T-t)} - [S + e^{-r(T-t)} \Delta s \{p_D(u) - p_D(d)\} + P]$

$= Xe^{-r(T-t)} - [S + e^{-r(T-t)} \Delta s \{p_U(u) - p_U(d)\} + (C + Xe^{-r(T-t)} - S)]$ … ***put-call parity***

$= e^{-r(T-t)} \Delta s \{p_D(d) - p_D(u)\} - C$

Therefore, $E(P_{put}) = \text{Max}\,[e^{-r(T-t)} \Delta s \{p_D(d) - p_D(u)\} - C, - P]$

$= \text{Max}\,[e^{-r(T-t)} \Delta s \{p_D(d) - p_D(u)\} - C, - (C + Xe^{-r(T-t)} - S)]$ **… (v)**

Event N:

$E_N(\text{Put}) = Xe^{-r(T-t)} - [\{(S + e^{-r(T-t)} \Delta s)\, p_N(u) + (S - e^{-r(T-t)} \Delta s)\, p_N(d) + S\, p_N(n)\} + P]$

$= Xe^{-r(T-t)} - [S + e^{-r(T-t)} \Delta s \{p_N(u) - p_N(d)\} + P]$

$= Xe^{-r(T-t)} - (S + P)$

$= (Xe^{-r(T-t)} - S) - \{C + (Xe^{-r(T-t)} - S)\}$ … ***put-call parity***

$= - C$

Therefore, $E(P_{put}) = \text{Max}\,[- C, - P]$

$= \text{Max}\,[-C, - (C + Xe^{-r(T-t)} - S)]$ **… (vi)**

From equations (4), (5) and (6) we see that $\mathbf{E_U}$ **(Put) <** $\mathbf{E_N}$ **(Put) <** $\mathbf{E_D}$ **(Put)** and hence it is proved why we have the *consistent preference relation* ***P_D is strictly preferred to P_N*** ***<u>and</u> P_N is strictly preferred to P_U*** from a put buyer's point of view. The call buyer's consistent preference relation is also explainable likewise.

We can now proceed to computationally derive the associated utility structures using a Monte Carlo discrete-event simulation approach to estimate the change in equity following a particular investment strategy under each of the aforementioned event spaces.

Computational derivation of investor's utility curves under a *protective put* strategy:

There is a more or less well-established theory of utility maximization in case of deductible insurance policy on non-financial assets whereby the basic underlying assumption is that cost of insurance is a convex function of the expected indemnification. Such an assumption has been showed to satisfy the sufficiency condition for expected utility maximization when individual preferences exhibit risk aversion. The final wealth function at end of the insurance period is given as follows:

$$\mathbf{Z_T = Z_0 + M - x + I\ (x) - C\ (D)} \qquad \textbf{... (vii)}$$

Here Z_T is the final wealth at time t = T, Z_0 is the initial wealth at time t = 0, x is a random loss variable, I (x) is the indemnification function, C (x) is the cost of insurance and $0 \leq D \leq M$ is the level of the deductible. However the parallels that can be drawn between ordinary insurance and portfolio insurance is different when the portfolio consists of financial assets being continuously traded on the floors of organized financial markets. While the form of an insurance contract might look familiar – an assured value in return for a price – the mechanism of providing such assurance will have to be quite different because unlike other tangible assets like houses or cars, when one portfolio of financial assets gets knocked down, virtually all others are likely to follow suit making "risk pooling", the typical method of insurance, quite inadequate for portfolio insurance. Derivative assets like options do provide a suitable mechanism for portfolio insurance.

If the market is likely to move adversely, holding a long put alongside ensures that the investor is better off than just holding a long position in the underlying asset. The long put offers the investor some kind of *price insurance* in case the market goes down. This strategy is known in derivatives parlance as a *protective put*. The strategy effectively puts a floor on the downside deviations without cutting off the upside by too much. From the expected changes in investor's equity we can computationally derive his or her utility curves under the strategies A_1 and A_2 in each of the three probability spaces D, N and U.

The following hypothetical data have been assumed to calculate the simulated put price:

S = \$50.00 (purchase price of the underlying security)

X = \$55.00 (put strike price)

(T – t) = 1 (single period investment horizon)

Risk-free rate = 5%

The put option pay-offs have been valued by Monte Carlo simulation of a trinomial tree using a customized MS-Excel spreadsheet for one hundred independent replications in each case.

Event space: D Strategy: A_1 (Long underlying asset)

<u>Instance (i):</u> (–)ΔS = \$5.00, (+)ΔS = \$15.00

Table 1

Price movement	Probability	Expected Δ Equity
Up (+ \$15.00)	0.1	\$1.50
Neutral (\$0.00)	0.3	\$0.00
Down (– \$5.00)	0.6	(\$3.00) Σ = (\$1.50)

To see how the expected change in investor's equity goes up with an increased upside potential we will double the possible up movement at each of the next two stages while keeping the down movement unaltered. This should enable us to account for any possible loss of investor utility by way of the cost of using a portfolio insurance strategy.

<u>Instance (ii):</u> (+) ΔS = \$30.00

Table 2

Price movement	Probability	Expected Δ Equity
Up (+ \$30.00)	0.1	\$3.00
Neutral (\$0.00)	0.3	\$0.00
Down (– \$5.00)	0.6	(\$3.00) Σ = \$0.00

<u>Instance (iii):</u> $(+)\Delta S = \$60.00$

Table 3

Price movement	Probability	Expected Δ Equity
Up (+ \$60.00)	0.1	\$6.00
Neutral (\$0.00)	0.3	\$0.00
Down (– \$5.00)	0.6	(\$3.00) Σ = \$3.00

Event space: D Strategy: A_2 (Long underlying asset + long put)

<u>Instance (i):</u> $(-)\Delta S = \$5.00$, $(+)\Delta S = \$15.00$

Table 4

Simulated put price	\$6.99
Variance	\$11.63
Simulated asset value	\$48.95
Variance	\$43.58

Table 5

Price movement	Probability	Expected Δ Equity	Expected excess equity	Utility index
Up (+ \$8.01)	0.1	\$0.801		
Neutral (– \$1.99)	0.3	(\$0.597)		
Down (– \$1.99)	0.6	(\$1.194)		
		Σ = (–\$0.99)	\$0.51	≈ 0.333

Instance (ii): (+)ΔS = \$30.00

Table 6

Simulated put price	\$6.75
Variance	\$13.33
Simulated asset value	\$52.15
Variance	\$164.78

Table 7

Price movement	Probability	Expected Δ Equity	Expected excess equity	Utility index
Up (+ $23.25)	0.1	$2.325		
Neutral (– $1.75)	0.3	($0.525)		
Down (– $1.75)	0.6	($1.05)		
		Σ = $0.75	$0.75	≈ 0.666

<u>Instance (iii):</u> (+)ΔS = $60.00

Table 8

Simulated put price	$6.71
Variance	$12.38
Simulated asset value	$56.20
Variance	$520.77

Table 9

Price movement	Probability	Expected Δ Equity	Expected excess equity	Utility index
Up (+ $53.29)	0.1	$5.329		
Neutral (– $1.71)	0.3	($0.513)		
Down (– $1.71)	0.6	($1.026)		
		Σ = $3.79	$0.79	≈ 0.999

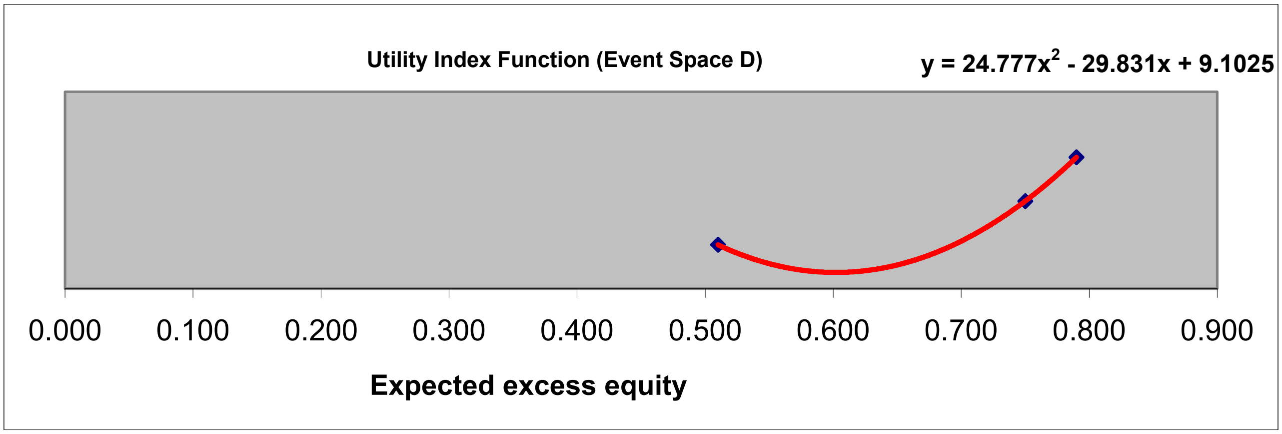

Figure 1

The utility function as obtained above is *convex* in probability space D, which indicates that the protective strategy can make the investor risk-loving even when the market is expected to move in an adverse direction, as the expected payoff from the put option largely neutralizes the likely erosion of security value at an affordable insurance cost! This seems in line with intuitive behavioral reasoning, as investors with a viable downside protection will become more aggressive in their approach than they would be without it implying markedly lowered risk averseness for the investors with insurance.

Event space: N Strategy: A_1 (Long underlying asset)

Instance (i): (–)ΔS = \$5.00, (+)ΔS = \$15.00

Table 10

Price movement	Probability	Expected Δ Equity
Up (+ \$15.00)	0.2	\$3.00
Neutral (\$0.00)	0.6	\$0.00
Down (– \$5.00)	0.2	(\$1.00) Σ = \$2.00

<u>Instance (ii):</u> (+)ΔS = \$30.00

Table 11

Price movement	Probability	Expected Δ Equity
Up (+ \$30.00)	0.2	\$6.00
Neutral (\$0.00)	0.6	\$0.00
Down (– \$5.00)	0.2	(\$1.00) Σ = \$5.00

<u>Instance (iii):</u> (+)ΔS = \$60.00

Table 12

Price movement	Probability	Expected Δ Equity
Up (+ \$60.00)	0.2	\$12.00
Neutral (\$0.00)	0.6	\$0.00
Down (– \$5)	0.2	(\$1.00) Σ = \$11.00

Event space: N Strategy: A_2 (Long underlying asset + long put)

<u>Instance (i):</u> (–)ΔS = \$5.00, (+)ΔS = \$15.00

Table 13

Simulated put price	\$4.85
Variance	\$9.59
Simulated asset value	\$51.90
Variance	\$47.36

Table 14

Price movement	Probability	Expected Δ Equity	Expected excess equity	Utility index
Up (+ \$11.15)	0.2	\$2.23		
Neutral (+ \$0.15)	0.6	\$0.09		
Down (+ \$0.15)	0.2	\$0.03		
		Σ = \$2.35	\$0.35	≈ 0.999

<u>Instance (ii):</u> (+)ΔS = \$30.00

Table 15

Simulated put price	$4.80
Variance	$9.82
Simulated asset value	$55.20
Variance	$169.15

Table 16

Price movement	Probability	Expected Δ Equity	Expected excess equity	Utility index
Up (+ $25.20)	0.2	$5.04		
Neutral (+ $0.20)	0.6	$0.12		
Down (+ $0.20)	0.2	$0.04		
		Σ = $5.20	$0.20	≈ 0.333

<u>Instance (iii): </u>(+)ΔS = $60.00

Table 17

Simulated put price	$4.76
Variance	$8.68
Simulated asset value	$60.45
Variance	$585.40

Table 18

Price movement	Probability	Expected Δ Equity	Expected excess equity	Utility index
Up (+ $55.24)	0.2	$11.048		
Neutral (+ $0.24)	0.6	$0.144		
Down (+ $0.24)	0.2	$0.048		
		Σ = $11.24	$0.24	≈ 0.666

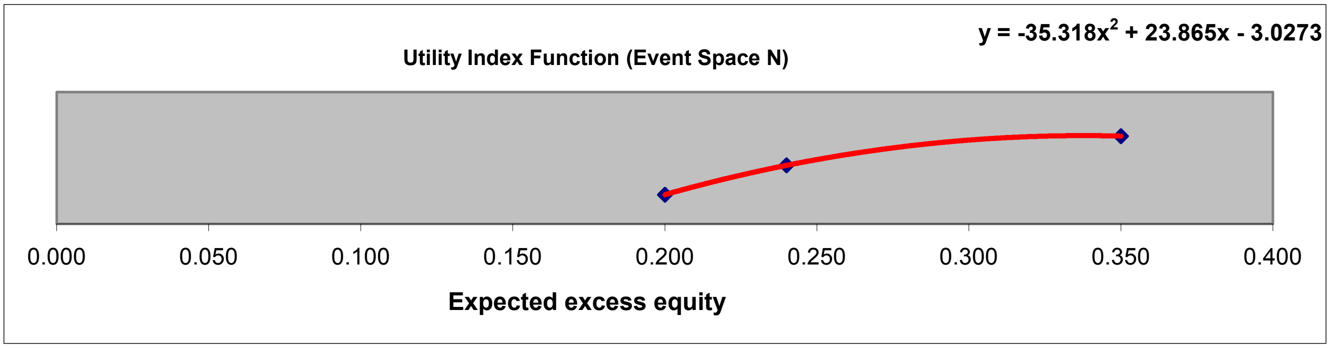

Figure 2

The utility function as obtained above is *concave* in probability space N, which indicates that the insurance provided by the protective strategy can no longer make the investor risk-loving as the expected value of the insurance is offset by the cost of buying the put! This is again in line with intuitive behavioral reasoning because if the market is equally likely to move up or down and more likely to stay unmoved the investor would deem himself or herself better off not buying the insurance because in order to have the insurance i.e. the put option it is necessary to pay an out-of-pocket cost, which may not be offset by the expected payoff from the put option under the prevalent market scenario.

Event space: U Strategy: A_1 (Long underlying asset)

Instance (i): $(-)\Delta S = \$5.00$, $(+)\Delta S = \$15.00$

Table 19

Price movement	Probability	Expected Δ Equity
Up (+ \$15.00)	0.6	\$9.00
Neutral (\$0.00)	0.3	\$0.00
Down (– \$5.00)	0.1	(\$0.50) Σ = \$8.50

<u>Instance (ii):</u> (+)ΔS = \$30.00

Table 20

Price movement	Probability	Expected Δ Equity
Up (+ \$30.00)	0.6	\$18.00
Neutral (\$0.00)	0.3	\$0.00
Down (– \$5.00)	0.1	(\$0.50) Σ = \$17.50

<u>Instance (iii):</u> (+) ΔS = \$60.00

Table 21

Price movement	Probability	Expected Δ Equity
Up (+ \$60.00)	0.6	\$36.00
Neutral (\$0.00)	0.3	\$0.00
Down (– \$5)	0.1	(\$0.50) Σ = \$35.50

Event space: U Strategy: A_2 (Long underlying asset + long put)

<u>Instance (i):</u> (–)ΔS = \$5.00, (+)ΔS = \$15.00

Table 22

Simulated put price	\$2.28
Variance	\$9.36
Simulated asset value	\$58.60
Variance	\$63.68

Table 23

Price movement	Probability	Expected Δ Equity	Expected excess equity	Utility index
Up (+ \$12.72)	0.6	\$7.632		
Neutral (+ \$2.72)	0.3	\$0.816		
Down (+ \$2.72)	0.1	\$0.272		
		Σ = \$8.72	\$0.22	≈ 0.333

<u>Instance (ii):</u> (+)ΔS = \$30.00

Table 24

Simulated put price	\$2.14
Variance	\$10.23
Simulated asset value	\$69.00
Variance	\$228.79

Table 25

Price movement	Probability	Expected Δ Equity	Expected excess equity	Utility index
Up (+ \$27.86)	0.6	\$16.716		
Neutral (+ \$2.86)	0.3	\$0.858		
Down (+ \$2.86)	0.1	\$0.286		
		Σ = \$17.86	\$0.36	≈ 0.666

Instance (iii): (+)ΔS = \$60.00

Table 26

Simulated put price	\$2.09
Variance	\$9.74
Simulated asset value	\$88.55
Variance	\$864.80

Table 27

Price movement	Probability	Expected Δ Equity	Expected excess equity	Utility index
Up (+ \$57.91)	0.6	\$34.746		
Neutral (+ \$2.91)	0.3	\$0.873		
Down (+ \$2.91)	0.1	\$0.291		
		Σ = \$35.91	\$0.41	≈ 0.999

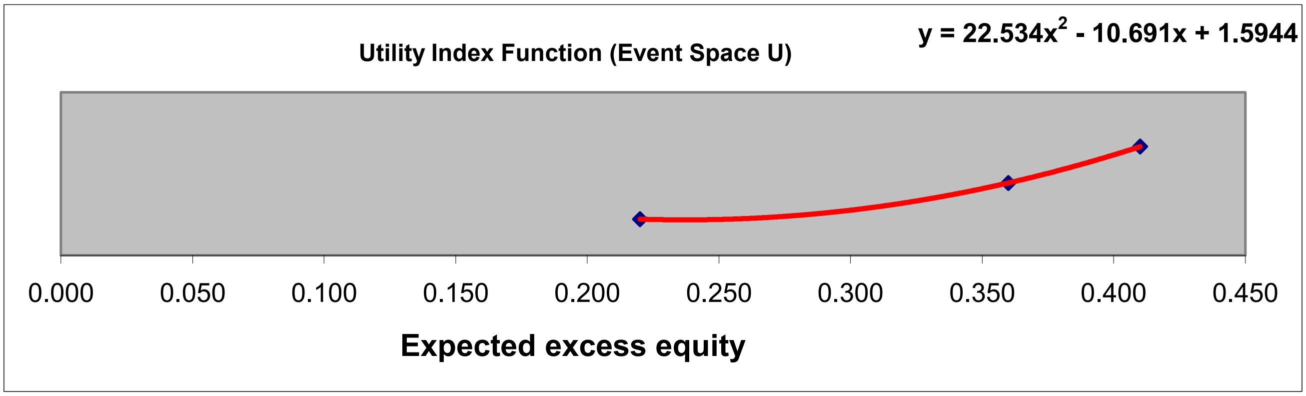

Figure 3

In accordance with intuitive, behavioral reasoning the utility function is again seen to be *convex* in the probability space U, which is probably attributable to the fact that while the market is expected to move in a favourable direction the put option nevertheless keeps the downside protected while costing less than the expected payoff on exercise thereby fostering a risk-loving attitude in the investor as he gets to enjoy the best of both worlds.

Note: Particular values assigned to the utility indices won't affect the essential mathematical structure of the utility curve – but only cause a scale shift in the parameters. For example, the indices could easily have been taken as (0.111, 0.555, 0.999) - these assigned values should not have any computational significance as long as all they all lie within the conventional interval (0, 1]. Repeated simulations have shown that the investor would be considered extremely unlucky to get an excess return less than the minimum excess return obtained or extremely lucky to get an excess return more than the maximum excess return obtained under each of the event spaces. Hence, the maximum and minimum expected excess equity within a particular event space should correspond to the lowest and highest utility indices and the utility derived from the median excess equity should then naturally occupy the middle position. As long as this is the case, there will be no alteration in the fundamental mathematical structure of the investor's utility functions no matter what index values are assigned to his or her utility from expected excess equity.

Extrapolating the ranges of investor's risk aversion within each probability space:

For a continuous, twice-differentiable utility function u (x), the ***Arrow-Pratt measure of absolute risk aversion (ARA)*** is given as follows:

$$\lambda (x) = -[d^2u (x)/dx^2][du (x)/dx]^{-1} \qquad \dots \text{(viii)}$$

λ (x) > 0 if u is monotonically increasing and *strictly concave* as in case of a risk-averse investor having u'' (x) < 0. Obviously, λ (x) = 0 for the risk-neutral investor with a *linear* utility function having u'' (x) = 0 while λ (x) < 0 for the risk-loving investor with a *strictly convex* utility function having u'' (x) > 0.

Case I: Probability Space D:

u (x) = $24.777x^2 - 29.831x + 9.1025$, u' (x) = $49.554x - 29.831$ and u''(x) = 49.554. Thus λ (x) = – 49.554/(49.554x – 29.831). Therefore, given the *convex utility function*, the defining range is λ (x) < 0 i.e. (49.554x – 29.831) < 0 or x < 0.60199.

Case II: Probability Space N:

u (x) = $-35.318\ x^2 + 23.865x - 3.0273$, u' (x) = $-70.636x + 23.865$ and u''(x) = –70.636. Thus, λ (x) = – [–70.636 /(–70.636x + 23.865)] = 70.636/(–70.636x + 23.865). Therefore, given the *concave utility function*, the defining range is λ (x) > 0, i.e. we have the denominator (-70.636x + 23.865) > 0 or x > 0.33786.

Case III: Probability Space U:

u (x) = $22.534x^2 - 10.691x + 1.5944$, u' (x) = $45.068x - 10.691$ and u''(x) = 45.068. Thus λ (x) = – 45.068/(45.068x – 10.691). Therefore, given the *convex utility function*, the defining range is λ (x) < 0 i.e. (45.068x – 10.691) < 0 or x < 0.23722.

These defining ranges as evaluated above will however depend on the parameters of the utility function and will therefore be different for different investors according to the values assigned to his or her utility indices corresponding to the expected excess equity.

In general, if we have a parabolic utility function $u(x) = a + bx - cx^2$, where $c > 0$ ensures concavity, then we have $u'(x) = b - 2cx$ and $u''(x) = -2c$. The Arrow-Pratt measure is given by $\lambda(x) = 2c/(b-2cx)$. Therefore, for $\lambda(x) \geq 0$, we need $b > 2cx$, thus it can only apply for a limited range of x. Notice that $\lambda'(x) > 0$ up to where $x = b/2c$. Beyond that, marginal utility is negative - i.e. beyond this level of equity, utility *declines*. One more implication is that there is an increasing apparent unwillingness to take risk as their equity increases, i.e. with larger excess equity investors are less willing to take risks as concave, parabolic utility functions exhibit increasing absolute risk aversion (IARA).

People sometimes use a past outcome as a critical factor in evaluating the likely outcome from a risky decision to be taken in the present. Also it has been experimentally demonstrated that decisions can be taken in violation of conditional preference relations. This has been the crux of a whole body of behavioral utility theory developed on the basis of what has come to be known as *non-expected utility* following the famous work in *prospect theory* (Kahneman and Tversky, 1979). It has been empirically demonstrated that people are willing to take more risks immediately following gains and take less risks immediately following losses with the probability distribution of the payoffs remaining unchanged. Also decisions are affected more by instantaneous utility resulting from immediate gains than by disutility resulting from the cumulative magnitude of likely losses as in the assessment of health risks from addictive alcohol consumption. It has also been seen in experimental psychology studies that generated explanations cause a greater degree of belief persistence than provided explanations. This is due to a psychological miscalibration whereby people tend to be guided by outcomes in their most recent memory. In the face of all these challenges to the expected utility paradigm, it must however be noted that the utility structures underlying the behavior of investors with loss insurance in the three different market scenarios as derived above are *independent of any psychological miscalibration* on the part of the individual based on prior history of positive or negative payoffs but rather are a direct statistical consequence of the portfolio insurance strategy itself and the expected payoffs likely to follow from such a strategy.

Extending the classical computational model to a Neutrosophical computational model:

Neutrosophy forms the philosophical foundation of a relatively new branch of mathematical logic that relates to the cause, structure and scope of neutralities as well as their interactions with different ideational spectra. In formal terms, neutrosophic logic is a generalization of fuzzy logic – whereas fuzzy logic deals with the imprecision regarding membership of a set X and its compliment X^c, neutrosophic logic recognizes and studies a non-standard, neutral subset of X and X^c.

If T, I, F are standard or non-standard real subsets of $^-]\ 0, 1\ [^+$, then T, I, F are referred to as *neutrosophic components* which represent truth value, indeterminacy value and falsity value of a proposition respectively. The governing principle of Neutrosophy is that if a set X exists to which there is a compliment X^c, then there exists a continuum-power spectrum of neutralities $_NX$. Then $x \in X$ by t%, $x \in {}_NX$ by i% and $x \in X^c$ by f%, where we have $(t, i, f) \subset (T, I, F)$.

The practical applicability of such a logical framework in the context of an option-based portfolio insurance strategy becomes immediately apparent when we consider imperfect markets and asymmetric flow of market information. Imprecision arises in financial markets, as it does in any other setting, out of incomplete information, inherent randomness of information source (stochasticity) and incorrect interpretation of subjective information. The neutrosophic components T, I, F, viewed dynamically as set-valued vector functions, can be said to depend at each instance on multiple parameters which may be spatial, temporal or even psychological. For example, the proposition "The market will break the resistance tomorrow" may be 50% true, 75% indeterminate and 40% false as of today at the close of trading but with new information coming overnight it might change to 80% true, 40% indeterminate and 15% false which may subsequently again change to 100% true, 0% indeterminate and 0% false when the trading starts the next day and the market really rises through the roof. Moreover, the evaluations may be different for different market analysts according to their inconsistent (or even conflicting) information sources and/or non-corresponding interpretations. For example, according to one analyst the proposition could be 50% true 75% indeterminate and 40% false while according to another (with more recent and/or more accurate information) it may be 80% true, 40% indeterminate and 15% false. However, as the trading starts the next day and the market actually breaks the resistance, all the individual assessments will ultimately converge to 100% true, 0% indeterminate and 0% false. How fast this convergence takes place will be dependent on the level of market efficiency. This is perhaps the closest representation of the human thought process. It characterizes the imprecision of knowledge due to asymmetric dissemination of information, acquisition errors, stochasticity and interpretational vagueness due to lack of clear contours of the defining subsets. The superior and inferior limits of these defining subsets have to be pre-specified in order to set up a workable computational model of a neutrosophic problem.

The simulation model we have employed here to explore the utility structures underlying a simple option-based portfolio insurance strategy can now be further extended in the light of neutrosophic reasoning. Instead of running the simulations individually under each of the probability spaces U, N and D, one can define a neutrosophic probability space where the market has u% chance of being up, n% chance of being neither up nor down and d% chance of being down. These probability assessments could of course be in the nature of set-valued vector functions defined over specific spatio-temporal domains so as to leave only stochasticity and interpretational variations as the major sources of change in the assessments. Then these two parameters may be separately simulated according to some suitable probability distributions and the results fed into the option-payoff simulation to yield a dynamic scenario whereby the neutrosophic components change according to changes in the parameters and the resulting effect on utility structure can be numerically explored.

Conclusion:

In this paper we have computationally examined the implications on investor's utility of a simple option strategy of portfolio insurance under alternative market scenarios, which we believe is novel both in content as well as context. We have found that such insurance strategies can indeed have quite interesting governing utility structures underlying them. The expected excess payoffs from an insurance strategy can make the investor risk-loving when it is known with a relatively high prior probability that the market will either move in an adverse direction or in a favourable direction. The investor seems to display risk-averseness only when the market is equally likely to move in either direction and has a relatively high prior probability of staying unmoved. We have further outlined a suggested computational methodology to apply neutrosophic reasoning to the problem of portfolio insurance. However, we leave the actual computational modeling of investor utility on a neutrosophic event space to a subsequent research endeavor. The door is now open for further research along these lines going deep into the governing utility structures that may underlie more complex derivative trading strategies, portfolio insurance schemes and structured financial products.

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A Proposed Artificial Neural Network Classifier to Identify Tumor Metastases Part I

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Abstract:
In this paper we propose a classification scheme to isolate truly benign tumors from those that initially start off as benign but subsequently show metastases. A non-parametric artificial neural network methodology has been chosen because of the analytical difficulties associated with extraction of closed-form stochastic-likelihood parameters given the extremely complicated and possibly non-linear behavior of the state variables. This is intended as the first of a three-part research output. In this paper, we have proposed and justified the computational schema. In the second part we shall set up a working model of our schema and pilot-test it with clinical data while in the concluding part we shall give an in-depth analysis of the numerical output and model findings and compare it to existing methods of tumor growth modeling and malignancy prediction.

Key words: Cell cycle, oncogenes, tumor suppressors, tumor metastases, Lebowitz-Rubinow models of continuous-time tumor growth, non-linear dynamics and chaos, multi-layer perceptrons

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Introduction - mechanics of the mammalian cell cycle:

The mammalian cell division cycle passes through four distinct phases with specific drivers, functions and critical checkpoints for each phase

Phase	Main drivers	Functions	Checkpoints
G_1 (*gap* 1)	Cell size, protein content, nutrient level	Preparatory biochemical metamorphosis	Tumor-suppressor gene p53
S (synthesization)	Replicator elements	New DNA synthesization	ATM gene (related to the MEC1 yeast gene)
G_2 (*gap* 2)	Cyclin B accumulation	Pre-mitosis preparatory changes	Levels of cyclin B/cdk1 – increased radiosensitivity
M (mitosis)	Mitosis Promoting Factor (MPF) – complex of cyclin B and cdk1	Entry to mitosis; metaphase-anaphase transition; exit	Mitotic spindle – control of metaphase-anaphase transition

The steady-state number of cells in a tissue is a function of the relative amount of cell proliferation and cell death. The principal determinant of cell proliferation is the residual effect of the interaction between oncogenes and tumor-suppressor genes. Cell death is determined by the residual effect of the interaction of proapoptotic and antiapoptotic genes. Therefore, the number of cells may increase due to either increased oncogenes activity or antiapoptotic genes activity or by decreased activity of the tumor-suppressor genes or the proapoptotic genes. This relationship may be shown as follows:

$$\mathbf{C_n = f\,(O, S, P, AP)}, \text{ such that } \mathbf{\{C_n'\,(O), C_n'\,(AP)\} > 0} \text{ and } \mathbf{\{C_n'\,(S), C_n'\,(P)\} < 0 \;\ldots\; (i)}$$

Here C_n is the steady-state number of cells, O is oncogenes activity, S is tumor-suppressor genes activity, P is proapoptotic genes activity and AP is antiapoptotic genes activity. The abnormal growth of tumor cells result from a combined effect of too few *cell-cycle decelerators* (tumor-suppressors) and too many *cell-cycle accelerators* (oncogenes). The most commonly mutated gene in human cancers is p53, which the cancerous tumors bring about either by overexpression of the p53 binding protein *mdm2* or through pathogens like the *human papilloma virus (HPV)*. Though not the objective of this paper, it could be an interesting and potentially rewarding epidemiological exercise to isolate the proportion of p53 mutation principally brought about by the overexpression of *mdm2* and the proportion of such mutation principally brought about by viral infection.

Brief review of some existing mathematical models of cell population growth:

Though the exact mechanism by which cancer kills a living body is not known till date, it nevertheless seems appropriate to link the severity of cancerous growth to the steady-state number of cells present, which again is a function of the number of oncogenes and tumor-suppressor genes. A number of mathematical models have been constructed studying tumor growth with respect to C_n, the simplest of which express C_n as a function of time without any cell classification scheme based on histological differences. An ***inherited cycle length model*** was implemented by Lebowitz and Rubinow (1974) as an alternative to the simpler age-structured models in which variation in cell cycle times is attributed to occurrence of a chance event. In the LR model, variation in cell-cycle times is attributed to a distribution in inherited generation traits and the determination of the cell cycle length is therefore endogenous to the model. The population density function in the LR model is of the form $\mathbf{C_n\ (a,\ t;\ \tau)}$ where τ is the inherited cycle length. The boundary condition for the model is given as follows:

$$\mathbf{C_n\ (0,\ t;\ \tau) = 2\,{}_0\!\int^{\infty} K\ (\tau,\tau')\ C_n\ (\tau',\ t;\ \tau')\ d\tau'} \qquad \textbf{... (ii)}$$

In the above equation, the kernel $\mathbf{K\ (\tau,\tau')}$ is referred to as the *transition probability function* and gives the probability that a parent cell of cycle length τ' produces a daughter cell of cycle length τ. It is the assumption that every dividing parent cell produces two daughters that yields the multiplier 2. The degree of correlation between the parent and daughter cells is ultimately decided by the choice of the kernel K. The LR model was further extended by Webb (1986) who chose to impose sufficiency conditions on the kernel K in order to ensure that the solutions asymptotically converge to a state of balanced exponential growth. He actually showed that the well-defined collection of mappings $\{S\ (t): t \geq 0\}$ from the Banach space B into itself forms a *strongly continuous semi-group of bounded linear operators*. Thus, for $t \geq 0$, $S\ (t)$ is the operator that transforms an initial distribution $\phi\ (a, \tau)$ into the corresponding solution $C_n\ (a, t; \tau)$ of the LR model at time t. Initially the model only allowed for a positive parent-daughter correlation in cycle times but keeping in tune with experimental evidence for such correlation possibly also being negative, a later; more general version of the Webb model has been developed which considers the sign of the correlation and allows for both cases.

There are also models that take C_n as a function of both time as well as some physiological structure variables. Rubinow (1968) suggested one such scheme where the age variable "a" is replaced by a structure variable "μ" representing some physiological measure of cell maturity with a varying rate of change over time $v = d\mu/dt$. If it is given that $C_n\ (\mu, t)$ represents the cell population density at time t with respect to the structure variable μ, then the population balance model of Rubinow takes the following form:

$$\partial C_n/\partial t + \partial(vC_n)/\partial\mu = -\lambda C_n \qquad \text{… (iii)}$$

Here $\lambda\ (\mu)$ is the maturity-dependent proportion of cells lost per unit of time due to non-mitotic causes. Either v depends on μ or on additional parameters like culture conditions.

Purpose of the present paper:

Growth in cell biology indicates changes in the size of a cell mass due to several interrelated causes the main ones among which are proliferation, differentiation and death. In a normal tissue, cell number remains constant because of a balance between proliferation, death and differentiation. In abnormal situations, increased steady-state cell number is attributable to either inhibited differentiation/death or increased proliferation with the other two properties remaining unchanged. Cancer can form along either route. Contrary to popular belief, cancer cells do not necessarily proliferate faster than the normal ones. Proliferation rates observed in well-differentiated tumors are not significantly higher from those seen in progenitor normal cells. Many normal cells hyperproliferate on occasions but otherwise retain their normal histological behavior. This is known as *hyperplasia*. In this paper, we propose a **non-parametric approach based on an artificial neural network classifier** to detect whether a hyperplasic cell proliferation could eventually become carcinogenic. That is, our model proposes to determine whether a tumor stays benign or subsequently undergoes metastases and becomes malignant as is rather prone to occur in certain forms of cancer.

Benign versus malignant tumors:

A benign tumor grows at a relatively slow rate, does not metastasize, bears histological resemblance to the cells of normal tissue, and tends to form a clearly defined mass. A malignant tumor consists of cancer cells that are highly irregular, grow at a much faster rate, and have a tendency to metastasize. Though benign tumors are usually not directly life threatening, some of the benign types do have the capability of becoming malignant. Therefore, viewed a stochastic process, a purely benign growth should approach some critical *steady-state mass* whereas any growth that subsequently becomes cancerous would fail to approach such a steady-state mass. One of the underlying premises of our model then is that cell population growth takes place according to the basic *Markov chain rule* such that the observed tumor mass in time t_{j+1} is dependent on the mass in time t_j.

Non-linear cellular biorhythms and chaos:

A major drawback of using a parametric stochastic-likelihood modeling approach is that often closed-form solutions become analytically impossible to obtain. The axiomatic approach involves deriving analytical solutions of *stiff stochastic differential-difference equation systems*. But these are often hard to extract especially if the *governing system is decidedly non-linear* like Rubinow's suggested physiological structure model with velocity v depending on the population density C_n. The best course to take in such cases is one using a non-parametric approach like that of artificial neural networks.

The idea of chaos and non-linearity in biochemical processes is not new. Perhaps the most widely referred study in this respect is the *Belousov-Zhabotinsky (BZ) reaction*. This chemical reaction is named after B. P. Belousov who discovered it for the first time and A. M. Zhabotinsky who continued Belousov´s early work. R. J. Field, Endre Körös, and R. M. Noyes published the mechanism of this oscillating reaction in 1972. Their work opened an entire new research area of *nonlinear chemical dynamics*.

Classically the BZ reaction consist of a one-electron redox catalyst, an organic substrate that can be easily brominated and oxidized, and sodium or potassium bromate ion in form of $NaBrO_3$ or $KBrO_3$ all dissolved in sulfuric or nitric acid and mostly using C_e (III)/C_e (IV) salts and M_n (II) salts as catalysts. Also Ruthenium complexes are now extensively studied, because of the reaction's extreme photosensitivity. There is no reason why the highly intricate intracellular biochemical processes, which are inherently of a much higher order of complexity in terms of molecular kinetics compared to the BZ reaction, could not be better viewed in this light. In fact, experimental studies investigating the physiological clock (of yeast) due to oscillating enzymatic breakdown of sugar, have revealed that the coupling to membrane transport could, under certain conditions, result in *chaotic biorhythms*. The yeast does provide a useful experimental model for histologists studying cancerous cell growth because the ATM gene, believed to be a critical checkpoint in the S stage of the cell cycle, is related to the MEC1 yeast gene. Zaguskin has further conjectured that all biorhythms have a *discrete fractal structure*.

The almost ubiquitous growth function used to model population dynamics has the following well-known *difference equation* form:

$$\mathbf{X_{t+1} = rX_t\,(1 - X_t/k)} \qquad \textbf{… (iv)}$$

Such models exhibit *period-doubling* and subsequently chaotic behavior for certain critical parameter values of r and k. The limit set becomes a fractal at the point where the model degenerates into pure chaos. We can easily deduce in a discrete form that the original Rubinow model is a linear one in the sense that $\mathbf{C_{nt+1}}$ is *linearly dependent* on $\mathbf{C_{nt}}$:

$$\Delta \mathbf{C_n}/\Delta \mathbf{t} + \Delta(\mathbf{vC_{nt}})/\Delta\mu = -\lambda \mathbf{C_{nt}}, \text{ that is}$$

$$(\Delta \mathbf{C_n}/\Delta \mathbf{t}) + (\Delta \mathbf{v}/\Delta\mu)\, \mathbf{C_{nt}} + (\Delta \mathbf{C_{nt}}/\Delta\mu)\, \mathbf{v} = -\lambda \mathbf{C_{nt}}$$

$$\Delta \mathbf{C_n} = -\,\mathbf{C_{nt}}\,(\lambda + \Delta \mathbf{v}/\Delta\mu)\,/\,(2/\Delta \mathbf{t}) \ldots \text{ as } v = \Delta\mu/\Delta t$$

Putting $\mathbf{k} = -\,[(2/\Delta \mathbf{t}) - 1 - (\lambda + \Delta \mathbf{v}/\Delta\mu)]^{-1}$ and $\mathbf{r} = (2/\Delta \mathbf{t})^{-1}$ we get;

$$\mathbf{C_{nt+1}} = \mathbf{rC_{nt}}\,(1 - 1/\mathbf{k}) \qquad \ldots \text{(v)}$$

Now this may be oversimplifying things and the true equation could indeed be analogous to the non-linear population growth model having a more recognizable form as follows:

$$\mathbf{C_{nt+1}} = \mathbf{rC_{nt}}\,(1 - \mathbf{C_{nt}}/\mathbf{k}) \qquad \ldots \text{(vi)}$$

Therefore, we take the conjectural position that very similar period-doubling limit cycles degenerating into chaos could explain some of the sudden "jumps" in cell population observed in malignancy when the standard linear models become drastically inadequate.

No linear classifier can identify a *chaotic attractor* if one is indeed operating as we surmise in the biochemical molecular dynamics of cell population growth. A non-linear and preferably non-parametric classifier is called for and for this very reason we have proposed artificial neural networks as a fundamental methodological building block here. Similar approach has paid off reasonably impressively in the case of complex systems modeling, especially with respect to weather forecasting and financial distress prediction.

Artificial neural networks primer:

Any artificial neural network is characterized by specifications on its *neurodynamics* and *architecture*. While neurodynamics refers to the input combinations, output generation, type of mapping function used and weighting schemes, architecture refers to the network configuration i.e. type and number of neuron interconnectivity and number of layers.

The input layer of an artificial neural network actually acts as a buffer for the inputs, as numeric data are transferred to the next layer. The output layer functions similarly except for the fact that the direction of dataflow is reversed. The transfer activation function is one that determines the output from the weighted inputs of a neuron by *mapping* the input data onto a suitable solution space. The output of neuron j after the summation of its weighted inputs from neuron 1 to i has been mapped by the transfer function f can be shown to be as follows:

$$\boldsymbol{O_j = f_j\ (\Sigma w_{ij} x_i)} \qquad \textbf{... (vii)}$$

A transfer function maps any real numbers into a domain normally bounded by 0 to 1 or –1 to 1. The most commonly used transfer functions are sigmoid, hypertan, and Gaussian.

A network is considered fully connected if the output from a neuron is connected to every other neuron in the next layer. A network may be *forward propagating* or *backward propagating* depending on whether outputs from one layer are passed unidirectionally to the succeeding or the preceding layer respectively. Networks working in closed loops are termed recurrent networks but the term is sometimes used interchangeably with backward propagating networks. Fully connected feed-forward networks are also called *multi-layer perceptrons* (*MLPs*) and as of now they are the most commonly used artificial neural network configuration. Our proposed artificial neural network classifier may also be conceptualized as a *recursive combination* of such *MLPs*.

Neural networks also come with something known as a hidden layer containing *hidden neurons* to deal with very complex, non-linear problems that cannot be resolved by merely the neurons in the input and output layers. There is no definite formula to determine the number of hidden layers required in a neural network set up. A useful heuristic approach would be to start with a small number of hidden layers with the numbers being allowed to increase gradually only if the learning is deemed inadequate. This should theoretically also address the regression problem of *over-fitting* i.e. the network performing very well with the training set data but poorly with the test set data. A neural network having no hidden layers at all basically becomes a *linear classifier* and is therefore statistically indistinguishable from the general linear regression model.

Model premises:

(1) The function governing the biochemical dynamics of cell population growth is inherently non-linear

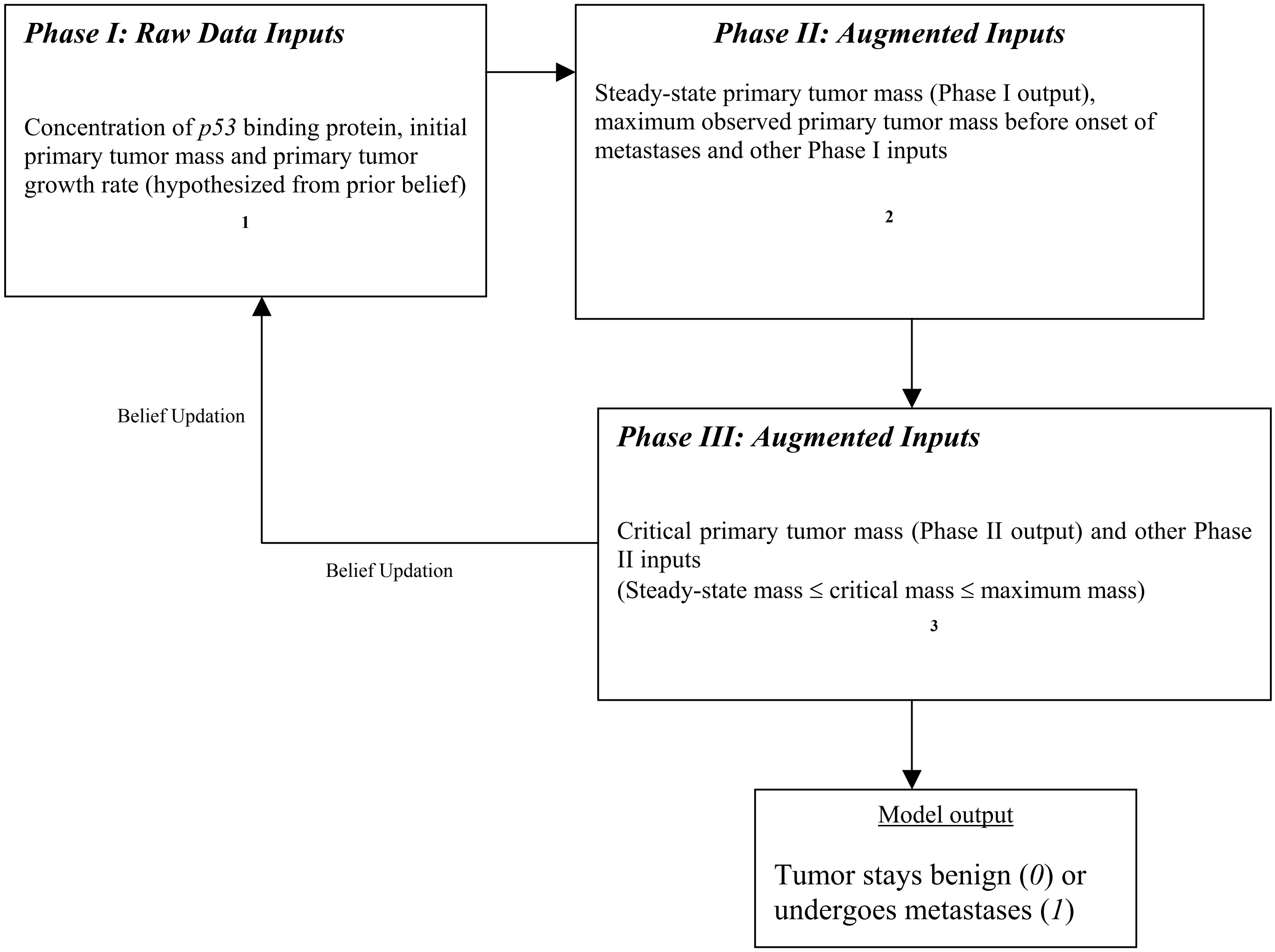
(2) The sudden and rapid degeneration of a benign cell growth to a malignant one may be attributed to an underlying chaotic attractor

(3) Given adequate training data, a non-linear binary classification technique such as that of Artificial Neural Networks can *learn* to detect this underlying chaotic attractor and thereby prove useful in predicting whether a benign cell growth may subsequently turn cancerous

Model structure:

We propose a nested approach where we treat the output generated by an earlier phase as an input in a latter phase. This will ensure that the artificial neural network virtually acts as a *knowledge-based system* as it takes its own predictions in the preceding phases into consideration as input data and tries to generate further predictions in succeeding phases.

This means that for a k-phase model, our set up will actually consist of k recursive networks having k phases such that the j^{th} phase will have input function $\mathbf{I_j = f \{O (p'_{j-1}), I (p_{j-1}), p_j\}}$, where the terms $\mathbf{O (p'_{j-1})}$ and $\mathbf{I (p_{j-1})}$ are the output and input functions of the previous phase and p_j is the vector of additional inputs for the j^{th} stage. The said recursive approach will have the following schema for a nested artificial neural network model with k = 3:



Phase I – target class variable: benign primary tumor mass

Phase II – target class variable: primary tumor mass at point of detection of malignancy

Phase III – target class variable: metastases (M) $\rightarrow$ 1, no metastases (B) $\rightarrow$ 0

As is apparent from the above schema, the model is intended to act as a sort of a *knowledge bank* that continuously keeps updating *prior* beliefs about tumor growth rate. The critical input variables are taken as concentration of p53 binding protein and observed tumor mass. The first one indicates the activity of the oncogenes vis-à-vis the tumor suppressors while the second one considers the extent of hyperplasia.

The model is proposed to be trained in phase I with histopathological data on concentration of p53 binding protein along with clinically observed data on tumor mass. The inputs and output of Phase I is proposed to be fed as input to Phase II along with additional clinical data on maximum tumor mass. The output and inputs of Phase 2 is finally to be fed into Phase III to generate the model output – a binary variable *M|B* that takes value of *1* if the tumor is predicted to metastasize or *0* otherwise. The recursive structure of the model is intended to pick up any underlying chaotic attractor that might be at work at the point where benign hyperplasia starts to degenerate into cancer. Issues regarding network configuration, learning rate, weighting scheme and mapping function are left open to experimentation. It is logical to start with a small number of hidden neurons and subsequently increase the number if the system shows inadequate learning.

Addressing the problem of training data unavailability:

While training a neural network, if no target class data is available, the complimentary class must be inferred by default. Training a network only on one class of inputs, with no counter-examples, causes the network to classify everything as the only class it has been shown. However, by training the network on randomly selected counter-examples during training can make it behave as a *novelty detector* in the test set. It will then pick up any deviation from the norm as an *abnormality*. For example, in our proposed model, if the clinical data for initially benign tumors subsequently turning malignant is unavailable, the network can be trained with the benign cases with random inputs of the malignant type so that it automatically picks up any deviation from the norm as a possible malignant case.

A mathematical justification for synthesizing unavailable training data with random numbers can be derived from the fact that network training seeks to minimize the sum squared of errors over the training set. In a binary classification scheme like the one we are interested in, where a single input k produces an output f (k), the desired outputs are 0 if the input is a benign tumor that has stayed benign (*B*) and 1 if the input is a benign tumor that has subsequently turned malignant (*M*). If the prior probability of any piece of data being a member of class *B* is P_B and that of class *M* is P_M; and if the probability distribution functions of the two classes expressed as functions of input k are p_B (k) and p_M (k), then the sum squared error, ε, over the entire training set will be given as follows:

$$\varepsilon = \int_{-\infty}^{\infty} \mathbf{P_B p_B (k)[f (k) - 0]^2 + P_M p_M (k)[f (k) - 1]^2 \, dk} \qquad \textbf{… (viii)}$$

Differentiating this equation with respect to the function f and equating to zero we get:

$$\partial\varepsilon/\partial \mathbf{f} = \mathbf{2p_B (k) P_B f (k) + 2p_M (k) P_M f (k) - 2p_M (k) P_M = 0 \text{ i.e.}}$$
$$\mathbf{f (k)^* = [p_M (k) P_M] / [p_B (k) P_B + p_M (k) P_M]} \qquad \textbf{… (ix)}$$

The above optimal value of f (k) is exactly the same as the probability of the correct classification being *M* given that the input was k. This shows that by training for minimization of sum squared error; and using as targets 0 for class *B* and 1 for class *M*, the output from the network converges to an identical value as the probability of class *M*.

Gazing upon the road ahead:

The main objective of our proposed model is to isolate truly benign tumors from those that initially start off as benign but subsequently show metastases. The non-parametric artificial neural network methodology has been chosen because of the analytical difficulties associated with extraction of closed-form stochastic likelihood parameters given the extremely complicated and possibly non-linear behavior of the state variables. This computational approach is proposed as a methodological alternative to the stochastic calculus techniques of tumor growth modeling commonly used in mathematical biology. Though how the approach actually performs with numerical data remains to be extensively tested, the proposed schema has been made as flexible as possible to suit most designed experiments to test its performance effectiveness and efficiency. In this paper we have just outlined a research approach – we shall test it out in a subsequent one.

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Utility of Choice: Information Theoretic Approach to Investment Decision-Making

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Abstract:

In this paper we have devised an alternative methodological approach for quantifying utility in terms of expected information content of the decision-maker's choice set. We have proposed an extension to the concept of utility by incorporating extrinsic utility; which we have defined as the utility derived from the element of choice afforded to the decision-maker by the availability of an object within his or her object set. We have subsequently applied this extended utility concept to the case of investor utility derived from a structured, financial product – an custom-made investment portfolio incorporating an endogenous capital-guarantee through inclusion of cash as a risk-free asset, based on the Black-Scholes derivative-pricing formulation. We have also provided instances of potential application of information and coding theory in the realms of financial decision-making with such structured portfolios, in terms of transmission of product information.

Key words: Utility theory, constrained optimization, entropy, Shannon-Fano information theory, structured financial products

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Introduction:

In early nineteenth century most economists conceptualized utility as a psychic reality – cardinally measurable in terms of *utils* like distance in kilometers or temperature in degrees centigrade. In the later part of nineteenth century Vilfredo Pareto discovered that all the important aspects of demand theory could be analyzed ordinally using geometric devices, which later came to be known as "indifference curves". The indifference curve approach effectively did away with the notion of a cardinally measurable utility and went on to form the methodological cornerstone of modern microeconomic theory.

An indifference curve for a two-commodity model is mathematically defined as the locus of all such points in $\boldsymbol{E^2}$ where different combinations of the two commodities give the same level of satisfaction to the consumer so as the consumer is indifferent to any particular combination. Such indifference curves are always *convex to the origin* because of the operation of the *law of substitution*. This law states that the scarcer a commodity becomes, the greater becomes its relative substitution value so that its marginal utility rises relative to the marginal utility of the other commodity that has become comparatively plentiful.

In terms of the indifference curves approach, the problem of utility maximization for an individual consumer may be expressed as a constrained non-linear programming problem that may be written in its general form for an n-commodity model as follows:

$$\begin{aligned} &\textbf{Maximize } U = U(C_1, C_2 \ldots C_n) \\ &\textbf{Subject to } \Sigma\, C_jP_j \leq B \\ &\textbf{and } C_j \geq 0, \textbf{ for } j = 1, 2 \ldots n \end{aligned} \qquad \textbf{(1)}$$

If the above problem is formulated with a strict equality constraint i.e. if the consumer is allowed to use up the entire budget on the n commodities, then the utility maximizing condition of consumer's equilibrium is derived as the following first-order condition:

$$\partial U/\partial C_j = (\partial U/\partial C_j) - \lambda P_j = 0 \text{ i.e.}$$

$$(\partial U/\partial C_j)/P_j = \lambda^* = \text{constant, for } j = 1, 2 \ldots n \quad (2)$$

This pertains to the classical economic theory that in order to maximize utility, individual consumers necessarily must allocate their budget so as to *equalize the ratio of marginal utility to price* for every commodity under consideration, with this ratio being found equal to the optimal value of the *Lagrangian multiplier* λ^*.

However a rather necessary pre-condition for the above indifference curve approach to work is $(U_{C1}, U_{C2} \ldots U_{Cn}) > 0$ i.e. the marginal utilities derived by the consumer from each of the n commodities must be positive. Otherwise of course the problem degenerates. To prevent this from happening one needs to strictly adhere to the law of substitution under all circumstances. This however, at times, could become an untenable proposition if measure of utility is strictly restricted to an intrinsic one. This is because, for the required condition to hold, each of the n commodities necessarily *must always have a positive intrinsic utility* for the consumer. However, this would invariably lead to anomalous reasoning like the intrinsic utility of a woolen jacket being independent of the temperature or the intrinsic utility of an umbrella being independent of rainfall.

Choice among alternative courses of action consist of trade-offs that confound subjective probabilities and marginal utilities and are almost always too coarse to allow for a meaningful separation of the two. From the viewpoint of a classical statistical decision theory like that of *Bayesian inference* for example, failure to obtain a correct representation of the underlying behavioral basis would be considered a major pitfall in the aforementioned analytical framework.

Choices among alternative courses of action are largely determined by the relative degrees of belief an individual attaches to the prevailing uncertainties. Following Vroom (Vroom; 1964), the *motivational strength* S_n of choice c_n among N alternative available choices from the choice set $C = \{c_1, c_2 \ldots c_N\}$ may be ranked with respect to the multiplicative product of the relative reward $r(c_n)$ that the individual attaches to the consequences resulting from the choice c_n, the likelihood that the choice set under consideration will yield a positive intrinsic utility and the respective probabilities p{r (c_n)} associated with r (c_n) such that:

$$\mathbf{S_{max} = Max_n\, [r\,(c_n)\ x\ p\,(U_{r(C)} > 0)\ x\ p\{r\,(c_n)\}],\ n = 1, 2 \ldots N} \qquad \mathbf{(3)}$$

Assuming for the time-being that the individual is calibrated with perfect certainty with respect to the intrinsic utility resulting from a choice set such that we have the condition $\mathbf{p\,(U_{r(C)} > 0) = \{0, 1\}}$, the above model can be reduced as follows:

$$\mathbf{S_{max} = Max_k\, [r\,(c_k)\ x\ p\{r\,(c_k)\}],\ k = 1, 2 \ldots K} \text{ such that } K < N \qquad \mathbf{(4)}$$

Therefore, choice A, which entails a large reward with a low probability of the reward being actualized could theoretically yield the same motivational strength as choice B, which entails a smaller reward with a higher probability of the reward being actualized.

However, we recognize the fact that the *information conveyed* to the decision-maker by the outcomes would be quite different for A and B though their values may have the same mathematical expectation. Therefore, whereas intrinsic utility could explain the ranking with respect to *expected value* of the outcomes, there really has to be another dimension to utility whereby the ***expected information*** is considered – that of *extrinsic utility*. So, though there is a very low probability of having an unusually cold day in summer, the information conveyed to the likely buyer of a woolen jacket by occurrence of such an aberration in the weather pattern would be quite substantial, thereby validating a extended substitution law based on an *expected information measure* of utility. The specific objective of this paper is to formulate a mathematically sound theoretical edifice for the formal induction of extrinsic utility into the folds of statistical decision theory.

A few essential working definitions

Object: Something with respect to which an individual may perform a specific goal-oriented behavior

Object set: The set O of a number of different objects available to an individual at any particular point in space and time with respect to achieving a goal where n {O} = K

Choice: A path towards the sought goal emanating from a particular course of action - for a single available object within the individual's object set, there are two available choices - either the individual takes that object or he or she does not take that object. Therefore, generalizing for an object set with K alternative objects, there can be 2^K alternative courses of action for the individual

Choice set: The set C of all available choices where C = ***P*** O, n {C} = 2^K

Outcome: The relative reward resulting from making a particular choice

Decision-making is nothing but goal-oriented behavior. According to the celebrated *theory of reasoned action* (Fishbain; 1979), the immediate determinant of human behavior is the intention to perform (or not to perform) the behavior. For example, the simplest way to determine whether an individual will invest in Acme Inc. equity shares is to ask whether he or she *intends* to do so. This does not necessarily mean that there will always be a perfect relationship between intension and behavior. However, there is no denying the fact that people usually tend to act in accordance with their intensions.

However, though intention may be shaped by a positive intrinsic utility expected to be derived from the outcome of a decision, the *ability* of the individual to actually act according to his or her intention also needs to be considered. For example, if an investor truly intends to buy a call option on the equity stock of Acme Inc. even then his or her intention cannot get translated into behavior if there is no exchange-traded call option available on that equity stock. Thus we may view the *additional element of choice* as a measure of extrinsic utility. ***Utility is not only to be measured by the intrinsic want-satisfying capacity of a commodity for an intending individual but also* by *the availability of the particular commodity at that point in space and time to enable that individual to act according to his or her intension***. Going back to our woolen jacket example, though the intrinsic utility of such a garment in summer is practically zero, the extrinsic utility afforded by its mere availability can nevertheless suffice to uphold the law of substitution.

Utility and thermodynamics

In our present paper we have attempted to extend the classical utility theory applying the entropy measure of information (Shannon, 1948), which by itself bears a direct constructional analogy to the ***Boltzmann equation*** in thermodynamics. There is some uniformity in views among economists as well as physicists that a functional correspondence exists between the formalisms of economic theory and classical thermodynamics. The laws of thermodynamics can be intuitively interpreted in an economic context and the correspondences do show that thermodynamic entropy and economic utility are related concepts sharing the same formal framework. Utility is said to arise from that component of thermodynamic entropy whose change is due to irreversible transformations. This is the standard *Carnot entropy* given by $\mathbf{dS = \delta Q/T}$ where S is the entropy measure, Q is the thermal energy of state transformation (irreversible) and T is the absolute temperature. In this paper however we will keep to the information theoretic definition of entropy rather than the purely thermodynamic one.

Underlying premises of our extrinsic utility model

1. Utility derived from making a choice can be distinctly categorized into two forms:

   (a) Intrinsic utility ($U_{r(C)}$) – the intrinsic, non-quantifiable capacity of the potential outcome from a particular choice set to satisfy a particular human want under given circumstances; in terms of expected utility theory $U_{r\,(C)} = \Sigma\, r\,(c_j)\, p\{r\,(c_j)\}$, where j = 1, 2 … K and

   (b) Extrinsic utility (U_X) – the additional possible choices afforded by the mere availability of a specific object within the object set of the individual

2. An choice set with n (C) = 1 (i.e. when K = 0) with respect to a particular individual corresponds to lowest (zero) extrinsic utility; so U_X cannot be negative

3. The law of diminishing marginal utility tends to hold in case of U_x when an individual repeatedly keeps making the same choice to the exclusion of other available choices within his or her choice set

Expressing the frequency of alternative choices in terms of the probability of getting an outcome r_j by making a choice c_j, the generalized extrinsic utility function can be framed as a modified version of *Shannon's entropy function* as follows:

$$\mathbf{U_X = - K \Sigma_j p \{r (c_j)\} \log_2 p \{r (c_j)\}, j = 1, 2 \ldots 2^K} \qquad \mathbf{(5)}$$

The multiplier -K = -n (O) is a scale factor somewhat analogous to the ***Boltzmann constant*** in classical thermodynamics with a reversed sign. Therefore general extrinsic utility maximization reduces to the following non-linear programming problem:

$$\begin{aligned} &\textbf{Maximize } \mathbf{U_X = - K \Sigma_j p \{r (c_j)\} \log_2 p \{r (c_j)\}} \\ &\textbf{Subject to } \mathbf{\Sigma p \{r (c_j)\} = 1,} \\ &\mathbf{p \{r (c_j)\} \geq 0;} \textbf{ and} \\ &\mathbf{j = 1, 2 \ldots 2^K} \end{aligned} \qquad \mathbf{(6)}$$

Putting the objective function into the usual Lagrangian multiplier form, we get

$$\mathbf{Z = - K \Sigma p \{r (c_j)\} \log_2 p \{r (c_j)\} + \lambda (\Sigma p \{r (c_j)\} - 1)} \qquad \mathbf{(7)}$$

Now, as per the first-order condition for maximization, we have

$$\mathbf{\partial Z / \partial p \{r (c_j)\} = - K (\log_2 p \{r (c_j)\} + 1) + \lambda = 0} \textbf{ i.e.}$$

$$\mathbf{\log_2 p \{r (c_j)\} = \lambda / K - 1} \qquad \mathbf{(8)}$$

Therefore; for a pre-defined K; p $\{r(c_j)\}$ is independent of j, i.e. all the probabilities are necessarily equalized to the constant value $\mathbf{p\{r(c_j)\}^* = 2^{-K}}$ at the point of maximum U_X.

It is also intuitively obvious that when p $\{r(c_j)\} = 2^{-K}$ for j = 1, 2, … 2^K, the individual has the *maximum element of choice in terms of the available objects within his or her object set*. For a choice set with a single available choice, the extrinsic utility function will be simply given as $\mathbf{U_X = -p\{r(c)\} \log_2 p\{r(c)\} - (1 - p\{r(c)\}) \log_2 (1 - p\{r(c)\})}$. Then the slope of the marginal extrinsic utility curve will as usual be given by $\mathbf{d^2U_X/dp\{r(c)\}^2 < 0}$, and this can additionally serve as an alternative basis for intuitively deriving the generalized, downward-sloping demand curve and is thus a valuable theoretical spin-off!

Therefore, though the mathematical expectation of a reward resulting from two mutually exclusive choices may be the same thereby giving them equal rank in terms of the intrinsic utility of the expected reward, the expected information content of the outcome from the two choices will be quite different given different probabilities of getting the relative rewards. The following vector will then give a *composite measure of total expected utility* from the object set:

$$\mathbf{U = [U_r, U_X] = [\Sigma r(c_j)\, p\{r(c_j)\}, -K\, \Sigma_j\, p\{r(c_j)\} \log_2 p\{r(c_j)\}],\ j = 1, 2 \ldots 2^K} \qquad \mathbf{(9)}$$

Now, having established the essential premise of formulating an extrinsic utility measure, we can proceed to let go of the assumption that an individual is calibrated with perfect certainty about the intrinsic utility resulting from the given choice set so that we now look at the full Vroom model rather than the reduced version. If we remove the restraining condition that $\mathbf{p(U_{r(C)} > 0) = \{0, 1\}}$ and instead we have the more general case of $\mathbf{0 \leq p(U_{r(C)} > 0) \leq 1}$, then we introduce another probabilistic dimension to our choice set whereby the individual is no longer certain about the nature of the impact the outcomes emanating from a specific choice will have on his intrinsic utility. This can be intuitively interpreted in terms of the *likely opportunity cost* of making a choice from within a given choice set to the exclusion of all other possible choice sets. For the particular choice set C, if the likely opportunity cost is less than the potential reward obtainable, then $\mathbf{U_{r\,(c)} > 0}$, if opportunity cost is equal to the potential reward obtainable, then $\mathbf{U_{r(C)} = 0}$, else if the opportunity cost is greater than the potential reward obtainable then $\mathbf{U_{r\,(C)} < 0}$.

Writing $\mathbf{U_{r(C)} = \Sigma_j\ r\ (c_j)\ p\{r\ (c_j)\}}$, j = 1, 2 … N, the *total expected utility vector* now becomes:

$$[\mathbf{U_{r(C)}, U_X}] = [\mathbf{\Sigma_j\ r\ (c_j)\ p\{r\ (c_j)\}, -K\ \Sigma\ p\ \{r\ (c_j) | U_{r(C)} > 0\}\ log_2\ p\ \{r\ (c_j) | U_{r(C)} > 0\}}],\ j = 1, 2 \dots N \qquad \mathbf{(10)}$$

Here $\mathbf{p\ \{r\ (c_j) | U_{r(C)} > 0\}}$ may be estimated by the standard ***Bayes criterion*** as under:

$$\mathbf{p\ \{r\ (c_j) | U_{r(c)} > 0\} = [p\ \{(U_{r(C)} \geq 0 | r\ (c_j)\}\ p\ \{(r\ (c_j)\}][\Sigma_j\ p\ \{(U_{r(C)} > 0 | r\ (c_j)\}\ p\ \{(r\ (c_j)\}]^{-1}} \qquad \mathbf{(11)}$$

A practical application in the realms of Behavioral Finance - Evaluating an investor's extrinsic utility from capital-guaranteed, structured financial products

Let a structured financial product be made up of a basket of n different assets such that the investor has the right to claim the return on the best-performing asset out of that basket after a stipulated holding period. Then, if one of the n assets in the basket is the risk-free asset then the investor gets assured of a minimum return equal to the risk-free rate i on his invested capital at the termination of the stipulated holding period. This effectively means that his or her investment becomes endogenously capital-guaranteed as the terminal wealth, even at its worst, cannot be lower in value to the initial wealth plus the return earned on the risk-free asset minus a finite cost of ***portfolio insurance***.

Therefore, with respect to each risky asset, we can have a *binary response from the investor in terms of his or her funds-allocation decision* whereby the investor either takes funds out of an asset or puts funds into an asset. Since the overall portfolio has to be *self-financing* in order to pertain to a ***Black-Scholes*** kind of pricing model, funds added to one asset will also mean same amount of funds removed from one or more of the other assets in that basket. If the basket consists of a single risky asset s (and of course cash as the risk-free asset) then, if η_s is the amount of re-allocation effected each time with respect to the risky asset s, the two alternative, mutually exclusive choices open to the investor with respect to the risky asset s are as follows:

(1) **C ($\eta_s \geq 0$)** (funds left in asset s), with associated outcome **r ($\eta_s \geq 0$)**; and

(2) **C ($\eta_s < 0$)** (funds removed from asset s), with associated outcome **r ($\eta_s < 0$)**

Therefore what the different assets are giving to the investor apart from their intrinsic utility in the form of higher expected terminal reward is some *extrinsic utility in the form of available re-allocation options*. Then the expected present value of the final return is given as follows:

$$\mathbf{E\ (r) = Max\ [w, Max_{j}\ \{e^{-it}\ E\ (r_j)_t\}],\ j = 1, 2 \ldots 2^{n-1}} \qquad \mathbf{(12)}$$

In the above equation i is the rate of return on the risk-free asset and t is the length of the investment horizon in continuous time and w is the initial wealth invested i.e. ignoring insurance cost, if the risk-free asset outperforms all other assets $\mathbf{E\ (r) = we^{it}/e^{it} = w}$.

Now what is the probability of each of the (n – 1) risky assets performing worse than the risk-free asset? Even if we assume that there are some cross-correlations present among the (n – 1) risky assets, given the statistical nature of the risk-return trade-off the joint probability of these assets performing worse than the risk-free asset will be very low over moderately long investment horizons. And this probability will keep going down with every additional risky asset added to the basket. Thus each additional asset will *empower the investor* with additional choices with regards to re-allocating his or her funds among the different assets according to their observed performances.

Intuitively we can make out that the extrinsic utility to the investor is indeed maximized when there is *an equal positive probability of actualizing each outcome r_j resulting from*

η_j *given that the intrinsic utility* $U_{r(C)}$ *is greater than zero.* By a purely economic rationale, *each additional asset introduced into the basket will be so introduced if and only if it significantly raises the expected monetary value of the potential terminal reward*. As already demonstrated, the extrinsic utility maximizing criterion will be given as under:

$$\mathbf{p\ (r_j \mid U_{r(C)} > 0)^* = 2^{-(n-1)}} \text{ for } j = 1, 2 \ldots 2^{n-1} \qquad \textbf{(13)}$$

The *composite utility vector from the multi-asset structured product* will be as follows:

$$\mathbf{[U_{r(C)}, U_X] = [E\ (\ r\ ), -\ (n-1)\Sigma\ p\ \{r_j \mid U_{r(C)} > 0\}\ \log_2 p\ \{r_j \mid U_{r(C)} > 0\}],\ j = 1, 2 \ldots 2^{n-1}} \qquad \textbf{(14)}$$

<u>Choice set with a structured product having two risky assets (and cash):</u>

0	0
1	0
0	1
1	1

That is, the investor can remove all funds from the two risky assets and convert it to cash (the risk-free asset), or the investor can take funds out of asset 2 and put it in asset 1, or the investor can take funds out of asset 1 and put it in asset 2, or the investor can convert some cash into funds and put it in both the risky assets. Thus there are 4 alternative choices for the investor when it comes to re-balancing his portfolio.

<u>Choice set with a structured product having three risky assets (and cash):</u>

0	0	0
0	0	1
0	1	0

0	1	1
1	0	0
1	0	1
1	1	0
1	1	1

That is, the investor can remove all funds from the three risky assets and convert it into cash (the risk-free asset), or the investor can take funds out of asset 1 and asset 2 and put it in asset 3, or the investor can take funds out from asset 1 and asset 3 and put it in asset 2, or the investor can take funds out from asset 2 and asset 3 and put it in asset 1, or the investor can take funds out from asset 1 and put it in asset 2 and asset 3, or the investor can take funds out of asset 2 and put it in asset 1 and asset 3, or the investor can take funds out of asset 3 and put it in asset 1 and asset 2, or the investor can convert some cash into funds and put it in all three of the assets. Thus there are 8 alternative choices for the investor when it comes to re-balancing his portfolio.

Of course, according to the Black-Scholes hedging principle, the re-balancing needs to be done each time by setting the optimal proportion of funds to be invested in each asset equal to the partial derivatives of the option valuation formula w.r.t. each of these assets. However, the total number of alternative choices available to the investor increases with every new risky asset that is added to the basket thereby contributing to the extrinsic utility in terms of the expected information content of the total portfolio.

Coding of product information about multi-asset, structured financial portfolios

Extending the entropy measure of extrinsic utility, we may conceptualize the interaction between the buyer and the vendor as a two-way communication flow whereby the vendor *informs* the buyer about the expected utility derivable from the product on offer and the buyer *informs* the seller about his or her individual expected utility criteria. An economic transaction goes through if the two sets of information are compatible. Of course, the greater expected information content of the vendor's communication, the higher is the extrinsic utility of the buyer. Intuitively, the expected information content of the vendor's communication will increase with increase in the variety of the product on offer, as that will increase the likelihood of matching the buyer's expected utility criteria.

The product information from vendor to potential buyer may be transferred through some medium e.g. the vendor's website on the Internet, a targeted e-mail or a telephonic promotion scheme. But such transmission of information is subject to noise and distractions brought about by environmental as well as psycho-cognitive factors. While a ***distraction*** is prima facie predictable, (e.g. the pop-up windows that keep on opening when some commercial websites are accessed), ***noise*** involves unpredictable perturbations (e.g. conflicting product information received from any competing sources).

Transmission of information calls for some kind of *coding*. Coding may be defined as a mapping of words from a source alphabet A to a code alphabet B. A ***discrete, finite memory-less channel*** with finite inputs and output alphabets is defined by a set of transition probabilities p_i (j), i = 1, 2 … a and j = 1,2 … b with Σ_j p_i (j) = 1 and pi (j) ≥ 0. Here p_i (j) is the probability that for an input letter i output letter j will be received.

A ***code word*** of length n is defined as a sequence of n input letters which are actually n integers chosen from 1,2 … a. A ***block code*** of length n having M words is a mapping of the message integers from 1 to M into a set of code words each having a fixed length n. Thus for a structured product with N component assets, a block code of length n having N words would be used to map message integers from 1 to N, corresponding to each of the N assets, into a set of a fixed-length code words. Then there would be a total number of $C = 2^N$ possible combinations such that $\log_2 C = N$ binary-state devises (flip-flops) would be needed.

A ***decoding system*** for a block code is the inverse mapping of all output words of length n into the original message integers from 1 to M. Assuming all message integers are used with same probability 1/M, the probability of error P_e for a code and decoding system ensemble is defined as the probability of an integer being transmitted and received as a word which is mapped into another integer i.e. P_e is the probability of wrongly decoding a message.

Therefore, in terms of our structured product set up, P_e might be construed as the probability of misclassifying the best performing asset. Say within a structured product consisting of three risky assets - a blue-chip equity portfolio, a market-neutral hedge fund and a commodity future (and cash as the risk-free asset), while the original transmitted information indicates the hedge fund to be the best performer, due to erroneous decoding of the encoded message, the equity portfolio is interpreted as the best performer. Such erroneous decoding could result in investment funds being allocated to the wrong asset at the wrong time.

The relevance of Shannon-Fano coding to product information transmission

By the well-known Kraft's inequality we have $\mathbf{K = \Sigma_n 2^{-li} \leq 1}$, where l_i stands for some definite code word lengths with a radix of 2 for binary encoding. For block codes, $l_i = l$ for i = 1, 2 … n. As per ***Shannon's coding theorem***, *it is possible to encode all sequences of n message integers into sequences of binary digits in such a way that the average number of binary digits per message symbol is approximately equally to the entropy of the source, the approximation increasing in accuracy with increase in n*. For efficient binary codes, K = 1 i.e. $\log_2 K = 0$ as it corresponds to the maximal entropy condition. Therefore the inequality occurs if and only if $\mathbf{p_i \neq 2^{-li}}$. Though the Shannon-Fano coding scheme is not strictly the most efficient, it has the advantage of directly deriving the code word length l_i from the corresponding probability p_i. With source symbols $s_1, s_2 \ldots s_n$ and their corresponding probabilities $p_1, p_2 \ldots p_n$, where for each p_i there is an integer l_i, then given that we have bounds that span an unit length, we have the following relationship:

$$\mathbf{\log_2 (p_i^{-1}) \leq l_i < \log_2 (p_i^{-1}) + 1} \qquad \mathbf{(15)}$$

Removing the logs, taking reciprocals and summing each term we therefore get,

$\mathbf{\Sigma_n\, p_i \geq \Sigma_n\, 2^{li} \geq p_i/2}$, that is,

$$\mathbf{1 \geq \Sigma_n\, 2^{li} \geq ½} \qquad \textbf{(16)}$$

Inequality (16) gets us back to the Kraft's inequality. This shows that there is an instantaneously decodable code having the Shannon-Fano lengths l_i. By multiplying inequality (15) by p_i and summing we get:

$\mathbf{\Sigma_n\, (p_i \log_2 p_i^{-1}) \leq \Sigma_n\, p_i l_i < \Sigma_n\, (p_i \log_2 p_i^{-1}) + 1}$, that is,

$$\mathbf{H_2\,(S) \leq L \leq H_2\,(S) + 1} \qquad \textbf{(17)}$$

That is, in terms of the average Shannon-Fano code length L, we have conditional entropy as an effective lower bound while it is also the non-integral component of the upper bound of L. This underlines the relevance of a Shannon-Fano form of coding to our structured product formulation as this implies that the average code word length used in this form of product information coding would be ***bounded by a measure of extrinsic utility*** to the potential investor of the structured financial product itself, which is definitely an intuitively appealing prospect.

Conceptualizing product information transmission as a Markov process

The Black-Scholes option-pricing model is based on the underlying assumption that asset prices evolve according to the geometric diffusion process of a Brownian motion. The Brownian motion model has the following fundamental assumptions:

(1). $W_0=0$

(2). W_t-W_s is a random variable that is normally distributed with mean 0 and variance t-s

(3). W_t-W_s is independent of W_v-W_u if (s, t) and (u, v) are non-overlapping time intervals

Property (3) implies that the Brownian motion is a *Markovian process* with no long-term memory. The switching behavior of asset prices from “high” (Bull state) to “low” (Bear state) and vice versa according to Markovian transition rule constitutes a well-researched topic in stochastic finance. It has in fact been proved that a steady-state equilibrium exists when the state probabilities are equalized for a stationary transition-probability matrix (Bhattacharya, 2001). This steady-state equilibrium corresponds to the condition of strong efficiency in the financial markets whereby no historical market information can result in arbitrage opportunities over any significant length of time.

By logical extension, considering a structured portfolio with n assets, the best performer may be hypothesized to be traceable by a first-order Markov process, whereby the best performing asset at time t+1 is dependent on the best performing asset at time t. For example, with n = 3, we have the following state-transition matrix:

	Asset 1	Asset 2	Asset 3
Asset 1	P (1 | 1)	P (2 | 1)	P (3 | 1)
Asset 2	P (2 | 1)	P (2 | 2)	P (3 | 2)
Asset 3	P (3 | 1)	P (3 | 2)	P (3 | 3)

In information theory also, a similar Markov structure is used to improve the encoding of a source alphabet. For each state in the Markov system, an appropriate code can be obtained from the corresponding transition probabilities of leaving that state. The efficiency gain will depend on how variable the probabilities are for each state. However, as the order of the Markov process is increased, the gain will tend to be less and less while the number of attainable states approach infinity.

The strength of the Markov formulation lies in its capacity of handling correlation between successive states. If S_1, S_2 … S_m are the first m states of a stochastic variable, what is the probability that the next state will be S_i? This is written as the conditional probability p (S_i | S_1, S_2 … S_m). Then, the Shannon measure of information from a state S_i is given as usual as follows:

$$\mathbf{I\ (S_i \mid S_1, S_2 \ldots S_m) = log_2\ \{p\ (S_i \mid S_1, S_2 \ldots S_m)\}^{-1}} \qquad \mathbf{(17)}$$

The entropy of a Markov process is then derived as follows:

$$\mathbf{H\ (S) = \sum_{S^{m+1}} p\ (S_1, S_2 \ldots S_m, S_i)\ I\ (S_i \mid S_1, S_2 \ldots S_m)} \qquad \mathbf{(18)}$$

Then the extrinsic utility to an investor from a structured financial product expressed in terms of the entropy of a Markov process governing the state-transition of the best performing asset over N component risky assets (and cash as the one risk-free asset) within the structured portfolio would be given as follows:

$$\mathbf{U_x = H\ (Portfolio) = \sum_{S^{N+1}} p\ (S_1, S_2 \ldots S_m, S_i)\ I\ (S_i \mid S_1, S_2 \ldots S_m)} \qquad \mathbf{(19)}$$

However, to find the entropy of a Markov source alphabet one needs to explicitly derive the stationary probabilities of being in each state of the Markov process. But these state probabilities may be hard to derive explicitly especially if there are a large number of allowable states (e.g. corresponding to a large number of elementary risky assets within a structured financial product). Using ***Gibbs inequality***, it can be show that the following limit can be imposed for bounding the entropy of the Markov process:

$\mathbf{\Sigma_j\ p\ (S_j)\ H\ (Portfolio \mid S_j) \leq H\ (S^*)}$, where **H (S*)** is termed the ***adjoint system*** **(20)**

The entropy of the original message symbols given by the zero memory source adjoint system with $\mathbf{p}\ (\mathbf{S_i}) = \mathbf{p_i}$ bound the entropy of the Markov process. The equality holds if and only if $\mathbf{p}\ (\mathbf{S_j}, \mathbf{S_i}) = \mathbf{p_j p_i}$ that is, in terms of the structured portfolio set up, the equality holds if and only if the joint probability of the best performer being the pair of assets i and j is equal to the product of their individual probabilities (Hamming, 1986). Thus a clear analogical parallel may be drawn between Markovian structure of the coding process and performances of financial assets contained within a structured investment portfolio.

Conclusion and scope for future research

In this paper we have basically outlined a novel methodological approach whereby expected information measure is used as a measure of utility derivable from a basket of commodities. We have illustrated the concepts with an applied finance perspective whereby we have used this methodological approach to derive a measure of investor utility from a structured financial portfolio consisting of many elementary risky assets combined with cash as the risk-free asset thereby giving the product a quasi - capital guarantee status. We have also borrowed concepts from mathematical information theory and coding to draw analogical parallels with the utility structures evolving out of multi-asset, structured financial products. In particular, principles of Shannon-Fano coding have been applied to the coding of product information for transmission from vendor (fund manager) to the potential buyer (investor). Finally we have dwelled upon the very similar Markovian structure of coding process and that of asset performances.

This paper in many ways is a curtain raiser on the different ways in which tools and concepts from mathematical information theory can be applied in utility analysis in general and to analyzing investor utility preferences in particular. It seeks to extend the normal peripheries of utility theory to a new domain – that of information theoretic utility. Thus a cardinal measure of utility is proposed in the form of the Shannon-Boltzmann entropy measure. Being a new methodological approach, the scope of future research is boundless especially in exploring the analogical Markovian properties of asset performances and message transmission and devising an efficient coding scheme to represent the two-way transfer of utility information from vendor to buyer and vice versa. The mathematical kinship between neoclassical utility theory and classical thermodynamics is also worth exploring, may be aimed at establishing some higher-dimensional, theoretical connectivity between the isotherms and the indifference curves!

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